SCHRIFTENREIHE DER FAKULTÄT FÜR MATHEMATIK

The Index of Some Mixed Order Dirac-Type Operators and Generalised Dirichlet-Neumann Tensor Fields

by

Dirk Pauly & Marcus Waurick

SM-UDE-823 2020

Received: June 8, 2020

THE INDEX OF SOME MIXED ORDER DIRAC-TYPE OPERATORS AND GENERALISED DIRICHLET-NEUMANN TENSOR FIELDS

DIRK PAULY AND MARCUS WAURICK

Dedicated to the Captain

Abstract. We revisit a construction principle of Fredholm operators using Hilbert complexes of densely defined, closed linear operators and apply this to particular choices of differential operators. The resulting index is then computed with the help of explicitly describing the dimension of the cohomology groups of generalised harmonic Dirichlet and Neumann tensor fields. The main results of this contribution are to compute the indices of the Dirac-type operators associated to the elasticity complex and the newly found biharmonic complex, relevant for the biharmonic equation, elasticity, and in the theory of general relativity. The differential operators are of mixed order and cannot be seen as leading order type with relatively compact perturbation. As a side product we present a comprehensive description of the underlying generalised 'harmonic' Dirichlet-Neumann vector and tensor fields defining the respective cohomology groups, including their dimensions and an explicit construction of bases in terms of topological invariants, which are of both analytical and numerical interest. For this we follow in close lines the work of Rainer Picard [23].

Contents

1. Introduction 2
2. The Construction Principle and the Index Theorem 7
2.1. Some More Results 9
3. The de Rham Complex and Its Indices 13
3.1. Picard's Extended Maxwell System 13
3.2. Some More Results 15
3.3. The Dirac Operator 17
4. The First Biharmonic Complex and Its Indices 18
4.1. Some More Results 20
5. The Second Biharmonic Complex and Its Indices 21
5.1. Some More Results 23
6. The Elasticity Complex and Its Indices 25
6.1. Some More Results 26
7. Conclusion 28

Date: June 8, 2020; *Corresponding Author*: Dirk Pauly.

2010 *Mathematics Subject Classification. Primary*: 47A53 (Fredholm operators; index theories), 58B15 (Fredholm structures); *Secondary*: 35G15 (Boundary value problems for linear higher-order equations), 35Q61 (Maxwell equations), 35Q41 (Time-dependent Schrödinger equations, Dirac equations), 35Q40 (PDEs in connection with quantum mechanics).

Key words and phrases. Dirac Operator, Picard's Extended Maxwell System, Fredholm Index, Cohomology, Hilbert Complex, Elasticity Complex, biharmonic Complex, Harmonic Dirichlet and Neumann Tensors.

Achnowledgements: We cordially thank Walter Zulehner, JKU Linz, for many inspiring and fruitful discussions and kind help related to the Poincaré maps in the Appendix.

References 28
Appendix. Dirichlet and Neumann Fields 29
Appendix A. Dirichlet Fields 30
A.1. Dirichlet Vector Fields of the Classical de Rham Complex 31
A.2. Dirichlet Tensor Fields of the First Biharmonic Complex 32
A.3. Dirichlet Tensor Fields of the Second Biharmonic Complex 35
A.4. Dirichlet Tensor Fields of the Elasticity Complex 37
Appendix B. Neumann Fields 39
B.1. Neumann Vector Fields of the Classical de Rham Complex 40
B.2. Neumann Tensor Fields of the First Biharmonic Complex 42
B.3. Neumann Tensor Fields of the Second Biharmonic Complex 50
B.4. Neumann Tensor Fields of the Elasticity Complex 57

1. Introduction

This article is concerned with the explicit computation of the Fredholm index if a differential operator is 'apparently' of mixed order. More precisely, we shall establish a collection of theorems like the following:

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^3$ be open, bounded with strong*[1] *Lipschitz boundary. Then there exists a subspace $\mathcal{V} \subseteq L^{2,3\times3}_{\mathbb{T}}(\Omega) \times L^2(\Omega)$ such that*

$$\mathcal{D} := \begin{pmatrix} \operatorname{Div} & 0 \\ \operatorname{symCurl} & \operatorname{Gradgrad} \end{pmatrix} : \mathcal{V} \subseteq L^{2,3\times3}_{\mathbb{T}}(\Omega) \times L^2(\Omega) \to L^{2,3}(\Omega) \times L^{2,3\times3}_{\mathbb{S}}(\Omega)$$

and $\mathcal{D}^*$ are densely defined and closed Fredholm operators, where $L^{2,3\times3}_{\mathbb{T}}(\Omega)$ and $L^{2,3\times3}_{\mathbb{S}}(\Omega)$ denote the sets of trace free and symmetric 3×3 matrices with entries in $L^2(\Omega)$, respectively. Moreover,

$$\operatorname{ind}\mathcal{D} = 4(p - m - n + 1), \qquad \operatorname{ind}\mathcal{D}^* = -\operatorname{ind}\mathcal{D}$$

where n is the number of connected components of Ω, m is the number of connected components of its complement $\mathbb{R}^3 \setminus \Omega$, and p is the number of handles, see Definition 3.5 and Assumption 3 for the precise notion.

In the course of the manuscript, we shall describe the subspace $\mathcal{V} = \operatorname{dom}\mathcal{D}$ explicitly, see Theorem 4.4 and Remark 4.5. A refined notation will indicate (full) natural boundary conditions by $\,^\circ\,$ and algebraic properties of the tensor fields belonging to the domain of definition of the repetitive operators by $\mathbb{S}$ and $\mathbb{T}$ (symmetric and trace free), e.g., the latter operators read

$$\mathcal{D} = \mathcal{D}^{\mathsf{bih},1} := \begin{pmatrix} \mathring{\operatorname{Div}}_{\mathbb{T}} & 0 \\ \operatorname{symCurl}_{\mathbb{T}} & \mathring{\operatorname{Gradgrad}} \end{pmatrix}, \qquad (\mathcal{D}^{\mathsf{bih},1})^* = \begin{pmatrix} -\operatorname{devGrad} & \mathring{\operatorname{Curl}}_{\mathbb{S}} \\ 0 & \operatorname{divDiv}_{\mathbb{S}} \end{pmatrix}.$$

These operators are related to the (primal and dual) first biharmonic complex, also called Gradgrad or divDiv complex, i.e.,

$$\{0\} \xrightarrow{\iota_{\{0\}}} L^2(\Omega) \xrightarrow{\mathring{\operatorname{Gradgrad}}} L^{2,3\times3}_{\mathbb{S}}(\Omega) \xrightarrow{\mathring{\operatorname{Curl}}_{\mathbb{S}}} L^{2,3\times3}_{\mathbb{T}}(\Omega) \xrightarrow{\mathring{\operatorname{Div}}_{\mathbb{T}}} L^{2,3}(\Omega) \xrightarrow{\pi_{\mathsf{RT}_{\mathsf{pw}}}} \mathsf{RT}_{\mathsf{pw}},$$
$$\{0\} \xleftarrow{\pi_{\{0\}}} L^2(\Omega) \xleftarrow{\operatorname{divDiv}_{\mathbb{S}}} L^{2,3\times3}_{\mathbb{S}}(\Omega) \xleftarrow{\operatorname{symCurl}_{\mathbb{T}}} L^{2,3\times3}_{\mathbb{T}}(\Omega) \xleftarrow{-\operatorname{devGrad}} L^{2,3}(\Omega) \xleftarrow{\iota_{\mathsf{RT}_{\mathsf{pw}}}} \mathsf{RT}_{\mathsf{pw}},$$

[1]The boundary of a strong Lipschitz domain is locally a graph of some Lipschitz function.

relevant for the biharmonic equation, elasticity, and in the theory of general relativity. In the second biharmonic complex the boundary conditions are interchanged, i.e.,

$$\{0\} \xrightarrow{\iota_{\{0\}}} L^{2,3}(\Omega) \xrightarrow{\mathring{\operatorname{devGrad}}} L^{2,3\times 3}_{\mathbb{T}}(\Omega) \xrightarrow{\mathring{\operatorname{symCurl}}_{\mathbb{T}}} L^{2,3\times 3}_{\mathbb{S}}(\Omega) \xrightarrow{\mathring{\operatorname{divDiv}}_{\mathbb{S}}} L^{2}(\Omega) \xrightarrow{\pi_{\mathsf{P}^1_{\mathsf{pw}}}} \mathsf{P}^1_{\mathsf{pw}},$$

$$\{0\} \xleftarrow{\pi_{\{0\}}} L^{2,3}(\Omega) \xleftarrow{-\operatorname{Div}_{\mathbb{T}}} L^{2,3\times 3}_{\mathbb{T}}(\Omega) \xleftarrow{\operatorname{Curl}_{\mathbb{S}}} L^{2,3\times 3}_{\mathbb{S}}(\Omega) \xleftarrow{\operatorname{Gradgrad}} L^{2}(\Omega) \xleftarrow{\iota_{\mathsf{P}^1_{\mathsf{pw}}}} \mathsf{P}^1_{\mathsf{pw}},$$

leading to the operators

$$\mathcal{D}^{\mathsf{bih},2} := \begin{pmatrix} \mathring{\operatorname{divDiv}}_{\mathbb{S}} & 0 \\ \operatorname{Curl}_{\mathbb{S}} & \mathring{\operatorname{devGrad}} \end{pmatrix}, \qquad (\mathcal{D}^{\mathsf{bih},2})^* = \begin{pmatrix} \operatorname{Gradgrad} & \mathring{\operatorname{symCurl}}_{\mathbb{T}} \\ 0 & -\operatorname{Div}_{\mathbb{T}} \end{pmatrix},$$

see Theorem 5.5 and Remark 5.6. Another interesting complex is the elasticity complex, also called CurlCurl complex, i.e.,

$$\{0\} \xrightarrow{\iota_{\{0\}}} L^{2,3}(\Omega) \xrightarrow{\mathring{\operatorname{symGrad}}} L^{2,3\times 3}_{\mathbb{S}}(\Omega) \xrightarrow{\mathring{\operatorname{CurlCurl}}^{\top}_{\mathbb{S}}} L^{2,3\times 3}_{\mathbb{S}}(\Omega) \xrightarrow{\mathring{\operatorname{Div}}_{\mathbb{S}}} L^{2,3}(\Omega) \xrightarrow{\pi_{\mathsf{RM}_{\mathsf{pw}}}} \mathsf{RM}_{\mathsf{pw}},$$

$$\{0\} \xleftarrow{\pi_{\{0\}}} L^{2,3}(\Omega) \xleftarrow{-\operatorname{Div}_{\mathbb{S}}} L^{2,3\times 3}_{\mathbb{S}}(\Omega) \xleftarrow{\operatorname{CurlCurl}^{\top}_{\mathbb{S}}} L^{2,3\times 3}_{\mathbb{S}}(\Omega) \xleftarrow{-\operatorname{symGrad}} L^{2,3}(\Omega) \xleftarrow{\iota_{\mathsf{RM}_{\mathsf{pw}}}} \mathsf{RM}_{\mathsf{pw}}.$$

Here, we shall discuss the operators

$$\mathcal{D}^{\mathsf{ela}} := \begin{pmatrix} \mathring{\operatorname{Div}}_{\mathbb{S}} & 0 \\ \operatorname{CurlCurl}^{\top}_{\mathbb{S}} & \mathring{\operatorname{symGrad}} \end{pmatrix}, \qquad (\mathcal{D}^{\mathsf{ela}})^* = \begin{pmatrix} -\operatorname{symGrad} & \mathring{\operatorname{CurlCurl}}^{\top}_{\mathbb{S}} \\ 0 & -\operatorname{Div}_{\mathbb{S}} \end{pmatrix},$$

being of the same type, see Theorem 6.4 and Remark 6.5. Here and throughout this paper, we denote by grad, curl, and div the classical operators from vector analysis. Moreover, Grad acts componentwise as $\operatorname{grad}^{\top}$ mapping vector fields to tensor fields. Curl and Div act row-wise as $\operatorname{curl}^{\top}$ and div mapping tensor fields to tensor and vector fields, respectively.

Before we come to more in depth description of the main results, we shall provide a small overview of Fredholm index theory for differential operators next.

It is one of the greatest mathematical achievements of the twentieth century to relate the analytic notion of the Fredholm index for operators defined on Hilbert spaces to particular elliptic operators and their corresponding geometric properties of the underlying compact manifold the operators are defined on. The corner stone of this insight is the celebrated Atijah-Singer index theorem, see e.g. [16]. The methods of proof led to the invention of K-theory, which has evolved ever since and is an active field of research. Albeit being a breakthrough in mathematics, K-theory is a rather difficult tool to work with when it comes to explicitly compute the index for particular examples. Hence, in any case there is a need to provide many examples, where it is possible to obtain an explicit index formula.

In particular, when it comes to explicitly computing the Witten index (a generalised version of the Fredholm index) there is a need to thoroughly understand the Fredholm case in particular situations. We refer to [8] for a preliminary version of an explicit index theorem properly justified in [6] and, using a similar pathway as in [8], to [10], where the transition from the Fredholm situtation to the Witten index has been performed in [10, Chapter 14]. The generalisation of the one-plus-one-dimensional situation of [8] has been addressed in the seminal paper [9].

The approach to compute the index in Theorem 1.1 (and in all the others) is based on a construction principle for Fredholm operators provided in [7]. The fundamental observation given in [7] is that it is possible to construct a Fredholm operator with the help of Hilbert complexes of densely defined and closed linear operators, i.e,

$$\cdots \xrightarrow{\cdots} H_0 \xrightarrow{A_0} H_1 \xrightarrow{A_1} H_2 \xrightarrow{A_2} H_3 \xrightarrow{\cdots} \cdots,$$

$$\cdots \xleftarrow{\cdots} H_0 \xleftarrow{A_0^*} H_1 \xleftarrow{A_1^*} H_2 \xleftarrow{A_2^*} H_3 \xleftarrow{\cdots} \cdots.$$

More precisely, if A_0, A_1, and A_2 are densely defined, closed linear operators defined on suitable Hilbert spaces H_l such that

$$\operatorname{ran} A_0 \subseteq \ker A_1, \qquad \operatorname{ran} A_1 \subseteq \ker A_2,$$

then the block matrix operator

$$\mathcal{D} := \begin{pmatrix} A_2 & 0 \\ A_1^* & A_0 \end{pmatrix}$$

with its natural domain of definition is closed and densely defined. It is Fredholm, if the ranges $\operatorname{ran} A_0$, $\operatorname{ran} A_1$, and $\operatorname{ran} A_2$ are closed and if both kernels

$$N_0 := \ker A_0, \qquad N_{2,*} := \ker A_2^*$$

and both cohomology groups

$$K_1 := \ker A_1 \cap \ker A_0^*, \qquad K_2 := \ker A_2 \cap \ker A_1^*$$

are finite-dimensional. In this case, its index is then given by

$$\operatorname{ind} \mathcal{D} = \dim N_0 - \dim K_1 + \dim K_2 - \dim N_{2,*}, \tag{1}$$

cf. Theorem 2.8. For its adjoint, which is then Fredholm as well, we simply have

$$\mathcal{D}^* := \begin{pmatrix} A_2^* & A_1 \\ 0 & A_0^* \end{pmatrix}, \qquad \operatorname{ind} \mathcal{D}^* = -\operatorname{ind} \mathcal{D}.$$

In a first application of this observation presented in this article, we look at the classical de Rham complex

$$\begin{array}{c} \{0\} \xrightarrow{A_{-1}=\iota_{\{0\}}} L^2(\Omega) \xrightarrow{A_0=\mathring{\operatorname{grad}}} L^{2,3}(\Omega) \xrightarrow{A_1=\mathring{\operatorname{curl}}} L^{2,3}(\Omega) \xrightarrow{A_2=\mathring{\operatorname{div}}} L^2(\Omega) \xrightarrow{A_3=\pi_{\mathbb{R}_{\mathsf{pw}}}} \mathbb{R}_{\mathsf{pw}}, \\ \{0\} \xleftarrow{A_{-1}^*=\pi_{\{0\}}} L^2(\Omega) \xleftarrow{A_0^*=-\operatorname{div}} L^{2,3}(\Omega) \xleftarrow{A_1^*=\operatorname{curl}} L^{2,3}(\Omega) \xleftarrow{A_2^*=-\operatorname{grad}} L^2(\Omega) \xleftarrow{A_3^*=\iota_{\mathbb{R}_{\mathsf{pw}}}} \mathbb{R}_{\mathsf{pw}}, \end{array} \tag{2}$$

where again the super index $^\circ$ signifies homogeneous Dirichlet boundary conditions, see Theorem 3.8. By (1) in order to compute the index it is necessary to calculate the dimension of the cohomology groups, i.e., the dimension of the harmonic Dirichlet and Neumann fields

$$\mathcal{H}_D^{\mathsf{Rhm}}(\Omega) := K_1 = \ker(\mathring{\operatorname{curl}}) \cap \ker(\operatorname{div}),$$
$$\mathcal{H}_N^{\mathsf{Rhm}}(\Omega) := K_2 = \ker(\mathring{\operatorname{div}}) \cap \ker(\operatorname{curl}),$$

respectively. In [23], this has been done by Picard. As it turns out these dimensions are related to topological properties of the underlying domain the differential operators are defined on, that is,

$$\dim \mathcal{H}_D^{\mathsf{Rhm}}(\Omega) = m - 1, \qquad \dim \mathcal{H}_N^{\mathsf{Rhm}}(\Omega) = p,$$

see Theorem 3.6. In consequence, it is possible to compute the indices for the block de Rham operators

$$\mathcal{D}^{\mathsf{Rhm}} := \begin{pmatrix} \mathring{\operatorname{div}} & 0 \\ \operatorname{curl} & \mathring{\operatorname{grad}} \end{pmatrix}, \qquad (\mathcal{D}^{\mathsf{Rhm}})^* := \begin{pmatrix} -\operatorname{grad} & \mathring{\operatorname{curl}} \\ 0 & -\operatorname{div} \end{pmatrix}$$

by (1) in terms of m, p, and n, i.e.,

$$\operatorname{ind} \mathcal{D}^{\mathsf{Rhm}} = p - m - n + 1, \qquad \operatorname{ind}(\mathcal{D}^{\mathsf{Rhm}})^* = -\operatorname{ind} \mathcal{D}^{\mathsf{Rhm}},$$

see Theorem 3.8. It is noteworthy that this index theorem provides an index theorem for the Dirac operator on open manifolds with boundary endowed with a particular boundary condition, see [25] and Section 3.3 below.

For a proof of Theorem 1.1 (and the others) we will combine the structural viewpoint outlined by [7] and ideas taken from the explicit computation of the dimension of the cohomolgy groups. The foundation for all of this to be applicable, however, is the newly found biharmonic complex, see [20, 21], and the more familiar elasticity complex, see [22]. In [20, 21] the crucial properties and compact embedding results have been found for the biharmonic Hilbert complex underlying the computation of the index in Theorem 1.1. In [22] the corresponding results are presented for the elasticity complex. These results also stress that the mixed order differential operators given in Theorem 1.1 (and the others) *cannot* be viewed as a leading order term subject to a relatively compact perturbation.

In Section 2, we briefly recall the notion of Hilbert complexes of densely defined and closed linear operators. Also, we provide a small introduction to the construction principle for Fredholm operators provided in [7]. As we slightly deviate from the approach presented there we recall some of the proofs for convenience of the reader. In order to have a non-trivial yet rather elementary example at hand, we present the so-called Picard's extended Maxwell system in Section 3. This sets the stage for the index theorem for the Dirac operator provided in Section 3.3. In Section 4, we recall the first biharmonic complex and provide the explicit formulation of our main result Theorem 1.1, see Theorem 4.4. Similar results will be presented in Section 5 for the second biharmonic complex and in Section 6 for the elasticity complex. The Appendix is concerned with the topological setting introduced in [23] and, in particular, with the computation of bases and hence the dimensions of the generalised Dirichlet and Neumann vector and tensor fields for the different complexes, respectively, and thus concluding the proofs of our main results.

Note that unlike to many research topics in the analysis of partial differential equations (and other topics), we shall use Ω being 'open' and a 'domain' as synonymous terms. In particular, we shall not imply Ω to satisfy any connectivity properties, when calling Ω a domain.

Recalling and introducing the cohomology groups

$$K_1 = \mathcal{H}_D^{\cdots}(\Omega), \qquad K_2 = \mathcal{H}_N^{\cdots}(\Omega),$$

i.e., the Dirichlet and Neumann fields

$$\begin{aligned}
\mathcal{H}_D^{\mathsf{Rhm}}(\Omega) &= \ker(\mathring{\mathrm{curl}}) \cap \ker(\mathrm{div}), & \mathcal{H}_N^{\mathsf{Rhm}}(\Omega) &= \ker(\mathring{\mathrm{div}}) \cap \ker(\mathrm{curl}),\\
\mathcal{H}_{D,\mathbb{S}}^{\mathsf{bih},1}(\Omega) &= \ker(\mathring{\mathrm{Curl}}_{\mathbb{S}}) \cap \ker(\mathrm{divDiv}_{\mathbb{S}}), & \mathcal{H}_{N,\mathbb{T}}^{\mathsf{bih},1}(\Omega) &= \ker(\mathring{\mathrm{Div}}_{\mathbb{T}}) \cap \ker(\mathrm{symCurl}_{\mathbb{T}}),\\
\mathcal{H}_{D,\mathbb{T}}^{\mathsf{bih},2}(\Omega) &= \ker(\mathrm{sym}\mathring{\mathrm{Curl}}_{\mathbb{T}}) \cap \ker(\mathrm{Div}_{\mathbb{T}}), & \mathcal{H}_{N,\mathbb{S}}^{\mathsf{bih},2}(\Omega) &= \ker(\mathrm{div}\mathring{\mathrm{Div}}_{\mathbb{S}}) \cap \ker(\mathrm{Curl}_{\mathbb{S}}),\\
\mathcal{H}_{D,\mathbb{S}}^{\mathsf{ela}}(\Omega) &= \ker(\mathrm{Curl}\mathring{\mathrm{Curl}}_{\mathbb{S}}^{\top}) \cap \ker(\mathrm{Div}_{\mathbb{S}}), & \mathcal{H}_{N,\mathbb{S}}^{\mathsf{ela}}(\Omega) &= \ker(\mathring{\mathrm{Div}}_{\mathbb{S}}) \cap \ker(\mathrm{CurlCurl}_{\mathbb{S}}^{\top}),
\end{aligned}$$

let us summarise some of the main results of this contribution (including our Appendix), such as the dimensions of the kernels N_0, $N_{2,*}$, i.e.,

$$\begin{aligned}
\dim\ker(\mathring{\mathrm{grad}}) &= 0, & \dim\ker(\mathrm{grad}) &= n,\\
\dim\ker(\mathrm{Grad}\mathring{\mathrm{grad}}) &= 0, & \dim\ker(\mathrm{devGrad}) &= 4n,\\
\dim\ker(\mathrm{dev}\mathring{\mathrm{Grad}}) &= 0, & \dim\ker(\mathrm{Gradgrad}) &= 4n,\\
\dim\ker(\mathrm{sym}\mathring{\mathrm{Grad}}) &= 0, & \dim\ker(\mathrm{symGrad}) &= 6n,
\end{aligned}$$

and the dimensions of the cohomology groups K_1, K_2, i.e.,

$$\begin{aligned}
\dim\mathcal{H}_D^{\mathsf{Rhm}}(\Omega) &= m-1, & \dim\mathcal{H}_N^{\mathsf{Rhm}}(\Omega) &= p,\\
\dim\mathcal{H}_{D,\mathbb{S}}^{\mathsf{bih},1}(\Omega) &= 4(m-1), & \dim\mathcal{H}_{N,\mathbb{T}}^{\mathsf{bih},1}(\Omega) &= 4p,
\end{aligned}$$

$$\dim \mathcal{H}^{\mathsf{bih},2}_{D,\mathbb{T}}(\Omega) = 4(m-1), \qquad \dim \mathcal{H}^{\mathsf{bih},2}_{N,\mathbb{S}}(\Omega) = 4p,$$
$$\dim \mathcal{H}^{\mathsf{ela}}_{D,\mathbb{S}}(\Omega) = 6(m-1), \qquad \dim \mathcal{H}^{\mathsf{ela}}_{N,\mathbb{S}}(\Omega) = 6p,$$

and the indices $\operatorname{ind}\mathcal{D}$, $\operatorname{ind}\mathcal{D}^*$ of the involved Fredholm operators, i.e.,

$$\operatorname{ind}\mathcal{D}^{\mathsf{Rhm}} = p-m-n+1, \qquad \operatorname{ind}(\mathcal{D}^{\mathsf{Rhm}})^* = -\operatorname{ind}\mathcal{D}^{\mathsf{Rhm}},$$
$$\operatorname{ind}\mathcal{D}^{\mathsf{bih},1} = 4(p-m-n+1), \qquad \operatorname{ind}(\mathcal{D}^{\mathsf{bih},1})^* = -\operatorname{ind}\mathcal{D}^{\mathsf{bih},1},$$
$$\operatorname{ind}\mathcal{D}^{\mathsf{bih},2} = 4(p-m-n+1), \qquad \operatorname{ind}(\mathcal{D}^{\mathsf{bih},2})^* = -\operatorname{ind}\mathcal{D}^{\mathsf{bih},2},$$
$$\operatorname{ind}\mathcal{D}^{\mathsf{ela}} = 6(p-m-n+1), \qquad \operatorname{ind}(\mathcal{D}^{\mathsf{ela}})^* = -\operatorname{ind}\mathcal{D}^{\mathsf{ela}}.$$

Remark 1.2. *We observe that in all of our examples, where generally the operators A_j carry the boundary condition and the adjoints A_j^* do not have boundary conditions, the dimensions of the first and second cohomology groups K_1 and K_2 ('Dirichlet fields' and 'Neumann fields') are given by*

$$\dim K_1 = \frac{\dim N_{2,*}}{n}\cdot(m-1), \qquad \dim K_2 = \frac{\dim N_{2,*}}{n}\cdot p,$$

respectively. The indices of $\mathcal{D}$ and $\mathcal{D}^*$ are

$$-\operatorname{ind}\mathcal{D}^* = \operatorname{ind}\mathcal{D} = \frac{\dim N_{2,*}}{n}\cdot(p-m-n+1).$$

For the construction of bases and to compute the dimensions of the latter Neumann fields it is crucial, that these are sufficiently regular, e.g., continuous in Ω. We even have the following local regularity results.

Lemma 1.3 (local regularity of the cohomology groups)**.** *Let $\Omega \subset \mathbb{R}^3$ be open. Then*

$$\mathcal{H}^{\mathsf{Rhm}}_{D}(\Omega), \mathcal{H}^{\mathsf{Rhm}}_{N}(\Omega) \subset C^{\infty,3}(\Omega) \cap L^{2,3}(\Omega),$$
$$\mathcal{H}^{\mathsf{bih},1}_{D,\mathbb{S}}(\Omega), \mathcal{H}^{\mathsf{ela}}_{D,\mathbb{S}}(\Omega), \mathcal{H}^{\mathsf{bih},2}_{N,\mathbb{S}}(\Omega), \mathcal{H}^{\mathsf{ela}}_{N,\mathbb{S}}(\Omega) \subset C^{\infty,3\times3}(\Omega) \cap L^{2,3\times3}_{\mathbb{S}}(\Omega),$$
$$\mathcal{H}^{\mathsf{bih},2}_{D,\mathbb{T}}(\Omega), \mathcal{H}^{\mathsf{bih},1}_{N,\mathbb{T}}(\Omega) \subset C^{\infty,3\times3}(\Omega) \cap L^{2,3\times3}_{\mathbb{T}}(\Omega).$$

Proof. Vector fields in $\mathcal{H}^{\mathsf{Rhm}}_{D}(\Omega) \cup \mathcal{H}^{\mathsf{Rhm}}_{N}(\Omega)$ are harmonic and thus belong to $C^{\infty,3}(\Omega)$. Tensor fields

$$S \in \mathcal{H}^{\mathsf{bih},1}_{D,\mathbb{S}}(\Omega) \cup \mathcal{H}^{\mathsf{bih},2}_{N,\mathbb{S}}(\Omega) \subset \ker(\operatorname{Curl}_{\mathbb{S}}) \cap \ker(\operatorname{divDiv}_{\mathbb{S}})$$

can be represented locally, e.g., in any topologically trivial and smooth subdomain $\widetilde{\Omega}$ of Ω, by $S = \operatorname{Gradgrad} u$ with $u \in H^2(\widetilde{\Omega})$, see [21, Theorem 3.10], which holds also without boundary conditions. Thus $\operatorname{divDiv}_{\mathbb{S}} \operatorname{Gradgrad} u = 0$ in $\widetilde{\Omega}$. Local regularity for the biharmonic equation shows $u \in C^\infty(\widetilde{\Omega})$ and hence $S = \operatorname{Gradgrad} u \in C^{\infty,3\times3}(\widetilde{\Omega})$, i.e., $S \in C^{\infty,3\times3}(\Omega)$. Tensor fields

$$T \in \mathcal{H}^{\mathsf{bih},2}_{D,\mathbb{T}}(\Omega) \cup \mathcal{H}^{\mathsf{bih},1}_{N,\mathbb{T}}(\Omega) \subset \ker(\operatorname{symCurl}_{\mathbb{T}}) \cap \ker(\operatorname{Div}_{\mathbb{T}})$$

can be represented locally by $T = \operatorname{devGrad} v$ with $v \in H^{1,3}(\widetilde{\Omega})$, see [21, Theorem 3.10]. Thus $\operatorname{Div}_{\mathbb{T}} \operatorname{devGrad} v = 0$ in $\widetilde{\Omega}$. Local elliptic regularity shows $v \in C^{\infty,3}(\widetilde{\Omega})$ and hence $T = \operatorname{devGrad} v \in C^{\infty,3\times3}(\widetilde{\Omega})$, i.e., $T \in C^{\infty,3\times3}(\Omega)$. Tensor fields

$$S \in \mathcal{H}^{\mathsf{ela}}_{D,\mathbb{S}}(\Omega) \cup \mathcal{H}^{\mathsf{ela}}_{N,\mathbb{S}}(\Omega) \subset \ker(\operatorname{CurlCurl}^{\top}_{\mathbb{S}}) \cap \ker(\operatorname{Div}_{\mathbb{S}})$$

can be represented locally by $S = \operatorname{symGrad} v$ with $v \in H^{1,3}(\widetilde{\Omega})$, see [22, Theorem 3.5]. Thus $\operatorname{Div}_{\mathbb{S}} \operatorname{symGrad} v = 0$ in $\widetilde{\Omega}$. Local elliptic regularity shows $v \in C^{\infty,3}(\widetilde{\Omega})$ and hence $S = \operatorname{symGrad} v \in C^{\infty,3\times3}(\widetilde{\Omega})$, i.e., $S \in C^{\infty,3\times3}(\Omega)$. $\square$

2. The Construction Principle and the Index Theorem

In this section, we provide the basic construction principle, which is the basis for the operators in question. The theory in more general terms has been developed already in [7]. Here, we rephrase the situation with a slightly more particular viewpoint. For the convenience of the reader, we carry out the necessary proofs here.

Throughout this section, we let H_0, H_1, H_2, H_3 be Hilbert spaces, and

$$\begin{aligned} A_0 &: \operatorname{dom} A_0 \subseteq H_0 \longrightarrow H_1, \\ A_1 &: \operatorname{dom} A_1 \subseteq H_1 \longrightarrow H_2, \\ A_2 &: \operatorname{dom} A_2 \subseteq H_2 \longrightarrow H_3 \end{aligned}$$

be densely defined and <u>closed</u> linear operators.

Definition 2.1. *Let A_0, A_1, A_2 be defined as above.*

- *We call a pair (A_0, A_1) a* complex *(*Hilbert complex*), if* $\operatorname{ran} A_0 \subseteq \ker A_1$.
- *We say a complex (A_0, A_1) is* closed*, if* $\operatorname{ran} A_0$ *and* $\operatorname{ran} A_1$ *are closed.*
- *A complex (A_0, A_1) is said to be* compact*, if the embedding* $\operatorname{dom} A_1 \cap \operatorname{dom} A_0^* \hookrightarrow H_1$ *is compact.*
- *The triple (A_0, A_1, A_2) is called a* (closed/compact) complex*, if both (A_0, A_1) and (A_1, A_2) are (closed/compact) complexes.*
- *We say that a complex (A_0, A_1, A_2) is* maximal compact*, if (A_0, A_1, A_2) is a compact complex and both embeddings* $\operatorname{dom} A_0 \hookrightarrow H_0$ *and* $\operatorname{dom} A_2^* \hookrightarrow H_3$ *are compact as well.*

Remark 2.2. *The 'FA-ToolBox' from* [17, 18, 19, 21, 22] *shows that (A_0, A_1) resp. (A_0, A_1, A_2) is a (closed/compact/maximal compact) complex, if and only if (A_1^*, A_0^*) resp. (A_2^*, A_1^*, A_0^*) is a (closed/compact/maximal compact) complex.*

Throughout this section, we assume that (A_0, A_1, A_2) is a complex, i.e.,

$$H_0 \xrightarrow{A_0} H_1 \xrightarrow{A_1} H_2 \xrightarrow{A_2} H_3,$$
$$H_0 \xleftarrow{A_0^*} H_1 \xleftarrow{A_1^*} H_2 \xleftarrow{A_2^*} H_3.$$

We define the operator

$$\begin{aligned} \mathcal{D} : (\operatorname{dom} A_2 \cap \operatorname{dom} A_1^*) \times \operatorname{dom} A_0 \subseteq H_2 \times H_0 &\longrightarrow H_3 \times H_1 \\ (x, y) &\longmapsto (A_2 x, A_1^* x + A_0 y). \end{aligned}$$

In block operator matrix notation, we have

$$\mathcal{D} = \begin{pmatrix} A_2 & 0 \\ A_1^* & A_0 \end{pmatrix}.$$

We gather some elementary facts about $\mathcal{D}$.

Proposition 2.3. *$\mathcal{D}$ is a densely defined and closed linear operator.*

Proof. For the closedness of $\mathcal{D}$, we let $\big((x_k, y_k)\big)$ be a sequence in $\operatorname{dom} \mathcal{D}$ with $\big((x_k, y_k)\big)$ converging to some (x, y) in $H_2 \times H_0$ and $(\mathcal{D}(x_k, y_k))$ converging to (w, z) in $H_3 \times H_1$. One readily sees using the closedness of A_2 that $x \in \operatorname{dom} A_2$ and $A_2 x = w$. Next, we observe that $\operatorname{ran} A_0 \subseteq \ker A_1 \perp_{H_1} \operatorname{ran} A_1^*$. Hence, $(A_1^* x_k)$ and $(A_0 y_k)$ are both convergent to some $z_1 \in H_1$ and $z_2 \in H_1$, respectively. By the closedness of both A_1^* and A_0, we thus deduce that $x \in \operatorname{dom} A_1^*$ and $y \in \operatorname{dom} A_0$ with $z_1 = A_1^* x$ and $z_2 = A_0 y$ as well as $z = z_1 + z_2 = A_1^* x + A_0 y$.

For $\mathcal{D}$ being densely defined, we see that by assumption, $\operatorname{dom} A_0$ is dense in H_0. Hence, it suffices to show that $\operatorname{dom} A_2 \cap \operatorname{dom} A_1^*$ is dense in H_2. Decompose

$$H_2 = \overline{\operatorname{ran} A_2^*} \oplus_{H_2} \ker A_2, \qquad H_2 = \ker A_1^* \oplus_{H_2} \overline{\operatorname{ran} A_1}. \tag{3}$$

Moreover, recalling $K_2 = \ker A_2 \cap \ker A_1^*$ and by the complex property we get

$$\ker A_2 = K_2 \oplus_{H_2} \overline{\operatorname{ran} A_1}$$

and hence

$$\begin{aligned} H_2 &= \overline{\operatorname{ran} A_2^*} \oplus_{H_2} K_2 \oplus_{H_2} \overline{\operatorname{ran} A_1}, \\ \operatorname{dom} A_2 \cap \operatorname{dom} A_1^* &= (\operatorname{dom} A_2 \cap \overline{\operatorname{ran} A_2^*}) \oplus_{H_2} K_2 \oplus_{H_2} (\operatorname{dom} A_1^* \cap \overline{\operatorname{ran} A_1}). \end{aligned} \tag{4}$$

Using the same decomposition arguments it is not difficult to see that $\operatorname{dom} A_2 \cap \overline{\operatorname{ran} A_2^*}$ is dense in $\overline{\operatorname{ran} A_2^*}$ and, similarly, that also $\operatorname{dom} A_1^* \cap \overline{\operatorname{ran} A_1}$ is dense in $\overline{\operatorname{ran} A_1}$, see, e.g., the so-called functional analysis 'FA-ToolBox' presented in [17, 18, 19, 21, 22]. Hence we deduce the density result. □

Theorem 2.4. $\mathcal{D}^* = \begin{pmatrix} A_2^* & A_1 \\ 0 & A_0^* \end{pmatrix}$. *More precisely,*

$$\begin{aligned} \mathcal{D}^* : \operatorname{dom} A_2^* \times (\operatorname{dom} A_1 \cap \operatorname{dom} A_0^*) \subseteq H_3 \times H_1 &\longrightarrow H_2 \times H_0 \\ (w, z) &\longmapsto (A_2^* w + A_1 z, A_0^* z). \end{aligned}$$

Proof. Note that

$$\begin{pmatrix} A_2^* & A_1 \\ 0 & A_0^* \end{pmatrix} \subseteq \mathcal{D}^*$$

holds by definition since for all $(x, y) \in \operatorname{dom} \mathcal{D} = (\operatorname{dom} A_2 \cap \operatorname{dom} A_1^*) \times \operatorname{dom} A_0$ and for all $(w, z) \in \operatorname{dom} A_2^* \times (\operatorname{dom} A_1 \cap \operatorname{dom} A_0^*)$

$$\begin{aligned} \big\langle \mathcal{D}(x, y), (w, z) \big\rangle_{H_3 \times H_1} &= \langle A_2 x, w \rangle_{H_3} + \langle A_1^* x + A_0 y, z \rangle_{H_1} \\ &= \langle x, A_2^* w + A_1 z \rangle_{H_2} + \langle y, A_0^* z \rangle_{H_0} = \big\langle (x, y), \mathcal{D}^*(w, z) \big\rangle_{H_2 \times H_0}. \end{aligned}$$

Let $(w, z) \in \operatorname{dom} \mathcal{D}^*$ and denote $(u, v) := \mathcal{D}^*(w, z)$. For all $y \in \operatorname{dom} A_0$ we have $(0, y) \in \operatorname{dom} \mathcal{D}$ and infer

$$\langle A_0 y, z \rangle_{H_1} = \big\langle \mathcal{D}(0, y), (w, z) \big\rangle_{H_3 \times H_1} = \big\langle (0, y), \mathcal{D}^*(w, z) \big\rangle_{H_2 \times H_0} = \langle y, v \rangle_{H_0}.$$

Hence, $z \in \operatorname{dom} A_0^*$ and $A_0^* z = v$.

For all $x \in \operatorname{dom} A_2 \cap \operatorname{dom} A_1^*$ we see $(x, 0) \in \operatorname{dom} \mathcal{D}$ and deduce that

$$\begin{aligned} \langle A_2 x, w \rangle_{H_3} + \langle A_1^* x, z \rangle_{H_1} &= \big\langle \mathcal{D}(x, 0), (w, z) \big\rangle_{H_3 \times H_1} \\ &= \big\langle (x, 0), \mathcal{D}^*(w, z) \big\rangle_{H_2 \times H_0} = \langle x, u \rangle_{H_2}. \end{aligned} \tag{5}$$

Let π_2 denote the orthonormal projector onto $\overline{\operatorname{ran} A_2^*}$ in (3). Then for $\widetilde{x} \in \operatorname{dom} A_2$ we have

$$x := \pi_2 \widetilde{x} \in \operatorname{dom} A_2 \cap \overline{\operatorname{ran} A_2^*} \subset \operatorname{dom} A_2 \cap \ker A_1^* \subset \operatorname{dom} A_2 \cap \operatorname{dom} A_1^*, \quad A_2 x = A_2 \widetilde{x}$$

and by (5)

$$\langle A_2 \widetilde{x}, w \rangle_{H_3} = \langle A_2 x, w \rangle_{H_3} + \langle A_1^* x, z \rangle_{H_1} = \langle x, u \rangle_{H_2} = \langle \widetilde{x}, \pi_2 u \rangle_{H_2}.$$

Thus $w \in \operatorname{dom} A_2^*$ and $A_2^* w = \pi_2 u$. Analogously, let π_1 denote the orthonormal projector onto $\overline{\operatorname{ran} A_1}$ in (3). Then for $\widetilde{x} \in \operatorname{dom} A_1^*$ we have

$$x := \pi_1 \widetilde{x} \in \operatorname{dom} A_1^* \cap \overline{\operatorname{ran} A_1} \subset \operatorname{dom} A_1^* \cap \ker A_2 \subset \operatorname{dom} A_2 \cap \operatorname{dom} A_1^*, \quad A_1^* x = A_1^* \widetilde{x}$$

and by (5)

$$\langle A_1^*\widetilde{x}, z\rangle_{H_1} = \langle A_2 x, w\rangle_{H_3} + \langle A_1^* x, z\rangle_{H_1} = \langle x, u\rangle_{H_2} = \langle \widetilde{x}, \pi_1 u\rangle_{H_2}.$$

Thus $z \in \operatorname{dom} A_1$ and $A_1 z = \pi_1 u$. Therefore, $(w, z) \in \operatorname{dom} A_2^* \times (\operatorname{dom} A_1 \cap \operatorname{dom} A_0^*)$. Moreover, using the orthonormal projector π_0 onto K_2 in (4) we see for $x \in K_2$ by (5)

$$\langle x, \pi_0 u\rangle_{H_2} = \langle \pi_0 x, u\rangle_{H_2} = \langle x, u\rangle_{H_2} = \langle A_2 x, w\rangle_{H_3} + \langle A_1^* x, z\rangle_{H_1} = 0,$$

yielding $\pi_0 u = 0$. Finally, by (4) we arrive at

$$\mathcal{D}^*(w, z) = (u, v) = (\pi_0 u + \pi_1 u + \pi_2 u, A_0^* z) = (A_1 z + A_2^* w, A_0^* z),$$

completing the proof. □

Lemma 2.5. *For the kernels it holds*

$$\ker \mathcal{D} = K_2 \times N_0 = (\ker A_2 \cap \ker A_1^*) \times \ker A_0,$$
$$\ker \mathcal{D}^* = N_{2,*} \times K_1 = \ker A_2^* \times (\ker A_1 \cap \ker A_0^*).$$

Proof. For $(x, y) \in \ker \mathcal{D}$ we have $A_2 x = 0$ and $A_1^* x + A_0 y = 0$. By orthogonality and the complex property, i.e., $\operatorname{ran} A_0 \subset \ker A_1 \perp_{H_1} \operatorname{ran} A_1^*$, we see $A_1^* x = A_0 y = 0$. The assertion about $\ker \mathcal{D}^*$ follows analogously. □

Corollary 2.6. *The closures of the ranges are given by*

$$\overline{\operatorname{ran} \mathcal{D}} = (\ker \mathcal{D}^*)^{\perp_{H_3 \times H_1}} = N_{2,*}^{\perp_{H_3}} \times K_1^{\perp_{H_1}},$$
$$\overline{\operatorname{ran} \mathcal{D}^*} = (\ker \mathcal{D})^{\perp_{H_2 \times H_0}} = K_2^{\perp_{H_2}} \times N_0^{\perp_{H_0}}.$$

Lemma 2.7. *Let (A_0, A_1, A_2) be a maximal compact Hilbert complex. Then the embedding $\operatorname{dom} \mathcal{D} \hookrightarrow H_2 \times H_0$ is compact, and so is the embedding $\operatorname{dom} \mathcal{D}^* \hookrightarrow H_3 \times H_1$.*

Proof. Let $\big((x_k, y_k)\big)$ be a $(\operatorname{dom} \mathcal{D})$-bounded sequence in $\operatorname{dom} \mathcal{D}$. Then, as in the proof of Lemma 2.5, by orthogonality and the complex property (x_k) is a $(\operatorname{dom} A_2 \cap \operatorname{dom} A_1^*)$-bounded sequence in $\operatorname{dom} A_2 \cap \operatorname{dom} A_1^*$ and (y_k) is a $(\operatorname{dom} A_0)$-bounded sequence in $\operatorname{dom} A_0$. Since (A_0, A_1, A_2) is maximal compact, we can extract converging subsequences of (x_k) and (y_k). Analogously, we see that also $\operatorname{dom} \mathcal{D}^* \hookrightarrow H_3 \times H_1$ is compact, finishing the proof. □

We now recall the abstract index theorem taken from [7] formulated for the present situation.

Theorem 2.8. *Let (A_0, A_1, A_2) be a maximal compact Hilbert complex. Then $\mathcal{D}$ and $\mathcal{D}^*$ are Fredholm operators with indices*

$$\operatorname{ind} \mathcal{D} = \dim N_0 - \dim K_1 + \dim K_2 - \dim N_{2,*}, \qquad \operatorname{ind} \mathcal{D}^* = -\operatorname{ind} \mathcal{D}.$$

Proof. Utilising the general 'FA-ToolBox' from, e.g., [17, 18, 19, 21, 22], and Lemma 2.7 we observe that both ranges $\operatorname{ran} \mathcal{D}$ and $\operatorname{ran} \mathcal{D}^*$ are closed and that both kernels $\ker \mathcal{D}$ and $\ker \mathcal{D}^*$ are finite dimensional. Therefore, both $\mathcal{D}$ and $\mathcal{D}^*$ are Fredholm operators. The index $\operatorname{ind} \mathcal{D} = \dim \ker \mathcal{D} - \dim \ker \mathcal{D}^*$ is then given by Lemma 2.5. □

2.1. Some More Results.

Let us mention some additional features of the 'FA-ToolBox' from [17, 18, 19, 21, 22]. Lemma 2.7 and Theorem 2.8 imply some additional results for the reduced operators

$$\mathcal{D}_{\mathsf{red}} := \mathcal{D}|_{\operatorname{ran} \mathcal{D}^*} = \mathcal{D}|_{(\ker \mathcal{D})^{\perp_{H_2 \times H_0}}}, \qquad \mathcal{D}^*_{\mathsf{red}} := \mathcal{D}^*|_{\operatorname{ran} \mathcal{D}} = \mathcal{D}^*|_{(\ker \mathcal{D}^*)^{\perp_{H_3 \times H_1}}}.$$

Corollary 2.9. *Let (A_0, A_1, A_2) be a maximal compact Hilbert complex. Then the inverse operators $\mathcal{D}_{\mathsf{red}}^{-1} : \operatorname{ran}\mathcal{D} \to \operatorname{ran}\mathcal{D}^*$ and $(\mathcal{D}_{\mathsf{red}}^*)^{-1} : \operatorname{ran}\mathcal{D}^* \to \operatorname{ran}\mathcal{D}$ are compact. Moreover, $\mathcal{D}_{\mathsf{red}}^{-1} : \operatorname{ran}\mathcal{D} \to \operatorname{dom}\mathcal{D}_{\mathsf{red}}$ and $(\mathcal{D}_{\mathsf{red}}^*)^{-1} : \operatorname{ran}\mathcal{D}^* \to \operatorname{dom}\mathcal{D}_{\mathsf{red}}^*$ are continuous and, equivalently, the Friedrichs-Poincaré type estimates*

$$\begin{aligned}
\big|(x,y)\big|_{H_2\times H_0} &\leq c_{\mathcal{D}}\big|\mathcal{D}(x,y)\big|_{H_3\times H_1} = c_{\mathcal{D}}\big(|A_2x|_{H_3}^2 + |A_1^*x|_{H_1}^2 + |A_0y|_{H_1}^2\big)^{1/2},\\
\big|(w,z)\big|_{H_3\times H_1} &\leq c_{\mathcal{D}}\big|\mathcal{D}^*(w,z)\big|_{H_2\times H_0} = c_{\mathcal{D}}\big(|A_2^*w|_{H_2}^2 + |A_1z|_{H_2}^2 + |A_0^*z|_{H_0}^2\big)^{1/2}
\end{aligned}$$

hold for all $(x,y) \in \operatorname{dom}\mathcal{D}_{\mathsf{red}}$ and for all $(w,z) \in \operatorname{dom}\mathcal{D}_{\mathsf{red}}^*$ with the same optimal constant $c_{\mathcal{D}} > 0$.

The latter estimates are additive combinations of the corresponding estimates for A_0 and (A_2, A_1^*) as well as A_2^* and (A_1, A_0^*), respectively.

Remark 2.10. *The compactness assumptions (maximal compact) are not needed to render $\mathcal{D}$ and $\mathcal{D}^*$ Fredholm operators. It suffices to assume that (A_0, A_1, A_2) is a closed Hilbert complex with finite-dimensional kernels N_0 and $N_{2,*}$ and finite-dimensional cohomology groups K_1 and K_2. In this case, the latter Friedrichs-Poincaré type estimates still hold and $\mathcal{D}_{\mathsf{red}}^{-1}$ and $(\mathcal{D}_{\mathsf{red}}^*)^{-1}$ are still continuous.*

Remark 2.11. *There are simple relations between the primal, dual, and adjoint complexes, when $\mathcal{D}$ is considered. More precisely, let us denote the latter primal operators $\mathcal{D}$ and $\mathcal{D}^*$ of the primal complex (A_0, A_1, A_2) by*

$$\mathcal{D} = \mathcal{D}^p = \begin{pmatrix} A_2 & 0 \\ A_1^* & A_0 \end{pmatrix}, \qquad \mathcal{D}^* = (\mathcal{D}^p)^* = \begin{pmatrix} A_2^* & A_1 \\ 0 & A_0^* \end{pmatrix},$$

and the dual operators corresponding to the dual complex (A_2^*, A_1^*, A_0^*) by

$$\mathcal{D}^d = \begin{pmatrix} A_0^* & 0 \\ A_1 & A_2^* \end{pmatrix}, \qquad (\mathcal{D}^d)^* = \begin{pmatrix} A_0 & A_1^* \\ 0 & A_2 \end{pmatrix}.$$

By Remark 2.2 (A_0, A_1, A_2) is a maximal compact complex, if and only if (A_2^*, A_1^*, A_0^*) is a maximal compact complex. Note that we may weaken the assumptions according to Remark 2.10. Theorem 2.8 shows that $\mathcal{D}^p$, $(\mathcal{D}^p)^*$, $\mathcal{D}^d$, $(\mathcal{D}^d)^*$ are Fredholm operators with indices

$$\begin{aligned}
\operatorname{ind}\mathcal{D}^p &= \dim N_0^p - \dim K_1^p + \dim K_2^p - \dim N_{2,*}^p, & \operatorname{ind}(\mathcal{D}^p)^* &= -\operatorname{ind}\mathcal{D}^p,\\
\operatorname{ind}\mathcal{D}^d &= \dim N_0^d - \dim K_1^d + \dim K_2^d - \dim N_{2,*}^d, & \operatorname{ind}(\mathcal{D}^d)^* &= -\operatorname{ind}\mathcal{D}^d.
\end{aligned}$$

Next we observe

$$\begin{aligned}
N_0^d &= \ker A_2^* = N_{2,*}^p, & N_{2,*}^d &= \ker A_0 = N_0^p,\\
K_1^d &= \ker A_1^* \cap \ker A_2 = K_2^p, & K_2^d &= \ker A_0^* \cap \ker A_1 = K_1^p.
\end{aligned}$$

Hence

$$-\operatorname{ind}(\mathcal{D}^d)^* = \operatorname{ind}\mathcal{D}^d = -\operatorname{ind}\mathcal{D}^p = \operatorname{ind}(\mathcal{D}^p)^*.$$

Note that basically $\mathcal{D}^d$ and $(\mathcal{D}^p)^*$ as well as $\mathcal{D}^p$ and $(\mathcal{D}^d)^*$ are the 'same' operators.

Note that the Hilbert space adjoints A_l^* depend on the particular choice of the inner products (metrics) of the underlying Hilbert spaces H_l. A typical example is simply given by 'weighted' inner products induced by 'weights' λ_l, $l = 0, 1, 2, 3$, i.e., symmetric and positive topological isomorphisms (symmetric and positive bijective bounded linear operators) $\lambda_l : H_l \to H_l$ inducing inner products

$$\langle\,\cdot\,,\,\cdot\,\rangle_{\widetilde{H}_l} := \langle\lambda_l\,\cdot\,,\,\cdot\,\rangle_{H_l} : \widetilde{H}_l \times \widetilde{H}_l \to \mathbb{C},$$

where $\widetilde{H}_l := H_l$ (as linear space) equipped with the inner product $\langle\,\cdot\,,\,\cdot\,\rangle_{\widetilde{H}_l}$. A sufficiently general situation is defined by $\lambda_0 := \mathrm{Id}$, $\lambda_3 := \mathrm{Id}$, and λ_1, λ_2 being symmetric and positive topological isomorphisms, as well as $\widetilde{H}_l := \big(H_l, \langle \lambda_l\,\cdot\,,\,\cdot\,\rangle_{H_l}\big)$, $l = 0,1,2,3$. Then the modified operators[2]

$$\begin{aligned}
\widetilde{A}_0 : \operatorname{dom}\widetilde{A}_0 &:= \operatorname{dom} A_0 \subseteq \widetilde{H}_0 \longrightarrow \widetilde{H}_1; & x &\longmapsto A_0 x,\\
\widetilde{A}_1 : \operatorname{dom}\widetilde{A}_1 &:= \operatorname{dom} A_1 \subseteq \widetilde{H}_1 \longrightarrow \widetilde{H}_2; & y &\longmapsto \lambda_2^{-1} A_1 y,\\
\widetilde{A}_2 : \operatorname{dom}\widetilde{A}_2 &:= \lambda_2^{-1}\operatorname{dom} A_2 \subseteq \widetilde{H}_2 \longrightarrow \widetilde{H}_3; & z &\longmapsto A_2 \lambda_2 z,\\
\widetilde{A}_0^* : \operatorname{dom}\widetilde{A}_0^* &= \lambda_1^{-1}\operatorname{dom} A_0^* \subseteq \widetilde{H}_1 \longrightarrow \widetilde{H}_0; & y &\longmapsto A_0^* \lambda_1 y,\\
\widetilde{A}_1^* : \operatorname{dom}\widetilde{A}_1^* &= \operatorname{dom} A_1^* \subseteq \widetilde{H}_2 \longrightarrow \widetilde{H}_1; & z &\longmapsto \lambda_1^{-1} A_1^* z,\\
\widetilde{A}_2^* : \operatorname{dom}\widetilde{A}_2^* &= \operatorname{dom} A_2^* \subseteq \widetilde{H}_3 \longrightarrow \widetilde{H}_2; & x &\longmapsto A_2^* x
\end{aligned}$$

form again a primal and dual Hilbert complex, i.e.,

$$\widetilde{H}_0 \xrightarrow{\widetilde{A}_0} \widetilde{H}_1 \xrightarrow{\widetilde{A}_1} \widetilde{H}_2 \xrightarrow{\widetilde{A}_2} \widetilde{H}_3,$$
$$\widetilde{H}_0 \xleftarrow{\widetilde{A}_0^*} \widetilde{H}_1 \xleftarrow{\widetilde{A}_1^*} \widetilde{H}_2 \xleftarrow{\widetilde{A}_2^*} \widetilde{H}_3,$$

and we can define

$$\widetilde{\mathcal{D}} := \begin{pmatrix} \widetilde{A}_2 & 0 \\ \widetilde{A}_1^* & \widetilde{A}_0 \end{pmatrix}, \qquad \widetilde{\mathcal{D}}^* = \begin{pmatrix} \widetilde{A}_2^* & \widetilde{A}_1 \\ 0 & \widetilde{A}_0^* \end{pmatrix}.$$

The closedness of the operators $\widetilde{A}_l$ and the complex properties are easily checked. Moreover, it is not hard to see that the closedness of $(\widetilde{A}_0, \widetilde{A}_1, \widetilde{A}_2)$ is implied by the closedness of (A_0, A_1, A_2). Remark 2.2, Proposition 2.3, Theorem 2.4, Lemma 2.5, and Corollary 2.6 are also valid for $(\widetilde{A}_0, \widetilde{A}_1, \widetilde{A}_2)$. In particular,

$$\begin{aligned}
\ker\widetilde{\mathcal{D}} &= \widetilde{K}_2 \times \widetilde{N}_0 = (\ker\widetilde{A}_2 \cap \ker\widetilde{A}_1^*) \times \ker\widetilde{A}_0 = \big((\lambda_2^{-1}\ker A_2) \cap \ker A_1^*\big) \times \ker A_0,\\
\ker\widetilde{\mathcal{D}}^* &= \widetilde{N}_{2,*} \times \widetilde{K}_1 = \ker\widetilde{A}_2^* \times (\ker\widetilde{A}_1 \cap \ker\widetilde{A}_0^*) = \ker A_2^* \times \big(\ker A_1 \cap (\lambda_1^{-1}\ker A_0^*)\big),\\
\overline{\operatorname{ran}\widetilde{\mathcal{D}}} &= (\ker\widetilde{\mathcal{D}}^*)^{\perp_{\widetilde{H}_3\times\widetilde{H}_1}} = \widetilde{N}_{2,*}^{\perp_{\widetilde{H}_3}} \times \widetilde{K}_1^{\perp_{\widetilde{H}_1}},\\
\overline{\operatorname{ran}\widetilde{\mathcal{D}}^*} &= (\ker\widetilde{\mathcal{D}})^{\perp_{\widetilde{H}_2\times\widetilde{H}_0}} = \widetilde{K}_2^{\perp_{\widetilde{H}_2}} \times \widetilde{N}_0^{\perp_{\widetilde{H}_0}}.
\end{aligned}$$

Of course, Lemma 2.7 and Theorem 2.8 hold as well. To relate these two main results to the original complex (A_0, A_1, A_2) we have the following:

Lemma 2.12. *The compactness properties and the dimensions of the kernels and cohomology groups of the latter complexes are independent of the weights λ_l. More precisely,*

- **(i)** $\widetilde{N}_0 = N_0$ *and* $\widetilde{N}_{2,*} = N_{2,*}$, *as* $\operatorname{dom}\widetilde{A}_0 = \operatorname{dom} A_0$ *and* $\operatorname{dom}\widetilde{A}_{2,*} = \operatorname{dom} A_{2,*}$,
- **(ii₁)** $\dim\big(\ker A_1 \cap (\lambda_1^{-1}\ker A_0^*)\big) = \dim\widetilde{K}_1 = \dim K_1 = \dim(\ker A_1 \cap \ker A_0^*)$,
- **(ii₂)** $\dim\big(\ker A_2 \cap (\lambda_2^{-1}\ker A_1^*)\big) = \dim\widetilde{K}_2 = \dim K_2 = \dim(\ker A_2 \cap \ker A_1^*)$,
- **(iii₁)** $\operatorname{dom}\widetilde{A}_1 \cap \operatorname{dom}\widetilde{A}_0^* = \operatorname{dom} A_1 \cap (\lambda_1^{-1}\operatorname{dom} A_0^*) \hookrightarrow \widetilde{H}_1 \quad\Leftrightarrow\quad \operatorname{dom} A_1 \cap \operatorname{dom} A_0^* \hookrightarrow H_1$,
- **(iii₂)** $\operatorname{dom}\widetilde{A}_2 \cap \operatorname{dom}\widetilde{A}_1^* = \operatorname{dom} A_2 \cap (\lambda_2^{-1}\operatorname{dom} A_1^*) \hookrightarrow \widetilde{H}_2 \quad\Leftrightarrow\quad \operatorname{dom} A_2 \cap \operatorname{dom} A_1^* \hookrightarrow H_2$.

---

[2]E.g., we compute $\widetilde{A}_0^*$. Let $y \in \operatorname{dom}\widetilde{A}_0^*$. Then for $x \in \operatorname{dom}\widetilde{A}_0 = \operatorname{dom} A_0$

$$\langle x, \widetilde{A}_0^* y\rangle_{H_0} = \langle x, \widetilde{A}_0^* y\rangle_{\widetilde{H}_0} = \langle \widetilde{A}_0 x, y\rangle_{\widetilde{H}_1} = \langle A_0 x, \lambda_1 y\rangle_{H_1},$$

showing that $\lambda_1 y \in \operatorname{dom} A_0^*$ and $A_0^* \lambda_1 y = \widetilde{A}_0^* y$.

Proof. For the proof we follow in close lines the ideas of [4, Theorem 6.1], where [4] is the extended version of [5].

(i) is trivial and it is sufficient to show only (ii$_1$) and (iii$_1$).

For (ii$_1$), let μ be another weight having the same properties as λ_1. Similar to (3), (4) we have by orthogonality in $\widetilde{H}_1$ and by the complex property

$$\begin{aligned} \widetilde{H}_1 &= \overline{\operatorname{ran}\widetilde{A}_0} \oplus_{\widetilde{H}_1} \ker\widetilde{A}_0^* = \overline{\operatorname{ran}A_0} \oplus_{\widetilde{H}_1} \lambda_1^{-1}\ker A_0^*, \\ \ker\widetilde{A}_1 &= \overline{\operatorname{ran}\widetilde{A}_0} \oplus_{\widetilde{H}_1} (\ker\widetilde{A}_1 \cap \ker\widetilde{A}_0^*) = \overline{\operatorname{ran}A_0} \oplus_{\widetilde{H}_1} \big(\ker A_1 \cap (\lambda_1^{-1}\ker A_0^*)\big), \end{aligned} \tag{6}$$

and we note that $\widetilde{H}_1 = H_1$ and $\ker\widetilde{A}_1 = \ker A_1$ as sets. Denoting the $\widetilde{H}_1$-orthonormal projector onto $\lambda_1^{-1}\ker A_0^*$ resp. $\ker A_1 \cap (\lambda_1^{-1}\ker A_0^*)$ by π, we consider the linear mapping

$$\widehat{\pi} : \ker A_1 \cap (\mu^{-1}\ker A_0^*) \longrightarrow \ker A_1 \cap (\lambda_1^{-1}\ker A_0^*); \qquad y \longrightarrow \pi y.$$

As $\pi y = 0$ implies $y \in (\mu^{-1}\ker A_0^*) \cap \overline{\operatorname{ran}A_0} = \{0\}$, which follows by H_1-orthogonality considering $\langle \mu y, y\rangle_{H_1}$, we see that $\widehat{\pi}$ is injective. Thus

$$\dim\big(\ker A_1 \cap (\mu^{-1}\ker A_0^*)\big) \leq \dim\big(\ker A_1 \cap (\lambda_1^{-1}\ker A_0^*)\big).$$

The other inequality $\geq$ is deduced by symmetry and hence equality holds.

For (iii$_1$), we use a similar decomposition strategy. Let μ be as before and let

$$\operatorname{dom}A_1 \cap (\lambda_1^{-1}\operatorname{dom}A_0^*) \hookrightarrow H_1 \tag{7}$$

be compact. Moreover, let us consider a bounded sequence

$$(y_k) \subset \operatorname{dom}A_1 \cap (\mu^{-1}\operatorname{dom}A_0^*),$$

i.e., (y_k), $(A_1 y_k)$, $(A_0^*\mu\, y_k)$ are bounded. Similar to (6) we get

$$\begin{aligned} \operatorname{dom}\widetilde{A}_1 &= \overline{\operatorname{ran}\widetilde{A}_0} \oplus_{\widetilde{H}_1} (\operatorname{dom}\widetilde{A}_1 \cap \ker\widetilde{A}_0^*) = \overline{\operatorname{ran}A_0} \oplus_{\widetilde{H}_1} \big(\operatorname{dom}A_1 \cap (\lambda_1^{-1}\ker A_0^*)\big), \\ \operatorname{dom}\widetilde{A}_0^* &= (\overline{\operatorname{ran}\widetilde{A}_0} \cap \operatorname{dom}\widetilde{A}_0^*) \oplus_{\widetilde{H}_1} \ker\widetilde{A}_0^* = \big(\overline{\operatorname{ran}A_0} \cap (\lambda_1^{-1}\operatorname{dom}A_0^*)\big) \oplus_{\widetilde{H}_1} \lambda_1^{-1}\ker A_0^*, \end{aligned} \tag{8}$$

and $\operatorname{dom}\widetilde{A}_1 = \operatorname{dom}A_1$ and $\operatorname{dom}\widetilde{A}_0^* = \lambda_1^{-1}\operatorname{dom}A_0^*$ as sets. Now, we apply both decompositions of (8) to (y_k). First, we $\widetilde{H}_1$-orthogonally decompose $y_k \in \operatorname{dom}A_1$ into

$$y_k = u_k + v_k, \qquad u_k \in \overline{\operatorname{ran}A_0} \subseteq \ker A_1, \quad v_k \in \operatorname{dom}A_1 \cap (\lambda_1^{-1}\ker A_0^*)$$

with $A_1 y_k = A_1 v_k$. Hence (v_k) is bounded in $\operatorname{dom}A_1 \cap (\lambda_1^{-1}\ker A_0^*)$ and by (7) we can extract a H_1-converging subsequence, again dented by (v_k). Second, we $\widetilde{H}_1$-orthogonally decompose $\lambda_1^{-1}\mu y_k \in \lambda_1^{-1}\operatorname{dom}A_0^*$ into

$$\lambda_1^{-1}\mu y_k = w_k + z_k, \qquad w_k \in \underbrace{\overline{\operatorname{ran}A_0}}_{\subseteq \ker A_1} \cap (\lambda_1^{-1}\operatorname{dom}A_0^*), \quad z_k \in \lambda_1^{-1}\ker A_0^*$$

with $A_0^*\mu y_k = A_0^*\lambda_1 w_k$. Hence (w_k) is bounded in $\ker A_1 \cap (\lambda_1^{-1}\operatorname{dom}A_0^*)$ and by (7) we can extract a H_1-converging subsequence, again dented by (w_k). Finally, by H_1-orthogonality, i.e., $u_k \in \overline{\operatorname{ran}A_0} \perp_{H_1} \ker A_0^* \ni \lambda_1 z_k$,

$$\begin{aligned} \big\langle \mu(y_k - y_l), y_k - y_l\big\rangle_{H_1} = &\underbrace{\big\langle \mu(y_k - y_l), u_k - v_l\big\rangle_{H_1}} + \big\langle \mu(y_k - y_l), v_k - v_l\big\rangle_{H_1} \\ &= \big\langle \lambda_1(w_k - w_l), u_k - u_l\big\rangle_{H_1} \\ &\leq c\big(|w_k - w_l|_{H_1} + |v_k - v_l|_{H_1}\big), \end{aligned}$$

which shows that (y_k) is a H_1-Cauchy sequence in H_1. Thus $\operatorname{dom}A_1 \cap (\mu^{-1}\operatorname{dom}A_0^*) \hookrightarrow H_1$ is compact. $\square$

Now we can formulate the counterparts of Lemma 2.7 and Theorem 2.8. The proofs follow immediately by Lemma 2.12.

Lemma 2.13. *Maximal compactness does not depend on the weights λ_l. More precisely: (A_0, A_1, A_2) is a maximal compact Hilbert complex, if and only if the Hilbert complex $(\widetilde{A}_0, \widetilde{A}_1, \widetilde{A}_2)$ is maximal compact. In this case, $\operatorname{dom}\widetilde{\mathcal{D}} \hookrightarrow \widetilde{H}_2 \times \widetilde{H}_0$ and $\operatorname{dom}\widetilde{\mathcal{D}}^* \hookrightarrow \widetilde{H}_3 \times \widetilde{H}_1$ are compact.*

Theorem 2.14. *The Fredholm indices do not depend on the weights λ_l. More precisely: Let (A_0, A_1, A_2) be a maximal compact Hilbert complex. Then $\mathcal{D}$, $\widetilde{\mathcal{D}}$, $\mathcal{D}^*$, and $\widetilde{\mathcal{D}}^*$ are Fredholm operators with indices*

$$\operatorname{ind}\widetilde{\mathcal{D}} = \operatorname{ind}\mathcal{D} = \dim N_0 - \dim K_1 + \dim K_2 - \dim N_{2,*}, \quad \operatorname{ind}\widetilde{\mathcal{D}}^* = \operatorname{ind}\mathcal{D}^* = -\operatorname{ind}\mathcal{D}.$$

3. The de Rham Complex and Its Indices

In this section, we specialise to a particular choice of the operators A_0, A_1, A_2. Also, we will show that the assumptions of Theorem 2.8 are satisfied for this particular choice of operators. We will, thus, obtain an index formula. The computations of the dimensions of the occurring cohomology groups date back to [23].

Definition 3.1. *Let $\Omega \subseteq \mathbb{R}^3$ be an open set. We put*

$$\begin{aligned}
\operatorname{grad}_c : C_c^\infty(\Omega) \subseteq L^2(\Omega) &\longrightarrow L^{2,3}(\Omega), & \phi &\longmapsto \operatorname{grad}\phi,\\
\operatorname{curl}_c : C_c^{\infty,3}(\Omega) \subseteq L^{2,3}(\Omega) &\longrightarrow L^{2,3}(\Omega), & \Phi &\longmapsto \operatorname{curl}\Phi,\\
\operatorname{div}_c : C_c^{\infty,3}(\Omega) \subseteq L^{2,3}(\Omega) &\longrightarrow L^{2}(\Omega), & \Phi &\longmapsto \operatorname{div}\Phi,
\end{aligned}$$

and further define the densely defined and closed linear operators

$$\begin{aligned}
\operatorname{grad} &:= -\operatorname{div}_c^*, & \operatorname{curl} &:= \operatorname{curl}_c^*, & \operatorname{div} &:= -\operatorname{grad}_c^*,\\
\mathring{\operatorname{grad}} &:= -\operatorname{div}^* = \overline{\operatorname{grad}_c}, & \mathring{\operatorname{curl}} &:= \operatorname{curl}^* = \overline{\operatorname{curl}_c}, & \mathring{\operatorname{div}} &:= -\operatorname{grad}^* = \overline{\operatorname{div}_c}.
\end{aligned}$$

In terms of classical definitions and notions, we record the following equalities (that are easily seen):

$$\begin{aligned}
\operatorname{dom}(\operatorname{grad}) &= H^1(\Omega), & \operatorname{dom}(\mathring{\operatorname{grad}}) &= \overline{C_c^\infty(\Omega)}^{H^1(\Omega)} = H_0^1(\Omega),\\
\operatorname{dom}(\operatorname{curl}) &= H(\operatorname{curl},\Omega), & \operatorname{dom}(\mathring{\operatorname{curl}}) &= \overline{C_c^{\infty,3}(\Omega)}^{H(\operatorname{curl},\Omega)} = H_0(\operatorname{curl},\Omega),\\
\operatorname{dom}(\operatorname{div}) &= H(\operatorname{div},\Omega), & \operatorname{dom}(\mathring{\operatorname{div}}) &= \overline{C_c^{\infty,3}(\Omega)}^{H(\operatorname{div},\Omega)} = H_0(\operatorname{div},\Omega).
\end{aligned}$$

3.1. **Picard's Extended Maxwell System.** We want to apply the index theorem in the following situation of the classical de Rham complex:

$$\begin{aligned}
A_0 &:= \mathring{\operatorname{grad}}, & A_1 &:= \mathring{\operatorname{curl}}, & A_2 &:= \mathring{\operatorname{div}},\\
A_0^* &= -\operatorname{div}, & A_1^* &= \operatorname{curl}, & A_2^* &= -\operatorname{grad},
\end{aligned}$$

$$\mathcal{D}^{\mathsf{Rhm}} := \begin{pmatrix} A_2 & 0 \\ A_1^* & A_0 \end{pmatrix} = \begin{pmatrix} \mathring{\operatorname{div}} & 0 \\ \operatorname{curl} & \mathring{\operatorname{grad}} \end{pmatrix}, \quad (\mathcal{D}^{\mathsf{Rhm}})^* = \begin{pmatrix} A_2^* & A_1 \\ 0 & A_0^* \end{pmatrix} = \begin{pmatrix} -\operatorname{grad} & \mathring{\operatorname{curl}} \\ 0 & -\operatorname{div} \end{pmatrix},$$

$$\begin{aligned}
&\{0\} \xrightarrow{A_{-1}=\iota_{\{0\}}} L^2(\Omega) \xrightarrow{A_0=\mathring{\operatorname{grad}}} L^{2,3}(\Omega) \xrightarrow{A_1=\mathring{\operatorname{curl}}} L^{2,3}(\Omega) \xrightarrow{A_2=\mathring{\operatorname{div}}} L^2(\Omega) \xrightarrow{A_3=\pi_{\mathbb{R}_{\mathsf{pw}}}} \mathbb{R}_{\mathsf{pw}},\\
&\{0\} \xleftarrow{A_{-1}^*=\pi_{\{0\}}} L^2(\Omega) \xleftarrow{A_0^*=-\operatorname{div}} L^{2,3}(\Omega) \xleftarrow{A_1^*=\operatorname{curl}} L^{2,3}(\Omega) \xleftarrow{A_2^*=-\operatorname{grad}} L^2(\Omega) \xleftarrow{A_3^*=\iota_{\mathbb{R}_{\mathsf{pw}}}} \mathbb{R}_{\mathsf{pw}}.
\end{aligned} \tag{9}$$

We note

$$\operatorname{dom}\mathcal{D}^{\mathsf{Rhm}} = (\operatorname{dom}A_2 \cap \operatorname{dom}A_1^*) \times \operatorname{dom}A_0 = (H_0(\operatorname{div},\Omega) \cap H(\operatorname{curl},\Omega)) \times H_0^1(\Omega).$$

The complex properties, i.e., $A_1A_0 \subseteq 0$ and $A_2A_1 \subseteq 0$, are based on Schwarz's lemma ensuring that $\operatorname{curl}_c \operatorname{grad}_c = 0$ and $\operatorname{div}_c \operatorname{curl}_c = 0$.

Proposition 3.2. *Let $\Omega \subseteq \mathbb{R}^3$ be open. Then*

$$\operatorname{ran}A_0 = \operatorname{ran}(\mathring{\operatorname{grad}}) \subseteq \ker(\mathring{\operatorname{curl}}) = \ker A_1,$$
$$\operatorname{ran}A_1 = \operatorname{ran}(\mathring{\operatorname{curl}}) \subseteq \ker(\mathring{\operatorname{div}}) = \ker A_2$$

and by Remark 2.2 the same holds for the adjoints (operators without homogeneous boundary conditions).

Proof. See, e.g., [26, Proposition 6.1.5]. □

Theorem 3.3 (Picard-Weber-Weck selection theorem, [24], [27], [29])**.** *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded weak*[3] *Lipschitz domain. Then*

$$\operatorname{dom}A_1 \cap \operatorname{dom}A_0^* = \operatorname{dom}(\mathring{\operatorname{curl}}) \cap \operatorname{dom}(\operatorname{div}),$$
$$\operatorname{dom}A_2 \cap \operatorname{dom}A_1^* = \operatorname{dom}(\mathring{\operatorname{div}}) \cap \operatorname{dom}(\operatorname{curl})$$

are both compactly embedded into $H_1 = H_2 = L^{2,3}(\Omega)$.

Remark 3.4. *Proposition 3.2 in conjunction with Theorem 3.3 and Rellich's selection theorems show that* $(\mathring{\operatorname{grad}}, \mathring{\operatorname{curl}}, \mathring{\operatorname{div}})$ *is a maximal compact complex. By Remark 2.2 so is the dual complex* $(-\operatorname{grad}, \operatorname{curl}, -\operatorname{div})$.

Note that

$$\begin{aligned} N_0^{\mathsf{Rhm}} &= \ker A_0 = \ker(\mathring{\operatorname{grad}}), \\ N_{2,*}^{\mathsf{Rhm}} &= \ker A_2^* = \ker(\operatorname{grad}), \\ K_1^{\mathsf{Rhm}} &= \ker A_1 \cap \ker A_0^* = \ker(\mathring{\operatorname{curl}}) \cap \ker(\operatorname{div}) =: \mathcal{H}_D^{\mathsf{Rhm}}(\Omega), \\ K_2^{\mathsf{Rhm}} &= \ker A_2 \cap \ker A_1^* = \ker(\mathring{\operatorname{div}}) \cap \ker(\operatorname{curl}) =: \mathcal{H}_N^{\mathsf{Rhm}}(\Omega), \end{aligned} \tag{10}$$

where we recall from the introduction the classical harmonic Dirichlet and Neumann fields $\mathcal{H}_D^{\mathsf{Rhm}}(\Omega)$ and $\mathcal{H}_N^{\mathsf{Rhm}}(\Omega)$, respectively.

Definition 3.5. *Let $\Omega \subset \mathbb{R}^3$ be bounded and open. Then we denote by*

- *n the number of connected components of Ω,*
- *m the number of connected components of the complement $\mathbb{R}^3 \setminus \overline{\Omega}$,*
- *p the number of handles of Ω, see Assumption 3 in Appendix B for details.*

For p to be well defined we suppose Assumption 3 to hold.

The dimensions of the cohomology groups are given as follows.

Theorem 3.6 ([23, Theorem 1])**.** *Let $\Omega \subseteq \mathbb{R}^3$ be open and bounded with continuous boundary. Moreover, suppose Assumption 3. Then*

$$\dim \mathcal{H}_D^{\mathsf{Rhm}}(\Omega) = m - 1, \qquad \dim \mathcal{H}_N^{\mathsf{Rhm}}(\Omega) = p.$$

Remark 3.7. *Note that for Ω to have a continuous boundary*[4] *is equivalent to have the segment property, see, e.g.,* [2].

[3] The boundary of a weak Lipschitz domain is a 2-dimensional submanifold of the 3-dimensional Lipschitz manifold $\overline{\Omega}$ with boundary.

[4] A boundary being locally representable as the graph of a continuous function.

Let us introduce the space of piecewise constants by

$$\mathbb{R}_{\mathsf{pw}} := \big\{u \in L^2(\Omega) : \forall\, C(\text{connect. comp.}) \subseteq \Omega \quad \exists\, \alpha_C \in \mathbb{R} : u|_C = \alpha_C\big\}.$$

Theorem 3.8. *Let $\Omega \subset \mathbb{R}^3$ be a bounded weak Lipschitz domain. Then $\mathcal{D}^{\mathsf{Rhm}}$ is a Fredholm operator with index*

$$\operatorname{ind}\mathcal{D}^{\mathsf{Rhm}} = \dim N_0^{\mathsf{Rhm}} - \dim K_1^{\mathsf{Rhm}} + \dim K_2^{\mathsf{Rhm}} - \dim N_{2,*}^{\mathsf{Rhm}}.$$

If additionally Γ is continuous and Assumption 3 holds, then

$$\operatorname{ind}\mathcal{D}^{\mathsf{Rhm}} = p - m - n + 1.$$

Proof. Recall Remark 3.4. Apply Theorem 2.8 together with (10), the observations

$$N_0^{\mathsf{Rhm}} = \ker(\mathring{\operatorname{grad}}) = \{0\}, \qquad N_{2,*}^{\mathsf{Rhm}} = \ker(\operatorname{grad}) = \mathbb{R}_{\mathsf{pw}}, \tag{11}$$

and Theorem 3.6. □

Remark 3.9. *By Theorem 2.8 the adjoint of the de Rham operator $(\mathcal{D}^{\mathsf{Rhm}})^*$ is Fredholm as well with index $\operatorname{ind}(\mathcal{D}^{\mathsf{Rhm}})^* = -\operatorname{ind}\mathcal{D}^{\mathsf{Rhm}}$. Moreover, Picard's extended Maxwell system is given by*

$$\mathcal{M}^{\mathsf{Rhm}} := \begin{pmatrix} 0 & \mathcal{D}^{\mathsf{Rhm}} \\ -(\mathcal{D}^{\mathsf{Rhm}})^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & A_2 & 0 \\ 0 & 0 & A_1^* & A_0 \\ -A_2^* & -A_1 & 0 & 0 \\ 0 & -A_0^* & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \mathring{\operatorname{div}} & 0 \\ 0 & 0 & \operatorname{curl} & \mathring{\operatorname{grad}} \\ \operatorname{grad} & -\mathring{\operatorname{curl}} & 0 & 0 \\ 0 & \operatorname{div} & 0 & 0 \end{pmatrix}$$

with $(\mathcal{M}^{\mathsf{Rhm}})^* = -\mathcal{M}^{\mathsf{Rhm}}$ and $\operatorname{ind}\mathcal{M}^{\mathsf{Rhm}} = \dim\ker\mathcal{M}^{\mathsf{Rhm}} - \dim\ker(\mathcal{M}^{\mathsf{Rhm}})^* = 0$.

3.2. **Some More Results.**

The construction of a maximal compact Hilbert complex is also possible for mixed boundary conditions as well as for inhomogeneous and anisotropic media, such as constitutive material laws, see, e.g., [3, 18, 19]. For mixed boundary conditions we note the following:

Remark 3.10. *In order to provide a greater variety of index theorems, it would be interesting to compute the dimensions of the harmonic Dirichlet and Neumann fields also in the situation of mixed boundary conditions. At least for the authors of this article it is completely beyond their expertise in geometry and topology and it appears to be an open problem as to which index formulas could be expected in terms of subcohomologies and related concepts.*

For inhomogeneous and anisotropic media (constitutive material laws) we have:

Remark 3.11. *As mentioned, a maximal compact Hilbert complex can also be constructed for inhomogeneous and anisotropic media. These may be considered as weights λ_l as presented in Theorem 2.14. For Maxwell's equations a typical situation is given by the choices $\lambda_0 := \mathrm{Id}$, $\lambda_3 := \mathrm{Id}$, and $\lambda_1 := \varepsilon, \lambda_2 := \mu : \Omega \to \mathbb{R}^{3\times 3}$ being symmetric and uniformly positive definite $L^\infty(\Omega)$-matrix (tensor) fields. Let us introduce the Hilbert spaces $L^{2,3}_\varepsilon(\Omega) := \widetilde{H}_1 := \big(L^{2,3}(\Omega), \langle\varepsilon\,\cdot\,,\,\cdot\,\rangle_{L^{2,3}(\Omega)}\big)$ and similarly $L^{2,3}_\mu(\Omega) := \widetilde{H}_2$ as well as $\widetilde{H}_0 = \widetilde{H}_3 = H_0 = H_3 = L^2(\Omega)$. We look at*

$$\widetilde{A}_0 := \mathring{\operatorname{grad}}, \qquad \widetilde{A}_1 := \mu^{-1}\mathring{\operatorname{curl}}, \qquad \widetilde{A}_2 := \mathring{\operatorname{div}}\,\mu,$$

$$\widetilde{A}_0^* = -\operatorname{div}\varepsilon, \qquad \widetilde{A}_1^* = \varepsilon^{-1}\operatorname{curl}, \qquad \widetilde{A}_2^* = -\operatorname{grad},$$

$$\widetilde{\mathcal{D}}^{\mathsf{Rhm}} := \begin{pmatrix} \widetilde{A}_2 & 0 \\ \widetilde{A}_1^* & \widetilde{A}_0 \end{pmatrix} = \begin{pmatrix} \mathring{\operatorname{div}}\,\mu & 0 \\ \varepsilon^{-1}\operatorname{curl} & \mathring{\operatorname{grad}} \end{pmatrix},$$

$$(\widetilde{\mathcal{D}}^{\mathsf{Rhm}})^* = \begin{pmatrix} \widetilde{A}_2^* & \widetilde{A}_1 \\ 0 & \widetilde{A}_0^* \end{pmatrix} = \begin{pmatrix} -\operatorname{grad} & \mu^{-1}\mathring{\operatorname{curl}} \\ 0 & -\operatorname{div}\varepsilon \end{pmatrix},$$

i.e., the de Rham complex, cf. (9),

$$\begin{gathered}\{0\} \xrightarrow{\widetilde{A}_{-1}=\iota_{\{0\}}} L^2(\Omega) \xrightarrow{\widetilde{A}_0=\mathring{\operatorname{grad}}} L^{2,3}_{\varepsilon}(\Omega) \xrightarrow{\widetilde{A}_1=\mu^{-1}\mathring{\operatorname{curl}}} L^{2,3}_{\mu}(\Omega) \xrightarrow{\widetilde{A}_2=\mathring{\operatorname{div}}\,\mu} L^2(\Omega) \xrightarrow{\widetilde{A}_3=\pi_{\mathbb{R}_{\mathsf{pw}}}} \mathbb{R}_{\mathsf{pw}}, \\ \{0\} \xleftarrow{\widetilde{A}_{-1}^*=\pi_{\{0\}}} L^2(\Omega) \xleftarrow{\widetilde{A}_0^*=-\operatorname{div}\varepsilon} L^{2,3}_{\varepsilon}(\Omega) \xleftarrow{\widetilde{A}_1^*=\varepsilon^{-1}\operatorname{curl}} L^{2,3}_{\mu}(\Omega) \xleftarrow{\widetilde{A}_2^*=-\operatorname{grad}} L^2(\Omega) \xleftarrow{\widetilde{A}_3^*=\iota_{\mathbb{R}_{\mathsf{pw}}}} \mathbb{R}_{\mathsf{pw}}.\end{gathered} \tag{12}$$

Lemma 2.12, Lemma 2.13, and Theorem 2.14 show that the compactness properties, the dimensions of the kernels and cohomology groups, the maximal compactness, and the Fredholm indices of the de Rham complex do not dependent of the material weights ε and μ. More precisely,

- $\dim\big(\ker(\mathring{\operatorname{curl}})\cap\big(\varepsilon^{-1}\ker(\operatorname{div})\big)\big) = \dim\big(\ker(\mathring{\operatorname{curl}})\cap\ker(\operatorname{div})\big) = \dim\mathcal{H}^{\mathsf{Rhm}}_D(\Omega) = m-1,$
- $\dim\big(\big(\mu^{-1}\ker(\mathring{\operatorname{div}})\big)\cap\ker(\operatorname{curl})\big) = \dim\big(\ker(\mathring{\operatorname{div}})\cap\ker(\operatorname{curl})\big) = \dim\mathcal{H}^{\mathsf{Rhm}}_N(\Omega) = p,$
- $\operatorname{dom}(\mathring{\operatorname{curl}})\cap\big(\varepsilon^{-1}\operatorname{dom}(\operatorname{div})\big) \hookrightarrow L^{2,3}_{\varepsilon}(\Omega) \quad\Leftrightarrow\quad \operatorname{dom}(\mathring{\operatorname{curl}})\cap\operatorname{dom}(\operatorname{div}) \hookrightarrow L^{2,3}(\Omega),$
- $\big(\mu^{-1}\operatorname{dom}(\mathring{\operatorname{div}})\big)\cap\operatorname{dom}(\operatorname{curl}) \hookrightarrow L^{2,3}_{\mu}(\Omega) \quad\Leftrightarrow\quad \operatorname{dom}(\mathring{\operatorname{div}})\cap\operatorname{dom}(\operatorname{curl}) \hookrightarrow L^{2,3}(\Omega),$
- $(\mathring{\operatorname{grad}}, \mu^{-1}\mathring{\operatorname{curl}}, \mathring{\operatorname{div}}\,\mu)$ *is maximal compact, iff* $(\mathring{\operatorname{grad}}, \mathring{\operatorname{curl}}, \mathring{\operatorname{div}})$ *is maximal compact,*
- $-\operatorname{ind}(\widetilde{\mathcal{D}}^{\mathsf{Rhm}})^* = \operatorname{ind}\widetilde{\mathcal{D}}^{\mathsf{Rhm}} = \operatorname{ind}\mathcal{D}^{\mathsf{Rhm}} = p-m-n+1.$

At this point, see Lemma 2.5, Corollary 2.6, and (11), we note that the kernels and ranges are given by

$$\begin{aligned}
\ker\mathcal{D}^{\mathsf{Rhm}} &= K_2^{\mathsf{Rhm}}\times N_0^{\mathsf{Rhm}} = \mathcal{H}^{\mathsf{Rhm}}_N(\Omega)\times\{0\},\\
\ker(\mathcal{D}^{\mathsf{Rhm}})^* &= N_{2,*}^{\mathsf{Rhm}}\times K_1^{\mathsf{Rhm}} = \mathbb{R}_{\mathsf{pw}}\times\mathcal{H}^{\mathsf{Rhm}}_D(\Omega),\\
\operatorname{ran}\mathcal{D}^{\mathsf{Rhm}} &= (\ker(\mathcal{D}^{\mathsf{Rhm}})^*)^{\bot_{L^2(\Omega)\times L^{2,3}(\Omega)}} = \mathbb{R}_{\mathsf{pw}}^{\bot_{L^2(\Omega)}}\times\mathcal{H}^{\mathsf{Rhm}}_D(\Omega)^{\bot_{L^{2,3}(\Omega)}},\\
\operatorname{ran}(\mathcal{D}^{\mathsf{Rhm}})^* &= (\ker\mathcal{D}^{\mathsf{Rhm}})^{\bot_{L^{2,3}(\Omega)\times L^2(\Omega)}} = \mathcal{H}^{\mathsf{Rhm}}_N(\Omega)^{\bot_{L^{2,3}(\Omega)}}\times L^2(\Omega).
\end{aligned}$$

Finally, Corollary 2.9 yields additional results for the corresponding reduced operators

$$\begin{aligned}
\mathcal{D}^{\mathsf{Rhm}}_{\mathsf{red}} &= \mathcal{D}^{\mathsf{Rhm}}\big|_{(\ker\mathcal{D}^{\mathsf{Rhm}})^{\bot_{H_2\times H_0}}} = \begin{pmatrix}\mathring{\operatorname{div}} & 0\\ \operatorname{curl} & \mathring{\operatorname{grad}}\end{pmatrix}\Big|_{\mathcal{H}^{\mathsf{Rhm}}_N(\Omega)^{\bot_{L^{2,3}(\Omega)}}\times L^2(\Omega)},\\
(\mathcal{D}^{\mathsf{Rhm}}_{\mathsf{red}})^* &= (\mathcal{D}^{\mathsf{Rhm}})^*\big|_{(\ker(\mathcal{D}^{\mathsf{Rhm}})^*)^{\bot_{H_3\times H_1}}} = \begin{pmatrix}-\operatorname{grad} & \mathring{\operatorname{curl}}\\ 0 & -\operatorname{div}\end{pmatrix}\Big|_{\mathbb{R}_{\mathsf{pw}}^{\bot_{L^2(\Omega)}}\times\mathcal{H}^{\mathsf{Rhm}}_D(\Omega)^{\bot_{L^{2,3}(\Omega)}}}.
\end{aligned}$$

Corollary 3.12. *Let $\Omega\subset\mathbb{R}^3$ be a bounded weak Lipschitz domain with continuous boundary. Then*

$$\begin{aligned}
(\mathcal{D}^{\mathsf{Rhm}}_{\mathsf{red}})^{-1} &: \operatorname{ran}\mathcal{D}^{\mathsf{Rhm}}\to\operatorname{ran}(\mathcal{D}^{\mathsf{Rhm}})^*,\\
((\mathcal{D}^{\mathsf{Rhm}}_{\mathsf{red}})^*)^{-1} &: \operatorname{ran}(\mathcal{D}^{\mathsf{Rhm}})^*\to\operatorname{ran}\mathcal{D}^{\mathsf{Rhm}}
\end{aligned}$$

are compact. Furthermore,

$$\begin{aligned}
(\mathcal{D}^{\mathsf{Rhm}}_{\mathsf{red}})^{-1} &: \operatorname{ran}\mathcal{D}^{\mathsf{Rhm}}\to\operatorname{dom}\mathcal{D}^{\mathsf{Rhm}}_{\mathsf{red}},\\
((\mathcal{D}^{\mathsf{Rhm}}_{\mathsf{red}})^*)^{-1} &: \operatorname{ran}(\mathcal{D}^{\mathsf{Rhm}})^*\to\operatorname{dom}(\mathcal{D}^{\mathsf{Rhm}}_{\mathsf{red}})^*
\end{aligned}$$

are continuous and, equivalently, the Friedrichs-Poincaré type estimate

$$\big|(E,u)\big|_{L^{2,3}(\Omega)\times L^2(\Omega)} \le c_{\mathcal{D}^{\mathsf{Rhm}}}\big(|\operatorname{grad}u|^2_{L^{2,3}(\Omega)} + |\operatorname{div}E|^2_{L^2(\Omega)} + |\operatorname{curl}E|^2_{L^{2,3}(\Omega)}\big)^{1/2}$$

holds for all (E, u) *in*

$$\operatorname{dom} \mathcal{D}^{\mathsf{Rhm}}_{\mathsf{red}} = \big(H_0(\operatorname{div}, \Omega) \cap H(\operatorname{curl}, \Omega) \cap \mathcal{H}^{\mathsf{Rhm}}_N(\Omega)^{\perp_{L^{2,3}(\Omega)}}\big) \times H^1_0(\Omega)$$

or (u, E) *in*

$$\operatorname{dom}(\mathcal{D}^{\mathsf{Rhm}}_{\mathsf{red}})^* = \big(H^1(\Omega) \cap \mathbb{R}_{\mathsf{pw}}^{\perp_{L^2(\Omega)}}\big) \times \big(H_0(\operatorname{curl}, \Omega) \cap H(\operatorname{div}, \Omega) \cap \mathcal{H}^{\mathsf{Rhm}}_D(\Omega)^{\perp_{L^{2,3}(\Omega)}}\big)$$

with some optimal constant $c_{\mathcal{D}^{\mathsf{Rhm}}} > 0$*.*

Note that the latter estimate is an additive combination of the well known Friedrichs-Poincaré estimates for grad and the well known Maxwell estimates for $(\operatorname{curl}, \operatorname{div})$.

3.3. **The Dirac Operator.** We will flag up a relationship of the Dirac operator and Picard's extended Maxwell system. Let the assumptions of Theorem 3.8 be satisfied. The extended Maxwell operator is an operator that is surprisingly close to the Dirac operator. We shall carry out this construction in the following. Recall from Remark 3.9 that Picard's extended Maxwell system is given by the operator

$$\mathcal{M} := \begin{pmatrix} 0 & \mathcal{D} \\ -\mathcal{D}^* & 0 \end{pmatrix}, \qquad \mathcal{D} := \mathcal{D}^{\mathsf{Rhm}}.$$

Next, we shall introduce the Dirac operator. For this, we define the Pauli matrices

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Introducing

$$\begin{aligned} \mathcal{Q} \colon \operatorname{dom} \mathcal{Q} \subseteq L^{2,2}(\Omega) &\longrightarrow L^{2,2}(\Omega) \\ \psi &\longmapsto \sum_{j=1}^{3} \partial_j\, \sigma_j \psi = \begin{pmatrix} \partial_3 & \partial_1 - i\,\partial_2 \\ \partial_1 + i\,\partial_2 & -\,\partial_3 \end{pmatrix} \psi, \end{aligned}$$

we define the Dirac operator

$$\mathcal{L} := \begin{pmatrix} 0 & \mathcal{Q} \\ -\mathcal{Q}^* & 0 \end{pmatrix}.$$

We have not specified the domain of definition of $\mathcal{Q}$, yet. For now, we shall assume $C^{\infty,2}_c(\Omega) \subseteq \operatorname{dom} \mathcal{Q}$. We shall find the domain of definition of $\mathcal{Q}$ corresponding to $\mathcal{M}$; see also Proposition 3.13 below. We introduce the unitary operators from $L^{2,4}(\Omega)$ into itself

$$W := \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad U := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then the operators $\mathcal{L}$ (Dirac operator) and $\mathcal{M}$ (Picard's extended Maxwell operator) are unitarily equivalent. More precisely, we have with V from Proposition 3.13

$$\mathcal{M} = \begin{pmatrix} U & 0 \\ 0 & W \end{pmatrix} \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix} \mathcal{L} \begin{pmatrix} V^* & 0 \\ 0 & V^* \end{pmatrix} \begin{pmatrix} U^* & 0 \\ 0 & W^* \end{pmatrix},$$

$$\operatorname{dom} \mathcal{Q}^* \times \operatorname{dom} \mathcal{Q} := \begin{pmatrix} V^* & 0 \\ 0 & V^* \end{pmatrix} \begin{pmatrix} U^* & 0 \\ 0 & W^* \end{pmatrix} \big(\operatorname{dom} \mathcal{D}^* \times \operatorname{dom} \mathcal{D}\big) \begin{pmatrix} U & 0 \\ 0 & W \end{pmatrix} \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix}$$

and, consequently, $\mathcal{Q}$ with domain $\operatorname{dom}(V^* U^* \mathcal{D} W V) = \operatorname{dom}(\mathcal{D} W V)$ is a Fredholm operator. Moreover, we have $\operatorname{ind} \mathcal{L} = 0$ and

$$\operatorname{ind} \mathcal{Q} = \operatorname{ind} \mathcal{D} = p - m - n + 1.$$

We conclude this section by stating the missing proposition used above. The proofs of which are straightforward and will therefore be omitted. In a slightly similar fashion, they can be found [25]. For the next result we use $L^2_{\mathbb{R}}(\Omega)$ and $L^2_{\mathbb{C}}(\Omega)$ to denote the Hilbert space $L^2(\Omega)$ with the reals and the complex numbers as respective underlying field.

Proposition 3.13 (Realification of $\mathcal{L}$)**.** *It holds:*

(i) $V : L^2_{\mathbb{C}}(\Omega) \to L^{2,2}_{\mathbb{R}}(\Omega)$ *with* $Vf := (\Re f, \Im f)$ *is unitary.*

(ii) $ViV^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

(iii) $$\widetilde{\mathcal{Q}} := V\mathcal{Q}V^* = \partial_1 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + \partial_2 \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} + \partial_3 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

with $\operatorname{dom} \widetilde{\mathcal{Q}} = V \operatorname{dom} \mathcal{Q} V^*$.

4. The First Biharmonic Complex and Its Indices

In this section, we focus on our first main result and properly introduce the operators involved in the formulation of Theorem 1.1. Thus, we introduce the first biharmonic complex (see [20, 21]) constructed for biharmonic problems and general relativity, but also relevant in problems for elasticity. It will be interesting to see that the differential operator is apparently of mixed order rather than just of first order. It it worth noting that the apparently leading order term is *not* dominating the lower order differential operators.

Definition 4.1. *Let* $\Omega \subseteq \mathbb{R}^3$ *be an open set. We put*

$$\begin{aligned} \operatorname{Gradgrad}_c &: C^\infty_c(\Omega) \subseteq L^2(\Omega) \longrightarrow L^{2,3\times 3}_{\mathbb{S}}(\Omega), & \phi &\longmapsto \operatorname{Gradgrad}\phi, \\ \operatorname{Curl}_c &: C^{\infty,3\times 3}_{c,\mathbb{S}}(\Omega) \subseteq L^{2,3\times 3}_{\mathbb{S}}(\Omega) \longrightarrow L^{2,3\times 3}_{\mathbb{T}}(\Omega), & \Phi &\longmapsto \operatorname{Curl}\Phi, \\ \operatorname{Div}_c &: C^{\infty,3\times 3}_{c,\mathbb{T}}(\Omega) \subseteq L^{2,3\times 3}_{\mathbb{T}}(\Omega) \longrightarrow L^{2,3}(\Omega), & \Phi &\longmapsto \operatorname{Div}\Phi, \end{aligned}$$

and further define the densely defined and closed linear operators

$$\begin{aligned} \operatorname{divDiv}_{\mathbb{S}} &:= \operatorname{Gradgrad}_c^*, & \mathring{\operatorname{Gradgrad}} &:= \operatorname{divDiv}_{\mathbb{S}}^* = \overline{\operatorname{Gradgrad}_c}, \\ \operatorname{symCurl}_{\mathbb{T}} &:= \operatorname{Curl}_c^*, & \mathring{\operatorname{Curl}}_{\mathbb{S}} &:= \operatorname{symCurl}_{\mathbb{T}}^* = \overline{\operatorname{Curl}_c}, \\ \operatorname{devGrad} &:= -\operatorname{Div}_c^*, & \mathring{\operatorname{Div}}_{\mathbb{T}} &:= -\operatorname{devGrad}^* = \overline{\operatorname{devGrad}_c}. \end{aligned}$$

We shall apply the index theorem in the following situation of the first biharmonic complex:

$$\begin{aligned} A_0 &:= \mathring{\operatorname{Gradgrad}}, & A_1 &:= \mathring{\operatorname{Curl}}_{\mathbb{S}}, & A_2 &:= \mathring{\operatorname{Div}}_{\mathbb{T}}, \\ A_0^* &= \operatorname{divDiv}_{\mathbb{S}}, & A_1^* &= \operatorname{symCurl}_{\mathbb{T}}, & A_2^* &= -\operatorname{devGrad}, \end{aligned}$$

$$\mathcal{D}^{\mathsf{bih},1} := \begin{pmatrix} A_2 & 0 \\ A_1^* & A_0 \end{pmatrix} = \begin{pmatrix} \mathring{\operatorname{Div}}_{\mathbb{T}} & 0 \\ \operatorname{symCurl}_{\mathbb{T}} & \mathring{\operatorname{Gradgrad}} \end{pmatrix},$$

$$(\mathcal{D}^{\mathsf{bih},1})^* = \begin{pmatrix} A_2^* & A_1 \\ 0 & A_0^* \end{pmatrix} = \begin{pmatrix} -\operatorname{devGrad} & \mathring{\operatorname{Curl}}_{\mathbb{S}} \\ 0 & \operatorname{divDiv}_{\mathbb{S}} \end{pmatrix},$$

$$\begin{aligned} &\{0\} \xrightarrow{\iota_{\{0\}}} L^2(\Omega) \xrightarrow{\mathring{\operatorname{Gradgrad}}} L^{2,3\times 3}_{\mathbb{S}}(\Omega) \xrightarrow{\mathring{\operatorname{Curl}}_{\mathbb{S}}} L^{2,3\times 3}_{\mathbb{T}}(\Omega) \xrightarrow{\mathring{\operatorname{Div}}_{\mathbb{T}}} L^{2,3}(\Omega) \xrightarrow{\pi_{\mathsf{RT}_{\mathsf{pw}}}} \mathsf{RT}_{\mathsf{pw}}, \\ &\{0\} \xleftarrow{\pi_{\{0\}}} L^2(\Omega) \xleftarrow{\operatorname{divDiv}_{\mathbb{S}}} L^{2,3\times 3}_{\mathbb{S}}(\Omega) \xleftarrow{\operatorname{symCurl}_{\mathbb{T}}} L^{2,3\times 3}_{\mathbb{T}}(\Omega) \xleftarrow{-\operatorname{devGrad}} L^{2,3}(\Omega) \xleftarrow{\iota_{\mathsf{RT}_{\mathsf{pw}}}} \mathsf{RT}_{\mathsf{pw}}. \end{aligned} \tag{13}$$

The foundation of the index theorem to hold is the following compactness result established by Pauly and Zulehner. Note that it holds $\operatorname{dom}(\mathring{\operatorname{Gradgrad}}) = H_0^2(\Omega)$ and $\operatorname{dom}(\operatorname{devGrad}) = H^{1,3}(\Omega)$.

Theorem 4.2 ([21, Lemma 3.22, Theorem 3.23])**.** *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded strong Lipschitz domain. Then* $(\mathring{\operatorname{Gradgrad}}, \mathring{\operatorname{Curl}}_{\mathbb{S}}, \mathring{\operatorname{Div}}_{\mathbb{T}})$ *is a maximal compact Hilbert complex.*

We observe and define

$$
\begin{aligned}
N_0^{\mathsf{bih},1} &= \ker A_0 = \ker(\mathring{\operatorname{Gradgrad}}),\\
N_{2,*}^{\mathsf{bih},1} &= \ker A_2^* = \ker(\operatorname{devGrad}),\\
K_1^{\mathsf{bih},1} &= \ker A_1 \cap \ker A_0^* = \ker(\mathring{\operatorname{Curl}}_{\mathbb{S}}) \cap \ker(\operatorname{divDiv}_{\mathbb{S}}) =: \mathcal{H}_{D,\mathbb{S}}^{\mathsf{bih},1}(\Omega),\\
K_2^{\mathsf{bih},1} &= \ker A_2 \cap \ker A_1^* = \ker(\mathring{\operatorname{Div}}_{\mathbb{T}}) \cap \ker(\operatorname{symCurl}_{\mathbb{T}}) =: \mathcal{H}_{N,\mathbb{T}}^{\mathsf{bih},1}(\Omega).
\end{aligned} \tag{14}
$$

The dimensions of the cohomology groups are given as follows.

Theorem 4.3. *Let $\Omega \subseteq \mathbb{R}^3$ be open and bounded with continuous boundary. Moreover, suppose Assumption 3. Then*

$$
\dim \mathcal{H}_{D,\mathbb{S}}^{\mathsf{bih},1}(\Omega) = 4(m-1), \qquad \dim \mathcal{H}_{N,\mathbb{T}}^{\mathsf{bih},1}(\Omega) = 4p.
$$

Proof. We postpone the proof to the Appendix. □

Let us introduce the space of piecewise Raviart-Thomas fields by

$$
\mathsf{RT}_{\mathsf{pw}} := \big\{ v \in L^{2,3}(\Omega) : \forall\, C(\text{con. cp.}) \subseteq \Omega \quad \exists\, \alpha_C \in \mathbb{R},\, \beta_C \in \mathbb{R}^3 : u|_C(x) = \alpha_C x + \beta_C \big\}.
$$

The proper formulation of the first main result, Theorem 1.1, reads as follows.

Theorem 4.4. *Let $\Omega \subset \mathbb{R}^3$ be a bounded strong Lipschitz domain. Then $\mathcal{D}^{\mathsf{bih},1}$ is a Fredholm operator with index*

$$
\operatorname{ind} \mathcal{D}^{\mathsf{bih},1} = \dim N_0^{\mathsf{bih},1} - \dim K_1^{\mathsf{bih},1} + \dim K_2^{\mathsf{bih},1} - \dim N_{2,*}^{\mathsf{bih},1}.
$$

If additionally Assumption 3 holds, then

$$
\operatorname{ind} \mathcal{D}^{\mathsf{bih},1} = 4(p - m - n + 1).
$$

Proof. Using Theorem 4.2 apply Theorem 2.8 together with (14), the observations

$$
N_0^{\mathsf{bih},1} = \ker(\mathring{\operatorname{Gradgrad}}) = \{0\}, \qquad N_{2,*}^{\mathsf{bih},1} = \ker(\operatorname{devGrad}) = \mathsf{RT}_{\mathsf{pw}}, \tag{15}
$$

see [21, Lemma 3.2, Lemma 3.3], and Theorem 4.3. □

Remark 4.5. *By Theorem 2.8 the adjoint $(\mathcal{D}^{\mathsf{bih},1})^*$ is Fredholm as well with index simply given by* $\operatorname{ind}(\mathcal{D}^{\mathsf{bih},1})^* = -\operatorname{ind} \mathcal{D}^{\mathsf{bih},1}$. *Similar to Remark 3.9 we define the extended first biharmonic operator*

$$
\mathcal{M}^{\mathsf{bih},1} := \begin{pmatrix} 0 & \mathcal{D}^{\mathsf{bih},1} \\ -(\mathcal{D}^{\mathsf{bih},1})^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \mathring{\operatorname{Div}}_{\mathbb{T}} & 0 \\ 0 & 0 & \operatorname{symCurl}_{\mathbb{T}} & \mathring{\operatorname{Gradgrad}} \\ \operatorname{devGrad} & -\mathring{\operatorname{Curl}}_{\mathbb{S}} & 0 & 0 \\ 0 & -\operatorname{divDiv}_{\mathbb{S}} & 0 & 0 \end{pmatrix}
$$

with $(\mathcal{M}^{\mathsf{bih},1})^* = -\mathcal{M}^{\mathsf{bih},1}$ *and* $\operatorname{ind} \mathcal{M}^{\mathsf{bih},1} = 0$.

4.1. **Some More Results.** Inhomogeneous and anisotropic media may also be considered for the first biharmonic complex, cf. Remark 3.11.

Remark 4.6. *Let* $\lambda_0 := \mathrm{Id}$, $\lambda_3 := \mathrm{Id}$, *and* $\lambda_1 := \varepsilon, \lambda_2 := \mu : \Omega \to \mathbb{R}^{3\times3\times3\times3}$ *being symmetric and uniformly positive definite* $L^\infty(\Omega)$*-tensor fields. Moreover, let us introduce* $L^{2,3\times3}_{\mathbb{S},\varepsilon}(\Omega) := \widetilde{H}_1 := \big(L^{2,3\times3}_{\mathbb{S}}(\Omega), \langle\varepsilon\,\cdot\,,\,\cdot\,\rangle_{L^{2,3\times3}_{\mathbb{S}}(\Omega)}\big)$ *and similarly* $L^{2,3\times3}_{\mathbb{T},\mu}(\Omega) := \widetilde{H}_2$ *as well as* $\widetilde{H}_0 = H_0 = L^2(\Omega)$, $\widetilde{H}_3 = H_3 = L^{2,3}(\Omega)$. *We look at*

$$\widetilde{A}_0 := \mathrm{Grad}\mathring{\mathrm{grad}}, \qquad \widetilde{A}_1 := \mu^{-1}\,\mathring{\mathrm{Curl}}_{\mathbb{S}}, \qquad \widetilde{A}_2 := \mathring{\mathrm{Div}}_{\mathbb{T}}\,\mu,$$
$$\widetilde{A}_0^* = \mathrm{divDiv}_{\mathbb{S}}\,\varepsilon, \qquad \widetilde{A}_1^* = \varepsilon^{-1}\,\mathrm{symCurl}_{\mathbb{T}}, \qquad \widetilde{A}_2^* = -\,\mathrm{devGrad},$$

$$\widetilde{\mathcal{D}}^{\mathsf{bih},1} := \begin{pmatrix} \widetilde{A}_2 & 0 \\ \widetilde{A}_1^* & \widetilde{A}_0 \end{pmatrix} = \begin{pmatrix} \mathring{\mathrm{Div}}_{\mathbb{T}}\,\mu & 0 \\ \varepsilon^{-1}\,\mathrm{symCurl}_{\mathbb{T}} & \mathrm{Grad}\mathring{\mathrm{grad}} \end{pmatrix},$$
$$(\widetilde{\mathcal{D}}^{\mathsf{bih},1})^* = \begin{pmatrix} \widetilde{A}_2^* & \widetilde{A}_1 \\ 0 & \widetilde{A}_0^* \end{pmatrix} = \begin{pmatrix} -\,\mathrm{devGrad} & \mu^{-1}\,\mathring{\mathrm{Curl}}_{\mathbb{S}} \\ 0 & \mathrm{divDiv}_{\mathbb{S}}\,\varepsilon \end{pmatrix},$$

i.e., the first biharmonic complex, cf. (13),

$$\{0\} \xrightarrow{\iota_{\{0\}}} L^2(\Omega) \xrightarrow{\mathrm{Grad}\mathring{\mathrm{grad}}} L^{2,3\times3}_{\mathbb{S},\varepsilon}(\Omega) \xrightarrow{\mu^{-1}\mathring{\mathrm{Curl}}_{\mathbb{S}}} L^{2,3\times3}_{\mathbb{T},\mu}(\Omega) \xrightarrow{\mathring{\mathrm{Div}}_{\mathbb{T}}\,\mu} L^{2,3}(\Omega) \xrightarrow{\pi_{\mathsf{RT}_{\mathsf{pw}}}} \mathsf{RT}_{\mathsf{pw}},$$
$$\{0\} \xleftarrow{\pi_{\{0\}}} L^2(\Omega) \xleftarrow{\mathrm{divDiv}_{\mathbb{S}}\,\varepsilon} L^{2,3\times3}_{\mathbb{S},\varepsilon}(\Omega) \xleftarrow{\varepsilon^{-1}\,\mathrm{symCurl}_{\mathbb{T}}} L^{2,3\times3}_{\mathbb{T},\mu}(\Omega) \xleftarrow{-\,\mathrm{devGrad}} L^{2,3}(\Omega) \xleftarrow{\iota_{\mathsf{RT}_{\mathsf{pw}}}} \mathsf{RT}_{\mathsf{pw}}. \tag{16}$$

Lemma 2.12, Lemma 2.13, and Theorem 2.14 show that the compactness properties, the dimensions of the kernels and cohomology groups, the maximal compactness, and the Fredholm indices of the first biharmonic complex do not dependent of the material weights ε *and* μ. *More precisely,*

- $$\dim\big(\ker(\mathring{\mathrm{Curl}}_{\mathbb{S}}) \cap (\varepsilon^{-1}\ker(\mathrm{divDiv}_{\mathbb{S}}))\big) = \dim\big(\ker(\mathring{\mathrm{Curl}}_{\mathbb{S}}) \cap \ker(\mathrm{divDiv}_{\mathbb{S}})\big) = \dim \mathcal{H}^{\mathsf{bih},1}_{D,\mathbb{S}}(\Omega) = 4(m-1),$$
- $$\dim\big((\mu^{-1}\ker(\mathring{\mathrm{Div}}_{\mathbb{T}})) \cap \ker(\mathrm{symCurl}_{\mathbb{T}})\big) = \dim\big(\ker(\mathring{\mathrm{Div}}_{\mathbb{T}}) \cap \ker(\mathrm{symCurl}_{\mathbb{T}})\big) = \dim \mathcal{H}^{\mathsf{bih},1}_{N,\mathbb{T}}(\Omega) = 4p,$$
- $$\mathrm{dom}(\mathring{\mathrm{Curl}}_{\mathbb{S}}) \cap (\varepsilon^{-1}\,\mathrm{dom}(\mathrm{divDiv}_{\mathbb{S}})) \hookrightarrow L^{2,3\times3}_{\mathbb{S},\varepsilon}(\Omega) \quad\Leftrightarrow\quad \mathrm{dom}(\mathring{\mathrm{Curl}}_{\mathbb{S}}) \cap \mathrm{dom}(\mathrm{divDiv}_{\mathbb{S}}) \hookrightarrow L^{2,3\times3}_{\mathbb{S}}(\Omega),$$
- $$(\mu^{-1}\,\mathrm{dom}(\mathring{\mathrm{Div}}_{\mathbb{T}})) \cap \mathrm{dom}(\mathrm{symCurl}_{\mathbb{T}}) \hookrightarrow L^{2,3\times3}_{\mathbb{T},\mu}(\Omega) \quad\Leftrightarrow\quad \mathrm{dom}(\mathring{\mathrm{Div}}_{\mathbb{T}}) \cap \mathrm{dom}(\mathrm{symCurl}_{\mathbb{T}}) \hookrightarrow L^{2,3\times3}_{\mathbb{T}}(\Omega),$$
- $(\mathrm{Grad}\mathring{\mathrm{grad}}, \mu^{-1}\,\mathring{\mathrm{Curl}}_{\mathbb{S}}, \mathring{\mathrm{Div}}_{\mathbb{T}}\,\mu)$ *max cpt, iff* $(\mathrm{Grad}\mathring{\mathrm{grad}}, \mathring{\mathrm{Curl}}_{\mathbb{S}}, \mathring{\mathrm{Div}}_{\mathbb{T}})$ *max cpt,*
- $$-\,\mathrm{ind}(\widetilde{\mathcal{D}}^{\mathsf{bih},1})^* = \mathrm{ind}\,\widetilde{\mathcal{D}}^{\mathsf{bih},1} = \mathrm{ind}\,\mathcal{D}^{\mathsf{bih},1} = 4(p-m-n+1).$$

Note that the kernels and ranges are given by

$$\ker\mathcal{D}^{\mathsf{bih},1} = K_2^{\mathsf{bih},1} \times N_0^{\mathsf{bih},1} = \mathcal{H}^{\mathsf{bih},1}_{N,\mathbb{T}}(\Omega) \times \{0\},$$
$$\ker(\mathcal{D}^{\mathsf{bih},1})^* = N_{2,*}^{\mathsf{bih},1} \times K_1^{\mathsf{bih},1} = \mathsf{RT}_{\mathsf{pw}} \times \mathcal{H}^{\mathsf{bih},1}_{D,\mathbb{S}}(\Omega),$$
$$\mathrm{ran}\,\mathcal{D}^{\mathsf{bih},1} = (\ker(\mathcal{D}^{\mathsf{bih},1})^*)^{\perp_{L^{2,3}(\Omega)\times L^{2,3\times3}_{\mathbb{S}}(\Omega)}} = \mathsf{RT}_{\mathsf{pw}}^{\perp_{L^{2,3}(\Omega)}} \times \mathcal{H}^{\mathsf{bih},1}_{D,\mathbb{S}}(\Omega)^{\perp_{L^{2,3\times3}_{\mathbb{S}}(\Omega)}},$$
$$\mathrm{ran}(\mathcal{D}^{\mathsf{bih},1})^* = (\ker\mathcal{D}^{\mathsf{bih},1})^{\perp_{L^{2,3\times3}_{\mathbb{T}}(\Omega)\times L^2(\Omega)}} = \mathcal{H}^{\mathsf{bih},1}_{N,\mathbb{T}}(\Omega)^{\perp_{L^{2,3\times3}_{\mathbb{T}}(\Omega)}} \times L^2(\Omega),$$

see Lemma 2.5, Corollary 2.6, and (15). Corollary 2.9 shows additional results for the corresponding reduced operators

$$\mathcal{D}^{\mathsf{bih},1}_{\mathsf{red}} = \mathcal{D}^{\mathsf{bih},1}|_{(\ker\mathcal{D}^{\mathsf{bih},1})^{\perp_{H_2\times H_0}}} = \begin{pmatrix} \mathring{\operatorname{Div}}_{\mathbb{T}} & 0 \\ \operatorname{symCurl}_{\mathbb{T}} & \mathring{\operatorname{Gradgrad}} \end{pmatrix}\Big|_{\mathcal{H}^{\mathsf{bih},1}_{N,\mathbb{T}}(\Omega)^{\perp_{L^{2,3\times 3}_{\mathbb{T}}(\Omega)}}\times L^2(\Omega)},$$

$$(\mathcal{D}^{\mathsf{bih},1}_{\mathsf{red}})^* = (\mathcal{D}^{\mathsf{bih},1})^*|_{(\ker(\mathcal{D}^{\mathsf{bih},1})^*)^{\perp_{H_3\times H_1}}} = \begin{pmatrix} -\operatorname{devGrad} & \mathring{\operatorname{Curl}}_{\mathbb{S}} \\ 0 & \operatorname{divDiv}_{\mathbb{S}} \end{pmatrix}\Big|_{\mathsf{RT}_{\mathsf{pw}}^{\perp_{L^{2,3}(\Omega)}}\times\mathcal{H}^{\mathsf{bih},1}_{D,\mathbb{S}}(\Omega)^{\perp_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)}}}.$$

Corollary 4.7. *Let $\Omega\subset\mathbb{R}^3$ be a bounded strong Lipschitz domain. Then*

$$(\mathcal{D}^{\mathsf{bih},1}_{\mathsf{red}})^{-1} : \operatorname{ran}\mathcal{D}^{\mathsf{bih},1} \to \operatorname{ran}(\mathcal{D}^{\mathsf{bih},1})^*,$$
$$((\mathcal{D}^{\mathsf{bih},1}_{\mathsf{red}})^*)^{-1} : \operatorname{ran}(\mathcal{D}^{\mathsf{bih},1})^* \to \operatorname{ran}\mathcal{D}^{\mathsf{bih},1}$$

are compact. Furthermore,

$$(\mathcal{D}^{\mathsf{bih},1}_{\mathsf{red}})^{-1} : \operatorname{ran}\mathcal{D}^{\mathsf{bih},1} \to \operatorname{dom}\mathcal{D}^{\mathsf{bih},1}_{\mathsf{red}},$$
$$((\mathcal{D}^{\mathsf{bih},1}_{\mathsf{red}})^*)^{-1} : \operatorname{ran}(\mathcal{D}^{\mathsf{bih},1})^* \to \operatorname{dom}(\mathcal{D}^{\mathsf{bih},1}_{\mathsf{red}})^*$$

are continuous and, equivalently, the Friedrichs-Poincaré type estimates

$$\big|(T,u)\big|_{L^{2,3\times 3}_{\mathbb{T}}(\Omega)\times L^2(\Omega)} \leq c_{\mathcal{D}^{\mathsf{bih},1}}\big(|\operatorname{Gradgrad}u|^2_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)} + |\operatorname{Div}T|^2_{L^{2,3}(\Omega)} + |\operatorname{symCurl}T|^2_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)}\big)^{1/2},$$

$$\big|(v,S)\big|_{L^{2,3}(\Omega)\times L^{2,3\times 3}_{\mathbb{S}}(\Omega)} \leq c_{\mathcal{D}^{\mathsf{bih},1}}\big(|\operatorname{devGrad}v|^2_{L^{2,3\times 3}_{\mathbb{T}}(\Omega)} + |\operatorname{divDiv}S|^2_{L^2(\Omega)} + |\operatorname{Curl}S|^2_{L^{2,3\times 3}_{\mathbb{T}}(\Omega)}\big)^{1/2}$$

hold for all (T,u) in

$$\operatorname{dom}\mathcal{D}^{\mathsf{bih},1}_{\mathsf{red}} = \big(\operatorname{dom}(\mathring{\operatorname{Div}}_{\mathbb{T}})\cap\operatorname{dom}(\operatorname{symCurl}_{\mathbb{T}})\cap\mathcal{H}^{\mathsf{bih},1}_{N,\mathbb{T}}(\Omega)^{\perp_{L^{2,3\times 3}_{\mathbb{T}}(\Omega)}}\big)\times H^2_0(\Omega)$$

for all (v,S) in

$$\operatorname{dom}(\mathcal{D}^{\mathsf{bih},1}_{\mathsf{red}})^* = \big(H^{1,3}(\Omega)\cap\mathsf{RT}_{\mathsf{pw}}^{\perp_{L^{2,3}(\Omega)}}\big) \times\big(\operatorname{dom}(\mathring{\operatorname{Curl}}_{\mathbb{S}})\cap\operatorname{dom}(\operatorname{divDiv}_{\mathbb{S}})\cap\mathcal{H}^{\mathsf{bih},1}_{D,\mathbb{S}}(\Omega)^{\perp_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)}}\big)$$

with some optimal constant $c_{\mathcal{D}^{\mathsf{bih},1}}>0$.

5. The Second Biharmonic Complex and Its Indices

Definition 5.1. *Let $\Omega\subseteq\mathbb{R}^3$ be an open set. We put*

$$\operatorname{devGrad}_c : C^{\infty,3}_c(\Omega)\subseteq L^{2,3}(\Omega)\longrightarrow L^{2,3\times 3}_{\mathbb{T}}(\Omega), \qquad \phi\longmapsto\operatorname{devGrad}\phi,$$
$$\operatorname{symCurl}_c : C^{\infty,3\times 3}_{c,\mathbb{T}}(\Omega)\subseteq L^{2,3\times 3}_{\mathbb{T}}(\Omega)\longrightarrow L^{2,3\times 3}_{\mathbb{S}}(\Omega), \qquad \Phi\longmapsto\operatorname{symCurl}\Phi,$$
$$\operatorname{divDiv}_c : C^{\infty,3\times 3}_{c,\mathbb{S}}(\Omega)\subseteq L^{2,3\times 3}_{\mathbb{S}}(\Omega)\longrightarrow L^2(\Omega), \qquad \Phi\longmapsto\operatorname{divDiv}\Phi,$$

and further define the densely defined and closed linear operators

$$\operatorname{Div}_{\mathbb{T}} := -\operatorname{devGrad}_c^*, \qquad \mathring{\operatorname{devGrad}} := -\operatorname{Div}_{\mathbb{T}}^* = \overline{\operatorname{devGrad}_c},$$
$$\operatorname{Curl}_{\mathbb{S}} := \operatorname{symCurl}_c^*, \qquad \mathring{\operatorname{symCurl}}_{\mathbb{T}} := \operatorname{Curl}_{\mathbb{S}}^* = \overline{\operatorname{symCurl}_c},$$
$$\operatorname{Gradgrad} := \operatorname{divDiv}_c^*, \qquad \mathring{\operatorname{divDiv}}_{\mathbb{S}} := \operatorname{Gradgrad}^* = \overline{\operatorname{divDiv}_c}.$$

We shall apply the index theorem in the following situation of the second biharmonic complex:

$$\begin{aligned}
A_0 &:= \operatorname{dev}\mathring{\operatorname{Grad}}, & A_1 &:= \operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}}, & A_2 &:= \operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}},\\
A_0^* &= -\operatorname{Div}_{\mathbb{T}}, & A_1^* &= \operatorname{Curl}_{\mathbb{S}}, & A_2^* &= \operatorname{Gradgrad},
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}^{\mathsf{bih},2} &:= \begin{pmatrix} A_2 & 0 \\ A_1^* & A_0 \end{pmatrix} = \begin{pmatrix} \operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}} & 0 \\ \operatorname{Curl}_{\mathbb{S}} & \operatorname{dev}\mathring{\operatorname{Grad}} \end{pmatrix},\\
(\mathcal{D}^{\mathsf{bih},2})^* &= \begin{pmatrix} A_2^* & A_1 \\ 0 & A_0^* \end{pmatrix} = \begin{pmatrix} \operatorname{Gradgrad} & \operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}} \\ 0 & -\operatorname{Div}_{\mathbb{T}} \end{pmatrix},
\end{aligned}$$

$$\begin{gathered}
\{0\} \xrightarrow{\iota_{\{0\}}} L^{2,3}(\Omega) \xrightarrow{\operatorname{dev}\mathring{\operatorname{Grad}}} L^{2,3\times 3}_{\mathbb{T}}(\Omega) \xrightarrow{\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}}} L^{2,3\times 3}_{\mathbb{S}}(\Omega) \xrightarrow{\operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}}} L^{2}(\Omega) \xrightarrow{\pi_{\mathsf{P}^1_{\mathsf{pw}}}} \mathsf{P}^1_{\mathsf{pw}},\\
\{0\} \xleftarrow{\pi_{\{0\}}} L^{2,3}(\Omega) \xleftarrow{-\operatorname{Div}_{\mathbb{T}}} L^{2,3\times 3}_{\mathbb{T}}(\Omega) \xleftarrow{\operatorname{Curl}_{\mathbb{S}}} L^{2,3\times 3}_{\mathbb{S}}(\Omega) \xleftarrow{\operatorname{Gradgrad}} L^{2}(\Omega) \xleftarrow{\iota_{\mathsf{P}^1_{\mathsf{pw}}}} \mathsf{P}^1_{\mathsf{pw}}.
\end{gathered} \tag{17}$$

Note that $\operatorname{dom}(\operatorname{dev}\mathring{\operatorname{Grad}}) = H^{1,3}_0(\Omega)$ by [21, Lemma 3.2].

Lemma 5.2. *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded strong Lipschitz domain. Then it holds that* $\operatorname{dom}(\operatorname{Gradgrad}) = H^2(\Omega)$ *and that there exists $c > 0$ such that for all $u \in H^2(\Omega)$*

$$c\,|u|_{H^2(\Omega)} \leq |u|_{L^2(\Omega)} + |\operatorname{Grad}\operatorname{grad} u|_{L^{2,3\times3}(\Omega)}.$$

Proof. Let $u \in \operatorname{dom}(\operatorname{Gradgrad})$. Then $\operatorname{grad} u \in H^{-1,3}(\Omega)$ and $\operatorname{Grad}\operatorname{grad} u \in L^{2,3\times3}(\Omega)$. Necas' regularity yields $\operatorname{grad} u \in L^{2,3}(\Omega)$ and thus $u \in H^1(\Omega)$ and $\operatorname{grad} u \in H^{1,3}(\Omega)$. Hence $u \in H^2(\Omega)$ and by Necas' inequality we have

$$\begin{aligned}
|\operatorname{grad} u|_{L^{2,3}(\Omega)} &\leq c\big(|\operatorname{grad} u|_{H^{-1,3}(\Omega)} + |\operatorname{Grad}\operatorname{grad} u|_{H^{-1,3\times3}(\Omega)}\big)\\
&\leq c\big(|u|_{L^2(\Omega)} + |\operatorname{Grad}\operatorname{grad} u|_{L^{2,3\times3}(\Omega)}\big),
\end{aligned}$$

showing the desired estimate. □

Theorem 5.3. *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded strong Lipschitz domain. Then the second biharmonic complex* $(\operatorname{dev}\mathring{\operatorname{Grad}}, \operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}}, \operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}})$ *is a maximal compact Hilbert complex.*

Proof. The assertions can be shown by using the 'FA-ToolBox' from [17, 18, 19, 21, 22]. The compact embeddings for topologically trivial domains can be proved by a combination of Helmholtz decompositions and regular potentials as in [21, Theorem 3.10, Theorem 3.12, Lemma 3.19] or in [22, Theorem 3.5, Corollary 3.6, Lemma 3.8]. For general strong Lipschitz domains we follow the proof of [21, Lemma 3.22] or [22, Theorem 3.17]. Due to the boundary condition attached to the 'second order' operator $\operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}}$ the proofs have to be modified at some places leading to some additional (but handable) difficulties. □

We observe and define

$$\begin{aligned}
N_0^{\mathsf{bih},2} &= \ker A_0 = \ker(\operatorname{dev}\mathring{\operatorname{Grad}}),\\
N_{2,*}^{\mathsf{bih},2} &= \ker A_2^* = \ker(\operatorname{Gradgrad}),\\
K_1^{\mathsf{bih},2} &= \ker A_1 \cap \ker A_0^* = \ker(\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}}) \cap \ker(\operatorname{Div}_{\mathbb{T}}) =: \mathcal{H}^{\mathsf{bih},2}_{D,\mathbb{T}}(\Omega),\\
K_2^{\mathsf{bih},2} &= \ker A_2 \cap \ker A_1^* = \ker(\operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}}) \cap \ker(\operatorname{Curl}_{\mathbb{S}}) =: \mathcal{H}^{\mathsf{bih},2}_{N,\mathbb{S}}(\Omega).
\end{aligned} \tag{18}$$

Let us introduce the space of piecewise first order polynomials by

$$\mathsf{P}^1_{\mathsf{pw}} := \big\{v \in L^2(\Omega) : \forall\, C(\text{con. cp.}) \subseteq \Omega \quad \exists\, \alpha_C \in \mathbb{R},\, \beta_C \in \mathbb{R}^3 : u|_C(x) = \alpha_C + \beta_C \cdot x\big\}.$$

Theorem 5.4. *Let $\Omega \subseteq \mathbb{R}^3$ be open and bounded with continuous boundary. Moreover, suppose Assumption 3. Then*

$$\dim \mathcal{H}^{\mathsf{bih},2}_{D,\mathbb{T}}(\Omega) = 4(m-1), \qquad \dim \mathcal{H}^{\mathsf{bih},2}_{N,\mathbb{S}}(\Omega) = 4p.$$

Proof. We postpone the proof to the Appendix. □

Theorem 5.5. *Let $\Omega \subset \mathbb{R}^3$ be a bounded strong Lipschitz domain. Then $\mathcal{D}^{\mathsf{bih},2}$ is a Fredholm operator with index*

$$\operatorname{ind} \mathcal{D}^{\mathsf{bih},2} = \dim N_0^{\mathsf{bih},2} - \dim K_1^{\mathsf{bih},2} + \dim K_2^{\mathsf{bih},2} - \dim N_{2,*}^{\mathsf{bih},2}.$$

If additionally Assumption 3 holds, then

$$\operatorname{ind} \mathcal{D}^{\mathsf{bih},2} = 4(p-m-n+1).$$

Proof. Using Theorem 5.3 apply Theorem 2.8 together with (18), the observations

$$N_0^{\mathsf{bih},2} = \ker(\operatorname{dev}\mathring{\operatorname{Grad}}) = \{0\}, \qquad N_{2,*}^{\mathsf{bih},2} = \ker(\operatorname{Grad}\operatorname{grad}) = \mathsf{P}^1_{\mathsf{pw}} \tag{19}$$

by using [21, Lemma 3.2 (i)], and Theorem 5.4. □

Remark 5.6. *By Theorem 2.8 the adjoint $(\mathcal{D}^{\mathsf{bih},2})^*$ is Fredholm as well with index simply given by* $\operatorname{ind}(\mathcal{D}^{\mathsf{bih},2})^* = -\operatorname{ind}\mathcal{D}^{\mathsf{bih},2}$. *Similar to Remark 3.9 and Remark 4.5 we define the extended second biharmonic operator*

$$\mathcal{M}^{\mathsf{bih},2} := \begin{pmatrix} 0 & \mathcal{D}^{\mathsf{bih},2} \\ -(\mathcal{D}^{\mathsf{bih},2})^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}} & 0 \\ 0 & 0 & \operatorname{Curl}_{\mathbb{S}} & \operatorname{dev}\mathring{\operatorname{Grad}} \\ -\operatorname{Grad}\operatorname{grad} & -\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}} & 0 & 0 \\ 0 & \operatorname{Div}_{\mathbb{T}} & 0 & 0 \end{pmatrix}$$

with $(\mathcal{M}^{\mathsf{bih},2})^* = -\mathcal{M}^{\mathsf{bih},2}$ *and* $\operatorname{ind}\mathcal{M}^{\mathsf{bih},2} = 0$.

5.1. **Some More Results.** Inhomogeneous and anisotropic media may also be considered for the second biharmonic complex, cf. Remark 3.11 and Remark 4.6.

Remark 5.7. *Recall the notations from Remark 4.6 and set $\lambda_0 := \mathrm{Id}$, $\lambda_3 := \mathrm{Id}$, $\lambda_1 := \varepsilon$, $\lambda_2 := \mu$, and $\widetilde{H}_1 := L^{2,3\times 3}_{\mathbb{T},\varepsilon}(\Omega)$, $\widetilde{H}_2 := L^{2,3\times 3}_{\mathbb{S},\mu}(\Omega)$, $\widetilde{H}_0 = H_0 = L^{2,3}(\Omega)$, $\widetilde{H}_3 = H_3 = L^2(\Omega)$. We look at*

$$\begin{aligned} \widetilde{A}_0 &:= \operatorname{dev}\mathring{\operatorname{Grad}}, & \widetilde{A}_1 &:= \mu^{-1}\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}}, & \widetilde{A}_2 &:= \operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}}\mu, \\ \widetilde{A}_0^* &= -\operatorname{Div}_{\mathbb{T}}\varepsilon, & \widetilde{A}_1^* &= \varepsilon^{-1}\operatorname{Curl}_{\mathbb{S}}, & \widetilde{A}_2^* &= \operatorname{Grad}\operatorname{grad}, \end{aligned}$$

$$\begin{aligned} \widetilde{\mathcal{D}}^{\mathsf{bih},2} &:= \begin{pmatrix} \widetilde{A}_2 & 0 \\ \widetilde{A}_1^* & \widetilde{A}_0 \end{pmatrix} = \begin{pmatrix} \operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}}\mu & 0 \\ \varepsilon^{-1}\operatorname{Curl}_{\mathbb{S}} & \operatorname{dev}\mathring{\operatorname{Grad}} \end{pmatrix}, \\ (\widetilde{\mathcal{D}}^{\mathsf{bih},2})^* &= \begin{pmatrix} \widetilde{A}_2^* & \widetilde{A}_1 \\ 0 & \widetilde{A}_0^* \end{pmatrix} = \begin{pmatrix} \operatorname{Grad}\operatorname{grad} & \mu^{-1}\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}} \\ 0 & -\operatorname{Div}_{\mathbb{T}}\varepsilon \end{pmatrix}, \end{aligned}$$

i.e., the second biharmonic complex, cf. (17),

$$\begin{array}{c} \{0\} \xrightarrow{\iota_{\{0\}}} L^{2,3}(\Omega) \xrightarrow{\operatorname{dev}\mathring{\operatorname{Grad}}} L^{2,3\times 3}_{\mathbb{T},\varepsilon}(\Omega) \xrightarrow{\mu^{-1}\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}}} L^{2,3\times 3}_{\mathbb{S},\mu}(\Omega) \xrightarrow{\operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}}\mu} L^2(\Omega) \xrightarrow{\pi_{\mathsf{P}^1_{\mathsf{pw}}}} \mathsf{P}^1_{\mathsf{pw}}, \\ \{0\} \xleftarrow{\pi_{\{0\}}} L^{2,3}(\Omega) \xleftarrow{-\operatorname{Div}_{\mathbb{T}}\varepsilon} L^{2,3\times 3}_{\mathbb{T},\varepsilon}(\Omega) \xleftarrow{\varepsilon^{-1}\operatorname{Curl}_{\mathbb{S}}} L^{2,3\times 3}_{\mathbb{S},\mu}(\Omega) \xleftarrow{\operatorname{Grad}\operatorname{grad}} L^2(\Omega) \xleftarrow{\iota_{\mathsf{P}^1_{\mathsf{pw}}}} \mathsf{P}^1_{\mathsf{pw}}. \end{array} \tag{20}$$

Lemma 2.12, Lemma 2.13, and Theorem 2.14 show that the compactness properties, the dimensions of the kernels and cohomology groups, the maximal compactness, and the Fredholm indices of the second biharmonic complex do not dependent of the material weights ε and μ. More precisely,

- $$\begin{aligned}\dim\big(\ker(\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}})\cap\big(\varepsilon^{-1}\ker(\operatorname{Div}_{\mathbb{T}})\big)\big)&=\dim\big(\ker(\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}})\cap\ker(\operatorname{Div}_{\mathbb{T}})\big)\\&=\dim\mathcal{H}^{\mathsf{bih},2}_{D,\mathbb{T}}(\Omega)=4(m-1),\end{aligned}$$
- $$\begin{aligned}\dim\big(\big(\mu^{-1}\ker(\operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}})\big)\cap\ker(\operatorname{Curl}_{\mathbb{S}})\big)&=\dim\big(\ker(\operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}})\cap\ker(\operatorname{Curl}_{\mathbb{S}})\big)\\&=\dim\mathcal{H}^{\mathsf{bih},2}_{N,\mathbb{S}}(\Omega)=4p,\end{aligned}$$
- $$\begin{aligned}\operatorname{dom}(\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}})\cap\big(\varepsilon^{-1}\operatorname{dom}(\operatorname{Div}_{\mathbb{T}})\big)&\hookrightarrow L^{2,3\times3}_{\mathbb{T},\varepsilon}(\Omega)\\\Leftrightarrow\quad\operatorname{dom}(\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}})\cap\operatorname{dom}(\operatorname{Div}_{\mathbb{T}})&\hookrightarrow L^{2,3\times3}_{\mathbb{T}}(\Omega),\end{aligned}$$
- $$\begin{aligned}\big(\mu^{-1}\operatorname{dom}(\operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}})\big)\cap\operatorname{dom}(\operatorname{Curl}_{\mathbb{S}})&\hookrightarrow L^{2,3\times3}_{\mathbb{S},\mu}(\Omega)\\\Leftrightarrow\quad\operatorname{dom}(\operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}})\cap\operatorname{dom}(\operatorname{Curl}_{\mathbb{S}})&\hookrightarrow L^{2,3\times3}_{\mathbb{S}}(\Omega),\end{aligned}$$
- $(\operatorname{dev}\mathring{\operatorname{Grad}},\mu^{-1}\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}},\operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}}\mu)$ *m cpt, iff* $(\operatorname{dev}\mathring{\operatorname{Grad}},\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}},\operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}})$ *m cpt,*
- $$-\operatorname{ind}(\widetilde{\mathcal{D}}^{\mathsf{bih},2})^*=\operatorname{ind}\widetilde{\mathcal{D}}^{\mathsf{bih},2}=\operatorname{ind}\mathcal{D}^{\mathsf{bih},2}=4(p-m-n+1).$$

Note that the kernels and ranges are given by

$$\begin{aligned}\ker\mathcal{D}^{\mathsf{bih},2}&=K^{\mathsf{bih},2}_2\times N^{\mathsf{bih},2}_0=\mathcal{H}^{\mathsf{bih},2}_{N,\mathbb{S}}(\Omega)\times\{0\},\\\ker(\mathcal{D}^{\mathsf{bih},2})^*&=N^{\mathsf{bih},2}_{2,*}\times K^{\mathsf{bih},2}_1=\mathsf{P}^1_{\mathsf{pw}}\times\mathcal{H}^{\mathsf{bih},2}_{D,\mathbb{T}}(\Omega),\\\operatorname{ran}\mathcal{D}^{\mathsf{bih},2}&=(\ker(\mathcal{D}^{\mathsf{bih},2})^*)^{\perp_{L^2(\Omega)\times L^{2,3\times3}_{\mathbb{T}}(\Omega)}}=(\mathsf{P}^1_{\mathsf{pw}})^{\perp_{L^2(\Omega)}}\times\mathcal{H}^{\mathsf{bih},2}_{D,\mathbb{T}}(\Omega)^{\perp_{L^{2,3\times3}_{\mathbb{T}}(\Omega)}},\\\operatorname{ran}(\mathcal{D}^{\mathsf{bih},2})^*&=(\ker\mathcal{D}^{\mathsf{bih},2})^{\perp_{L^{2,3\times3}_{\mathbb{S}}(\Omega)\times L^{2,3}(\Omega)}}=\mathcal{H}^{\mathsf{bih},2}_{N,\mathbb{S}}(\Omega)^{\perp_{L^{2,3\times3}_{\mathbb{S}}(\Omega)}}\times L^{2,3}(\Omega),\end{aligned}$$

see Lemma 2.5, Corollary 2.6, and (19). Corollary 2.9 shows additional results for the corresponding reduced operators

$$\begin{aligned}\mathcal{D}^{\mathsf{bih},2}_{\mathsf{red}}=\mathcal{D}^{\mathsf{bih},2}|_{(\ker\mathcal{D}^{\mathsf{bih},2})^{\perp_{H_2\times H_0}}}&=\begin{pmatrix}\operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}}&0\\\operatorname{Curl}_{\mathbb{S}}&\operatorname{dev}\mathring{\operatorname{Grad}}\end{pmatrix}\Big|_{\mathcal{H}^{\mathsf{bih},2}_{N,\mathbb{S}}(\Omega)^{\perp_{L^{2,3\times3}_{\mathbb{S}}(\Omega)}}\times L^{2,3}(\Omega)},\\(\mathcal{D}^{\mathsf{bih},2}_{\mathsf{red}})^*=(\mathcal{D}^{\mathsf{bih},2})^*|_{(\ker(\mathcal{D}^{\mathsf{bih},2})^*)^{\perp_{H_3\times H_1}}}&=\begin{pmatrix}\operatorname{Gradgrad}&\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}}\\0&-\operatorname{Div}_{\mathbb{T}}\end{pmatrix}\Big|_{(\mathsf{P}^1_{\mathsf{pw}})^{\perp_{L^2(\Omega)}}\times\mathcal{H}^{\mathsf{bih},2}_{D,\mathbb{T}}(\Omega)^{\perp_{L^{2,3\times3}_{\mathbb{T}}(\Omega)}}}.\end{aligned}$$

Corollary 5.8. *Let $\Omega\subset\mathbb{R}^3$ be a bounded strong Lipschitz domain. Then*

$$\begin{aligned}(\mathcal{D}^{\mathsf{bih},2}_{\mathsf{red}})^{-1}&:\operatorname{ran}\mathcal{D}^{\mathsf{bih},2}\to\operatorname{ran}(\mathcal{D}^{\mathsf{bih},2})^*,\\((\mathcal{D}^{\mathsf{bih},2}_{\mathsf{red}})^*)^{-1}&:\operatorname{ran}(\mathcal{D}^{\mathsf{bih},2})^*\to\operatorname{ran}\mathcal{D}^{\mathsf{bih},2}\end{aligned}$$

are compact. Furthermore,

$$\begin{aligned}(\mathcal{D}^{\mathsf{bih},2}_{\mathsf{red}})^{-1}&:\operatorname{ran}\mathcal{D}^{\mathsf{bih},2}\to\operatorname{dom}\mathcal{D}^{\mathsf{bih},2}_{\mathsf{red}},\\((\mathcal{D}^{\mathsf{bih},2}_{\mathsf{red}})^*)^{-1}&:\operatorname{ran}(\mathcal{D}^{\mathsf{bih},2})^*\to\operatorname{dom}(\mathcal{D}^{\mathsf{bih},2}_{\mathsf{red}})^*\end{aligned}$$

are continuous and, equivalently, the Friedrichs-Poincaré type estimates

$$\begin{aligned}\big|(S,v)\big|_{L^{2,3\times3}_{\mathbb{S}}(\Omega)\times L^{2,3}(\Omega)}&\leq c_{\mathcal{D}^{\mathsf{bih},2}}\big(|\operatorname{devGrad}v|^2_{L^{2,3\times3}_{\mathbb{T}}(\Omega)}\\&\qquad+|\operatorname{divDiv}S|^2_{L^2(\Omega)}+|\operatorname{Curl}S|^2_{L^{2,3\times3}_{\mathbb{T}}(\Omega)}\big)^{1/2},\\\big|(u,T)\big|_{L^2(\Omega)\times L^{2,3\times3}_{\mathbb{T}}(\Omega)}&\leq c_{\mathcal{D}^{\mathsf{bih},2}}\big(|\operatorname{Gradgrad}u|^2_{L^{2,3\times3}_{\mathbb{S}}(\Omega)}\end{aligned}$$

$$+ |\operatorname{Div} T|^2_{L^{2,3}(\Omega)} + |\operatorname{symCurl} T|^2_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)}\Big)^{1/2}$$

hold for all (S, v) *in*

$$\operatorname{dom}\mathcal{D}^{\mathsf{bih},2}_{\mathsf{red}} = \Big(\operatorname{dom}(\operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}}) \cap \operatorname{dom}(\operatorname{Curl}_{\mathbb{S}}) \cap \mathcal{H}^{\mathsf{bih},2}_{N,\mathbb{S}}(\Omega)^{\perp_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)}}\Big) \times H^{1,3}_0(\Omega)$$

for all (u, T) *in*

$$\operatorname{dom}(\mathcal{D}^{\mathsf{bih},2}_{\mathsf{red}})^* = \big(H^2(\Omega) \cap (\mathsf{P}^1_{\mathsf{pw}})^{\perp_{L^2(\Omega)}}\big) \times \Big(\operatorname{dom}(\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}}) \cap \operatorname{dom}(\operatorname{Div}_{\mathbb{T}}) \cap \mathcal{H}^{\mathsf{bih},2}_{D,\mathbb{T}}(\Omega)^{\perp_{L^{2,3\times 3}_{\mathbb{T}}(\Omega)}}\Big)$$

with some optimal constant $c_{\mathcal{D}^{\mathsf{bih},2}} > 0$.

6. The Elasticity Complex and Its Indices

This section is devoted to adapt our main results Theorem 1.1, Theorem 4.4, and Theorem 5.5, to the elasticity complex, see [22] for details. Its elasticity differential operator is of mixed order as well, this time in the center of the complex. As before for the biharmonic operators, the leading order term is *not* dominating the lower order differential operators.

Definition 6.1. *Let* $\Omega \subset \mathbb{R}^3$ *be an open set. We put*

$$\begin{aligned}
\operatorname{symGrad}_c : C^{\infty,3}_c(\Omega) \subseteq L^{2,3}(\Omega) &\to L^{2,3\times 3}_{\mathbb{S}}(\Omega), \quad \phi \mapsto \operatorname{sym}\operatorname{Grad}\phi,\\
\operatorname{CurlCurl}^\top_c : C^{\infty,3\times 3}_{c,\mathbb{S}}(\Omega) \subseteq L^{2,3\times 3}_{\mathbb{S}}(\Omega) &\to L^{2,3\times 3}_{\mathbb{S}}(\Omega), \quad \Phi \mapsto \operatorname{CurlCurl}^\top\Phi := \operatorname{Curl}(\operatorname{Curl}\Phi)^\top,\\
\operatorname{Div}_c : C^{\infty,3\times 3}_{c,\mathbb{S}}(\Omega) \subseteq L^{2,3\times 3}_{\mathbb{S}}(\Omega) &\to L^{2,3}(\Omega), \qquad \Phi \mapsto \operatorname{Div}\Phi,
\end{aligned}$$

and further define the densely defined and closed linear operators

$$\begin{aligned}
\operatorname{Div}_{\mathbb{S}} &:= -\operatorname{symGrad}^*_c, & \operatorname{sym}\mathring{\operatorname{Grad}} &:= -\operatorname{Div}^*_{\mathbb{S}} = \overline{\operatorname{symGrad}_c},\\
\operatorname{CurlCurl}^\top_{\mathbb{S}} &:= (\operatorname{CurlCurl}^\top_c)^*, & \operatorname{Curl}\mathring{\operatorname{Curl}}^\top_{\mathbb{S}} &:= (\operatorname{CurlCurl}^\top_{\mathbb{S}})^* = \overline{\operatorname{CurlCurl}^\top_c},\\
\operatorname{symGrad} &:= -\operatorname{Div}^*_c, & \mathring{\operatorname{Div}}_{\mathbb{S}} &:= -\operatorname{symGrad}^* = \overline{\operatorname{Div}_c}.
\end{aligned}$$

We want to apply the index theorem in the following situation of the elasticity complex:

$$\begin{aligned}
A_0 &:= \operatorname{sym}\mathring{\operatorname{Grad}}, & A_1 &:= \operatorname{Curl}\mathring{\operatorname{Curl}}^\top_{\mathbb{S}}, & A_2 &:= \mathring{\operatorname{Div}}_{\mathbb{S}},\\
A^*_0 &= -\operatorname{Div}_{\mathbb{S}}, & A^*_1 &= \operatorname{CurlCurl}^\top_{\mathbb{S}}, & A^*_2 &= -\operatorname{symGrad},
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}^{\mathsf{ela}} &:= \begin{pmatrix} A_2 & 0 \\ A^*_1 & A_0 \end{pmatrix} = \begin{pmatrix} \mathring{\operatorname{Div}}_{\mathbb{S}} & 0 \\ \operatorname{CurlCurl}^\top_{\mathbb{S}} & \operatorname{sym}\mathring{\operatorname{Grad}} \end{pmatrix},\\
(\mathcal{D}^{\mathsf{ela}})^* &= \begin{pmatrix} A^*_2 & A_1 \\ 0 & A^*_0 \end{pmatrix} = \begin{pmatrix} -\operatorname{symGrad} & \operatorname{Curl}\mathring{\operatorname{Curl}}^\top_{\mathbb{S}} \\ 0 & -\operatorname{Div}_{\mathbb{S}} \end{pmatrix},
\end{aligned}$$

$$\begin{gathered}
\{0\} \xrightarrow{\iota_{\{0\}}} L^{2,3}(\Omega) \xrightarrow{\operatorname{sym}\mathring{\operatorname{Grad}}} L^{2,3\times 3}_{\mathbb{S}}(\Omega) \xrightarrow{\operatorname{Curl}\mathring{\operatorname{Curl}}^\top_{\mathbb{S}}} L^{2,3\times 3}_{\mathbb{S}}(\Omega) \xrightarrow{\mathring{\operatorname{Div}}_{\mathbb{S}}} L^{2,3}(\Omega) \xrightarrow{\pi_{\mathsf{RM}_{\mathsf{pw}}}} \mathsf{RM}_{\mathsf{pw}},\\
\{0\} \xleftarrow{\pi_{\{0\}}} L^{2,3}(\Omega) \xleftarrow{-\operatorname{Div}_{\mathbb{S}}} L^{2,3\times 3}_{\mathbb{S}}(\Omega) \xleftarrow{\operatorname{CurlCurl}^\top_{\mathbb{S}}} L^{2,3\times 3}_{\mathbb{S}}(\Omega) \xleftarrow{-\operatorname{symGrad}} L^{2,3}(\Omega) \xleftarrow{\iota_{\mathsf{RM}_{\mathsf{pw}}}} \mathsf{RM}_{\mathsf{pw}}.
\end{gathered} \tag{21}$$

The foundation of the index theorem to follow is the following compactness result established by Pauly and Zulehner. Note that we have $\operatorname{dom}(\operatorname{sym}\mathring{\operatorname{Grad}}) = H^{1,3}_0(\Omega)$ and $\operatorname{dom}(\operatorname{symGrad}) = H^{1,3}(\Omega)$.

Theorem 6.2 ([22, Theorem 3.17])**.** *Let* $\Omega \subseteq \mathbb{R}^3$ *be a bounded strong Lipschitz domain. Then* $(\operatorname{sym}\mathring{\operatorname{Grad}}, \operatorname{Curl}\mathring{\operatorname{Curl}}^\top_{\mathbb{S}}, \mathring{\operatorname{Div}}_{\mathbb{S}})$ *is a maximal compact Hilbert complex.*

We observe and define

$$
\begin{aligned}
N_0^{\mathsf{ela}} &= \ker A_0 = \ker(\operatorname{sym}\mathring{\operatorname{Grad}}),\\
N_{2,*}^{\mathsf{ela}} &= \ker A_2^* = \ker(\operatorname{sym}\operatorname{Grad}),\\
K_1^{\mathsf{ela}} &= \ker A_1 \cap \ker A_0^* = \ker(\mathring{\operatorname{Curl}}\operatorname{Curl}_{\mathbb{S}}^\top) \cap \ker(\operatorname{Div}_{\mathbb{S}}) =: \mathcal{H}_{D,\mathbb{S}}^{\mathsf{ela}}(\Omega),\\
K_2^{\mathsf{ela}} &= \ker A_2 \cap \ker A_1^* = \ker(\mathring{\operatorname{Div}}_{\mathbb{S}}) \cap \ker(\operatorname{Curl}\operatorname{Curl}_{\mathbb{S}}^\top) =: \mathcal{H}_{N,\mathbb{S}}^{\mathsf{ela}}(\Omega).
\end{aligned}
\tag{22}
$$

The dimensions of the cohomology groups are given as follows.

Theorem 6.3. *Let $\Omega \subseteq \mathbb{R}^3$ be open and bounded with continuous boundary. Moreover, suppose Assumption 3. Then*

$$
\dim \mathcal{H}_{D,\mathbb{S}}^{\mathsf{ela}}(\Omega) = 6(m-1), \qquad \dim \mathcal{H}_{N,\mathbb{S}}^{\mathsf{ela}}(\Omega) = 6p.
$$

Proof. We postpone the proof to the Appendix. □

Let us introduce the space of piecewise rigid motions by

$$
\mathsf{RM}_{\mathsf{pw}} := \big\{ v \in L^{2,3}(\Omega) : \forall\, C(\text{con. cp.}) \subseteq \Omega \quad \exists\, \alpha_C, \beta_C \in \mathbb{R}^3 : u|_C(x) = \alpha_C \times x + \beta_C \big\}.
$$

Theorem 6.4. *Let $\Omega \subset \mathbb{R}^3$ be a bounded strong Lipschitz domain. Then $\mathcal{D}^{\mathsf{ela}}$ is a Fredholm operator with index*

$$
\operatorname{ind} \mathcal{D}^{\mathsf{ela}} = \dim N_0^{\mathsf{ela}} - \dim K_1^{\mathsf{ela}} + \dim K_2^{\mathsf{ela}} - \dim N_{2,*}^{\mathsf{ela}}.
$$

If additionally Assumption 3 holds, then

$$
\operatorname{ind} \mathcal{D}^{\mathsf{ela}} = 6(p - m - n + 1).
$$

Proof. Using Theorem 6.2 apply Theorem 2.8 together with (22), the observations

$$
N_0^{\mathsf{ela}} = \ker(\operatorname{sym}\mathring{\operatorname{Grad}}) = \{0\}, \qquad N_{2,*}^{\mathsf{ela}} = \ker(\operatorname{sym}\operatorname{Grad}) = \mathsf{RM}_{\mathsf{pw}},
\tag{23}
$$

see [22, Lemma 3.2], and Theorem 6.3. □

Remark 6.5. *By Theorem 2.8 the adjoint $(\mathcal{D}^{\mathsf{ela}})^*$ is Fredholm as well with index simply given by $\operatorname{ind}(\mathcal{D}^{\mathsf{ela}})^* = -\operatorname{ind}\mathcal{D}^{\mathsf{ela}}$. Similar to Remark 3.9, Remark 4.5, and Remark 5.6 we define the extended elasticity operator*

$$
\mathcal{M}^{\mathsf{ela}} := \begin{pmatrix} 0 & \mathcal{D}^{\mathsf{ela}} \\ -(\mathcal{D}^{\mathsf{ela}})^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \mathring{\operatorname{Div}}_{\mathbb{S}} & 0 \\ 0 & 0 & \operatorname{Curl}\operatorname{Curl}_{\mathbb{S}}^\top & \operatorname{sym}\mathring{\operatorname{Grad}} \\ \operatorname{sym}\operatorname{Grad} & -\mathring{\operatorname{Curl}}\operatorname{Curl}_{\mathbb{S}}^\top & 0 & 0 \\ 0 & \operatorname{Div}_{\mathbb{S}} & 0 & 0 \end{pmatrix}
$$

with $(\mathcal{M}^{\mathsf{ela}})^* = -\mathcal{M}^{\mathsf{ela}}$ and $\operatorname{ind}\mathcal{M}^{\mathsf{ela}} = 0$.

6.1. **Some More Results.** Inhomogeneous and anisotropic media may also be considered for the elasticity complex, cf. Remark 3.11, Remark 4.6, and Remark 5.7.

Remark 6.6. *Recall the notations from Remark 4.6 and Remark 5.7 and set $\lambda_0 := \mathrm{Id}$, $\lambda_3 := \mathrm{Id}$, $\lambda_1 := \varepsilon$, $\lambda_2 := \mu$, and $\widetilde{H}_3 = \widetilde{H}_0 = H_3 = H_0 = L^{2,3}(\Omega)$, $\widetilde{H}_1 := L^{2,3\times 3}_{\mathbb{S},\varepsilon}(\Omega)$, $\widetilde{H}_2 := L^{2,3\times 3}_{\mathbb{S},\mu}(\Omega)$. We look at*

$$
\begin{aligned}
\widetilde{A}_0 &:= \operatorname{sym}\mathring{\operatorname{Grad}}, & \widetilde{A}_1 &:= \mu^{-1}\mathring{\operatorname{Curl}}\operatorname{Curl}_{\mathbb{S}}^\top, & \widetilde{A}_2 &:= \mathring{\operatorname{Div}}_{\mathbb{S}}\,\mu,\\
\widetilde{A}_0^* &= -\operatorname{Div}_{\mathbb{S}}\,\varepsilon, & \widetilde{A}_1^* &= \varepsilon^{-1}\operatorname{Curl}\operatorname{Curl}_{\mathbb{S}}^\top, & \widetilde{A}_2^* &= -\operatorname{sym}\operatorname{Grad},
\end{aligned}
$$

$$
\widetilde{\mathcal{D}}^{\mathsf{ela}} := \begin{pmatrix} \widetilde{A}_2 & 0 \\ \widetilde{A}_1^* & \widetilde{A}_0 \end{pmatrix} = \begin{pmatrix} \mathring{\operatorname{Div}}_{\mathbb{S}}\,\mu & 0 \\ \varepsilon^{-1}\operatorname{Curl}\operatorname{Curl}_{\mathbb{S}}^\top & \operatorname{sym}\mathring{\operatorname{Grad}} \end{pmatrix},
$$

$$(\widetilde{\mathcal{D}}^{\mathsf{ela}})^* = \begin{pmatrix} \widetilde{A}_2^* & \widetilde{A}_1 \\ 0 & \widetilde{A}_0^* \end{pmatrix} = \begin{pmatrix} -\operatorname{symGrad} & \mu^{-1}\operatorname{Curl}\mathring{\operatorname{Curl}}_{\mathbb{S}}^{\top} \\ 0 & -\operatorname{Div}_{\mathbb{S}}\varepsilon \end{pmatrix},$$

i.e., the elasticity complex, cf. (21),

$$\begin{array}{c} \{0\} \xrightarrow{\iota_{\{0\}}} L^{2,3}(\Omega) \xrightarrow{\operatorname{sym}\mathring{\operatorname{Grad}}} L^{2,3\times3}_{\mathbb{S},\varepsilon}(\Omega) \xrightarrow{\mu^{-1}\operatorname{Curl}\mathring{\operatorname{Curl}}_{\mathbb{S}}^{\top}} L^{2,3\times3}_{\mathbb{S},\mu}(\Omega) \xrightarrow{\mathring{\operatorname{Div}}_{\mathbb{S}}\mu} L^{2,3}(\Omega) \xrightarrow{\pi_{\mathsf{RM}_{\mathsf{pw}}}} \mathsf{RM}_{\mathsf{pw}}, \\ \{0\} \xleftarrow{\pi_{\{0\}}} L^{2,3}(\Omega) \xleftarrow{-\operatorname{Div}_{\mathbb{S}}\varepsilon} L^{2,3\times3}_{\mathbb{S},\varepsilon}(\Omega) \xleftarrow{\varepsilon^{-1}\operatorname{CurlCurl}_{\mathbb{S}}^{\top}} L^{2,3\times3}_{\mathbb{S},\mu}(\Omega) \xleftarrow{-\operatorname{symGrad}} L^{2,3}(\Omega) \xleftarrow{\iota_{\mathsf{RM}_{\mathsf{pw}}}} \mathsf{RM}_{\mathsf{pw}}. \end{array} \tag{24}$$

Lemma 2.12, Lemma 2.13, and Theorem 2.14 show that the compactness properties, the dimensions of the kernels and cohomology groups, the maximal compactness, and the Fredholm indices of the elasticity complex do not dependent of the material weights ε and μ. More precisely,

- $$\begin{aligned} \dim\big(\ker(\operatorname{Curl}\mathring{\operatorname{Curl}}_{\mathbb{S}}^{\top}) \cap \big(\varepsilon^{-1}\ker(\operatorname{Div}_{\mathbb{S}})\big)\big) &= \dim\big(\ker(\operatorname{Curl}\mathring{\operatorname{Curl}}_{\mathbb{S}}^{\top}) \cap \ker(\operatorname{Div}_{\mathbb{S}})\big) \\ &= \dim \mathcal{H}^{\mathsf{ela}}_{D,\mathbb{S}}(\Omega) = 6(m-1), \end{aligned}$$
- $$\begin{aligned} \dim\big(\big(\mu^{-1}\ker(\mathring{\operatorname{Div}}_{\mathbb{S}})\big) \cap \ker(\operatorname{CurlCurl}_{\mathbb{S}}^{\top})\big) &= \dim\big(\ker(\mathring{\operatorname{Div}}_{\mathbb{S}}) \cap \ker(\operatorname{CurlCurl}_{\mathbb{S}}^{\top})\big) \\ &= \dim \mathcal{H}^{\mathsf{ela}}_{N,\mathbb{S}}(\Omega) = 6p, \end{aligned}$$
- $$\begin{aligned} &\operatorname{dom}(\operatorname{Curl}\mathring{\operatorname{Curl}}_{\mathbb{S}}^{\top}) \cap \big(\varepsilon^{-1}\operatorname{dom}(\operatorname{Div}_{\mathbb{S}})\big) \hookrightarrow L^{2,3\times3}_{\mathbb{S},\varepsilon}(\Omega) \\ \Leftrightarrow\quad &\operatorname{dom}(\operatorname{Curl}\mathring{\operatorname{Curl}}_{\mathbb{S}}^{\top}) \cap \operatorname{dom}(\operatorname{Div}_{\mathbb{S}}) \hookrightarrow L^{2,3\times3}_{\mathbb{S}}(\Omega), \end{aligned}$$
- $$\begin{aligned} &\big(\mu^{-1}\operatorname{dom}(\mathring{\operatorname{Div}}_{\mathbb{S}})\big) \cap \operatorname{dom}(\operatorname{CurlCurl}_{\mathbb{S}}^{\top}) \hookrightarrow L^{2,3\times3}_{\mathbb{S},\mu}(\Omega) \\ \Leftrightarrow\quad &\operatorname{dom}(\mathring{\operatorname{Div}}_{\mathbb{S}}) \cap \operatorname{dom}(\operatorname{CurlCurl}_{\mathbb{S}}^{\top}) \hookrightarrow L^{2,3\times3}_{\mathbb{S}}(\Omega), \end{aligned}$$
- $(\operatorname{sym}\mathring{\operatorname{Grad}}, \mu^{-1}\operatorname{Curl}\mathring{\operatorname{Curl}}_{\mathbb{S}}^{\top}, \mathring{\operatorname{Div}}_{\mathbb{S}}\mu)$ *m cpt, iff* $(\operatorname{sym}\mathring{\operatorname{Grad}}, \operatorname{Curl}\mathring{\operatorname{Curl}}_{\mathbb{S}}^{\top}, \mathring{\operatorname{Div}}_{\mathbb{S}})$ *m cpt,*
- $$-\operatorname{ind}(\widetilde{\mathcal{D}}^{\mathsf{ela}})^* = \operatorname{ind}\widetilde{\mathcal{D}}^{\mathsf{ela}} = \operatorname{ind}\mathcal{D}^{\mathsf{ela}} = 6(p-m-n+1).$$

Note that the kernels and ranges are given by

$$\begin{aligned} \ker\mathcal{D}^{\mathsf{ela}} &= K_2^{\mathsf{ela}} \times N_0^{\mathsf{ela}} = \mathcal{H}^{\mathsf{ela}}_{N,\mathbb{S}}(\Omega) \times \{0\}, \\ \ker(\mathcal{D}^{\mathsf{ela}})^* &= N_{2,*}^{\mathsf{ela}} \times K_1^{\mathsf{ela}} = \mathsf{RM}_{\mathsf{pw}} \times \mathcal{H}^{\mathsf{ela}}_{D,\mathbb{S}}(\Omega), \\ \operatorname{ran}\mathcal{D}^{\mathsf{ela}} &= (\ker(\mathcal{D}^{\mathsf{ela}})^*)^{\perp_{L^{2,3}(\Omega)\times L^{2,3\times3}_{\mathbb{S}}(\Omega)}} = \mathsf{RM}_{\mathsf{pw}}^{\perp_{L^{2,3}(\Omega)}} \times \mathcal{H}^{\mathsf{ela}}_{D,\mathbb{S}}(\Omega)^{\perp_{L^{2,3\times3}_{\mathbb{S}}(\Omega)}}, \\ \operatorname{ran}(\mathcal{D}^{\mathsf{ela}})^* &= (\ker\mathcal{D}^{\mathsf{ela}})^{\perp_{L^{2,3\times3}_{\mathbb{S}}(\Omega)\times L^{2,3}(\Omega)}} = \mathcal{H}^{\mathsf{ela}}_{N,\mathbb{S}}(\Omega)^{\perp_{L^{2,3\times3}_{\mathbb{S}}(\Omega)}} \times L^{2,3}(\Omega), \end{aligned}$$

see Lemma 2.5, Corollary 2.6, and (23). Corollary 2.9 shows additional results for the corresponding reduced operators

$$\begin{aligned} \mathcal{D}^{\mathsf{ela}}_{\mathsf{red}} = \mathcal{D}^{\mathsf{ela}}|_{(\ker\mathcal{D}^{\mathsf{ela}})^{\perp_{H_2\times H_0}}} &= \begin{pmatrix} \mathring{\operatorname{Div}}_{\mathbb{S}} & 0 \\ \operatorname{CurlCurl}_{\mathbb{S}}^{\top} & \operatorname{sym}\mathring{\operatorname{Grad}} \end{pmatrix}\Big|_{\mathcal{H}^{\mathsf{ela}}_{N,\mathbb{S}}(\Omega)^{\perp_{L^{2,3\times3}_{\mathbb{S}}(\Omega)}}\times L^{2,3}(\Omega)}, \\ (\mathcal{D}^{\mathsf{ela}}_{\mathsf{red}})^* = (\mathcal{D}^{\mathsf{ela}})^*|_{(\ker(\mathcal{D}^{\mathsf{ela}})^*)^{\perp_{H_3\times H_1}}} &= \begin{pmatrix} -\operatorname{symGrad} & \operatorname{Curl}\mathring{\operatorname{Curl}}_{\mathbb{S}}^{\top} \\ 0 & -\operatorname{Div}_{\mathbb{S}} \end{pmatrix}\Big|_{\mathsf{RM}_{\mathsf{pw}}^{\perp_{L^{2,3}(\Omega)}}\times\mathcal{H}^{\mathsf{ela}}_{D,\mathbb{S}}(\Omega)^{\perp_{L^{2,3\times3}_{\mathbb{S}}(\Omega)}}}. \end{aligned}$$

Corollary 6.7. *Let $\Omega \subset \mathbb{R}^3$ be a bounded strong Lipschitz domain. Then*

$$\begin{aligned} (\mathcal{D}^{\mathsf{ela}}_{\mathsf{red}})^{-1} &: \operatorname{ran}\mathcal{D}^{\mathsf{ela}} \to \operatorname{ran}(\mathcal{D}^{\mathsf{ela}})^*, \\ ((\mathcal{D}^{\mathsf{ela}}_{\mathsf{red}})^*)^{-1} &: \operatorname{ran}(\mathcal{D}^{\mathsf{ela}})^* \to \operatorname{ran}\mathcal{D}^{\mathsf{ela}} \end{aligned}$$

are compact. Furthermore,

$$(\mathcal{D}^{\mathsf{ela}}_{\mathsf{red}})^{-1} : \operatorname{ran}\mathcal{D}^{\mathsf{ela}} \to \operatorname{dom}\mathcal{D}^{\mathsf{ela}}_{\mathsf{red}},$$

$$((\mathcal{D}^{\mathsf{ela}}_{\mathsf{red}})^*)^{-1} : \operatorname{ran}(\mathcal{D}^{\mathsf{ela}})^* \to \operatorname{dom}(\mathcal{D}^{\mathsf{ela}}_{\mathsf{red}})^*$$

are continuous and, equivalently, the Friedrichs-Poincaré type estimate

$$\begin{aligned}\big|(S,v)\big|_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)\times L^{2,3}(\Omega)} \le c_{\mathcal{D}^{\mathsf{ela}}}\big(|\operatorname{symGrad} v|^2_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)} \\ + |\operatorname{Div} S|^2_{L^{2,3}(\Omega)} + |\operatorname{CurlCurl}^\top S|^2_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)}\big)^{1/2}\end{aligned}$$

holds for all (S,v) in

$$\operatorname{dom}\mathcal{D}^{\mathsf{ela}}_{\mathsf{red}} = \big(\operatorname{dom}(\mathring{\operatorname{Div}}_{\mathbb{S}}) \cap \operatorname{dom}(\operatorname{CurlCurl}^\top_{\mathbb{S}}) \cap \mathcal{H}^{\mathsf{ela}}_{N,\mathbb{S}}(\Omega)^{\perp_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)}}\big) \times H^{1,3}_0(\Omega)$$

or (v,S) in

$$\begin{aligned}\operatorname{dom}(\mathcal{D}^{\mathsf{ela}}_{\mathsf{red}})^* = \big(H^{1,3}(\Omega) \cap \mathsf{RM}_{\mathsf{pw}}^{\perp_{L^{2,3}(\Omega)}}\big) \\ \times \big(\operatorname{dom}(\mathring{\operatorname{CurlCurl}}^\top_{\mathbb{S}}) \cap \operatorname{dom}(\operatorname{Div}_{\mathbb{S}}) \cap \mathcal{H}^{\mathsf{ela}}_{D,\mathbb{S}}(\Omega)^{\perp_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)}}\big)\end{aligned}$$

with some optimal constant $c_{\mathcal{D}^{\mathsf{ela}}} > 0$.

7. Conclusion

The index theorems presented rest on the abstract construction principle provided in [7] and the results on the newly found biharmonic complex from [20, 21] and the elasticity complex from [22]. With this insight it is possible to construct basis fields for the generalised harmonic Dirichlet and Neumann tensor fields, see Appendix. This construction heavily relies on the choice of boundary conditions and we emphasise that the considered mixed order operators *cannot* be viewed as leading order plus relatively compact perturbation, when it comes to computation of the Fredholm index. In particular, techniques from pseudo-differential calculus successfully applied to obtain index formulas for operators defined on non-compact manifolds or compact manifolds without boundary, see e.g. [11, 12], are likely to be very difficult to be applicable in the present situation. It would be interesting to see, whether the operators considered above defined on an unbounded domain enjoy similar index formulas (maybe a comparable Witten index of some sort) even though the operator itself might not be of Fredholm type anymore.

References

[1] A. Agmon. *Lectures on elliptic boundary value problems.* Van Nostrand Mathematical Studies, No. 2, Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London, 1965.

[2] W. Arendt, H. Vogt, J. Voigt. *Form methods and applications.* 18th Internetseminar, `https://www.mat.tuhh.de/veranstaltungen/isem18/pdf/LectureNotes.pdf`.

[3] S. Bauer, D. Pauly, and M. Schomburg. The Maxwell compactness property in bounded weak Lipschitz domains with mixed boundary conditions. *SIAM J. Math. Anal.*, 48:2912-2943, 2016.

[4] S. Bauer, D. Pauly, and M. Schomburg. Weck's Selection Theorem: The Maxwell Compactness Property for Bounded Weak Lipschitz Domains with Mixed Boundary Conditions in Arbitrary Dimensions. 2016, `https://arxiv.org/abs/1609.05873`.

[5] S. Bauer, D. Pauly, and M. Schomburg. Weck's Selection Theorem: The Maxwell Compactness Property for Bounded Weak Lipschitz Domains with Mixed Boundary Conditions in Arbitrary Dimensions. *Maxwell's Equations: Analysis and Numerics (Radon Series on Computational and Applied Mathematics 24)*, 77-104, De Gruyter, 2019.

[6] R. Bott and R. Seeley. Some remarks on the paper of Callias. *Comm. Math. Phys.*, 62:235-245, 1978.

[7] J. Brüning and M. Lesch. Hilbert complexes. *Journal of Functional Analysis*, 108(1):88-132, 1992.

[8] C. Callias. Axial anomalies and index theorems on open spaces. *Commun. Math. Phys.*, 62:213-234, 1978.

[9] F. Gesztesy, Y. Latushkin, K. A. Makarov, F. Sukochev, and Y. Tomilov. The index formula and the spectral shift function for relatively trace class perturbations. *Adv. Math.*, 227:319-420, 2011.

[10] F. Gesztesy and M. Waurick. *The Callias Index Formula Revisited.* Springer Lecture Notes in Mathematics 2157, 2016.
[11] L. Hörmander. The Weyl calculus of pseudo-differential operators. *Commun. Pure Appl.Math.*, 32:359-443, 1979.
[12] R. Illner. On algebras of pseudodifferential operators in $L^p(\mathbb{R}^n)$. *Commun. PDE*, 2:359-393, 1977.
[13] P. Kuhn and D. Pauly. Regularity Results for Generalized Electro-Magnetic Problems. *Analysis (Munich)*, 30:225-252, 2010.
[14] R. Leis. *Initial-boundary value problems in mathematical physics.* B. G. Teubner, Stuttgart; John Wiley & Sons, Ltd., Chichester, 1986.
[15] R. Leis. *Vorlesungen über Funktionalanalysis.* `https://www.iam.uni-bonn.de/fileadmin/user_upload/leis/Funktionalanalysis.pdf`, 1997.
[16] A. Mukherjee. *Atiyah-Singer Index Theorem. An Introduction.* Hindustan Book Agency, New Delhi, 2013.
[17] D. Pauly. On the Maxwell and Friedrichs/Poincare Constants in ND. *Math. Z.*, 293(3):957-987, 2019.
[18] D. Pauly. A global div-curl-lemma for mixed boundary conditions in weak Lipschitz domains and a corresponding generalized A_0^*-A_1-lemma in Hilbert spaces. *Analysis (Berlin)*, 39:33-58, 2019.
[19] D. Pauly. Solution theory, variational formulations, and functional a posteriori error estimates for general first order systems with applications to electro-magneto-statics and more. *Numer. Funct. Anal. Optim.*, 41(1):16-112, 2020.
[20] D. Pauly and W. Zulehner. On Closed and Exact Grad-grad- and div-Div-Complexes, Corresponding Compact Embeddings for Tensor Rotations, and a Related Decomposition Result for Biharmonic Problems in 3D. 2016, `https://arxiv.org/abs/1609.05873`.
[21] D. Pauly and W. Zulehner. The divDiv-Complex and Applications to Biharmonic Equations. *Applicable Analysis*, 2020.
[22] D. Pauly and W. Zulehner. The Elasticity Complex. *submitted*, 2020, `https://arxiv.org/abs/2001.11007`.
[23] R. Picard. On the boundary value problems of electro- and magnetostatics. *Proc. Roy. Soc. Edinburgh Sect. A*, 92:165-174, 1982.
[24] R. Picard. An elementary proof for a compact imbedding result in generalized electromagnetic theory. *Math. Z.*, 187:151-164, 1984.
[25] R. Picard, S. Trostorff, and M. Waurick. On a Connection between the Maxwell System, the Extended Maxwell System, the Dirac Operator and Gravito-Electromagnetism. *Math. Meth. Appl. Sci.*, 40(2):415-434, 2017.
[26] C. Seifert, S. Trostorff, and M. Waurick. *Evolutionary Equations.* 23rd Internetseminar, `https://www.mat.tuhh.de/veranstaltungen/isem23/_media/main_lectures.pdf`.
[27] C. Weber. A local compactness theorem for Maxwell's equations. *Math. Meth. Appl. Sci.*, 2:12-25, 1980.
[28] C. Weber. Regularity theorems for Maxwell's equations. *Math. Meth. Appl. Sci.*, 3:523-536, 1981.
[29] N. Weck. Maxwell's boundary value problems on Riemannian manifolds with nonsmooth boundaries. *J. Math. Anal. Appl.*, 46:410-437, 1974.
[30] J. Wloka. *Partial differential equations.* Cambridge University Press, Cambridge, 1987.

Appendix. Dirichlet and Neumann Fields

In Theorem 3.6, Theorem 4.3, Theorem 5.4, and Theorem 6.3 we have seen that the dimensions of the harmonic Dirichlet and Neumann fields are given by the topological invariants of the open and bounded set Ω and its complement

$$\Xi := \mathbb{R}^3 \setminus \overline{\Omega},$$

i.e., by

- n, the number of connected components Ω_k of Ω, i.e., $\Omega = \dot{\bigcup}_{k=1}^{n} \Omega_k$,
- m, the number of connected components Ξ_ℓ of Ξ, i.e., $\Xi = \dot{\bigcup}_{\ell=0}^{m-1} \Xi_\ell$,
- p, the number of handles of Ω, see Assumption 3.

More precisely, we recall

$$\begin{aligned} \dim \mathcal{H}_{D}^{\mathsf{Rhm}}(\Omega) &= m-1, & \dim \mathcal{H}_{N}^{\mathsf{Rhm}}(\Omega) &= p, \\ \dim \mathcal{H}_{D,\mathbb{S}}^{\mathsf{bih},1}(\Omega) &= 4(m-1), & \dim \mathcal{H}_{N,\mathbb{T}}^{\mathsf{bih},1}(\Omega) &= 4p, \\ \dim \mathcal{H}_{D,\mathbb{T}}^{\mathsf{bih},2}(\Omega) &= 4(m-1), & \dim \mathcal{H}_{N,\mathbb{S}}^{\mathsf{bih},2}(\Omega) &= 4p, \\ \dim \mathcal{H}_{D,\mathbb{S}}^{\mathsf{ela}}(\Omega) &= 6(m-1), & \dim \mathcal{H}_{N,\mathbb{S}}^{\mathsf{ela}}(\Omega) &= 6p. \end{aligned}$$

This appendix provides the corresponding proofs in detail. For the de Rham complex we follow in close lines the arguments of Picard in [23] introducing some simplifications for bounded domains and trivial material tensors ε and μ. These ideas will be adapted and modified for the proofs of the corresponding results of the other Hilbert complexes.

Assumption 1. $\Omega \subset \mathbb{R}^3$ *is open and bounded with segment property, i.e.,* Ω *has a continuous boundary* $\Gamma := \partial\,\Omega$*, see Remark 3.7.*

Assumption 2. $\Omega \subset \mathbb{R}^3$ *is open, bounded, and* Γ *is strong Lipschitz.*

In view of Assumption 1 and Assumption 2 we note:

- Assumption 1 guarantees that $m, n \in \mathbb{N}$ are well defined. So does Assumption 3 for $p \in \mathbb{N}_0$. In particular, $\operatorname{int} \Xi_\ell \neq \emptyset$ for all $\ell = 0, \dots, m-1$.
- Assumption 2 implies Assumption 1.
- Assumption 2 simplifies some arguments, in particular, all ranges in the crucial Helmholtz type decompositions used in our proofs are closed, cf. Remark B.2, Remark B.11, Remark B.18, and Remark B.24. We emphasise that all our results presented in this appendix still hold with Assumption 2 replaced by the weaker Assumption 1. In this case it is not clear if the mentioned ranges are closed and in some of our arguments we need to use some additional density and approximation arguments.

Let us recall from Lemma 1.3 the local regularities

$$\begin{aligned} \mathcal{H}_{D}^{\mathsf{Rhm}}(\Omega), \mathcal{H}_{N}^{\mathsf{Rhm}}(\Omega) &\subset C^{\infty,3}(\Omega) \cap L^{2,3}(\Omega), \\ \mathcal{H}_{D,\mathbb{S}}^{\mathsf{bih},1}(\Omega), \mathcal{H}_{D,\mathbb{S}}^{\mathsf{ela}}(\Omega), \mathcal{H}_{N,\mathbb{S}}^{\mathsf{bih},2}(\Omega), \mathcal{H}_{N,\mathbb{S}}^{\mathsf{ela}}(\Omega) &\subset C^{\infty,3\times3}(\Omega) \cap L_{\mathbb{S}}^{2,3\times3}(\Omega), \\ \mathcal{H}_{D,\mathbb{T}}^{\mathsf{bih},2}(\Omega), \mathcal{H}_{N,\mathbb{T}}^{\mathsf{bih},1}(\Omega) &\subset C^{\infty,3\times3}(\Omega) \cap L_{\mathbb{T}}^{2,3\times3}(\Omega). \end{aligned} \tag{25}$$

In particular, all Dirichlet and Neumann fields of the respective cohomology groups are continuous and square integrable.

Appendix A. Dirichlet Fields

Let us denote the unbounded connected component of Ξ by Ξ_0 and its boundary by $\Gamma_0 := \partial\,\Xi_0$. The remaining connected components of Ξ are $\Xi_1, \dots, \Xi_{m-1}$ with boundaries $\Gamma_\ell := \partial\,\Xi_\ell$. Note that none of $\Gamma_0, \dots, \Gamma_{m-1}$ need to be connected. Furthermore, let us introduce an open (and bounded) ball $B \supset \overline{\Omega}$ and set $\widetilde{\Xi}_0 := B \cap \Xi_0$. Then the connected components of $B \setminus \overline{\Omega}$ are $\widetilde{\Xi}_0$ and $\Xi_1, \dots, \Xi_{m-1}$. Moreover, let

$$\xi_\ell \in C_c^\infty(\mathbb{R}^3), \qquad \ell = 1, \dots, m-1, \tag{26}$$

with disjoint supports such that $\xi_\ell = 0$ in a neighbourhood of Ξ_0 and in a neighbourhood of Ξ_k for all $\ell \neq k \in \{1, \dots, m-1\}$ as well as $\xi_\ell = 1$ in a neighbourhood of Ξ_ℓ. In particular, $\xi_\ell = 0$ in a neighbourhood of Γ_0 and in a neighbourhood of Γ_k for all $\ell \neq k \in \{1, \dots, m-1\}$ and $\xi_\ell = 1$ in a neighbourhood of Γ_ℓ. Theses indicator type functions ξ_ℓ will be used to construct a basis for the respective Dirichlet fields.

A.1. **Dirichlet Vector Fields of the Classical de Rham Complex.** For the de Rham complex, simliar to (3) and (4), we have the orthogonal decompositions

$$\begin{aligned} L^{2,3}(\Omega) = H_1 &= \operatorname{ran} A_0 \oplus_{H_1} \ker A_0^* = \operatorname{ran}(\mathring{\operatorname{grad}}, \Omega) \oplus_{L^{2,3}(\Omega)} \ker(\operatorname{div}, \Omega), \\ \ker(\mathring{\operatorname{curl}}, \Omega) = \ker(A_1) &= \operatorname{ran} A_0 \oplus_{H_1} K_1 = \operatorname{ran}(\mathring{\operatorname{grad}}, \Omega) \oplus_{L^{2,3}(\Omega)} \mathcal{H}_D^{\mathsf{Rhm}}(\Omega). \end{aligned} \tag{27}$$

Remark A.1. *It holds* $\operatorname{dom}(\mathring{\operatorname{grad}}, \Omega) = H_0^1(\Omega)$*. Moreover, the range in* (27) *is closed by the Friedrichs estimate*

$$\exists\, c > 0 \quad \forall\, \phi \in H_0^1(\Omega) \qquad |\phi|_{L^2(\Omega)} \leq c|\operatorname{grad}\phi|_{L^{2,3}(\Omega)},$$

which holds by Assumption 1. Note that Ω *open and bounded is already sufficient.*

Let us denote in (27) the orthogonal projector onto $\ker(\operatorname{div}, \Omega)$ resp. $\mathcal{H}_D^{\mathsf{Rhm}}(\Omega)$ by π. Moreover, recall the functions ξ_ℓ from (26). Then for $\ell = 1, \dots, m-1$

$$\operatorname{grad}\xi_\ell \in C_c^{\infty,3}(\Omega) \cap \ker(\operatorname{curl}, \Omega) \subset \ker(\mathring{\operatorname{curl}}, \Omega)$$

and there exists some $\psi_\ell \in H_0^1(\Omega)$ such that

$$\mathcal{H}_D^{\mathsf{Rhm}}(\Omega) \ni \pi \operatorname{grad}\xi_\ell = \operatorname{grad}(\xi_\ell - \psi_\ell) = \operatorname{grad} u_\ell, \qquad u_\ell := \xi_\ell - \psi_\ell \in H^1(\Omega).$$

We shall show that

$$\mathcal{B}_D^{\mathsf{Rhm}} := \{\operatorname{grad} u_1, \dots, \operatorname{grad} u_{m-1}\} \subset \mathcal{H}_D^{\mathsf{Rhm}}(\Omega) \tag{28}$$

defines a basis of $\mathcal{H}_D^{\mathsf{Rhm}}(\Omega)$.

Note that $\psi_\ell \in H_0^1(\Omega)$ can be found by the standard variational formulation

$$\forall\, \phi \in H_0^1(\Omega) \qquad \langle \operatorname{grad}\psi_\ell, \operatorname{grad}\phi\rangle_{L^{2,3}(\Omega)} = \langle \operatorname{grad}\xi_\ell, \operatorname{grad}\phi\rangle_{L^{2,3}(\Omega)},$$

i.e., $\psi_\ell = \mathring{\Delta}^{-1}\Delta\xi_\ell$. Therefore, $u_\ell = \xi_\ell - \psi_\ell = (1 - \mathring{\Delta}^{-1}\Delta)\xi_\ell \in H^1(\Omega)$ and

$$\begin{aligned} \operatorname{grad} u_\ell &= \operatorname{grad}(1 - \mathring{\Delta}^{-1}\Delta)\xi_\ell \\ &= (\operatorname{grad} - \operatorname{grad}\mathring{\Delta}^{-1}\Delta)\xi_\ell \\ &= (1 - \operatorname{grad}\mathring{\Delta}^{-1}\operatorname{div})\operatorname{grad}\xi_\ell. \end{aligned}$$

Let us also mention that u_ℓ solves in classical terms the Dirichlet Laplace problem

$$\begin{aligned} -\Delta u_\ell = -\operatorname{div}\operatorname{grad} u_\ell &= 0 \quad \text{in } \Omega, \\ u_\ell &= 1 \quad \text{on } \Gamma_\ell, \\ u_\ell &= 0 \quad \text{on } \Gamma_k,\ \ell \neq k = 0, \dots, m-1, \end{aligned} \tag{29}$$

which is uniquely solvable. In particular, for all $\ell = 1, \dots, m-1$ it holds $u_\ell = 0$ on Γ_0.

Lemma A.2. *Let Assumption 1 be satisfied. Then* $\mathcal{H}_D^{\mathsf{Rhm}}(\Omega) = \operatorname{lin}\mathcal{B}_D^{\mathsf{Rhm}}$*.*

Proof. Let $H \in \mathcal{H}_D^{\mathsf{Rhm}}(\Omega) = \ker(\mathring{\operatorname{curl}}, \Omega) \cap \ker(\operatorname{div}, \Omega)$. In particular, by the homogeneous boundary condition its extension by zero $\widetilde{H}$ to B belongs to $\ker(\mathring{\operatorname{curl}}, B)$. As B is topologically trivial (and smooth and bounded), there exists (a unique) $u \in H_0^1(B)$ such that $\operatorname{grad} u = \widetilde{H}$ in B, see, e.g., [21, Lemma 2.24]. As $\operatorname{grad} u = \widetilde{H} = 0$ in $B \setminus \overline{\Omega}$, u must be constant in each connected component $\widetilde{\Xi}_0, \Xi_1, \dots, \Xi_{m-1}$ of $B \setminus \overline{\Omega}$. Due to the homogenous boundary condition at ∂B, u vanishes in $\widetilde{\Xi}_0$. Therefore, $H = \operatorname{grad} u$ in Ω and $u \in H_0^1(B)$ such that $u|_{\Xi_{\widetilde{\Xi}_0}} = 0$ and $u|_{\Xi_\ell} =: \alpha_\ell \in \mathbb{R}$ for all $\ell = 1, \dots, m-1$. Let us consider

$$\widehat{H} := H - \sum_{\ell=1}^{m-1} \alpha_\ell \operatorname{grad} u_\ell = \operatorname{grad}\widehat{u} \in \mathcal{H}_D^{\mathsf{Rhm}}(\Omega), \qquad \widehat{u} := u - \sum_{\ell=1}^{m-1} \alpha_\ell u_\ell \in H^1(\Omega)$$

with u_ℓ from (28). The extension by zero of ψ_ℓ to $\widetilde{\psi}_\ell$ belongs to $H_0^1(B)$. Hence as an element of $H^1(B)$ we see that

$$\widehat{u}_B := u - \sum_{\ell=1}^{m-1} \alpha_\ell \xi_\ell + \sum_{\ell=1}^{m-1} \alpha_\ell \widetilde{\psi}_\ell \in H_0^1(B)$$

vanishes in all Ξ_ℓ. Thus $\widehat{u} = \widehat{u}_B|_\Omega \in H_0^1(\Omega)$ by Assumption 1, and we compute

$$|\widehat{H}|^2_{L^{2,3}(\Omega)} = \langle \operatorname{grad} \widehat{u}, \widehat{H} \rangle_{L^{2,3}(\Omega)} = 0,$$

finishing the proof. □

Note that, in classical terms, u from the later proof solves the linear Dirichlet Laplace problem

$$\begin{aligned} -\Delta u = -\operatorname{div} \operatorname{grad} u = -\operatorname{div} H = 0 &\qquad \text{in } \Omega, \\ u = 0 &\qquad \text{on } \Gamma_0, \\ u = \alpha_\ell \in \mathbb{R} &\qquad \text{on } \Gamma_\ell,\ \ell = 1, \dots, m-1, \end{aligned}$$

which is uniquely solvable as long as the constants are prescribed.

Lemma A.3. *Let Assumption 1 be satisfied. Then $\mathcal{B}_D^{\mathsf{Rhm}}$ is linear independent.*

Proof. Let

$$\sum_{\ell=1}^{m-1} \alpha_\ell \operatorname{grad} u_\ell = 0, \qquad u := \sum_{\ell=1}^{m-1} \alpha_\ell u_\ell.$$

Then $\operatorname{grad} u = 0$ in Ω, i.e., u is constant in each connected component of Ω. We show $u = 0$. Recall $u_\ell = \xi_\ell - \psi_\ell$ in Ω. Extension by zero of ψ_ℓ to $\widetilde{\psi}_\ell$ shows $\widetilde{u}_\ell \in H_0^1(B)$, where

$$\widetilde{u}_\ell := \begin{cases} u_\ell & \text{in } \Omega, \\ \xi_\ell & \text{in } B \setminus \overline{\Omega}, \end{cases} \qquad \operatorname{grad} \widetilde{u}_\ell = \begin{cases} \operatorname{grad} u_\ell & \text{in } \Omega, \\ \operatorname{grad} \xi_\ell = 0 & \text{in } B \setminus \overline{\Omega}. \end{cases}$$

Note that $\widetilde{u}_\ell = \xi_\ell = 0$ in $\widetilde{\Xi}_0$ and in Ξ_k for all $\ell \neq k = 1, \dots, m-1$ and that $\widetilde{u}_\ell = \xi_\ell = 1$ in Ξ_ℓ. Then

$$\widetilde{u} := \sum_{\ell=1}^{m-1} \alpha_\ell \widetilde{u}_\ell \in H_0^1(B)$$

with $\widetilde{u} = 0$ in $\widetilde{\Xi}_0$ and $\operatorname{grad} \widetilde{u} = 0$ in $B \setminus \overline{\Omega}$ as well as $\operatorname{grad} \widetilde{u} = \operatorname{grad} u = 0$ in Ω by assumption. Hence, $\operatorname{grad} \widetilde{u} = 0$ in B, showing $\widetilde{u} = 0$ in B. In particular, $u = 0$ in Ω, and $\alpha_\ell = \widetilde{u}|_{\Xi_\ell} = 0$ for all $\ell = 1, \dots, m-1$, finishing the proof. □

Theorem A.4. *Let Assumption 1 be satisfied. Then* $\dim \mathcal{H}_D^{\mathsf{Rhm}}(\Omega) = m - 1$ *and a basis of* $\mathcal{H}_D^{\mathsf{Rhm}}(\Omega)$ *is given by* (28).

Proof. Use Lemma A.2 and Lemma A.3. □

A.2. **Dirichlet Tensor Fields of the First Biharmonic Complex.** For the first biharmonic complex, simliar to (3), (4), and (27), we have the orthogonal decompositions

$$\begin{aligned} L^{2,3\times 3}_{\mathbb{S}}(\Omega) &= \operatorname{ran}(\mathring{\operatorname{Grad}}\operatorname{grad}, \Omega) \oplus_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)} \ker(\operatorname{divDiv}_{\mathbb{S}}, \Omega), \\ \ker(\mathring{\operatorname{Curl}}_{\mathbb{S}}, \Omega) &= \operatorname{ran}(\mathring{\operatorname{Grad}}\operatorname{grad}, \Omega) \oplus_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)} \mathcal{H}^{\mathsf{bih},1}_{D,\mathbb{S}}(\Omega). \end{aligned} \tag{30}$$

Remark A.5. *It holds* $\operatorname{dom}(\mathring{\operatorname{Gradgrad}}, \Omega) = H_0^2(\Omega)$ *by* [21, Lemma 3.3]. *Moreover, the range in* (30) *is closed by the Friedrichs type estimate*

$$\exists\, c > 0 \quad \forall\, \phi \in H_0^2(\Omega) \qquad |\phi|_{H^1(\Omega)} \leq c |\operatorname{Gradgrad} \phi|_{L^{2,3\times 3}(\Omega)},$$

which holds by Assumption 1. Note that Ω *open and bounded is already sufficient.*

Let us denote in (30) the orthogonal projector onto $\ker(\operatorname{divDiv}_{\mathbb{S}}, \Omega)$ resp. $\mathcal{H}_{D,\mathbb{S}}^{\mathsf{bih},1}(\Omega)$ by π and recall the functions ξ_ℓ from (26). We introduce polynomials $\widehat{p}_j$ given by $\widehat{p}_0(x) := 1$ and $\widehat{p}_j(x) := x_j$ for $j = 1, 2, 3$ and define $\xi_{\ell,j} := \xi_\ell \widehat{p}_j$ for all $\ell \in \{1, \ldots, m-1\}$ and all $j = 0, \ldots, 3$. In particular, for all $j = 0, \ldots, 3$ we have $\xi_{\ell,j} = 0$ in a neighbourhood of Ξ_0 and in a neighbourhood of Ξ_k for all $\ell \neq k \in \{1, \ldots, m-1\}$ and $\xi_{\ell,j} = \widehat{p}_j$ in a neighbourhood of Ξ_ℓ. Then

$$\operatorname{Gradgrad} \xi_{\ell,j} \in C_{c,\mathbb{S}}^{\infty,3\times 3}(\Omega) \cap \ker(\operatorname{Curl}_{\mathbb{S}}, \Omega) \subset \ker(\mathring{\operatorname{Curl}}_{\mathbb{S}}, \Omega)$$

and there exists some $\psi_{\ell,j} \in H_0^2(\Omega)$ such that

$$\mathcal{H}_{D,\mathbb{S}}^{\mathsf{bih},1}(\Omega) \ni \pi \operatorname{Gradgrad} \xi_{\ell,j} = \operatorname{Gradgrad}(\xi_{\ell,j} - \psi_{\ell,j}) = \operatorname{Gradgrad} u_{\ell,j}$$

with $u_{\ell,j} := \xi_{\ell,j} - \psi_{\ell,j} \in H^2(\Omega)$. We shall show that

$$\mathcal{B}_D^{\mathsf{bih},1} := \{\operatorname{Gradgrad} u_{\ell,j}\}_{\substack{\ell=1,\ldots,m-1, \\ j=0,\ldots,3}} \subset \mathcal{H}_{D,\mathbb{S}}^{\mathsf{bih},1}(\Omega) \tag{31}$$

defines a basis of $\mathcal{H}_{D,\mathbb{S}}^{\mathsf{bih},1}(\Omega)$.

Note that $\psi_{\ell,j} \in H_0^2(\Omega)$ can be found by the standard variational formulation

$$\forall\, \phi \in H_0^2(\Omega) \quad \langle \operatorname{Gradgrad} \psi_{\ell,j}, \operatorname{Gradgrad} \phi \rangle_{L_{\mathbb{S}}^{2,3\times 3}(\Omega)} = \langle \operatorname{Gradgrad} \xi_{\ell,j}, \operatorname{Gradgrad} \phi \rangle_{L_{\mathbb{S}}^{2,3\times 3}(\Omega)},$$

i.e., $\psi_{\ell,j} = (\mathring{\Delta}^2)^{-1} \Delta^2 \xi_{\ell,j}$. Therefore, $u_{\ell,j} = \xi_{\ell,j} - \psi_{\ell,j} = \big(1 - (\mathring{\Delta}^2)^{-1} \Delta^2\big) \xi_{\ell,j} \in H^2(\Omega)$ and

$$\begin{aligned}
\operatorname{Gradgrad} u_{\ell,j} &= \operatorname{Gradgrad} \big(1 - (\mathring{\Delta}^2)^{-1} \Delta^2\big) \xi_{\ell,j} \\
&= \big(\operatorname{Gradgrad} - \operatorname{Gradgrad} (\mathring{\Delta}^2)^{-1} \Delta^2\big) \xi_{\ell,j} \\
&= \big(1 - \operatorname{Gradgrad} (\mathring{\Delta}^2)^{-1} \operatorname{divDiv}_{\mathbb{S}}\big) \operatorname{Gradgrad} \xi_{\ell,j}.
\end{aligned}$$

Let us also mention that $u_{\ell,j}$ solves in classical terms the biharmonic Dirichlet problem

$$\begin{aligned}
\Delta^2 u_{\ell,j} := \operatorname{divDiv}_{\mathbb{S}} \operatorname{Gradgrad} u_{\ell,j} &= 0 && \text{in } \Omega, \\
u_{\ell,j} = \widehat{p}_j, \quad \operatorname{grad} u_{\ell,j} = \operatorname{grad} \widehat{p}_j &= e^j && \text{on } \Gamma_\ell, \\
u_{\ell,j} = 0, \quad \operatorname{grad} u_{\ell,j} &= 0 && \text{on } \Gamma_k,\ \ell \neq k = 0, \ldots, m-1,
\end{aligned} \tag{32}$$

which is uniquely solvable. In particular, for all $\ell = 1, \ldots, m-1$ and all $j = 0, \ldots, 3$ it holds $u_{\ell,j} = 0$ and $\operatorname{grad} u_{\ell,j} = 0$ on Γ_0. Here, we denote by e^j, $j = 1, 2, 3$, the Euclidean unit vectors in $\mathbb{R}^3$ and set $e^0 := 0 \in \mathbb{R}^3$.

Lemma A.6. *Let Assumption 1 be satisfied. Then* $\mathcal{H}_{D,\mathbb{S}}^{\mathsf{bih},1}(\Omega) = \operatorname{lin} \mathcal{B}_D^{\mathsf{bih},1}$.

Proof. We follow in close lines the arguments used in the proof of Lemma A.2. For this, let $S \in \mathcal{H}_{D,\mathbb{S}}^{\mathsf{bih},1}(\Omega) = \ker(\mathring{\operatorname{Curl}}_{\mathbb{S}}, \Omega) \cap \ker(\operatorname{divDiv}_{\mathbb{S}}, \Omega)$. In particular, by the homogeneous boundary condition its extension by zero $\widetilde{S}$ to B belongs to $\ker(\mathring{\operatorname{Curl}}_{\mathbb{S}}, B)$. As B is topologically trivial (and smooth and bounded), there exists (a unique) $u \in H_0^2(B)$ such that $\operatorname{Gradgrad} u = \widetilde{S}$ in B, see [21, Theorem 3.10 (i)]. As $\operatorname{Gradgrad} u = \widetilde{S} = 0$ in $B \setminus \overline{\Omega}$, u must belong to P_1, the polynomials of order 1, in each connected component $\widetilde{\Xi}_0, \Xi_1, \ldots, \Xi_{m-1}$ of $B \setminus \overline{\Omega}$. Due to the homogenous boundary condition at ∂B, u vanishes in $\widetilde{\Xi}_0$. Therefore, $S = \operatorname{Gradgrad} u$ in Ω and $u \in H_0^2(B)$ is such that $u|_{\widetilde{\Xi}_0} = 0$ and $u|_{\Xi_\ell} =: p_\ell =: \sum_{j=0}^3 \alpha_{\ell,j}\widehat{p}_j \in \mathsf{P}^1$, $\alpha_{\ell,j} \in \mathbb{R}$, for all $\ell = 1, \dots, m-1$. Let us consider

$$\widehat{S} := S - \sum_{\ell=1}^{m-1}\sum_{j=0}^{3} \alpha_{\ell,j} \operatorname{Gradgrad} u_{\ell,j} = \operatorname{Gradgrad} \widehat{u} \in \mathcal{H}_{D,\mathbb{S}}^{\mathsf{bih},1}(\Omega),$$

$$\widehat{u} := u - \sum_{\ell=1}^{m-1}\sum_{j=0}^{3} \alpha_{\ell,j} u_{\ell,j} \in H^2(\Omega)$$

with $u_{\ell,j}$ from (31). The extension by zero of $\psi_{\ell,j}$ to $\widetilde{\psi}_{\ell,j}$ belongs to $H_0^2(B)$. Hence as an element of $H^2(B)$ we see that

$$\widehat{u}_B := u - \sum_{\ell=1}^{m-1}\sum_{j=0}^{3} \alpha_{\ell,j}\xi_{\ell,j} + \sum_{\ell=1}^{m-1}\sum_{j=0}^{3} \alpha_{\ell,j}\widetilde{\psi}_{\ell,j} \in H_0^2(B)$$

vanishes in all Ξ_ℓ. Thus $\widehat{u} = \widehat{u}_B|_\Omega \in H_0^2(\Omega)$ by Assumption 1, and we compute

$$|\widehat{S}|^2_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)} = \langle \operatorname{Gradgrad} \widehat{u}, \widehat{S} \rangle_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)} = 0,$$

finishing the proof. □

Note that, in classical terms, u from the latter proof solves the linear biharmonic Dirichlet problem

$$\begin{aligned} \Delta^2 u = \operatorname{divDiv}_{\mathbb{S}} \operatorname{Gradgrad} u = \operatorname{divDiv}_{\mathbb{S}} S = 0 \qquad & \text{in } \Omega, \\ u = 0, \quad \operatorname{grad} u = 0 \qquad & \text{on } \Gamma_0, \\ u = p_\ell \in \mathsf{P}_1, \quad \operatorname{grad} u = \operatorname{grad} p_\ell \in \mathbb{R}^3 \qquad & \text{on } \Gamma_\ell,\ \ell = 1, \dots, m-1, \end{aligned}$$

which is uniquely solvable as long as the polynomials p_ℓ in P_1 are prescribed.

Lemma A.7. *Let Assumption 1 be satisfied. Then $\mathcal{B}_D^{\mathsf{bih},1}$ is linear independent.*

Proof. Let

$$\sum_{\ell=1}^{m-1}\sum_{j=0}^{3} \alpha_{\ell,j} \operatorname{Gradgrad} u_{\ell,j} = 0, \qquad u := \sum_{\ell=1}^{m-1}\sum_{j=0}^{3} \alpha_{\ell,j} u_{\ell,j}.$$

Then $\operatorname{Gradgrad} u = 0$ in Ω, i.e., u belongs to P_1 in each connected component of Ω. We show $u = 0$. Recall $u_{\ell,j} = \xi_{\ell,j} - \psi_{\ell,j}$ in Ω. Extension by zero of $\psi_{\ell,j}$ to $\widetilde{\psi}_{\ell,j}$ shows $\widetilde{u}_{\ell,j} \in H_0^2(B)$, where

$$\widetilde{u}_{\ell,j} := \begin{cases} u_{\ell,j} & \text{in } \Omega, \\ \xi_{\ell,j} & \text{in } B \setminus \overline{\Omega}, \end{cases} \qquad \operatorname{Gradgrad} \widetilde{u}_{\ell,j} = \begin{cases} \operatorname{Gradgrad} u_{\ell,j} & \text{in } \Omega, \\ \operatorname{Gradgrad} \xi_{\ell,j} = 0 & \text{in } B \setminus \overline{\Omega}. \end{cases}$$

Note that $\widetilde{u}_{\ell,j} = \xi_{\ell,j} = 0$ in $\widetilde{\Xi}_0$ and in Ξ_k for all $\ell \neq k = 1, \dots, m-1$ and $j = 0, \dots, 3$, and that $\widetilde{u}_{\ell,j} = \xi_{\ell,j} = \widehat{p}_j$ in Ξ_ℓ. Then

$$\widetilde{u} := \sum_{\ell=1}^{m-1}\sum_{j=0}^{3} \alpha_{\ell,j}\widetilde{u}_{\ell,j} \in H_0^2(B)$$

with $\widetilde{u} = 0$ in $\widetilde{\Xi}_0$ and $\operatorname{Gradgrad} \widetilde{u} = 0$ in $B \setminus \overline{\Omega}$ as well as $\operatorname{Gradgrad} \widetilde{u} = \operatorname{Gradgrad} u = 0$ in Ω by assumption. Hence, $\operatorname{Gradgrad} \widetilde{u} = 0$ in B, showing $\widetilde{u} = 0$ in B. In particular, $u = 0$ in Ω, and $\sum_{j=0}^3 \alpha_{\ell,j}\widehat{p}_j = \widetilde{u}|_{\Xi_\ell} = 0$ for all $\ell = 1, \dots, m-1$. We conclude $\alpha_{\ell,j} = 0$ for all $j = 0, \dots, 3$ and all ℓ, finishing the proof. □

Theorem A.8. *Let Assumption 1 be satisfied. Then* $\dim \mathcal{H}^{\mathsf{bih},1}_{D,\mathbb{S}}(\Omega) = 4(m-1)$ *and a basis of* $\mathcal{H}^{\mathsf{bih},1}_{D,\mathbb{S}}(\Omega)$ *is given by* (31).

Proof. Use Lemma A.6 and Lemma A.7. □

A.3. **Dirichlet Tensor Fields of the Second Biharmonic Complex.** For the second biharmonic complex, simliar to (3), (4), and (27), (30), we have the orthogonal decompositions

$$\begin{aligned} L^{2,3\times 3}_{\mathbb{T}}(\Omega) &= \operatorname{ran}(\operatorname{dev}\mathring{\operatorname{Grad}},\Omega) \oplus_{L^{2,3\times3}_{\mathbb{T}}(\Omega)} \ker(\operatorname{Div}_{\mathbb{T}},\Omega), \\ \ker(\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}},\Omega) &= \operatorname{ran}(\operatorname{dev}\mathring{\operatorname{Grad}},\Omega) \oplus_{L^{2,3\times3}_{\mathbb{T}}(\Omega)} \mathcal{H}^{\mathsf{bih},2}_{D,\mathbb{T}}(\Omega). \end{aligned} \tag{33}$$

Remark A.9. *It holds* $\operatorname{dom}(\operatorname{dev}\mathring{\operatorname{Grad}},\Omega) = H^{1,3}_0(\Omega)$ *by* [21, Lemma 3.2]. *Moreover, the range in* (33) *is closed by the Friedrichs type estimate*[5]

$$\exists\, c>0 \quad \forall\, \phi \in H^{1,3}_0(\Omega) \qquad |\phi|_{L^{2,3}(\Omega)} \le c |\operatorname{devGrad}\phi|_{L^{2,3\times3}(\Omega)},$$

which holds by Assumption 1. Note that Ω *open and bounded is already sufficient.*

Let us denote the orthogonal projector onto $\ker(\operatorname{Div}_{\mathbb{T}},\Omega)$ resp. $\mathcal{H}^{\mathsf{bih},2}_{D,\mathbb{T}}(\Omega)$ by π and recall $\xi_\ell \in C^\infty_c(\mathbb{R}^3)$ from (26). We introduce Raviart-Thomas fields $\widehat{r}_j$ given by $\widehat{r}_0(x) := x$ and $\widehat{r}_j(x) := e^j$ for $j=1,2,3$ and define $\xi_{\ell,j} := \xi_\ell \widehat{r}_j$ for all $\ell \in \{1,\dots,m-1\}$ and all $j=0,\dots,3$. In particular, for all $j=0,\dots,3$ we have $\xi_{\ell,j}=0$ in a neighbourhood of Ξ_0 and in a neighbourhood of Ξ_k for all $\ell \neq k \in \{1,\dots,m-1\}$ and $\xi_{\ell,j} = \widehat{r}_j$ in a neighbourhood of Ξ_ℓ. Then

$$\operatorname{devGrad}\xi_{\ell,j} \in C^{\infty,3\times3}_{c,\mathbb{T}}(\Omega) \cap \ker(\operatorname{symCurl}_{\mathbb{T}},\Omega) \subset \ker(\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}},\Omega)$$

and there exists some $\psi_{\ell,j} \in H^{1,3}_0(\Omega)$ such that

$$\mathcal{H}^{\mathsf{bih},2}_{D,\mathbb{T}}(\Omega) \ni \pi \operatorname{devGrad}\xi_{\ell,j} = \operatorname{devGrad}(\xi_{\ell,j} - \psi_{\ell,j}) = \operatorname{devGrad} v_{\ell,j}$$

with $v_{\ell,j} := \xi_{\ell,j} - \psi_{\ell,j} \in H^{1,3}(\Omega)$. We shall show that

$$\mathcal{B}^{\mathsf{bih},2}_D := \{\operatorname{devGrad} v_{\ell,j}\}_{\substack{\ell=1,\dots,m-1,\\ j=0,\dots,3}} \subset \mathcal{H}^{\mathsf{bih},2}_{D,\mathbb{T}}(\Omega) \tag{34}$$

defines a basis of $\mathcal{H}^{\mathsf{bih},2}_{D,\mathbb{T}}(\Omega)$.

Note that $\psi_{\ell,j} \in H^{1,3}_0(\Omega)$ can be found by the standard variational formulation

$$\forall\, \phi \in H^{1,3}_0(\Omega) \quad \langle \operatorname{devGrad}\psi_{\ell,j}, \operatorname{devGrad}\phi\rangle_{L^{2,3\times3}_{\mathbb{T}}(\Omega)} = \langle \operatorname{devGrad}\xi_{\ell,j}, \operatorname{devGrad}\phi\rangle_{L^{2,3\times3}_{\mathbb{T}}(\Omega)},$$

i.e., $\psi_{\ell,j} = \mathring{\Delta}_{\mathbb{T}}^{-1}\Delta_{\mathbb{T}}\xi_{\ell,j}$. Therefore, $u_{\ell,j} = \xi_{\ell,j} - \psi_{\ell,j} = (1 - \mathring{\Delta}_{\mathbb{T}}^{-1}\Delta_{\mathbb{T}})\xi_{\ell,j} \in H^{1,3}(\Omega)$ and

$$\begin{aligned} \operatorname{devGrad} v_{\ell,j} &= \operatorname{devGrad}(1 - \mathring{\Delta}_{\mathbb{T}}^{-1}\Delta_{\mathbb{T}})\xi_{\ell,j} \\ &= (\operatorname{devGrad} - \operatorname{devGrad}\mathring{\Delta}_{\mathbb{T}}^{-1}\Delta_{\mathbb{T}})\xi_{\ell,j} \\ &= (1 - \operatorname{devGrad}\mathring{\Delta}_{\mathbb{T}}^{-1}\operatorname{Div}_{\mathbb{T}})\operatorname{devGrad}\xi_{\ell,j}. \end{aligned}$$

Let us also mention that $v_{\ell,j}$ solves in classical terms the elasticity type Dirichlet problem

$$\begin{aligned} -\Delta_{\mathbb{T}} v_{\ell,j} := -\operatorname{Div}_{\mathbb{T}}\operatorname{devGrad} v_{\ell,j} &= 0 \quad \text{in } \Omega, \\ v_{\ell,j} &= \widehat{r}_j \quad \text{on } \Gamma_\ell, \\ v_{\ell,j} &= 0 \quad \text{on } \Gamma_k,\ \ell \neq k = 0,\dots,m-1, \end{aligned} \tag{35}$$

[5]Note that by $|T|^2 = |\operatorname{dev} T|^2 + \frac{1}{3}|\operatorname{tr} T|^2$ (pointwise) and $|\operatorname{Grad} v|^2_{L^{2,3\times3}(\Omega)} = |\operatorname{curl} v|^2_{L^{2,3}(\Omega)} + |\operatorname{div} v|^2_{L^2(\Omega)}$ for all $v \in H^{1,3}_0(\Omega)$, we have $2|\operatorname{Grad} v|^2_{L^{2,3\times3}(\Omega)} \le 3|\operatorname{devGrad} v|^2_{L^{2,3\times3}(\Omega)}$.

which is uniquely solvable. In particular, for all $\ell = 1, \dots, m-1$ and all $j = 0, \dots, 3$ it holds $v_{\ell,j} = 0$ on Γ_0.

Lemma A.10. *Let Assumption 1 be satisfied. Then $\mathcal{H}^{\mathsf{bih},2}_{D,\mathbb{T}}(\Omega) = \operatorname{lin} \mathcal{B}^{\mathsf{bih},2}_{D}$.*

Proof. We follow in close lines the arguments used in the proofs of Lemma A.2 and Lemma A.6. Let $T \in \mathcal{H}^{\mathsf{bih},2}_{D,\mathbb{T}}(\Omega) = \ker(\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}}, \Omega) \cap \ker(\operatorname{Div}_{\mathbb{T}}, \Omega)$. In particular, by the homogeneous boundary condition its extension by zero $\widetilde{T}$ to B belongs to $\ker(\operatorname{sym}\mathring{\operatorname{Curl}}_{\mathbb{T}}, B)$. As B is topologically trivial (and smooth and bounded), there exists (a unique vector field) $v \in H^{1,3}_0(B)$ such that $\operatorname{devGrad} v = \widetilde{T}$ in B. This follows analogously to [21, Theorem 3.10 (iv)]. As $\operatorname{devGrad} v = \widetilde{T} = 0$ in $B \setminus \overline{\Omega}$, v must be a Raviart-Thomas vector field, i.e., $v \in \mathsf{RT}$, in each connected component $\widetilde{\Xi}_0, \Xi_1, \dots, \Xi_{m-1}$ of $B \setminus \overline{\Omega}$. Due to the homogenous boundary condition at ∂B, v vanishes in $\widetilde{\Xi}_0$. Therefore, $T = \operatorname{devGrad} v$ in Ω and $v \in H^{1,3}_0(B)$ is such that $v|_{\widetilde{\Xi}_0} = 0$ and $v|_{\Xi_\ell} =: r_\ell =: \sum_{j=0}^{3} \alpha_{\ell,j} \widehat{r}_j \in \mathsf{RT}$, $\alpha_{\ell,j} \in \mathbb{R}$, for all $\ell = 1, \dots, m-1$. Let us consider

$$\widehat{T} := T - \sum_{\ell=1}^{m-1} \sum_{j=0}^{3} \alpha_{\ell,j} \operatorname{devGrad} v_{\ell,j} = \operatorname{devGrad} \widehat{v} \in \mathcal{H}^{\mathsf{bih},2}_{D,\mathbb{T}}(\Omega),$$

$$\widehat{v} := v - \sum_{\ell=1}^{m-1} \sum_{j=0}^{3} \alpha_{\ell,j} v_{\ell,j} \in H^{1,3}(\Omega)$$

with $v_{\ell,j}$ from (34). The extension by zero of $\psi_{\ell,j}$ to $\widetilde{\psi}_{\ell,j}$ belongs to $H^{1,3}_0(B)$. Hence as an element of $H^{1,3}(B)$ we see that

$$\widehat{v}_B := v - \sum_{\ell=1}^{m-1} \sum_{j=0}^{3} \alpha_{\ell,j} \xi_{\ell,j} + \sum_{\ell=1}^{m-1} \sum_{j=0}^{3} \alpha_{\ell,j} \widetilde{\psi}_{\ell,j} \in H^{1,3}_0(B)$$

vanishes in all Ξ_ℓ. Thus $\widehat{v} = \widehat{v}_B|_\Omega \in H^{1,3}_0(\Omega)$ by Assumption 1, and we compute

$$|\widehat{T}|^2_{L^{2,3\times3}_{\mathbb{T}}(\Omega)} = \langle \operatorname{devGrad} \widehat{v}, \widehat{T} \rangle_{L^{2,3\times3}_{\mathbb{T}}(\Omega)} = 0,$$

finishing the proof. $\square$

Note that, in classical terms, v from the latter proof solves the linear elasticity type Dirichlet problem

$$\begin{aligned} -\Delta_{\mathbb{T}} v = -\operatorname{Div}_{\mathbb{T}} \operatorname{devGrad} v = -\operatorname{Div}_{\mathbb{T}} T &= 0 && \text{in } \Omega, \\ v &= 0 && \text{on } \Gamma_0, \\ v &= r_\ell \in \mathsf{RT} && \text{on } \Gamma_\ell,\ \ell = 1, \dots, m-1, \end{aligned}$$

which is uniquely solvable as long as the Raviart-Thomas fields r_ℓ in RT are prescribed.

Lemma A.11. *Let Assumption 1 be satisfied. Then $\mathcal{B}^{\mathsf{bih},2}_{D}$ is linear independent.*

Proof. Let

$$\sum_{\ell=1}^{m-1} \sum_{j=0}^{3} \alpha_{\ell,j} \operatorname{devGrad} v_{\ell,j} = 0, \qquad v := \sum_{\ell=1}^{m-1} \sum_{j=0}^{3} \alpha_{\ell,j} v_{\ell,j}.$$

Then $\operatorname{devGrad} v = 0$ in Ω, i.e., $v \in \mathsf{RT}$ in each connected component of Ω. We show $v = 0$. Recall $v_{\ell,j} = \xi_{\ell,j} - \psi_{\ell,j}$ in Ω. Extension by zero of $\psi_{\ell,j}$ to $\widetilde{\psi}_{\ell,j}$ shows $\widetilde{v}_{\ell,j} \in H^{1,3}_0(B)$, where

$$\widetilde{v}_{\ell,j} := \begin{cases} v_{\ell,j} & \text{in } \Omega, \\ \xi_{\ell,j} & \text{in } B \setminus \overline{\Omega}, \end{cases} \qquad \operatorname{devGrad} \widetilde{v}_{\ell,j} = \begin{cases} \operatorname{devGrad} v_{\ell,j} & \text{in } \Omega, \\ \operatorname{devGrad} \xi_{\ell,j} = 0 & \text{in } B \setminus \overline{\Omega}. \end{cases}$$

Note that $\widetilde{v}_{\ell,j} = \xi_{\ell,j} = 0$ in $\widetilde{\Xi}_0$ and in Ξ_k for all $\ell \neq k = 1, \ldots, m-1$ and $j = 0, \ldots, 3$, and that $\widetilde{v}_{\ell,j} = \xi_{\ell,j} = \widehat{r}_j$ in Ξ_ℓ. Then

$$\widetilde{v} := \sum_{\ell=1}^{m-1} \sum_{j=0}^{3} \alpha_{\ell,j} \widetilde{v}_{\ell,j} \in H_0^{1,3}(B)$$

with $\widetilde{v} = 0$ in $\widetilde{\Xi}_0$ and $\operatorname{devGrad} \widetilde{v} = 0$ in $B \setminus \overline{\Omega}$ as well as $\operatorname{devGrad} \widetilde{v} = \operatorname{devGrad} v = 0$ in Ω by assumption. Hence, $\operatorname{devGrad} \widetilde{v} = 0$ in B, showing $\widetilde{v} = 0$ in B. In particular, $v = 0$ in Ω, and $\sum_{j=0}^{3} \alpha_{\ell,j} \widehat{r}_j = \widetilde{v}|_{\Xi_\ell} = 0$ for all $\ell = 1, \ldots, m-1$. We conclude $\alpha_{\ell,j} = 0$ for all $j = 0, \ldots, 3$ and all ℓ, finishing the proof. $\square$

Theorem A.12. *Let Assumption 1 be satisfied. Then* $\dim \mathcal{H}_{D,\mathbb{T}}^{\mathsf{bih},2}(\Omega) = 4(m-1)$ *and a basis of* $\mathcal{H}_{D,\mathbb{T}}^{\mathsf{bih},2}(\Omega)$ *is given by* (34).

Proof. Use Lemma A.10 and Lemma A.11. $\square$

A.4. **Dirichlet Tensor Fields of the Elasticity Complex.** For the elasticity complex, simliar to (3), (4), and (27), (30), (33), we have the orthogonal decompositions

$$\begin{aligned} L_{\mathbb{S}}^{2,3\times3}(\Omega) &= \operatorname{ran}(\operatorname{sym}\mathring{\operatorname{Grad}}, \Omega) \oplus_{L_{\mathbb{S}}^{2,3\times3}(\Omega)} \ker(\operatorname{Div}_{\mathbb{S}}, \Omega), \\ \ker(\operatorname{Curl}\mathring{\operatorname{Curl}}_{\mathbb{S}}^{\top}, \Omega) &= \operatorname{ran}(\operatorname{sym}\mathring{\operatorname{Grad}}, \Omega) \oplus_{L_{\mathbb{S}}^{2,3\times3}(\Omega)} \mathcal{H}_{D,\mathbb{S}}^{\mathsf{ela}}(\Omega). \end{aligned} \tag{36}$$

Remark A.13. *It holds* $\operatorname{dom}(\operatorname{sym}\mathring{\operatorname{Grad}}, \Omega) = H_0^{1,3}(\Omega)$ *by* [22, Lemma 3.2]. *Moreover, the range in* (36) *is closed by the Friedrichs type estimate*[6]

$$\exists\, c > 0 \quad \forall\, \phi \in H_0^{1,3}(\Omega) \qquad |\phi|_{L^{2,3}(\Omega)} \leq c |\operatorname{symGrad} \phi|_{L^{2,3\times3}(\Omega)},$$

which holds by Assumption 1. Note that Ω open and bounded is already sufficient.

Let us denote the orthogonal projector onto $\ker(\operatorname{Div}_{\mathbb{S}}, \Omega)$ resp. $\mathcal{H}_{D,\mathbb{S}}^{\mathsf{ela}}(\Omega)$ by π and recall $\xi_\ell \in C_c^\infty(\mathbb{R}^3)$ from (26). We introduce rigid motions $\widehat{r}_j$ given by $\widehat{r}_j(x) := e^j \times x$ and $\widehat{r}_{j+3}(x) := e^j$ for $j = 1, 2, 3$ and define $\xi_{\ell,j} := \xi_\ell \widehat{r}_j$ for all $\ell \in \{1, \ldots, m-1\}$ and for all $j = 1, \ldots, 6$. In particular, for all $j = 1, \ldots, 6$ we have $\xi_{\ell,j} = 0$ in a neighbourhood of Ξ_0 and in a neighbourhood of Ξ_k for all $\ell \neq k \in \{1, \ldots, m-1\}$ and $\xi_{\ell,j} = \widehat{r}_j$ in a neighbourhood of Ξ_ℓ. Then

$$\operatorname{symGrad} \xi_{\ell,j} \in C_{c,\mathbb{S}}^{\infty,3\times3}(\Omega) \cap \ker(\operatorname{CurlCurl}_{\mathbb{S}}^{\top}, \Omega) \subset \ker(\operatorname{Curl}\mathring{\operatorname{Curl}}_{\mathbb{S}}^{\top}, \Omega)$$

and there exists some $\psi_{\ell,j} \in H_0^{1,3}(\Omega)$ such that

$$\mathcal{H}_{D,\mathbb{S}}^{\mathsf{ela}}(\Omega) \ni \pi \operatorname{symGrad} \xi_{\ell,j} = \operatorname{symGrad}(\xi_{\ell,j} - \psi_{\ell,j}) = \operatorname{symGrad} v_{\ell,j}$$

with $v_{\ell,j} := \xi_{\ell,j} - \psi_{\ell,j} \in H^{1,3}(\Omega)$. We shall show that

$$\mathcal{B}_D^{\mathsf{ela}} := \{\operatorname{symGrad} v_{\ell,j}\}_{\substack{\ell=1,\ldots,m-1, \\ j=1,\ldots,6}} \subset \mathcal{H}_{D,\mathbb{S}}^{\mathsf{ela}}(\Omega) \tag{37}$$

defines a basis of $\mathcal{H}_{D,\mathbb{S}}^{\mathsf{ela}}(\Omega)$.

[6]Note that by $|\operatorname{Grad} v|^2 = |\operatorname{symGrad} v|^2 + |\operatorname{skw}\operatorname{Grad} v|^2 = |\operatorname{symGrad} v|^2 + \frac{1}{2}|\operatorname{curl} v|^2$ (pointwise) and by $|\operatorname{Grad} v|^2_{L^{2,3\times3}(\Omega)} = |\operatorname{curl} v|^2_{L^{2,3}(\Omega)} + |\operatorname{div} v|^2_{L^2(\Omega)}$ for all $v \in H_0^{1,3}(\Omega)$, we get Korn's inequality $|\operatorname{Grad} v|^2_{L^{2,3\times3}(\Omega)} \leq 2|\operatorname{symGrad} v|^2_{L^{2,3\times3}(\Omega)}$.

Note that $\psi_{\ell,j} \in H_0^{1,3}(\Omega)$ can be found by the standard variational formulation

$$\forall\, \phi \in H_0^{1,3}(\Omega) \quad \langle \operatorname{symGrad} \psi_{\ell,j}, \operatorname{symGrad} \phi \rangle_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)} = \langle \operatorname{symGrad} \xi_{\ell,j}, \operatorname{symGrad} \phi \rangle_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)},$$

i.e., $\psi_{\ell,j} = \mathring{\Delta}_{\mathbb{S}}^{-1} \Delta_{\mathbb{S}} \xi_{\ell,j}$. Therefore, $u_{\ell,j} = \xi_{\ell,j} - \psi_{\ell,j} = (1 - \mathring{\Delta}_{\mathbb{S}}^{-1}\Delta_{\mathbb{S}})\xi_{\ell,j} \in H^{1,3}(\Omega)$ and

$$\begin{aligned} \operatorname{symGrad} v_{\ell,j} &= \operatorname{symGrad}(1 - \mathring{\Delta}_{\mathbb{S}}^{-1}\Delta_{\mathbb{S}})\xi_{\ell,j} \\ &= (\operatorname{symGrad} - \operatorname{symGrad} \mathring{\Delta}_{\mathbb{S}}^{-1}\Delta_{\mathbb{S}})\xi_{\ell,j} \\ &= (1 - \operatorname{symGrad} \mathring{\Delta}_{\mathbb{S}}^{-1} \operatorname{Div}_{\mathbb{S}}) \operatorname{symGrad} \xi_{\ell,j}. \end{aligned}$$

Let us also mention that $v_{\ell,j}$ solves in classical terms the linear elasticity Dirichlet problem

$$\begin{aligned} -\Delta_{\mathbb{S}} v_{\ell,j} := -\operatorname{Div}_{\mathbb{S}} \operatorname{symGrad} v_{\ell,j} &= 0 && \text{in } \Omega, \\ v_{\ell,j} &= \widehat{r}_j && \text{on } \Gamma_\ell, \\ v_{\ell,j} &= 0 && \text{on } \Gamma_k,\ \ell \neq k = 0, \dots, m-1, \end{aligned} \tag{38}$$

which is uniquely solvable. In particular, for all $\ell = 1, \dots, m-1$ and all $j = 1, \dots, 6$ it holds $v_{\ell,j} = 0$ on Γ_0.

Lemma A.14. *Let Assumption 1 be satisfied. Then* $\mathcal{H}^{\mathsf{ela}}_{D,\mathbb{S}}(\Omega) = \operatorname{lin} \mathcal{B}^{\mathsf{ela}}_D$.

Proof. We follow in close lines the arguments used in the proofs of Lemma A.2, Lemma A.6, and Lemma A.10. Let $S \in \mathcal{H}^{\mathsf{ela}}_{D,\mathbb{S}}(\Omega) = \ker(\operatorname{Curl}\mathring{\operatorname{Curl}}_{\mathbb{S}}^{\top}, \Omega) \cap \ker(\operatorname{Div}_{\mathbb{S}}, \Omega)$. In particular, by the homogeneous boundary condition its extension by zero $\widetilde{S}$ to B belongs to $\ker(\operatorname{Curl}\mathring{\operatorname{Curl}}_{\mathbb{S}}^{\top}, B)$. As B is topologically trivial (and smooth and bounded), there exists (a unique vector field) $v \in H_0^{1,3}(B)$ such that $\operatorname{symGrad} v = \widetilde{S}$ in B, see [22, Theorem 3.5]. As $\operatorname{symGrad} v = \widetilde{S} = 0$ in $B \setminus \overline{\Omega}$, v must be a rigid motion, i.e., $v \in \mathsf{RM}$, in each connected component $\widetilde{\Xi}_0, \Xi_1, \dots, \Xi_{m-1}$ of $B \setminus \overline{\Omega}$. Due to the homogenous boundary condition at ∂B, v vanishes in $\widetilde{\Xi}_0$. Therefore, $S = \operatorname{symGrad} v$ in Ω and $v \in H_0^{1,3}(B)$ is such that $v|_{\widetilde{\Xi}_0} = 0$ and $v|_{\Xi_\ell} =: r_\ell =: \sum_{j=1}^{6} \alpha_{\ell,j} \widehat{r}_j \in \mathsf{RM}$, $\alpha_{\ell,j} \in \mathbb{R}$, for all $\ell = 1, \dots, m-1$. Let us consider

$$\begin{aligned} \widehat{S} &:= S - \sum_{\ell=1}^{m-1} \sum_{j=1}^{6} \alpha_{\ell,j} \operatorname{symGrad} v_{\ell,j} = \operatorname{symGrad} \widehat{v} \in \mathcal{H}^{\mathsf{ela}}_{D,\mathbb{S}}(\Omega), \\ \widehat{v} &:= v - \sum_{\ell=1}^{m-1} \sum_{j=1}^{6} \alpha_{\ell,j} v_{\ell,j} \in H^{1,3}(\Omega) \end{aligned}$$

with $v_{\ell,j}$ from (37). The extension by zero of $\psi_{\ell,j}$ to $\widetilde{\psi}_{\ell,j}$ belongs to $H_0^{1,3}(B)$. Hence as an element of $H^{1,3}(B)$ we see that

$$\widehat{v}_B := v - \sum_{\ell=1}^{m-1} \sum_{j=1}^{6} \alpha_{\ell,j} \xi_{\ell,j} + \sum_{\ell=1}^{m-1} \sum_{j=1}^{6} \alpha_{\ell,j} \widetilde{\psi}_{\ell,j} \in H_0^{1,3}(B)$$

vanishes in all Ξ_ℓ. Thus $\widehat{v} = \widehat{v}_B|_\Omega \in H_0^{1,3}(\Omega)$ by Assumption 1, and we compute

$$|\widehat{S}|^2_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)} = \langle \operatorname{symGrad} \widehat{v}, \widehat{S} \rangle_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)} = 0,$$

finishing the proof. □

Note that, in classical terms, v from the latter proof solves the linear elasticity Dirichlet problem

$$-\Delta_{\mathbb{S}} v = -\operatorname{Div}_{\mathbb{S}} \operatorname{symGrad} v = -\operatorname{Div}_{\mathbb{S}} S = 0 \qquad \text{in } \Omega,$$

$$
\begin{aligned}
v &= 0 && \text{on } \Gamma_0, \\
v &= r_\ell \in \mathsf{RM} && \text{on } \Gamma_\ell,\ \ell = 1, \dots, m-1,
\end{aligned}
$$

which is uniquely solvable as long as the rigid motions r_ℓ in RM are prescribed.

Lemma A.15. *Let Assumption 1 be satisfied. Then $\mathcal{B}_D^{\mathsf{ela}}$ is linear independent.*

Proof. Let

$$
\sum_{\ell=1}^{m-1}\sum_{j=1}^{6} \alpha_{\ell,j} \operatorname{symGrad} v_{\ell,j} = 0, \qquad v := \sum_{\ell=1}^{m-1}\sum_{j=1}^{6} \alpha_{\ell,j} v_{\ell,j}.
$$

Then $\operatorname{symGrad} v = 0$ in Ω, i.e., $v \in \mathsf{RM}$ in each connected component of Ω. We show $v = 0$. Recall $v_{\ell,j} = \xi_{\ell,j} - \psi_{\ell,j}$ in Ω. Extension by zero of $\psi_{\ell,j}$ to $\widetilde{\psi}_{\ell,j}$ shows $\widetilde{v}_{\ell,j} \in H_0^{1,3}(B)$, where

$$
\widetilde{v}_{\ell,j} := \begin{cases} v_{\ell,j} & \text{in } \Omega, \\ \xi_{\ell,j} & \text{in } B \setminus \overline{\Omega}, \end{cases} \qquad \operatorname{symGrad} \widetilde{v}_{\ell,j} = \begin{cases} \operatorname{symGrad} v_{\ell,j} & \text{in } \Omega, \\ \operatorname{symGrad} \xi_{\ell,j} = 0 & \text{in } B \setminus \overline{\Omega}. \end{cases}
$$

Note that $\widetilde{v}_{\ell,j} = \xi_{\ell,j} = 0$ in $\widetilde{\Xi}_0$ and in Ξ_k for all $\ell \neq k = 1, \dots, m-1$ and $j = 1, \dots, 6$ and that $\widetilde{v}_{\ell,j} = \xi_{\ell,j} = \widehat{r}_j$ in Ξ_ℓ. Then

$$
\widetilde{v} := \sum_{\ell=1}^{m-1}\sum_{j=1}^{6} \alpha_{\ell,j} \widetilde{v}_{\ell,j} \in H_0^{1,3}(B)
$$

with $\widetilde{v} = 0$ in $\widetilde{\Xi}_0$ and $\operatorname{symGrad} \widetilde{v} = 0$ in $B \setminus \overline{\Omega}$ as well as $\operatorname{symGrad} \widetilde{v} = \operatorname{symGrad} v = 0$ in Ω by assumption. Hence, $\operatorname{symGrad} \widetilde{v} = 0$ in B, showing $\widetilde{v} = 0$ in B. In particular, $v = 0$ in Ω, and $\sum_{j=1}^{6} \alpha_{\ell,j} \widehat{r}_j = \widetilde{v}|_{\Xi_\ell} = 0$ for all $\ell = 1, \dots, m-1$. We conclude $\alpha_{\ell,j} = 0$ for all $j = 1, \dots, 6$ and all ℓ, finishing the proof. □

Theorem A.16. *Let Assumption 1 be satisfied. Then $\dim \mathcal{H}_{D,\mathbb{S}}^{\mathsf{ela}}(\Omega) = 6(m-1)$ and a basis of $\mathcal{H}_{D,\mathbb{S}}^{\mathsf{ela}}(\Omega)$ is given by (37).*

Proof. Use Lemma A.14 and Lemma A.15. □

Appendix B. Neumann Fields

The key topological assumptions to be satisfied by Ω to compute a basis for the Neumann fields and for p to be well defined, is described in detail next. For this, we recall the construction in [23].

Assumption 3 ([23, Section 1])**.** *Let $\Omega \subseteq \mathbb{R}^3$ be open and bounded. There are $p \in \mathbb{N}_0$ piecewise C^1-curves ζ_j and p C^2-surfaces F_j, $j \in \{1, \dots, p\}$, with the following properties:*

- (A1) *The sets* $\operatorname{ran} \zeta_j$, $j \in \{1, \dots, p\}$, *are pairwise disjoint and given any closed piecewise C^1-curve ζ in Ω there exists uniquely determined $\alpha_j \in \mathbb{Z}$, $j \in \{1, \dots, p\}$, such that for all* $\Phi \in \ker(\operatorname{curl})$ *being continuously differentiable we have*

$$
\int_\zeta \langle \Phi, \mathrm{d}\,\lambda \rangle = \sum_{j=1}^{p} \alpha_j \int_{\zeta_j} \langle \Phi, \mathrm{d}\,\lambda \rangle.
$$

- (A2) $\operatorname{ran} F_j$, $j \in \{1, \dots, p\}$, *are pairwise disjoint and* $\operatorname{ran} F_j \cap \operatorname{ran} \zeta_k$ *is a singelton, if $j = k$, and empty, if $j \neq k$.*
- (A3) *If $\Omega_c \subseteq \Omega$ is a connected component, then $\Omega_c \setminus \bigcup_{j=1}^{p} \operatorname{ran} F_j$ is simply connected.*

p is called the topological genus of Ω and the curves ζ_j are said to represent a basis of the respective homology group of Ω. Let us recall from the beginning of this appendix, that Ω consists of the connected components Ω_k, i.e., $\Omega = \dot{\bigcup}_{k=1}^{n} \Omega_k$. In particular, for all $k = 1, \ldots, n$ we have that $\Omega_k \setminus \bigcup_{j=1}^{p} \operatorname{ran} F_j$ is simply connected. Moreover, we set

$$\Omega_F := \Omega \setminus \bigcup_{j=1}^{p} \operatorname{ran} F_j.$$

Let us introduce $\theta_j \in C^\infty(\Omega_F)$, $j = 1, \ldots, p$, with support in a small neighbourhood Υ_j of F_j on one side of F_j, such that $\theta_j = 1$ in a neighbourhood $\Upsilon_{j,1} \subset \Upsilon_j$ of the latter side of F_j and $\theta_j = 0$ in a neighbourhood $\Upsilon_{j,0}$ of the other side of F_j. Moreover, we assume that the supports of θ_j are disjoint and that θ_j together with all derivatives are bounded. In particular, $\theta_j = 0$ in all neighbourhoods $\Upsilon_{l,1} \cup F_l \cup \Upsilon_{l,0}$ of F_l for all $j \neq l = 1, \ldots, p$. Additionally, for all $l = 1, \ldots, p$ we pick curves

$$\zeta_{x_{l,0},x_{l,1}} \subset \zeta_l$$

with fixed starting points $x_{l,0} \in \Upsilon_{l,0}$ and fixed endpoints $x_{l,1} \in \Upsilon_{l,1}$. Note that $\theta_l(x_{l,0}) = 0$ and $\theta_l(x_{l,1}) = 1$ as well as $\theta_j(x_{l,1}) = \theta_j(x_{l,0}) = 0$ for all $l \neq j = 1, \ldots, p$.

B.1. **Neumann Vector Fields of the Classical de Rham Complex.** By definition $\theta_j = 0$ outside of a neighbourhood of F_j and θ_j is constant in the two neighbourhoods $\Upsilon_{j,1}$ and $\Upsilon_{j,0}$ of both sides of F_j. Hence $\operatorname{grad} \theta_j = 0$ in the two neighbourhoods $\Upsilon_{j,1}, \Upsilon_{j,0}$ of F_j and also in all other $\Upsilon_{l,1}, \Upsilon_{l,0}$ of F_l, $j \neq l = 1, \ldots, p$. Thus $\operatorname{grad} \theta_j$ can be continuously extended by zero to $\Theta_j \in C^{\infty,3}(\Omega) \cap L^{2,3}(\Omega)$ with $\Theta_j = 0$ in all the latter neighbourhoods $\widetilde{\Upsilon}_l := \Upsilon_{l,1} \cup F_l \cup \Upsilon_{l,0}$ of all the surfaces F_l.

Lemma B.1. *Let Assumption 3 be satisfied. Then* $\Theta_j \in \ker(\operatorname{curl}, \Omega)$.

Proof. Let $\Phi \in C_c^{\infty,3}(\Omega)$. As $\operatorname{supp} \Theta_j \subset \overline{\Upsilon}_j \setminus \widetilde{\Upsilon}_j$ we can pick another cut-off function $\varphi \in C_c^\infty(\Omega_F)$ with $\varphi|_{\operatorname{supp} \Theta_j \cap \operatorname{supp} \Phi} = 1$. Then

$$\langle \Theta_j, \operatorname{curl} \Phi \rangle_{L^{2,3}(\Omega)} = \langle \Theta_j, \operatorname{curl} \Phi \rangle_{L^{2,3}(\operatorname{supp} \Theta_j \cap \operatorname{supp} \Phi)} = \big\langle \operatorname{grad} \theta_j, \operatorname{curl}(\varphi \Phi) \big\rangle_{L^{2,3}(\Omega_F)} = 0,$$

as $\varphi \Phi \in C_c^{\infty,3}(\Omega_F)$. □

Note again that $\operatorname{supp} \Theta_j \subset \overline{\Upsilon}_j \setminus \widetilde{\Upsilon}_j$ and thus

$$\int_{\zeta_l} \langle \Theta_j, \mathrm{d}\,\lambda \rangle = \int_{\zeta_l \setminus \widetilde{\Upsilon}_j} \langle \operatorname{grad} \theta_j, \mathrm{d}\,\lambda \rangle = \int_{\zeta_{x_{l,0},x_{l,1}}} \langle \operatorname{grad} \theta_j, \mathrm{d}\,\lambda \rangle = \underbrace{\theta_j(x_{l,1})}_{=\delta_{l,j}} - \underbrace{\theta_j(x_{l,0})}_{=0},$$

where we recall the curves $\zeta_{x_{l,0},x_{l,1}} \subset \zeta_l$, with chosen starting points $x_{l,0}$ in $\Upsilon_{l,0}$ and respective endpoints $x_{l,1}$ in $\Upsilon_{l,1}$. Hence we have functionals β_l such that

$$\beta_l(\Theta_j) := \int_{\zeta_l} \langle \Theta_j, \mathrm{d}\,\lambda \rangle = \delta_{l,j}, \qquad l, j = 1, \ldots, p. \tag{39}$$

Let Assumption 1 be satisfied. For the de Rham complex, simliar to (3), (4), and (27), we have the orthogonal decompositions

$$\begin{aligned} L^{2,3}(\Omega) = H_2 &= \operatorname{ran} A_2^* \oplus_{H_2} \ker A_2 = \operatorname{ran}(\operatorname{grad}, \Omega) \oplus_{L^{2,3}(\Omega)} \ker(\mathring{\operatorname{div}}, \Omega), \\ \ker(\operatorname{curl}, \Omega) = \ker(A_1^*) &= \operatorname{ran} A_2^* \oplus_{H_2} K_2 = \operatorname{ran}(\operatorname{grad}, \Omega) \oplus_{L^{2,3}(\Omega)} \mathcal{H}_N^{\mathsf{Rhm}}(\Omega). \end{aligned} \tag{40}$$

Remark B.2. *It holds* $\operatorname{dom}(\operatorname{grad},\Omega)=H^1(\Omega)$. *Moreover, the range in* (40) *is closed by the Poincaré estimate*

$$\exists\, c>0 \quad \forall\, \phi\in H^1(\Omega)\cap\mathbb{R}_{\mathsf{pw}}^{\perp_{L^2(\Omega)}} \qquad |\phi|_{L^2(\Omega)}\le c|\operatorname{grad}\phi|_{L^{2,3}(\Omega)},$$

which is implied by Rellich's selection theorem as Assumption 1 holds.

Let us denote in (40) the orthogonal projector onto $\ker(\mathring{\operatorname{div}},\Omega)$ resp. $\mathcal{H}_N^{\mathsf{Rhm}}(\Omega)$ by π. By Lemma B.1 there exists some $\psi_j\in H^1(\Omega)$ such that

$$\mathcal{H}_N^{\mathsf{Rhm}}(\Omega)\ni\pi\Theta_j=\Theta_j-\operatorname{grad}\psi_j,\qquad (\Theta_j-\operatorname{grad}\psi_j)\big|_{\Omega_F}=\operatorname{grad}(\theta_j-\psi_j).$$

Since fields in $\mathcal{H}_N^{\mathsf{Rhm}}(\Omega)$ are harmonic, we emphasise that we have $\mathcal{H}_N^{\mathsf{Rhm}}(\Omega)\subset C^{\infty,3}(\Omega)$, cf. (25). As $\Theta_j\in C^{\infty,3}(\Omega)$, we see that also $\operatorname{grad}\psi_j\in C^{\infty,3}(\Omega)$ holds, yielding that $\psi_j\in H^1(\Omega)\cap C^\infty(\Omega)$. Therefore, all path integrals are well defined and we observe by (39)

$$\beta_l(\pi\Theta_j)=\int_{\zeta_l}\langle\pi\Theta_j,\mathrm{d}\,\lambda\rangle=\int_{\zeta_l}\langle\Theta_j,\mathrm{d}\,\lambda\rangle-\underbrace{\int_{\zeta_l}\langle\operatorname{grad}\psi_j,\mathrm{d}\,\lambda\rangle}_{=0}=\delta_{l,j},\quad l,j=1,\dots,p. \tag{41}$$

We shall show that

$$\mathcal{B}_N^{\mathsf{Rhm}}:=\{\pi\Theta_1,\dots,\pi\Theta_p\}\subset\mathcal{H}_N^{\mathsf{Rhm}}(\Omega) \tag{42}$$

defines a basis of $\mathcal{H}_N^{\mathsf{Rhm}}(\Omega)$.

Note that $\psi_j\in H^1(\Omega)\cap\mathbb{R}_{\mathsf{pw}}^{\perp_{L^2(\Omega)}}$ can be found by the standard variational formulation

$$\forall\,\phi\in H^1(\Omega)\qquad \langle\operatorname{grad}\psi_j,\operatorname{grad}\phi\rangle_{L^{2,3}(\Omega)}=\langle\Theta_j,\operatorname{grad}\phi\rangle_{L^{2,3}(\Omega)},$$

i.e., $\psi_j=\Delta^{-1}\operatorname{div}\Theta_j$. Therefore,

$$\pi\Theta_j=\Theta_j-\operatorname{grad}\psi_j=(1-\operatorname{grad}\Delta^{-1}\operatorname{div})\Theta_j.$$

Let us also mention that ψ_j solves in classical terms the Neumann Laplace problem

$$\begin{aligned}-\Delta\psi_j&=-\operatorname{div}\Theta_j &&\text{in }\Omega,\\ \nu\cdot\operatorname{grad}\psi_j&=\nu\cdot\Theta_j &&\text{on }\Gamma,\\ \int_{\Omega_k}\psi_j&=0 &&\text{for }k=1,\dots,n,\end{aligned} \tag{43}$$

which is uniquely solvable.

Lemma B.3. *Let Assumption 1 and Assumption 3 be satisfied. Then* $\mathcal{H}_N^{\mathsf{Rhm}}(\Omega)=\operatorname{lin}\mathcal{B}_N^{\mathsf{Rhm}}$.

Proof. Let $H\in\mathcal{H}_N^{\mathsf{Rhm}}(\Omega)=\ker(\mathring{\operatorname{div}},\Omega)\cap\ker(\operatorname{curl},\Omega)\subset C^{\infty,3}(\Omega)$, cf. (25), and define the numbers

$$\gamma_j:=\gamma_j(H):=\beta_j(H)=\int_{\zeta_j}\langle H,\mathrm{d}\,\lambda\rangle\in\mathbb{R},\qquad j=1,\dots,p.$$

We shall show that

$$\mathcal{H}_N^{\mathsf{Rhm}}(\Omega)\ni\widehat{H}:=H-\sum_{j=1}^p\gamma_j\pi\Theta_j=0\quad\text{in }\Omega.$$

The aim is to prove that there exists $u\in H^1(\Omega)$ such that $\operatorname{grad}u=\widehat{H}$, since then

$$|\widehat{H}|^2_{L^{2,3}(\Omega)}=\langle\operatorname{grad}u,\widehat{H}\rangle_{L^{2,3}(\Omega)}=0.$$

Observing by (41)

$$\int_{\zeta_l} \langle \widehat{H}, \mathrm{d}\,\lambda \rangle = \underbrace{\int_{\zeta_l} \langle H, \mathrm{d}\,\lambda \rangle}_{=\gamma_l} - \sum_{j=1}^{p} \gamma_j \underbrace{\int_{\zeta_l} \langle \pi\Theta_j, \mathrm{d}\,\lambda \rangle}_{=\beta_l(\pi\Theta_j)=\delta_{l,j}} = 0,$$

we have by Assumption 3 (A.1) for any closed piecewise C^1-curve ζ in Ω

$$\int_{\zeta} \langle \widehat{H}, \mathrm{d}\,\lambda \rangle = 0. \tag{44}$$

Recall the connected components $\Omega_1, \ldots, \Omega_n$ of Ω. For $1 \le k \le n$ let Ω_k and some $x_0 \in \Omega_k$ be fixed. By (44) the function $u : \Omega \to \mathbb{R}$ given by

$$u(x) := \int_{\zeta(x_0,x)} \langle \widehat{H}, \mathrm{d}\,\lambda \rangle, \qquad x \in \Omega_k,$$

where $\zeta(x_0, x)$ is any piecewise C^1-curve connecting x_0 with x, is well defined, i.e., independent of the respective curve $\zeta(x_0, x)$, and belongs to $C^\infty(\Omega_k)$ with $\operatorname{grad} u = \widehat{H} \in L^{2,3}(\Omega_k)$. Thus[7] $u \in L^2(\Omega_k)$, see, e.g., [14, Theorem 2.6 (1)] or [1, Theorem 3.2 (2)], and hence $u \in H^1(\Omega_k)$, showing $u \in H^1(\Omega)$. □

Remark B.4. *Note that in the latter proof the existence of* $u \in H^1(\Omega_k)$ *with* $\operatorname{grad} u = \widehat{H}$ *in* Ω_k *is well known, if the connected component* Ω_k *of* Ω *is even simply connected. In this case, namely, we know that* $\ker(\operatorname{curl}, \Omega_k) = \operatorname{ran}(\operatorname{grad}, \Omega_k)$.

Lemma B.5. *Let Assumption 1 and Assumption 3 be satisfied. Then* $\mathcal{B}_N^{\mathsf{Rhm}}$ *is linear independent.*

Proof. Let $\sum_{j=1}^{p} \gamma_j \pi\Theta_j = 0$, $\gamma_j \in \mathbb{R}$. (41) implies $0 = \sum_{j=1}^{p} \gamma_j \underbrace{\int_{\zeta_l} \langle \pi\Theta_j, \mathrm{d}\,\lambda \rangle}_{=\beta_l(\pi\Theta_j)=\delta_{l,j}} = \gamma_l$ for all l. □

Theorem B.6. *Let Assumption 1 and Assumption 3 be satisfied. Then* $\dim \mathcal{H}_N^{\mathsf{Rhm}}(\Omega) = p$ *and a basis of* $\mathcal{H}_N^{\mathsf{Rhm}}(\Omega)$ *is given by* (42).

Proof. Use Lemma B.3 and Lemma B.5. □

B.2. **Neumann Tensor Fields of the First Biharmonic Complex.** Recall from the latter section that by definition $\theta_j = 0$ outside of a neighbourhood of F_j and that θ_j is constant in the two neighbourhoods $\Upsilon_{j,1}$ and $\Upsilon_{j,0}$ of both sides of F_j. Moreover, let $\widehat{r}_k$ be the Raviart-Thomas fields from Section A.3 given by $\widehat{r}_0(x) := x$ and $\widehat{r}_k(x) := e^k$ for $k = 1, 2, 3$. We define the vector fields $\theta_{j,k} := \theta_j \widehat{r}_k$ and note $\operatorname{devGrad} \theta_{j,k} = 0$ in the two neighbourhoods $\Upsilon_{j,1}, \Upsilon_{j,0}$ of F_j and also in all other $\Upsilon_{l,1}, \Upsilon_{l,0}$ of F_l, $j \neq l = 1, \ldots, p$. Thus $\operatorname{devGrad} \theta_{j,k}$ can be continuously extended by zero to $\Theta_{j,k} \in C^{\infty,3\times3}(\Omega) \cap L^{2,3\times3}_{\mathbb{T}}(\Omega)$ with $\Theta_{j,k} = 0$ in all the latter neighbourhoods $\widetilde{\Upsilon}_l = \Upsilon_{l,1} \cup F_l \cup \Upsilon_{l,0}$ of all the surfaces F_l.

Lemma B.7. *Let Assumption 3 be satisfied. Then* $\Theta_{j,k} \in \ker(\operatorname{symCurl}_{\mathbb{T}}, \Omega)$.

Proof. Let $\Phi \in C^{\infty,3\times3}_{c,\mathbb{S}}(\Omega)$. As $\operatorname{supp} \Theta_{j,k} \subset \overline{\Upsilon}_j \setminus \widetilde{\Upsilon}_j$ we can pick another cut-off function $\varphi \in C^\infty_c(\Omega_F)$ with $\varphi|_{\operatorname{supp}\Theta_{j,k} \cap \operatorname{supp}\Phi} = 1$. Then

$$\langle \Theta_{j,k}, \operatorname{Curl}_{\mathbb{S}} \Phi \rangle_{L^{2,3\times3}_{\mathbb{T}}(\Omega)} = \langle \Theta_{j,k}, \operatorname{Curl}_{\mathbb{S}} \Phi \rangle_{L^{2,3\times3}_{\mathbb{T}}(\operatorname{supp}\Theta_{j,k} \cap \operatorname{supp}\Phi)}$$

[7] Indeed, it is sufficient to assume $u \in L^2_{\mathsf{loc}}(\Omega_k)$, see, e.g., [15, Satz 6.6.26, Beweis; Folgerung 6.3.2] or [30, Theorem 7.4].

$$= \left\langle \operatorname{devGrad}\theta_{j,k}, \operatorname{Curl}_{\mathbb{S}}(\varphi\Phi)\right\rangle_{L^{2,3\times3}_{\mathbb{T}}(\Omega_F)} = \underbrace{\left\langle \operatorname{Grad}\theta_{j,k}, \operatorname{dev}\operatorname{Curl}_{\mathbb{S}}(\varphi\Phi)\right\rangle_{L^{2,3\times3}_{\mathbb{T}}(\Omega_F)}}_{=\left\langle \operatorname{Grad}\theta_{j,k}, \operatorname{Curl}(\varphi\Phi)\right\rangle_{L^{2,3\times3}(\Omega_F)}} = 0$$

as $\varphi\Phi \in C^{\infty,3\times3}_c(\Omega_F)$. □

Before proceeding we need the following two lemmas:

Lemma B.8. *Let $u \in C^\infty_c(\mathbb{R}^3,\mathbb{R})$, $v, w \in C^\infty_c(\mathbb{R}^3,\mathbb{R}^3)$, and $S \in C^\infty_c(\mathbb{R}^3,\mathbb{R}^{3\times3})$. Then:*

- $(\operatorname{spn} v)\, w = v \times w = -(\operatorname{spn} w)\, v$ *and*[8] $(\operatorname{spn} v)(\operatorname{spn}^{-1} S) = -Sv$, *if* $\operatorname{sym} S = 0$
- $\operatorname{sym}\operatorname{spn} v = 0$ *and* $\operatorname{dev}(u\operatorname{Id}) = 0$
- $\operatorname{tr}\operatorname{Grad} v = \operatorname{div} v$ *and* $2\operatorname{skw}\operatorname{Grad} v = \operatorname{spn}\operatorname{curl} v$
- $\operatorname{Div}(u\operatorname{Id}) = \operatorname{grad} u$ *and* $\operatorname{Curl}(u\operatorname{Id}) = -\operatorname{spn}\operatorname{grad} u$,
  in particular, $\operatorname{curl}\operatorname{Div}(u\operatorname{Id}) = 0$ *and* $\operatorname{curl}\operatorname{spn}^{-1}\operatorname{Curl}(u\operatorname{Id}) = 0$
  and $\operatorname{sym}\operatorname{Curl}(u\operatorname{Id}) = 0$
- $\operatorname{Div}\operatorname{spn} v = -\operatorname{curl} v$ *and* $\operatorname{Div}\operatorname{skw} S = -\operatorname{curl}\operatorname{spn}^{-1}\operatorname{skw} S$,
  in particular, $\operatorname{div}\operatorname{Div}\operatorname{skw} S = 0$
- $\operatorname{Curl}\operatorname{spn} v = (\operatorname{div} v)\operatorname{Id} - (\operatorname{Grad} v)^\top$
  and $\operatorname{Curl}\operatorname{skw} S = (\operatorname{div}\operatorname{spn}^{-1}\operatorname{skw} S)\operatorname{Id} - (\operatorname{Grad}\operatorname{spn}^{-1}\operatorname{skw} S)^\top$
- $\operatorname{dev}\operatorname{Curl}\operatorname{spn} v = -(\operatorname{dev}\operatorname{Grad} v)^\top$
- $-2\operatorname{Curl}\operatorname{sym}\operatorname{Grad} v = 2\operatorname{Curl}\operatorname{skw}\operatorname{Grad} v = -(\operatorname{Grad}\operatorname{curl} v)^\top$
- $2\operatorname{spn}^{-1}\operatorname{skw}\operatorname{Curl} S = \operatorname{Div} S^\top - \operatorname{grad}\operatorname{tr} S = \operatorname{Div}\big(S - (\operatorname{tr} S)\operatorname{Id}\big)^\top$,
  in particular, $\operatorname{curl}\operatorname{Div} S^\top = 2\operatorname{curl}\operatorname{spn}^{-1}\operatorname{skw}\operatorname{Curl} S$
  and $2\operatorname{skw}\operatorname{Curl} S = \operatorname{spn}\operatorname{Div} S^\top$, *if* $\operatorname{tr} S = 0$
- $\operatorname{tr}\operatorname{Curl} S = 2\operatorname{div}\operatorname{spn}^{-1}\operatorname{skw} S$, *in particular,* $\operatorname{tr}\operatorname{Curl} S = 0$, *if* $\operatorname{skw} S = 0$,
  and $\operatorname{tr}\operatorname{Curl}\operatorname{sym} S = 0$ *and* $\operatorname{tr}\operatorname{Curl}\operatorname{skw} S = \operatorname{tr}\operatorname{Curl} S$
- $2(\operatorname{Grad}\operatorname{spn}^{-1}\operatorname{skw} S)^\top = (\operatorname{tr}\operatorname{Curl}\operatorname{skw} S)\operatorname{Id} - 2\operatorname{Curl}\operatorname{skw} S$
- $3\operatorname{Div}(\operatorname{dev}\operatorname{Grad} v)^\top = 2\operatorname{grad}\operatorname{div} v$
- $2\operatorname{Curl}\operatorname{sym}\operatorname{Grad} v = -2\operatorname{Curl}\operatorname{skw}\operatorname{Grad} v = -\operatorname{Curl}\operatorname{spn}\operatorname{curl} v = (\operatorname{Grad}\operatorname{curl} v)^\top$
- $2\operatorname{Div}\operatorname{sym}\operatorname{Curl} S = -2\operatorname{Div}\operatorname{skw}\operatorname{Curl} S = \operatorname{curl}\operatorname{Div} S^\top$
- $\operatorname{Curl}(\operatorname{Curl}\operatorname{sym} S)^\top = \operatorname{sym}\operatorname{Curl}(\operatorname{Curl} S)^\top$
- $\operatorname{Curl}(\operatorname{Curl}\operatorname{skw} S)^\top = \operatorname{skw}\operatorname{Curl}(\operatorname{Curl} S)^\top$

All formulas extend to distributions as well.

Proof. Almost all formulas can be found in [21, Lemma 3.9] and [21, Lemma A.1]. To show the last two formulas we note by [22, Appendix B] that $\operatorname{skw} T = 0$ implies $\operatorname{skw}\operatorname{Curl}(\operatorname{Curl} T)^\top = 0$, and that $\operatorname{sym} T = 0$ implies $\operatorname{sym}\operatorname{Curl}(\operatorname{Curl} T)^\top = 0$. Hence sym commutes with $\operatorname{Curl}\operatorname{Curl}^\top$ as

$$\operatorname{Curl}(\operatorname{Curl}\operatorname{sym} T)^\top = \operatorname{sym}\operatorname{Curl}(\operatorname{Curl}\operatorname{sym} T)^\top = \operatorname{sym}\operatorname{Curl}(\operatorname{Curl} T)^\top,$$

and so does skw. □

Lemma B.9. *Let $x, x_0 \in \Omega$ and let $\zeta_{x_0,x} \subset \Omega$ be a piecewise C^1-curve connecting x_0 to x.*

(i) *Let $v \in C^\infty(\Omega,\mathbb{R}^3)$. Then v and its divergence $\operatorname{div} v$ can be represented by*

$$v(x) - v(x_0) - \frac{1}{3}\operatorname{div} v(x_0)(x - x_0)$$

[8]Here, we introduce the skew-symmetric matrix $\operatorname{spn} v := \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}$ and the corresponding isometric mapping $\operatorname{spn} : \mathbb{R}^3 \to \mathbb{R}^{3\times3}_{\mathrm{skw}}$.

$$= \int_{\zeta_{x_0,x}} \operatorname{devGrad} v \,\mathrm{d}\,\lambda + \frac{1}{2}\int_{\zeta_{x_0,x}} \Big(\int_{\zeta_{x_0,y}} \big\langle \operatorname{Div}(\operatorname{devGrad} v)^\top, \mathrm{d}\,\lambda\big\rangle\Big) \operatorname{Id} \mathrm{d}\,\lambda_y,$$

$$\operatorname{div} v(x) - \operatorname{div} v(x_0) = \frac{3}{2}\int_{\zeta_{x_0,x}} \big\langle \operatorname{Div}(\operatorname{devGrad} v)^\top, \mathrm{d}\,\lambda\big\rangle.$$

(ii) *For all $T \in C^\infty(\Omega, \mathbb{R}^{3\times 3})$ it holds*

$$\int_{\zeta_{x_0,x}} \Big(\int_{\zeta_{x_0,y}} \langle \operatorname{Div} T^\top, \mathrm{d}\,\lambda\rangle\Big) \operatorname{Id} \mathrm{d}\,\lambda_y = \int_{\zeta_{x_0,x}} (x-y)\big\langle (\operatorname{Div} T^\top)(y), \mathrm{d}\,\lambda_y\big\rangle.$$

(iii) *Let $T \in C^\infty(\Omega, \mathbb{R}^{3\times 3})$ and define*

$$u(x) := \int_{\zeta_{x_0,x}} \langle \operatorname{Div} T^\top, \mathrm{d}\,\lambda\rangle, \qquad S := T + \frac{1}{2} u \operatorname{Id}, \qquad v(x) := \int_{\zeta_{x_0,x}} S \,\mathrm{d}\,\lambda.$$

Then $u \in C^\infty(\Omega, \mathbb{R})$, $S \in C^\infty(\Omega, \mathbb{R}^{3\times 3})$, and $v \in C^\infty(\Omega, \mathbb{R}^3)$ are well defined, i.e., independent of the respective curve, with

$$\operatorname{grad} u = \operatorname{Div} T^\top, \qquad \operatorname{Grad} v = S, \qquad \operatorname{devGrad} v = T,$$

if and only if $\operatorname{tr} T = 0$ *and* $\operatorname{symCurl}_{\mathbb{T}} T = 0$ *as well as*

$$\int_\zeta \langle \operatorname{Div} T^\top, \mathrm{d}\,\lambda\rangle = 0, \qquad \int_\zeta S \,\mathrm{d}\,\lambda = 0$$

hold for any closed piecewise C^1-curve $\zeta \subset \Omega$. In this case,

$$\operatorname{grad} u = \operatorname{Div} T^\top = \frac{2}{3} \operatorname{grad} \operatorname{div} v.$$

Remark B.10. *In Lemma B.9 (iii) for $T \in C^\infty_{\mathbb{T}}(\Omega, \mathbb{R}^{3\times 3})$ and $S := T + \frac{1}{2} u \operatorname{Id}$ with* $\operatorname{grad} u = \operatorname{Div} T^\top$ *the formulas*

$$\operatorname{curl} \operatorname{Div} T^\top = 2 \operatorname{Div} \operatorname{symCurl}_{\mathbb{T}} T, \qquad \operatorname{Curl} S = \operatorname{symCurl}_{\mathbb{T}} T$$

are crucial. These will be derived in the upcoming proof and follow by Lemma B.8.

In Lemma B.9 for a tensor field T the operation $T \,\mathrm{d}\,\lambda := \big(\langle \operatorname{row}_\ell T, \mathrm{d}\,\lambda\rangle\big)_{\ell=1,2,3}$ has to be understood row-wise, i.e., the transpose of the ℓth row of T is denoted by $\operatorname{row}_\ell T$, giving then the vector object $T \,\mathrm{d}\,\lambda$. More precisely,

$$\Big(\int_{\zeta_{x_0,x}} T \,\mathrm{d}\,\lambda\Big)_\ell = \int_{\zeta_{x_0,x}} \langle \operatorname{row}_\ell T, \mathrm{d}\,\lambda\rangle = \int_0^1 \Big\langle (\operatorname{row}_\ell T)\big(\varphi(t)\big), \varphi'(t)\Big\rangle \,\mathrm{d}\,t$$

with some parametrisation $\varphi \in C^1_{\mathsf{pw}}\big([0,1], \mathbb{R}^3\big)$ of $\zeta_{x_0,x}$. Furthermore, we define

$$\int_{\zeta_{x_0,x}} (x-y)\big\langle (\operatorname{Div} T^\top)(y), \mathrm{d}\,\lambda_y\big\rangle := \int_0^1 \big(x - \varphi(t)\big)\Big\langle (\operatorname{Div} T^\top)\big(\varphi(t)\big), \varphi'(t)\Big\rangle \,\mathrm{d}\,t.$$

Proof of Lemma B.9. For (i), let

$$T := \operatorname{devGrad} v = \operatorname{Grad} v - \frac{1}{3}(\operatorname{tr} \operatorname{Grad} v) \operatorname{Id} = \operatorname{Grad} v - \frac{1}{3}(\operatorname{div} v) \operatorname{Id}$$

and observe $3 \operatorname{Div} T^\top = 2 \operatorname{grad} \operatorname{div} v$ by Lemma B.8. Thus

$$v_k(x) - v_k(x_0) = \int_{\zeta_{x_0,x}} \langle \operatorname{grad} v_k, \mathrm{d}\,\lambda\rangle, \qquad k = 1,2,3,$$

$$\operatorname{div} v(x) - \operatorname{div} v(x_0) = \int_{\zeta_{x_0,x}} \langle \operatorname{grad} \operatorname{div} v, \mathrm{d}\,\lambda\rangle = \frac{3}{2}\int_{\zeta_{x_0,x}} \langle \operatorname{Div} T^\top, \mathrm{d}\,\lambda\rangle.$$

Therefore,

$$v(x)-v(x_0)=\int_{\zeta_{x_0,x}}\operatorname{Grad} v\,\mathrm{d}\,\lambda=\int_{\zeta_{x_0,x}}\operatorname{devGrad} v\,\mathrm{d}\,\lambda+\frac{1}{3}\underbrace{\int_{\zeta_{x_0,x}}\operatorname{div} v\operatorname{Id}\mathrm{d}\,\lambda}_{=\int_{\zeta_{x_0,x}}\operatorname{div} v(y)\operatorname{Id}\mathrm{d}\,\lambda_y}$$

$$=\int_{\zeta_{x_0,x}}T\,\mathrm{d}\,\lambda+\frac{1}{3}\operatorname{div} v(x_0)\int_{\zeta_{x_0,x}}\operatorname{Id}\mathrm{d}\,\lambda_y$$
$$+\frac{1}{2}\int_{\zeta_{x_0,x}}\Big(\int_{\zeta_{x_0,y}}\langle\operatorname{Div} T^\top,\mathrm{d}\,\lambda\rangle\Big)\operatorname{Id}\mathrm{d}\,\lambda_y.$$

Moreover,[9] $\int_{\zeta_{x_0,x}}\operatorname{Id}\mathrm{d}\,\lambda_y=\int_{\zeta_{x_0,x}}\operatorname{Grad} y\,\mathrm{d}\,\lambda_y=x-x_0$.

For (ii) we compute with φ from above

$$\int_{\zeta_{x_0,x}}\Big(\int_{\zeta_{x_0,y}}\langle\operatorname{Div} T^\top,\mathrm{d}\,\lambda\rangle\Big)\operatorname{Id}\mathrm{d}\,\lambda_y=\int_0^1\Big(\underbrace{\int_{\zeta_{x_0,\varphi(s)}}\langle\operatorname{Div} T^\top,\mathrm{d}\,\lambda\rangle}_{=\int_0^s\big\langle(\operatorname{Div} T^\top)\big(\varphi(t)\big),\varphi'(t)\big\rangle\mathrm{d}\,t}\Big)\underbrace{\operatorname{Id}\varphi'(s)}_{=\varphi'(s)}\mathrm{d}\,s$$

$$=\int_0^1\underbrace{\int_t^1\varphi'(s)\,\mathrm{d}\,s}_{=x-\varphi(t)}\Big\langle(\operatorname{Div} T^\top)\big(\varphi(t)\big),\varphi'(t)\Big\rangle\mathrm{d}\,t$$

$$=\int_{\zeta_{x_0,x}}(x-y)\big\langle(\operatorname{Div} T^\top)(y),\mathrm{d}\,\lambda_y\big\rangle.$$

For (iii), let $T\in C^\infty(\Omega,\mathbb{R}^{3\times3})$ and let u, S, and v be defined as stated. Moreover, let $\operatorname{tr} T=0$ and $\operatorname{symCurl}_{\mathbb{T}} T=0$ with

$$\int_\zeta\langle\operatorname{Div} T^\top,\mathrm{d}\,\lambda\rangle=0,\qquad\int_\zeta S\,\mathrm{d}\,\lambda=0$$

for any closed piecewise C^1-curve $\zeta\subset\Omega$. Note that

$$\begin{array}{llcc} & u\text{ well defined (indep. of }\zeta_{x_0,x}) & \wedge & \operatorname{grad} u=\operatorname{Div} T^\top\\ \Leftrightarrow & \forall\,\zeta\,(\text{cl pw }C^1)\ \int_\zeta\langle\operatorname{Div} T^\top,\mathrm{d}\,\lambda\rangle=0 & \wedge & \operatorname{curl}\operatorname{Div} T^\top=0,\end{array}$$

and

$$\begin{array}{llcc} & v\text{ well defined (indep. of }\zeta_{x_0,x}) & \wedge & \operatorname{Grad} v=S\\ \Leftrightarrow & \forall\,\zeta\,(\text{cl pw }C^1)\ \int_\zeta S\,\mathrm{d}\,\lambda=0 & \wedge & \operatorname{Curl} S=0.\end{array}$$

By Lemma B.8 we have

$\operatorname{curl}\operatorname{Div} T^\top=2\operatorname{Div}\operatorname{sym}\operatorname{Curl} T=0,$

[9]Alternatively, note with φ from above $\int_{\zeta_{x_0,x}}\operatorname{Id}\mathrm{d}\,\lambda_y=\int_0^1\operatorname{Id}\varphi'(s)\,\mathrm{d}\,s=\int_0^1\varphi'(s)\,\mathrm{d}\,s=x-x_0$.

i.e., u is well defined and $\operatorname{grad} u = \operatorname{Div} T^\top$, and

$$\operatorname{Curl} S = \operatorname{Curl} T + \frac{1}{2}\underbrace{\operatorname{Curl}(u\operatorname{Id})}_{=-\operatorname{spn}\operatorname{grad} u} = \operatorname{Curl} T - \underbrace{\frac{1}{2}\operatorname{spn}\operatorname{Div} T^\top}_{=\operatorname{skw}\operatorname{Curl} T} = \operatorname{symCurl} T = 0,$$

as $\operatorname{tr} T = 0$ and $\operatorname{symCurl}_{\mathbb{T}} T = 0$. Hence u, S, and v are well defined. Moreover, $\operatorname{Grad} v = S$ and $\operatorname{devGrad} v = \operatorname{dev} S = \operatorname{dev} T = T$ (since $\operatorname{dev}(u\operatorname{Id}) = 0$ and $\operatorname{tr} T = 0$) as well as $\operatorname{grad} u = \operatorname{Div} T^\top = \frac{2}{3}\operatorname{grad}\operatorname{div} v$ by Lemma B.8. Furthermore, $u \in C^\infty(\Omega,\mathbb{R})$, $S \in C^\infty(\Omega,\mathbb{R}^{3\times3})$, and $v \in C^\infty(\Omega,\mathbb{R}^3)$. On the other hand, let $T \in C^\infty(\Omega,\mathbb{R}^{3\times3})$, $u \in C^\infty(\Omega,\mathbb{R})$, $S \in C^\infty(\Omega,\mathbb{R}^{3\times3})$, and $v \in C^\infty(\Omega,\mathbb{R}^3)$ be given with

$$\operatorname{grad} u = \operatorname{Div} T^\top, \qquad \operatorname{Grad} v = S, \qquad \operatorname{devGrad} v = T.$$

Then $\operatorname{tr} T = 0$, $\operatorname{symCurl}_{\mathbb{T}} T = 0$, and $\operatorname{grad} u = \operatorname{Div} T^\top = \frac{2}{3}\operatorname{grad}\operatorname{div} v$ by Lemma B.8, as well as

$$\int_\zeta \langle \operatorname{Div} T^\top, \mathrm{d}\,\lambda\rangle = \int_\zeta \langle \operatorname{grad} u, \mathrm{d}\,\lambda\rangle = 0, \qquad \int_\zeta S\,\mathrm{d}\,\lambda = \int_\zeta \operatorname{Grad} v\,\mathrm{d}\,\lambda = 0,$$

completing the proof. □

Note that for $l, j = 1,\dots,p$ and $k = 0,\dots,3$ and for the curves $\zeta_{x_{l,0},x_{l,1}} \subset \zeta_l$ with the chosen starting points $x_{l,0} \in \Upsilon_{l,0}$ and respective endpoints $x_{l,1} \in \Upsilon_{l,1}$ we can compute by Lemma B.9

$$\begin{aligned}
\mathbb{R} \ni \beta_{l,0}(\Theta_{j,k}) &:= \frac{1}{2}\int_{\zeta_l}\langle \operatorname{Div}\Theta_{j,k}^\top, \mathrm{d}\,\lambda\rangle = \frac{1}{2}\int_{\zeta_{x_{l,0},x_{l,1}}}\big\langle \operatorname{Div}(\operatorname{devGrad}\theta_{j,k})^\top, \mathrm{d}\,\lambda\big\rangle \\
&= \frac{1}{3}\operatorname{div}\theta_{j,k}(x_{l,1}) - \frac{1}{3}\underbrace{\operatorname{div}\theta_{j,k}(x_{l,0})}_{=0} \\
&= \frac{1}{3}\delta_{l,j}\operatorname{div}\widehat{r}_k(x_{l,1}) = \delta_{l,j}\begin{cases} 1, & \text{if } k = 0, \\ 0, & \text{if } k = 1,2,3, \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{R}^3 \ni b_l(\Theta_{j,k}) &:= \int_{\zeta_l}\Theta_{j,k}\,\mathrm{d}\,\lambda + \frac{1}{2}\int_{\zeta_l}(x_{l,1}-y)\big\langle(\operatorname{Div}\Theta_{j,k}^\top)(y), \mathrm{d}\,\lambda_y\big\rangle \\
&= \int_{\zeta_{x_{l,0},x_{l,1}}}\operatorname{devGrad}\theta_{j,k}\,\mathrm{d}\,\lambda \\
&\qquad + \frac{1}{2}\int_{\zeta_{x_{l,0},x_{l,1}}}(x_{l,1}-y)\Big\langle\big(\operatorname{Div}(\operatorname{devGrad}\theta_{j,k})^\top\big)(y), \mathrm{d}\,\lambda_y\Big\rangle \\
&= \int_{\zeta_{x_{l,0},x_{l,1}}}\bigg(\operatorname{devGrad}\theta_{j,k}(y) \\
&\qquad + \frac{1}{2}\Big(\int_{\zeta_{x_{l,0},y}}\big\langle \operatorname{Div}(\operatorname{devGrad}\theta_{j,k})^\top, \mathrm{d}\,\lambda\big\rangle\Big)\operatorname{Id}\bigg)\,\mathrm{d}\,\lambda_y \\
&= \theta_{j,k}(x_{l,1})\underbrace{-\theta_{j,k}(x_{l,0}) - \frac{1}{3}\operatorname{div}\theta_{j,k}(x_{l,0})(x_{l,1}-x_{l,0})}_{=0} \\
&= \delta_{l,j}\widehat{r}_k(x_{l,1}) = \delta_{l,j}\begin{cases} x_{l,1}, & \text{if } k = 0, \\ e^k, & \text{if } k = 1,2,3. \end{cases}
\end{aligned}$$

Thus, we have functionals $\beta_{l,\ell}$ for $l = 1, \dots, p$ and $\ell = 0, \dots, 3$ given by

$$\beta_{l,0}(\Theta_{j,k}) = \delta_{l,j}\delta_{0,k}$$

for $l, j = 1, \dots, p$ and $k = 0, \dots, 3$, as well as

$$\beta_{l,\ell}(\Theta_{j,k}) := \big\langle b_l(\Theta_{j,k}), e^\ell \big\rangle = \delta_{l,j} \begin{cases} \langle x_{l,1}, e^\ell \rangle = (x_{l,1})_\ell, & \text{if } k = 0, \\ \langle e^k, e^\ell \rangle = \delta_{\ell,k}, & \text{if } k = 1, 2, 3, \end{cases}$$

for $l, j = 1, \dots, p$ and $\ell = 1, 2, 3$ and $k = 0, \dots, 3$. Therefore, we have

$$\beta_{l,\ell}(\Theta_{j,k}) = \delta_{l,j}\delta_{\ell,k} + (1 - \delta_{\ell,0})\delta_{0,k}\delta_{l,j}(x_{l,1})_\ell, \qquad l, j = 1, \dots, p, \quad k, \ell = 0, 1, 2, 3. \tag{45}$$

Let Assumption 2 be satisfied. For the first biharmonic complex, simliar to (3), (4), and (27), (40), we have the orthogonal decompositions

$$\begin{aligned} L^{2,3\times 3}_{\mathbb{T}}(\Omega) &= \operatorname{ran}(\operatorname{devGrad}, \Omega) \oplus_{L^{2,3\times 3}_{\mathbb{T}}(\Omega)} \ker(\mathring{\operatorname{Div}}_{\mathbb{T}}, \Omega), \\ \ker(\operatorname{symCurl}_{\mathbb{T}}, \Omega) &= \operatorname{ran}(\operatorname{devGrad}, \Omega) \oplus_{L^{2,3\times 3}_{\mathbb{T}}(\Omega)} \mathcal{H}^{\mathsf{bih},1}_{N,\mathbb{T}}(\Omega). \end{aligned} \tag{46}$$

Remark B.11. *It holds* $\operatorname{dom}(\operatorname{devGrad}, \Omega) = H^{1,3}(\Omega)$ *by* [21, Lemma 3.2]*. Moreover, the range in* (46) *is closed by the Poincaré type estimate*

$$\exists\, c > 0 \quad \forall\, \phi \in H^{1,3}(\Omega) \cap \mathsf{RT}_{\mathsf{pw}}^{\perp_{L^{2,3}(\Omega)}} \qquad |\phi|_{L^{2,3}(\Omega)} \leq c |\operatorname{devGrad} \phi|_{L^{2,3\times 3}(\Omega)},$$

which is implied by Rellich's selection theorem and [21, Lemma 3.2] *as Assumption 2 holds.*

Let us denote in (46) the orthogonal projector onto $\ker(\mathring{\operatorname{Div}}_{\mathbb{T}}, \Omega)$ resp. $\mathcal{H}^{\mathsf{bih},1}_{N,\mathbb{T}}(\Omega)$ by π. By Lemma B.7 there exists some $\psi_{j,k} \in H^{1,3}(\Omega)$ such that

$$\mathcal{H}^{\mathsf{bih},1}_{N,\mathbb{T}}(\Omega) \ni \pi\Theta_{j,k} = \Theta_{j,k} - \operatorname{devGrad}\psi_{j,k}, \quad (\Theta_{j,k} - \operatorname{devGrad}\psi_{j,k})\big|_{\Omega_F} = \operatorname{devGrad}(\theta_{j,k} - \psi_{j,k}).$$

As $\mathcal{H}^{\mathsf{bih},1}_{N,\mathbb{T}}(\Omega) \subset C^{\infty,3\times 3}(\Omega)$, cf. (25), we conclude by $\pi\Theta_{j,k}, \Theta_{j,k} \in C^{\infty,3\times 3}(\Omega)$ that also $\operatorname{devGrad}\psi_{j,k} \in C^{\infty,3\times 3}(\Omega)$ and hence $\psi_{j,k} \in C^{\infty,3}(\Omega)$. Thus all path integrals over the closed curves ζ_l are well defined. Furthermore, we observe by Lemma B.9

$$\begin{aligned} \beta_{l,0}(\operatorname{devGrad}\psi_{j,k}) &= \frac{1}{2}\int_{\zeta_l} \big\langle \operatorname{Div}(\operatorname{devGrad}\psi_{j,k})^\top, \mathrm{d}\,\lambda \big\rangle \\ &= \frac{1}{3}\operatorname{div}\psi_{j,k}(x_{l,1}) - \frac{1}{3}\operatorname{div}\psi_{j,k}(x_{l,1}) = 0 \end{aligned}$$

and

$$\begin{aligned} &b_l(\operatorname{devGrad}\psi_{j,k}) \\ &= \int_{\zeta_l} \operatorname{devGrad}\psi_{j,k}\,\mathrm{d}\,\lambda + \frac{1}{2}\int_{\zeta_l} (x_{l,1} - y)\Big\langle \big(\operatorname{Div}(\operatorname{devGrad}\psi_{j,k})^\top\big)(y), \mathrm{d}\,\lambda_y \Big\rangle \\ &= \int_{\zeta_{x_{l,1},x_{l,1}}} \left(\operatorname{devGrad}\psi_{j,k}(y) + \frac{1}{2}\Big(\int_{\zeta_{x_{l,1},y}} \big\langle \operatorname{Div}(\operatorname{devGrad}\psi_{j,k})^\top, \mathrm{d}\,\lambda \big\rangle \Big) \operatorname{Id} \right) \mathrm{d}\,\lambda_y \\ &= \psi_{j,k}(x_{l,1}) - \psi_{j,k}(x_{l,1}) - \frac{1}{3}\operatorname{div}\psi_{j,k}(x_{l,1})(x_{l,1} - x_{l,1}) = 0. \end{aligned}$$

Therefore, by (45)

$$\beta_{l,\ell}(\pi\Theta_{j,k}) = \beta_{l,\ell}(\Theta_{j,k}) - \underbrace{\beta_{l,\ell}(\operatorname{devGrad}\psi_{j,k})}_{=0} = \delta_{l,j}\delta_{\ell,k} + (1 - \delta_{\ell,0})\delta_{0,k}\delta_{l,j}(x_{l,1})_\ell \tag{47}$$

for all $l, j = 1, \dots, p$ and all $\ell, k = 0, 1, 2, 3$. We shall show that

$$\mathcal{B}_N^{\mathsf{bih},1} := \{\pi\Theta_{j,k}\}_{\substack{j=1,\dots,p,\\ k=0,1,2,3}} \subset \mathcal{H}_{N,\mathbb{T}}^{\mathsf{bih},1}(\Omega) \tag{48}$$

defines a basis of $\mathcal{H}_{N,\mathbb{T}}^{\mathsf{bih},1}(\Omega)$.

Note that $\psi_{j,k} \in H^{1,3}(\Omega) \cap \mathsf{RT}_{\mathsf{pw}}^{\perp_{L^{2,3}(\Omega)}}$ can be found by the variational formulation

$$\forall\, \phi \in H^{1,3}(\Omega) \qquad \langle \operatorname{devGrad} \psi_{j,k}, \operatorname{devGrad} \phi \rangle_{L^{2,3\times 3}(\Omega)} = \langle \Theta_{j,k}, \operatorname{devGrad} \phi \rangle_{L^{2,3\times 3}(\Omega)},$$

i.e., $\psi_{j,k} = \Delta_{\mathbb{T}}^{-1} \operatorname{Div}_{\mathbb{T}} \Theta_{j,k}$. Therefore,

$$\pi\Theta_{j,k} = \Theta_{j,k} - \operatorname{devGrad} \psi_{j,k} = (1 - \operatorname{devGrad} \Delta_{\mathbb{T}}^{-1} \operatorname{Div}_{\mathbb{T}})\Theta_{j,k}.$$

Let us also mention that $\psi_{j,k}$ solves in classical terms the Neumann elasticity type problem

$$\begin{aligned} -\Delta_{\mathbb{T}} \psi_{j,k} &= -\operatorname{Div}_{\mathbb{T}} \Theta_{j,k} && \text{in } \Omega, \\ (\operatorname{Grad} \psi_{j,k})\nu &= \Theta_{j,k}\nu && \text{on } \Gamma, \\ \int_{\Omega_l} (\psi_{j,k})_\ell &= 0 && \text{for } l = 1, \dots, n, \quad \ell = 1, 2, 3, \\ \int_{\Omega_l} x \cdot \psi_{j,k}(x) \,\mathrm{d}\,\lambda_x &= 0 && \text{for } l = 1, \dots, n, \end{aligned} \tag{49}$$

which is uniquely solvable.

Lemma B.12. *Let Assumption 2 as well as Assumption 3 be satisfied. Then it holds* $\mathcal{H}_{N,\mathbb{T}}^{\mathsf{bih},1}(\Omega) = \operatorname{lin} \mathcal{B}_N^{\mathsf{bih},1}$.

Proof. Let $H \in \mathcal{H}_{N,\mathbb{T}}^{\mathsf{bih},1}(\Omega) = \ker(\mathring{\operatorname{Div}}_{\mathbb{T}}, \Omega) \cap \ker(\operatorname{symCurl}_{\mathbb{T}}, \Omega) \subset C_{\mathbb{T}}^{\infty,3\times 3}(\Omega)$, cf. (25). With the above introduced functionals $\beta_{l,0}$ and b_l we recall

$$\begin{aligned} \mathbb{R} \ni \beta_{l,0}(H) &= \frac{1}{2} \int_{\zeta_l} \langle \operatorname{Div} H^\top, \mathrm{d}\,\lambda \rangle, \\ \mathbb{R}^3 \ni b_l(H) &= \int_{\zeta_l} H \,\mathrm{d}\,\lambda + \frac{1}{2} \int_{\zeta_l} (x_{l,1} - y) \big\langle (\operatorname{Div} H^\top)(y), \mathrm{d}\,\lambda_y \big\rangle, \end{aligned}$$

and define for $l = 1, \dots, p$ the numbers

$$\begin{aligned} \gamma_{l,0} &:= \gamma_{l,0}(H) := \beta_{l,0}(H), \\ \gamma_{l,\ell} &:= \gamma_{l,\ell}(H) := \big\langle b_l(H) - \beta_{l,0}(H) x_{l,1}, e^\ell \big\rangle = \beta_{l,\ell}(H) - \beta_{l,0}(H)(x_{l,1})_\ell, \quad \ell = 1, 2, 3. \end{aligned}$$

We shall show that

$$\mathcal{H}_{N,\mathbb{T}}^{\mathsf{bih},1}(\Omega) \ni \widehat{H} := H - \sum_{j=1}^{p} \sum_{k=0}^{3} \gamma_{j,k} \pi\Theta_{j,k} = 0 \quad \text{in } \Omega.$$

Similar to the proof of Lemma B.3, the aim is to prove that there exists $v \in H^{1,3}(\Omega)$ such that $\operatorname{devGrad} v = \widehat{H}$, since then

$$|\widehat{H}|^2_{L^{2,3\times 3}_{\mathbb{T}}(\Omega)} = \langle \operatorname{devGrad} v, \widehat{H} \rangle_{L^{2,3\times 3}_{\mathbb{T}}(\Omega)} = 0.$$

By (47) we observe

$$\frac{1}{2} \int_{\zeta_l} \langle \operatorname{Div} \widehat{H}^\top, \mathrm{d}\,\lambda \rangle = \beta_{l,0}(\widehat{H}) = \underbrace{\beta_{l,0}(H)}_{=\gamma_{l,0}} - \sum_{j=1}^{p} \sum_{k=0}^{3} \gamma_{j,k} \underbrace{\beta_{l,0}(\pi\Theta_{j,k})}_{=\delta_{l,j}\delta_{0,k}} = 0,$$

and thus by Assumption 3 (A.1) for any closed piecewise C^1-curve ζ in Ω

$$\int_\zeta \langle \operatorname{Div} \widehat{H}^\top, \mathrm{d}\,\lambda\rangle = 0. \tag{50}$$

Recall the connected components $\Omega_1, \dots, \Omega_n$ of Ω. For $1 \le k \le n$ let some $x_0 \in \Omega_k$ be fixed. By (50) and $\operatorname{curl} \operatorname{Div} \widehat{H}^\top = 2 \operatorname{Div} \operatorname{symCurl}_{\mathbb{T}} \widehat{H} = 0$, see Lemma B.8, cf. Lemma B.9 and Remark B.10, the function $u : \Omega \to \mathbb{R}$ and the tensor field $S : \Omega \to \mathbb{R}^{3\times 3}$ given by

$$u(x) := \int_{\zeta(x_0,x)} \langle \operatorname{Div} \widehat{H}^\top, \mathrm{d}\,\lambda\rangle, \quad S := \widehat{H} + \frac{1}{2} u \operatorname{Id}, \qquad x \in \Omega_k,$$

where $\zeta(x_0, x)$ is any piecewise C^1-curve connecting x_0 with x, are well defined, i.e., independent of the respective curve $\zeta(x_0, x)$, and belong to $C^\infty(\Omega_k)$ and $C^{\infty,3\times 3}(\Omega_k)$, respectively. Moreover, $\operatorname{grad} u = \operatorname{Div} \widehat{H}^\top$ and $\operatorname{Curl} S = \operatorname{symCurl}_{\mathbb{T}} \widehat{H} = 0$ by Remark B.10. Note that for $\zeta_{x_{l,0},x_{l,1}} \subset \zeta_l \subset \Omega_k$ we have with $c := u(x_{l,1}) \in \mathbb{R}$

$$u(x) = \underbrace{u(x) - u(x_{l,1})}_{= \int_{\zeta_{x_{l,1},x}} \langle \operatorname{grad} u, \mathrm{d}\,\lambda\rangle} + c = \int_{\zeta_{x_{l,1},x}} \langle \operatorname{Div} \widehat{H}^\top, \mathrm{d}\,\lambda\rangle + c, \qquad x \in \zeta_l,$$

and

$$\int_{\zeta_l} (c \operatorname{Id}) \,\mathrm{d}\,\lambda = c \int_{\zeta_l} \operatorname{Grad} x \,\mathrm{d}\,\lambda_x = 0.$$

Moreover, the closed curve ζ_l may be considered as the closed curve $\zeta_{x_{l,1},x_{l,1}}$ with circulation 1 along ζ_l. By Lemma B.9 and the definition of b_l we have

$$\begin{aligned}
\int_{\zeta_l} S \,\mathrm{d}\,\lambda &= \int_{\zeta_l} \widehat{H} \,\mathrm{d}\,\lambda + \frac{1}{2} \int_{\zeta_l} (u \operatorname{Id}) \,\mathrm{d}\,\lambda \\
&= \int_{\zeta_l} \widehat{H} \,\mathrm{d}\,\lambda + \frac{1}{2} \int_{\zeta_{x_{l,1},x_{l,1}}} \Big(\int_{\zeta_{x_{l,1},y}} \langle \operatorname{Div} \widehat{H}^\top, \mathrm{d}\,\lambda\rangle \Big) \operatorname{Id} \mathrm{d}\,\lambda_y \\
&= \int_{\zeta_l} \widehat{H} \,\mathrm{d}\,\lambda + \frac{1}{2} \int_{\zeta_l} (x_{l,1} - y) \big\langle (\operatorname{Div} \widehat{H}^\top)(y), \mathrm{d}\,\lambda \big\rangle \,\mathrm{d}\,\lambda_y = b_l(\widehat{H}).
\end{aligned}$$

Hence, for $\ell = 1, 2, 3$ we get by (47)

$$\begin{aligned}
&\Big(\int_{\zeta_l} S \,\mathrm{d}\,\lambda \Big)_\ell = \Big\langle \int_{\zeta_l} S \,\mathrm{d}\,\lambda, e^\ell \Big\rangle = \langle b_l(\widehat{H}), e^\ell\rangle = \beta_{l,\ell}(\widehat{H}) \\
&= \beta_{l,\ell}(H) - \sum_{j=1}^p \sum_{k=0}^3 \gamma_{j,k} \underbrace{\beta_{l,\ell}(\pi \Theta_{j,k})}_{=\delta_{l,j}\delta_{\ell,k} + (1-\delta_{\ell,0})\delta_{0,k}\delta_{l,j}(x_{l,1})_\ell} = \beta_{l,\ell}(H) - \underbrace{\gamma_{l,0}}_{=\beta_{l,0}(H)} (x_{l,1})_\ell - \gamma_{l,\ell} = 0.
\end{aligned}$$

Therefore, $\int_{\zeta_l} S \,\mathrm{d}\,\lambda = 0$ and thus by Assumption 3 (A.1) for any closed piecewise C^1-curve ζ in Ω

$$\int_\zeta S \,\mathrm{d}\,\lambda = 0. \tag{51}$$

By (51), cf. Lemma B.9, the vector field $v : \Omega \to \mathbb{R}^3$ given by

$$v(x) := \int_{\zeta_{x_0,x}} S \,\mathrm{d}\,\lambda, \qquad x \in \Omega_k,$$

where $\zeta(x_0,x)$ is any piecewise C^1-curve connecting x_0 with x, is well defined, i.e., independent of the respective curve $\zeta(x_0,x)$. Moreover, v belongs to $C^{\infty,3}(\Omega_k)$ and satisfies $\operatorname{Grad} v = S \in C^{\infty,3\times 3}(\Omega_k)$ as well as

$$\operatorname{devGrad} v = \operatorname{dev} S = \operatorname{dev} \widehat{H} = \widehat{H} \in C^{\infty,3\times 3}(\Omega_k) \cap L^{2,3\times 3}_{\mathbb{T}}(\Omega_k).$$

Similar to the end of the proof of Lemma B.3, elliptic regularity and, e.g., [14, Theorem 2.6 (1)] or [1, Theorem 3.2 (2)] show that $v \in C^{\infty,3}(\Omega_k)$ with $\operatorname{devGrad} v \in L^{2,3\times 3}_{\mathbb{T}}(\Omega_k)$ implies $v \in H^{1,3}(\Omega_k)$ and thus $v \in H^{1,3}(\Omega)$, completing the proof. Let us note that $v \in H^{1,3}(\Omega)$ implies also $S \in L^{2,3\times 3}(\Omega)$ and hence $u \in L^2(\Omega)$. □

Lemma B.13. *Let Assumption 2 and Assumption 3 be satisfied. Then* $\mathcal{B}^{\mathsf{bih},1}_N$ *is linear independent.*

Proof. Let $\sum_{j=1}^{p}\sum_{k=0}^{3}\gamma_{j,k}\pi\Theta_{j,k} = 0$, $\gamma_{j,k} \in \mathbb{R}$. (47) implies for $l = 1,\dots,p$

$$0 = \sum_{j=1}^{p}\sum_{k=0}^{3}\gamma_{j,k}\beta_{l,\ell}(\pi\Theta_{j,k}) = \gamma_{l,0}, \qquad \ell = 0,$$

$$0 = \sum_{j=1}^{p}\sum_{k=0}^{3}\gamma_{j,k}\beta_{l,\ell}(\pi\Theta_{j,k}) = \gamma_{l,\ell} + \gamma_{l,0}(x_{l,1})_\ell = \gamma_{l,\ell}, \qquad \ell = 1,2,3,$$

finishing the proof. □

Theorem B.14. *Let Assumption 2 as well as Assumption 3 be satisfied. Then it holds* $\dim \mathcal{H}^{\mathsf{bih},1}_{N,\mathbb{T}}(\Omega) = 4p$ *and a basis of* $\mathcal{H}^{\mathsf{bih},1}_{N,\mathbb{T}}(\Omega)$ *is given by* (48).

Proof. Use Lemma B.12 and Lemma B.13. □

B.3. **Neumann Tensor Fields of the Second Biharmonic Complex.** Again, recall from the latter section that by definition $\theta_j = 0$ outside a neighbourhood of F_j and θ_j is constant in the two neighbourhoods $\Upsilon_{j,1}$ and $\Upsilon_{j,0}$ of both sides of F_j. Moreover, let $\widehat{p}_k$ be the polynomials from Section A.2 given by $\widehat{p}_0(x) := 1$ and $\widehat{p}_k(x) := x_k$ for $k = 1,2,3$. We define the functions $\theta_{j,k} := \theta_j\widehat{p}_k$ and note $\operatorname{Gradgrad}\theta_{j,k} = 0$ in the two neighbourhoods $\Upsilon_{j,1}, \Upsilon_{j,0}$ of F_j and also in all other $\Upsilon_{l,1}, \Upsilon_{l,0}$ of F_l, $j \neq l = 1,\dots,p$. Thus $\operatorname{Gradgrad}\theta_{j,k}$ can be continuously extended by zero to $\Theta_{j,k} \in C^{\infty,3\times 3}(\Omega) \cap L^{2,3\times 3}_{\mathbb{S}}(\Omega)$ with $\Theta_{j,k} = 0$ in all the latter neighbourhoods $\widetilde{\Upsilon}_l = \Upsilon_{l,1} \cup F_l \cup \Upsilon_{l,0}$ of all the surfaces F_l.

Lemma B.15. *Let Assumption 3 be satisfied. Then* $\Theta_{j,k} \in \ker(\operatorname{Curl}_{\mathbb{S}},\Omega)$.

Proof. Let $\Phi \in C^{\infty,3\times 3}_{c,\mathbb{T}}(\Omega)$. As $\operatorname{supp}\Theta_{j,k} \subset \overline{\Upsilon}_j \setminus \widetilde{\Upsilon}_j$ we can pick another cut-off function $\varphi \in C^\infty_c(\Omega_F)$ with $\varphi|_{\operatorname{supp}\Theta_{j,k}\cap\operatorname{supp}\Phi} = 1$. Then

$$\langle\Theta_{j,k}, \operatorname{symCurl}_{\mathbb{T}}\Phi\rangle_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)} = \langle\Theta_{j,k}, \operatorname{symCurl}_{\mathbb{T}}\Phi\rangle_{L^{2,3\times 3}_{\mathbb{S}}(\operatorname{supp}\Theta_{j,k}\cap\operatorname{supp}\Phi)}$$
$$= \big\langle \operatorname{Gradgrad}\theta_{j,k}, \operatorname{symCurl}_{\mathbb{T}}(\varphi\Phi)\big\rangle_{L^{2,3\times 3}_{\mathbb{S}}(\Omega_F)} = \big\langle \operatorname{Grad}(\operatorname{grad}\theta_{j,k}), \operatorname{Curl}(\varphi\Phi)\big\rangle_{L^{2,3\times 3}(\Omega_F)} = 0$$

as $\varphi\Phi, \operatorname{Curl}(\varphi\Phi) \in C^{\infty,3\times 3}_c(\Omega_F)$. □

Before proceeding, we recall Lemma B.8 and we need the following lemma:

Lemma B.16. *Let $x, x_0 \in \Omega$ and let $\zeta_{x_0,x} \subset \Omega$ be a piecewise C^1-curve connecting x_0 to x.*

(i) *Let $u \in C^\infty(\Omega, \mathbb{R})$. Then u and its gradient* $\operatorname{grad} u$ *can be represented by*

$$u(x) - u(x_0) - \big\langle \operatorname{grad} u(x_0), x - x_0 \big\rangle = \int_{\zeta_{x_0,x}} \Big\langle \int_{\zeta_{x_0,y}} \operatorname{Gradgrad} u \,\mathrm{d}\,\lambda, \mathrm{d}\,\lambda_y \Big\rangle,$$

$$\operatorname{grad} u(x) - \operatorname{grad} u(x_0) = \int_{\zeta_{x_0,x}} \operatorname{Gradgrad} u \,\mathrm{d}\,\lambda.$$

(ii) *For all $S \in C^\infty(\Omega, \mathbb{R}^{3\times 3})$ it holds*

$$\int_{\zeta_{x_0,x}} \Big\langle \int_{\zeta_{x_0,y}} S \,\mathrm{d}\,\lambda, \mathrm{d}\,\lambda_y \Big\rangle = \int_{\zeta_{x_0,x}} \big\langle x - y, S(y) \,\mathrm{d}\,\lambda_y \big\rangle.$$

(iii) *Let $S \in C^\infty(\Omega, \mathbb{R}^{3\times 3})$ and define*

$$v(x) := \int_{\zeta_{x_0,x}} S \,\mathrm{d}\,\lambda, \qquad u(x) := \int_{\zeta_{x_0,x}} \langle v, \mathrm{d}\,\lambda \rangle.$$

Then $u \in C^\infty(\Omega, \mathbb{R})$ and $v \in C^\infty(\Omega, \mathbb{R}^3)$ are well defined, i.e., independent of the respective curve, with

$$\operatorname{grad} u = v, \qquad \operatorname{Gradgrad} u = \operatorname{Grad} v = S,$$

if and only if $\operatorname{skw} S = 0$ *and* $\operatorname{Curl}_{\mathbb{S}} S = 0$ *as well as*

$$\int_\zeta S \,\mathrm{d}\,\lambda = 0, \qquad \int_\zeta \langle v, \mathrm{d}\,\lambda \rangle = 0$$

hold for any closed piecewise C^1-curve $\zeta \subset \Omega$.

Remark B.17. *In Lemma B.16 (iii) for $S \in C^\infty_{\mathbb{S}}(\Omega, \mathbb{R}^{3\times 3})$ with* $\operatorname{Grad} v = S$ *the formula*

$$\operatorname{curl} v = 2 \operatorname{spn}^{-1} \operatorname{skw} S = 0$$

is crucial.

In Lemma B.16 for a tensor field S and a parametrisation $\varphi \in C^1_{\mathsf{pw}}\big([0,1], \mathbb{R}^3\big)$ of $\zeta_{x_0,x}$ we define

$$\int_{\zeta_{x_0,x}} \big\langle x - y, S(y) \,\mathrm{d}\,\lambda_y \big\rangle := \int_0^1 \Big\langle x - \varphi(t), S\big(\varphi(t)\big)\varphi'(t) \Big\rangle \,\mathrm{d}\,t.$$

Proof of Lemma B.16. For (i), we have

$$u(x) - u(x_0) = \int_{\zeta_{x_0,x}} \langle \operatorname{grad} u, \mathrm{d}\,\lambda \rangle,$$

$$\partial_k u(x) - \partial_k u(x_0) = \int_{\zeta_{x_0,x}} \langle \operatorname{grad} \partial_k u, \mathrm{d}\,\lambda \rangle, \qquad k = 1, 2, 3,$$

i.e.,

$$\operatorname{grad} u(x) - \operatorname{grad} u(x_0) = \int_{\zeta_{x_0,x}} \operatorname{Grad} \operatorname{grad} u \,\mathrm{d}\,\lambda.$$

Therefore,

$$\begin{aligned} u(x) - u(x_0) &= \int_{\zeta_{x_0,x}} \big\langle \operatorname{grad} u(y), \mathrm{d}\,\lambda_y \big\rangle \\ &= \int_{\zeta_{x_0,x}} \Big\langle \int_{\zeta_{x_0,y}} \operatorname{Grad} \operatorname{grad} u \,\mathrm{d}\,\lambda, \mathrm{d}\,\lambda_y \Big\rangle \end{aligned}$$

$$+ \underbrace{\int_{\zeta_{x_0,x}} \big\langle \operatorname{grad} u(x_0), \mathrm{d}\,\lambda_y \big\rangle}_{= \int_0^1 \langle \operatorname{grad} u(x_0), \varphi'(t) \rangle \, \mathrm{d}\, t = \langle \operatorname{grad} u(x_0), x - x_0 \rangle} .$$

For (ii) we compute

$$\begin{aligned}
\int_{\zeta_{x_0,x}} \Big\langle \int_{\zeta_{x_0,y}} S \,\mathrm{d}\,\lambda, \mathrm{d}\,\lambda_y \Big\rangle &= \int_0^1 \Big\langle \underbrace{\int_{\zeta_{x_0,\varphi(s)}} S \,\mathrm{d}\,\lambda}_{= \int_0^s S(\varphi(t))\varphi'(t)\,\mathrm{d}\,t}, \varphi'(s) \Big\rangle \,\mathrm{d}\, s \\
&= \int_0^1 \Big\langle S\big(\varphi(t)\big)\varphi'(t), \underbrace{\int_t^1 \varphi'(s)\,\mathrm{d}\,s}_{=x-\varphi(t)} \Big\rangle \,\mathrm{d}\, t \\
&= \int_{\zeta_{x_0,x}} \big\langle x - y, S(y)\,\mathrm{d}\,\lambda_y \big\rangle.
\end{aligned}$$

For (iii), let $S \in C^\infty(\Omega, \mathbb{R}^{3\times 3})$ and let v and u be defined as stated. Moreover, let $\operatorname{skw} S = 0$ and $\operatorname{Curl}_{\mathbb{S}} S = 0$ with

$$\int_\zeta S \,\mathrm{d}\,\lambda = 0, \qquad \int_\zeta \langle v, \mathrm{d}\,\lambda\rangle = 0$$

for any closed piecewise C^1-curve $\zeta \subset \Omega$. Note that

$$\begin{aligned}
& v \text{ well defined (indep. of } \zeta_{x_0,x}) && \wedge && \operatorname{Grad} v = S \\
\Leftrightarrow \quad & \forall\, \zeta \,(\text{cl pw } C^1) \ \int_\zeta S \,\mathrm{d}\,\lambda = 0 && \wedge && \operatorname{Curl} S = 0,
\end{aligned}$$

and

$$\begin{aligned}
& u \text{ well defined (indep. of } \zeta_{x_0,x}) && \wedge && \operatorname{grad} u = v \\
\Leftrightarrow \quad & \forall\, \zeta \,(\text{cl pw } C^1) \ \int_\zeta \langle v, \mathrm{d}\,\lambda\rangle = 0 && \wedge && \operatorname{curl} v = 0.
\end{aligned}$$

Hence v is well defined with $\operatorname{Grad} v = S$. By Lemma B.8 we have

$$\operatorname{curl} v = 2 \operatorname{spn}^{-1} \operatorname{skw} \operatorname{Grad} v = 2 \operatorname{spn}^{-1} \operatorname{skw} S = 0,$$

showing that u is well defined as well with $\operatorname{grad} u = v$ and thus $\operatorname{Gradgrad} u = \operatorname{Grad} v = S$. Furthermore, $u \in C^\infty(\Omega, \mathbb{R})$ and $v \in C^\infty(\Omega, \mathbb{R}^3)$. On the other hand, let $u \in C^\infty(\Omega, \mathbb{R})$ and $v \in C^\infty(\Omega, \mathbb{R}^3)$ be given with

$$\operatorname{grad} u = v, \qquad \operatorname{Gradgrad} u = \operatorname{Grad} v = S.$$

Then $\operatorname{skw} S = 0$, $\operatorname{Curl}_{\mathbb{S}} S = 0$, and

$$\int_\zeta \langle v, \mathrm{d}\,\lambda\rangle = \int_\zeta \langle \operatorname{grad} u, \mathrm{d}\,\lambda\rangle = 0, \qquad \int_\zeta S \,\mathrm{d}\,\lambda = \int_\zeta \operatorname{Grad} v \,\mathrm{d}\,\lambda = 0,$$

completing the proof. □

Note that for $l, j = 1, \dots, p$ and $k = 0, \dots, 3$ and for the curves $\zeta_{x_{l,0},x_{l,1}} \subset \zeta_l$ with the chosen starting points $x_{l,0} \in \Upsilon_{l,0}$ and respective endpoints $x_{l,1} \in \Upsilon_{l,1}$ we can compute by Lemma B.16

$$\begin{aligned}
\mathbb{R}^3 \ni b_l(\Theta_{j,k}) &:= \int_{\zeta_l} \Theta_{j,k} \,\mathrm{d}\,\lambda = \int_{\zeta_{x_{l,0},x_{l,1}}} \operatorname{Gradgrad} \theta_{j,k} \,\mathrm{d}\,\lambda \\
&= \operatorname{grad} \theta_{j,k}(x_{l,1}) - \underbrace{\operatorname{grad} \theta_{j,k}(x_{l,0})}_{=0} \\
&= \delta_{l,j} \operatorname{grad} \widehat{p}_k(x_{l,1}) = \delta_{l,j} \begin{cases} 0, & \text{if } k = 0, \\ e^k, & \text{if } k = 1, 2, 3, \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{R} \ni \beta_{l,0}(\Theta_{j,k}) &:= \int_{\zeta_l} \big\langle x_{l,1} - y, \Theta_{j,k}(y) \,\mathrm{d}\,\lambda_y \big\rangle \\
&= \int_{\zeta_{x_{l,0},x_{l,1}}} \big\langle x_{l,1} - y, \operatorname{Gradgrad} \theta_{j,k}(y) \,\mathrm{d}\,\lambda_y \big\rangle \\
&= \int_{\zeta_{x_{l,0},x_{l,1}}} \Big\langle \int_{\zeta_{x_{l,0},y}} \operatorname{Gradgrad} \theta_{j,k} \,\mathrm{d}\,\lambda, \mathrm{d}\,\lambda_y \Big\rangle \\
&= \theta_{j,k}(x_{l,1}) \underbrace{-\theta_{j,k}(x_{l,0}) - \big\langle \operatorname{grad} \theta_{j,k}(x_{l,0}), x_{l,1} - x_{l,0} \big\rangle}_{=0} \\
&= \delta_{l,j} \widehat{p}_k(x_{l,1}) = \delta_{l,j} \begin{cases} 1, & \text{if } k = 0, \\ (x_{l,1})_k, & \text{if } k = 1, 2, 3. \end{cases}
\end{aligned}$$

Thus, we have functionals $\beta_{l,\ell}$ for $l = 1, \dots, p$ and $\ell = 0, \dots, 3$ given by

$$\beta_{l,\ell}(\Theta_{j,k}) := \big\langle b_l(\Theta_{j,k}), e^\ell \big\rangle = \delta_{l,j} \begin{cases} 0, & \text{if } k = 0, \\ \delta_{\ell,k}, & \text{if } k = 1, 2, 3, \end{cases}$$

for $l, j = 1, \dots, p$ and $\ell = 1, 2, 3$ and $k = 0, \dots, 3$, as well as

$$\beta_{l,0}(\Theta_{j,k}) = \delta_{l,j}\delta_{0,k} + \delta_{l,j}(1 - \delta_{0,k})(x_{l,1})_k$$

for $l, j = 1, \dots, p$ and $k = 0, \dots, 3$. Therefore, we have

$$\beta_{l,\ell}(\Theta_{j,k}) = \delta_{l,j}\delta_{\ell,k} + (1 - \delta_{0,k})\delta_{\ell,0}\delta_{l,j}(x_{l,1})_k, \qquad l, j = 1, \dots, p, \quad k, \ell = 0, 1, 2, 3. \tag{52}$$

Let Assumption 2 be satisfied. For the second biharmonic complex, simliar to (3), (4), (27), (40), and (46), we have the orthogonal decompositions

$$\begin{aligned}
L^{2,3\times3}_{\mathbb{S}}(\Omega) &= \operatorname{ran}(\operatorname{Gradgrad}, \Omega) \oplus_{L^{2,3\times3}_{\mathbb{S}}(\Omega)} \ker(\operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}}, \Omega), \\
\ker(\operatorname{Curl}_{\mathbb{S}}, \Omega) &= \operatorname{ran}(\operatorname{Gradgrad}, \Omega) \oplus_{L^{2,3\times3}_{\mathbb{S}}(\Omega)} \mathcal{H}^{\mathsf{bih},2}_{N,\mathbb{S}}(\Omega).
\end{aligned} \tag{53}$$

Remark B.18. *It holds* $\operatorname{dom}(\operatorname{Gradgrad}, \Omega) = H^2(\Omega)$ *by Lemma 5.2. Moreover, the range in* (53) *is closed by the Poincaré type estimate*

$$\exists\, c > 0 \quad \forall\, \phi \in H^2(\Omega) \cap (\mathsf{P}^1_{\mathsf{pw}})^{\perp_{L^2(\Omega)}} \qquad |\phi|_{L^2(\Omega)} \leq c |\operatorname{Grad}\operatorname{grad} \phi|_{L^{2,3\times3}(\Omega)},$$

which is implied by Rellich's selection theorem and Lemma 5.2 as Assumption 2 holds.

Let us denote in (53) the orthogonal projector onto $\ker(\operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}},\Omega)$ resp. $\mathcal{H}^{\mathsf{bih},2}_{N,\mathbb{S}}(\Omega)$ by π. By Lemma B.15 there exists some $\psi_{j,k}\in H^2(\Omega)$ such that

$$\begin{aligned}\mathcal{H}^{\mathsf{bih},2}_{N,\mathbb{S}}(\Omega)\ni\pi\Theta_{j,k}&=\Theta_{j,k}-\operatorname{Gradgrad}\psi_{j,k},\\ (\Theta_{j,k}-\operatorname{Gradgrad}\psi_{j,k})\big|_{\Omega_F}&=\operatorname{Gradgrad}(\theta_{j,k}-\psi_{j,k}).\end{aligned}$$

As $\mathcal{H}^{\mathsf{bih},2}_{N,\mathbb{S}}(\Omega)\subset C^{\infty,3\times3}(\Omega)$, cf. (25), we conclude by $\pi\Theta_{j,k},\Theta_{j,k}\in C^{\infty,3\times3}(\Omega)$ that also $\operatorname{Gradgrad}\psi_{j,k}\in C^{\infty,3\times3}(\Omega)$ and hence $\psi_{j,k}\in C^{\infty}(\Omega)$. Hence all path integrals over the closed curves ζ_l are well defined. Furthermore, we observe by Lemma B.16

$$b_l(\operatorname{Gradgrad}\psi_{j,k})=\int_{\zeta_l}\operatorname{Gradgrad}\psi_{j,k}\,\mathrm{d}\,\lambda=\operatorname{grad}\psi_{j,k}(x_{l,1})-\operatorname{grad}\psi_{j,k}(x_{l,1})=0$$

and

$$\begin{aligned}\beta_{l,0}(\operatorname{Gradgrad}\psi_{j,k})&=\int_{\zeta_l}\big\langle x_{l,1}-y,\operatorname{Gradgrad}\psi_{j,k}(y)\,\mathrm{d}\,\lambda_y\big\rangle\\ &=\int_{\zeta_{x_{l,1},x_{l,1}}}\Big\langle\int_{\zeta_{x_{l,1},y}}\operatorname{Gradgrad}\psi_{j,k}\,\mathrm{d}\,\lambda,\mathrm{d}\,\lambda_y\Big\rangle\\ &=\psi_{j,k}(x_{l,1})-\psi_{j,k}(x_{l,1})-\big\langle\operatorname{grad}\psi_{j,k}(x_{l,1}),x_{l,1}-x_{l,1}\big\rangle=0.\end{aligned}$$

Therefore, by (52)

$$\beta_{l,\ell}(\pi\Theta_{j,k})=\beta_{l,\ell}(\Theta_{j,k})-\underbrace{\beta_{l,\ell}(\operatorname{Gradgrad}\psi_{j,k})}_{=0}=\delta_{l,j}\delta_{\ell,k}+(1-\delta_{0,k})\delta_{\ell,0}\delta_{l,j}(x_{l,1})_k\tag{54}$$

for all $l,j=1,\dots,p$ and all $\ell,k=0,1,2,3$. We shall show that

$$\mathcal{B}^{\mathsf{bih},2}_{N}:=\{\pi\Theta_{j,k}\}_{\substack{j=1,\dots,p,\\ k=0,1,2,3}}\subset\mathcal{H}^{\mathsf{bih},2}_{N,\mathbb{S}}(\Omega)\tag{55}$$

defines a basis of $\mathcal{H}^{\mathsf{bih},2}_{N,\mathbb{S}}(\Omega)$.

Note that $\psi_{j,k}\in H^2(\Omega)\cap(\mathsf{P}^1_{\mathsf{pw}})^{\perp_{L^2(\Omega)}}$ can be found by the variational formulation

$$\forall\,\phi\in H^2(\Omega)\qquad\langle\operatorname{Gradgrad}\psi_{j,k},\operatorname{Gradgrad}\phi\rangle_{L^{2,3\times3}(\Omega)}=\langle\Theta_{j,k},\operatorname{Gradgrad}\phi\rangle_{L^{2,3\times3}(\Omega)},$$

i.e., $\psi_{j,k}=(\Delta^2)^{-1}\operatorname{divDiv}_{\mathbb{S}}\Theta_{j,k}$. Therefore,

$$\pi\Theta_{j,k}=\Theta_{j,k}-\operatorname{Gradgrad}\psi_{j,k}=\big(1-\operatorname{Gradgrad}(\Delta^2)^{-1}\operatorname{divDiv}_{\mathbb{S}}\big)\Theta_{j,k}.$$

Let us also mention that $\psi_{j,k}$ solves in classical terms the biharmonic Neumann problem

$$\begin{aligned}\Delta^2\psi_{j,k}&=\operatorname{divDiv}_{\mathbb{S}}\Theta_{j,k}&&\text{in }\Omega,\\ (\operatorname{Gradgrad}\psi_{j,k})\nu&=\Theta_{j,k}\nu&&\text{on }\Gamma,\\ \nu\cdot\operatorname{Div}\operatorname{Gradgrad}\psi_{j,k}&=\nu\cdot\operatorname{Div}\Theta_{j,k}&&\text{on }\Gamma,\\ \int_{\Omega_l}\psi_{j,k}&=0&&\text{for }l=1,\dots,n,\\ \int_{\Omega_l}x_\ell\psi_{j,k}(x)\,\mathrm{d}\,\lambda_x&=0&&\text{for }l=1,\dots,n,\quad\ell=1,2,3,\end{aligned}\tag{56}$$

which is uniquely solvable.

Lemma B.19. *Let Assumption 2 as well as Assumption 3 be satisfied. Then it holds* $\mathcal{H}^{\mathsf{bih},2}_{N,\mathbb{S}}(\Omega)=\operatorname{lin}\mathcal{B}^{\mathsf{bih},2}_{N}$.

Proof. Let $H \in \mathcal{H}^{\mathsf{bih},2}_{N,\mathbb{S}}(\Omega) = \ker(\operatorname{div}\mathring{\operatorname{Div}}_{\mathbb{S}}, \Omega) \cap \ker(\operatorname{Curl}_{\mathbb{S}}, \Omega) \subset C^{\infty,3\times 3}_{\mathbb{S}}(\Omega)$, cf. (25). With the above introduced functionals $\beta_{l,0}$ and b_l we recall

$$\mathbb{R}^3 \ni b_l(H) = \int_{\zeta_l} H \,\mathrm{d}\,\lambda,$$
$$\mathbb{R} \ni \beta_{l,0}(H) = \int_{\zeta_l} \big\langle x_{l,1} - y, H(y)\,\mathrm{d}\,\lambda_y \big\rangle,$$

and define for $l = 1, \dots, p$ the numbers

$$\gamma_{l,\ell} := \gamma_{l,\ell}(H) := \big\langle b_l(H), e^\ell \big\rangle = \beta_{l,\ell}(H), \qquad \ell = 1, 2, 3,$$
$$\gamma_{l,0} := \gamma_{l,0}(H) := \beta_{l,0}(H) - \sum_{k=1}^{3} \beta_{l,k}(H)(x_{l,1})_k.$$

We shall show that

$$\mathcal{H}^{\mathsf{bih},2}_{N,\mathbb{S}}(\Omega) \ni \widehat{H} := H - \sum_{j=1}^{p}\sum_{k=0}^{3} \gamma_{j,k}\pi\Theta_{j,k} = 0 \quad \text{in } \Omega.$$

Similar to the proof of Lemma B.3 and Lemma B.12, the aim is to prove that there exists $u \in H^2(\Omega)$ such that $\operatorname{Gradgrad} u = \widehat{H}$, since then

$$|\widehat{H}|^2_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)} = \langle \operatorname{Gradgrad} u, \widehat{H} \rangle_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)} = 0.$$

By (54) we observe for $\ell = 1, 2, 3$

$$\Big(\int_{\zeta_l} \widehat{H}\,\mathrm{d}\,\lambda\Big)_\ell = \Big\langle \underbrace{\int_{\zeta_l} \widehat{H}\,\mathrm{d}\,\lambda}_{=b_l(\widehat{H})}, e^\ell \Big\rangle = \beta_{l,\ell}(\widehat{H}) = \underbrace{\beta_{l,\ell}(H)}_{=\gamma_{l,\ell}} - \sum_{j=1}^{p}\sum_{k=0}^{3} \gamma_{j,k} \underbrace{\beta_{l,\ell}(\pi\Theta_{j,k})}_{=\delta_{l,j}\delta_{\ell,k}} = 0,$$

and thus by Assumption 3 (A.1) for any closed piecewise C^1-curve ζ in Ω

$$\int_{\zeta} \widehat{H}\,\mathrm{d}\,\lambda = 0. \tag{57}$$

Recall the connected components $\Omega_1, \dots, \Omega_n$ of Ω. For $1 \le k \le n$ let some $x_0 \in \Omega_k$ be fixed. By (57) the vector field $v : \Omega \to \mathbb{R}^3$ given by

$$v(x) := \int_{\zeta(x_0,x)} \widehat{H}\,\mathrm{d}\,\lambda, \qquad x \in \Omega_k,$$

where $\zeta(x_0, x)$ is any piecewise C^1-curve connecting x_0 with x, is well defined, i.e., independent of the respective curve $\zeta(x_0, x)$, and belongs to $C^{\infty,3}(\Omega_k)$. Moreover, $\operatorname{Grad} v = \widehat{H}$ and $\operatorname{curl} v = 2\operatorname{spn}^{-1}\operatorname{skw}\widehat{H} = 0$ by Remark B.17. Note that for $\zeta_{x_{l,0},x_{l,1}} \subset \zeta_l \subset \Omega_k$ we have with $c := v(x_{l,1}) \in \mathbb{R}^3$

$$v(x) = \underbrace{v(x) - v(x_{l,1})}_{=\int_{\zeta_{x_{l,1},x}} \operatorname{Grad} v\,\mathrm{d}\,\lambda} + c = \int_{\zeta_{x_{l,1},x}} \widehat{H}\,\mathrm{d}\,\lambda + c, \qquad x \in \zeta_l,$$

and

$$\int_{\zeta_l} \langle c, \mathrm{d}\,\lambda \rangle = \sum_{\ell=1}^{3} c_\ell \int_{\zeta_l} \langle \operatorname{grad} x_\ell, \mathrm{d}\,\lambda \rangle = 0.$$

Moreover, the closed curve ζ_l may be considered as the closed curve $\zeta_{x_{l,1},x_{l,1}}$ with circulation 1 along ζ_l. By Lemma B.16, the definition of $\beta_{l,0}$, and (54) we have

$$\begin{aligned}\int_{\zeta_l}\langle v, \mathrm{d}\,\lambda\rangle &= \int_{\zeta_l}\Big\langle \int_{\zeta_{x_{l,1},y}} \widehat{H}\,\mathrm{d}\,\lambda, \mathrm{d}\,\lambda_y\Big\rangle = \int_{\zeta_{x_{l,1},x_{l,1}}}\Big\langle \int_{\zeta_{x_{l,1},y}} \widehat{H}\,\mathrm{d}\,\lambda, \mathrm{d}\,\lambda_y\Big\rangle \\ &= \int_{\zeta_l}\big\langle x_{l,1}-y, \widehat{H}(y)\,\mathrm{d}\,\lambda_y\big\rangle \\ &= \beta_{l,0}(\widehat{H}) = \beta_{l,0}(H) - \sum_{j=1}^{p}\sum_{k=0}^{3}\gamma_{j,k}\underbrace{\beta_{l,0}(\pi\Theta_{j,k})}_{=\delta_{l,j}\delta_{0,k}+(1-\delta_{0,k})\delta_{l,j}(x_{l,1})_k} \\ &= \beta_{l,0}(H) - \gamma_{l,0} - \sum_{k=1}^{3}\underbrace{\gamma_{l,k}}_{=\beta_{l,k}(H)}(x_{l,1})_k = 0.\end{aligned}$$

Therefore, by Assumption 3 (A.1) for any closed piecewise C^1-curve ζ in Ω

$$\int_{\zeta}\langle v, \mathrm{d}\,\lambda\rangle = 0. \tag{58}$$

By (58), cf. Lemma B.16, the function $u : \Omega \to \mathbb{R}$ given by

$$u(x) := \int_{\zeta_{x_0,x}}\langle v, \mathrm{d}\,\lambda\rangle, \qquad x \in \Omega_k,$$

where $\zeta(x_0, x)$ is any piecewise C^1-curve connecting x_0 with x, is well defined, i.e., independent of the respective curve $\zeta(x_0, x)$, and belongs to $C^\infty(\Omega_k)$ with $\operatorname{grad} u = v \in C^{\infty,3}(\Omega_k)$ and

$$\operatorname{Gradgrad} u = \operatorname{Grad} v = \widehat{H} \in C^{\infty,3\times3}(\Omega_k) \cap L^{2,3\times3}_{\mathbb{S}}(\Omega_k).$$

Similar to the end of the proof of Lemma B.3 and Lemma B.12, elliptic regularity and, e.g., [14, Theorem 2.6 (1)] or [1, Theorem 3.2 (2)] show that $v \in C^{\infty,3}(\Omega_k)$ together with $\operatorname{Grad} v \in L^{2,3\times3}_{\mathbb{S}}(\Omega_k)$ implies $v \in H^{1,3}(\Omega_k)$. Then, analogously, $u \in C^\infty(\Omega_k)$ with $\operatorname{grad} u = v \in L^{2,3}(\Omega_k)$ implies $u \in H^1(\Omega_k)$ and hence $u \in H^2(\Omega_k)$, i.e., $u \in H^2(\Omega)$, completing the proof. □

Lemma B.20. *Let Assumption 2 and Assumption 3 be satisfied. Then $\mathcal{B}^{\mathsf{bih},2}_{N}$ is linear independent.*

Proof. Let $\sum_{j=1}^{p}\sum_{k=0}^{3}\gamma_{j,k}\pi\Theta_{j,k} = 0$, $\gamma_{j,k} \in \mathbb{R}$. (54) implies for $l = 1, \dots, p$

$$0 = \sum_{j=1}^{p}\sum_{k=0}^{3}\gamma_{j,k}\beta_{l,\ell}(\pi\Theta_{j,k}) = \gamma_{l,\ell}, \qquad \ell = 1,2,3,$$

$$0 = \sum_{j=1}^{p}\sum_{k=0}^{3}\gamma_{j,k}\beta_{l,\ell}(\pi\Theta_{j,k}) = \gamma_{l,0} + \sum_{k=1}^{3}\gamma_{l,k}(x_{l,1})_k = \gamma_{l,0}, \qquad \ell = 0,$$

finishing the proof. □

Theorem B.21. *Let Assumption 2 as well as Assumption 3 be satisfied. Then it holds $\dim \mathcal{H}^{\mathsf{bih},2}_{N,\mathbb{S}}(\Omega) = 4p$ and a basis of $\mathcal{H}^{\mathsf{bih},2}_{N,\mathbb{S}}(\Omega)$ is given by* (55).

Proof. Use Lemma B.19 and Lemma B.20. □

B.4. **Neumann Tensor Fields of the Elasticity Complex.** Recall from the latter sections that by definition $\theta_j = 0$ outside of a neighbourhood of F_j and θ_j is constant in the two neighbourhoods $\Upsilon_{j,1}$ and $\Upsilon_{j,0}$ of both sides of F_j. Moreover, let $\widehat{r}_k$ be the rigid motions (Nedelec fields) from Section A.4 given by $\widehat{r}_k(x) := e^k \times x = \operatorname{spn}(e^k)\,x$ and $\widehat{r}_{k+3}(x) := e^k$ for $k = 1, 2, 3$. We define the vector fields $\theta_{j,k} := \theta_j \widehat{r}_k$ and note $\operatorname{symGrad}\theta_{j,k} = 0$ in the two neighbourhoods $\Upsilon_{j,1}, \Upsilon_{j,0}$ of F_j and also in all other $\Upsilon_{l,1}, \Upsilon_{l,0}$ of F_l, $j \neq l = 1, \ldots, p$. Thus $\operatorname{symGrad}\theta_{j,k}$ can be continuously extended by zero to $\Theta_{j,k} \in C^{\infty,3\times 3}(\Omega) \cap L^{2,3\times 3}_{\mathbb{S}}(\Omega)$ with $\Theta_{j,k} = 0$ in all the latter neighbourhoods $\widetilde{\Upsilon}_l = \Upsilon_{l,1} \cup F_l \cup \Upsilon_{l,0}$ of all the surfaces F_l.

Lemma B.22. *Let Assumption 3 be satisfied. Then* $\Theta_{j,k} \in \ker(\operatorname{CurlCurl}^{\top}_{\mathbb{S}}, \Omega)$.

Proof. Let $\Phi \in C^{\infty,3\times 3}_{c,\mathbb{S}}(\Omega)$. As $\operatorname{supp}\Theta_{j,k} \subset \overline{\Upsilon}_j \setminus \widetilde{\Upsilon}_j$ we can pick another cut-off function $\varphi \in C^{\infty}_{c}(\Omega_F)$ with $\varphi|_{\operatorname{supp}\Theta_{j,k}\cap\operatorname{supp}\Phi} = 1$. Then

$$\langle \Theta_{j,k}, \operatorname{CurlCurl}^{\top}_{\mathbb{S}} \Phi\rangle_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)} = \langle \Theta_{j,k}, \operatorname{CurlCurl}^{\top}_{\mathbb{S}} \Phi\rangle_{L^{2,3\times 3}_{\mathbb{S}}(\operatorname{supp}\Theta_{j,k}\cap\operatorname{supp}\Phi)}$$
$$= \big\langle \operatorname{symGrad}\theta_{j,k}, \operatorname{CurlCurl}^{\top}_{\mathbb{S}}(\varphi\Phi)\big\rangle_{L^{2,3\times 3}_{\mathbb{S}}(\Omega_F)} = \big\langle \operatorname{Grad}\theta_{j,k}, \operatorname{CurlCurl}^{\top}_{\mathbb{S}}(\varphi\Phi)\big\rangle_{L^{2,3\times 3}_{\mathbb{S}}(\Omega_F)}$$
$$= \Big\langle \operatorname{Grad}\theta_{j,k}, \operatorname{Curl}\big(\operatorname{Curl}(\varphi\Phi)\big)^{\top}\Big\rangle_{L^{2,3\times 3}(\Omega_F)} = 0$$

as $\varphi\Phi, \operatorname{CurlCurl}^{\top}_{\mathbb{S}}(\varphi\Phi) \in C^{\infty,3\times 3}_{c,\mathbb{S}}(\Omega_F)$ by Lemma B.8. □

Before proceeding we need the following lemma:

Lemma B.23. *Let $x, x_0 \in \Omega$ and let $\zeta_{x_0,x} \subset \Omega$ be a piecewise C^1-curve connecting x_0 to x.*

(i) *Let $v \in C^{\infty}(\Omega, \mathbb{R}^3)$. Then v and its rotation* $\operatorname{curl} v$ *can be represented by*

$$v(x) - v(x_0) - \frac{1}{2}\big(\operatorname{curl} v(x_0)\big) \times (x - x_0)$$
$$= \int_{\zeta_{x_0,x}} \operatorname{symGrad} v \,\mathrm{d}\,\lambda + \int_{\zeta_{x_0,x}} \int_{\zeta_{x_0,y}} \operatorname{spn}\big((\operatorname{Curl}\operatorname{symGrad} v)^{\top}\,\mathrm{d}\,\lambda\big)\,\mathrm{d}\,\lambda_y,$$

$$\operatorname{curl} v(x) - \operatorname{curl} v(x_0) = 2\int_{\zeta_{x_0,x}} (\operatorname{Curl}\operatorname{symGrad} v)^{\top}\,\mathrm{d}\,\lambda.$$

(ii) *For all $S \in C^{\infty}(\Omega, \mathbb{R}^{3\times 3})$ it holds*

$$\int_{\zeta_{x_0,x}} \int_{\zeta_{x_0,y}} \operatorname{spn}\big((\operatorname{Curl} S)^{\top}\,\mathrm{d}\,\lambda\big)\,\mathrm{d}\,\lambda_y = \int_{\zeta_{x_0,x}} \operatorname{spn}\big((\operatorname{Curl} S)^{\top}(y)\,\mathrm{d}\,\lambda_y\big)(x - y).$$

(iii) *Let $S \in C^{\infty}(\Omega, \mathbb{R}^{3\times 3})$ and define*

$$w(x) := \int_{\zeta_{x_0,x}} (\operatorname{Curl} S)^{\top}\,\mathrm{d}\,\lambda, \qquad T := S + \operatorname{spn} w, \qquad v(x) := \int_{\zeta_{x_0,x}} T\,\mathrm{d}\,\lambda.$$

Then $w, v \in C^{\infty}(\Omega, \mathbb{R}^3)$ and $T \in C^{\infty}(\Omega, \mathbb{R}^{3\times 3})$ are well defined, i.e., independent of the respective curve, with

$$\operatorname{Grad} w = (\operatorname{Curl} S)^{\top}, \qquad \operatorname{Grad} v = T, \qquad \operatorname{symGrad} v = S,$$

if and only if $\operatorname{skw} S = 0$ *and* $\operatorname{CurlCurl}^{\top}_{\mathbb{S}} S = 0$ *as well as*

$$\int_{\zeta} (\operatorname{Curl} S)^{\top}\,\mathrm{d}\,\lambda = 0, \qquad \int_{\zeta} T\,\mathrm{d}\,\lambda = 0$$

hold for any closed piecewise C^1-curve $\zeta \subset \Omega$. In this case,

$$\operatorname{Grad} w = (\operatorname{Curl} S)^\top = \frac{1}{2} \operatorname{Grad} \operatorname{curl} v.$$

In Lemma B.23 for a tensor field S and a parametrisation $\varphi \in C^1_{\mathsf{pw}}\big([0,1], \mathbb{R}^3\big)$ of $\zeta_{x_0,x}$ we define

$$\int_{\zeta_{x_0,x}} \operatorname{spn}\big((\operatorname{Curl} S)^\top(y)\, \mathrm{d}\,\lambda_y\big)(x-y) := \int_0^1 \operatorname{spn}\Big((\operatorname{Curl} S)^\top\big(\varphi(t)\big)\varphi'(t)\Big)\big(x-\varphi(t)\big)\, \mathrm{d}\, t.$$

Proof of Lemma B.23. For (i), let

$$S := \operatorname{symGrad} v = \operatorname{Grad} v - \operatorname{skw} \operatorname{Grad} v$$

and observe $2 \operatorname{Curl} S = -2 \operatorname{Curl} \operatorname{skw} \operatorname{Grad} v = (\operatorname{Grad} \operatorname{curl} v)^\top$ by Lemma B.8. Thus

$$\begin{aligned}
v_k(x) - v_k(x_0) &= \int_{\zeta_{x_0,x}} \langle \operatorname{grad} v_k, \mathrm{d}\,\lambda\rangle, \qquad k = 1,2,3,\\
v(x) - v(x_0) &= \int_{\zeta_{x_0,x}} \operatorname{Grad} v \, \mathrm{d}\,\lambda,\\
\operatorname{curl} v(x) - \operatorname{curl} v(x_0) &= \int_{\zeta_{x_0,x}} \operatorname{Grad} \operatorname{curl} v \, \mathrm{d}\,\lambda = 2 \int_{\zeta_{x_0,x}} (\operatorname{Curl} S)^\top \, \mathrm{d}\,\lambda.
\end{aligned}$$

Therefore, by Lemma B.8

$$\begin{aligned}
v(x) - v(x_0) &= \int_{\zeta_{x_0,x}} \operatorname{Grad} v \, \mathrm{d}\,\lambda = \int_{\zeta_{x_0,x}} \operatorname{symGrad} v \, \mathrm{d}\,\lambda + \underbrace{\int_{\zeta_{x_0,x}} \operatorname{skw} \operatorname{Grad} v \, \mathrm{d}\,\lambda}_{= \frac{1}{2} \int_{\zeta_{x_0,x}} \operatorname{spn} \operatorname{curl} v(y) \, \mathrm{d}\,\lambda_y}\\
&= \int_{\zeta_{x_0,x}} S \, \mathrm{d}\,\lambda + \frac{1}{2} \int_{\zeta_{x_0,x}} \operatorname{spn} \operatorname{curl} v(x_0) \, \mathrm{d}\,\lambda_y\\
&\qquad + \underbrace{\int_{\zeta_{x_0,x}} \operatorname{spn}\Big(\int_{\zeta_{x_0,y}} (\operatorname{Curl} S)^\top \, \mathrm{d}\,\lambda\Big) \mathrm{d}\,\lambda_y}_{= \int_{\zeta_{x_0,x}} \int_{\zeta_{x_0,y}} \operatorname{spn}\big((\operatorname{Curl} S)^\top \, \mathrm{d}\,\lambda\big) \mathrm{d}\,\lambda_y} .
\end{aligned}$$

Moreover, with φ from above[10]

$$\begin{aligned}
&\int_{\zeta_{x_0,x}} \operatorname{spn} \operatorname{curl} v(x_0) \, \mathrm{d}\,\lambda_y\\
&= \int_0^1 \big(\operatorname{spn} \operatorname{curl} v(x_0)\big)\varphi'(s)\, \mathrm{d}\, s = \big(\operatorname{spn} \operatorname{curl} v(x_0)\big)(x - x_0) = \big(\operatorname{curl} v(x_0)\big) \times (x - x_0).
\end{aligned}$$

[10]Alternatively, we can compute with $\mathrm{Id} = \operatorname{Grad} y$

$$\int_{\zeta_{x_0,x}} \underbrace{\operatorname{spn} \operatorname{curl} v(x_0)}_{= (\operatorname{spn} \operatorname{curl} v(x_0)) \, \mathrm{Id}} \mathrm{d}\,\lambda_y = \operatorname{spn} \operatorname{curl} v(x_0) \int_{\zeta_{x_0,x}} \operatorname{Grad} y \, \mathrm{d}\,\lambda_y = \big(\operatorname{spn} \operatorname{curl} v(x_0)\big)(x - x_0).$$

For (ii) we compute with φ from above

$$\begin{aligned}
\int_{\zeta_{x_0,x}}\int_{\zeta_{x_0,y}} \operatorname{spn}\big((\operatorname{Curl} S)^\top \,\mathrm{d}\,\lambda\big)\,\mathrm{d}\,\lambda_y &= \int_0^1 \Big(\underbrace{\int_{\zeta_{x_0,\varphi(s)}} \operatorname{spn}\big((\operatorname{Curl} S)^\top \,\mathrm{d}\,\lambda\big)} \Big)\varphi'(s)\,\mathrm{d}\,s \\
&\qquad\qquad = \int_0^s \operatorname{spn}\Big((\operatorname{Curl} S)^\top\big(\varphi(t)\big)\varphi'(t)\Big)\,\mathrm{d}\,t \\
&= \int_0^1 \operatorname{spn}\Big((\operatorname{Curl} S)^\top\big(\varphi(t)\big)\varphi'(t)\Big)\underbrace{\int_t^1 \varphi'(s)\,\mathrm{d}\,s}_{=x-\varphi(t)}\,\mathrm{d}\,t \\
&= \int_{\zeta_{x_0,x}} \operatorname{spn}\big((\operatorname{Curl} S)^\top(y)\,\mathrm{d}\,\lambda_y\big)(x-y).
\end{aligned}$$

For (iii), let $S \in C^\infty(\Omega, \mathbb{R}^{3\times 3})$ and let w, T, and v be defined as stated. Moreover, let $\operatorname{skw} S = 0$ and $\operatorname{CurlCurl}_{\mathbb{S}}^\top S = 0$ with

$$\int_\zeta (\operatorname{Curl} S)^\top \,\mathrm{d}\,\lambda = 0, \qquad \int_\zeta T\,\mathrm{d}\,\lambda = 0$$

for any closed piecewise C^1-curve $\zeta \subset \Omega$. Note that

$$\begin{array}{llcc}
 & w \text{ well defined (indep. of } \zeta_{x_0,x}) & \wedge & \operatorname{Grad} w = (\operatorname{Curl} S)^\top \\
\Leftrightarrow & \forall\,\zeta\,(\text{cl pw } C^1) \quad \displaystyle\int_\zeta (\operatorname{Curl} S)^\top \,\mathrm{d}\,\lambda = 0 & \wedge & \operatorname{Curl}(\operatorname{Curl} S)^\top = 0,
\end{array}$$

and

$$\begin{array}{llcc}
 & v \text{ well defined (indep. of } \zeta_{x_0,x}) & \wedge & \operatorname{Grad} v = T \\
\Leftrightarrow & \forall\,\zeta\,(\text{cl pw } C^1) \quad \displaystyle\int_\zeta T\,\mathrm{d}\,\lambda = 0 & \wedge & \operatorname{Curl} T = 0.
\end{array}$$

Hence w is well defined with $\operatorname{Grad} w = (\operatorname{Curl} S)^\top$. By Lemma B.8 we have

$$\begin{aligned}
\operatorname{Curl} T = \operatorname{Curl} S + \operatorname{Curl}\operatorname{spn} w &= \operatorname{Curl} S + (\operatorname{div} w)\operatorname{Id} - (\operatorname{Grad} w)^\top \\
&= (\operatorname{tr}\operatorname{Grad} w)\operatorname{Id} = (\operatorname{tr}\operatorname{Curl} S)\operatorname{Id} = 0,
\end{aligned}$$

as $\operatorname{skw} S = 0$. Hence v is also well defined with $\operatorname{Grad} v = T$. Moreover, $v, w \in C^\infty(\Omega, \mathbb{R}^3)$ and $T \in C^\infty(\Omega, \mathbb{R}^{3\times 3})$ as well as $\operatorname{sym}\operatorname{Grad} v = \operatorname{sym} T = \operatorname{sym} S = S$ and

$$\operatorname{Grad} w = (\operatorname{Curl} S)^\top = (\operatorname{Curl}\operatorname{sym}\operatorname{Grad} v)^\top = \frac{1}{2}\operatorname{Grad}\operatorname{curl} v.$$

On the other hand, let $w, v \in C^\infty(\Omega, \mathbb{R}^3)$ and $S, T \in C^\infty(\Omega, \mathbb{R}^{3\times 3})$ be given with

$$\operatorname{Grad} w = (\operatorname{Curl} S)^\top, \qquad \operatorname{Grad} v = T, \qquad \operatorname{symGrad} v = S.$$

Then $\operatorname{skw} S = 0$,

$$\operatorname{CurlCurl}_{\mathbb{S}}^\top S = \operatorname{Curl}(\operatorname{Curl} S)^\top = \operatorname{Curl}\operatorname{Grad} w = 0$$

and $2\operatorname{Grad} w = \operatorname{Grad}\operatorname{curl} v$ by Lemma B.8, as well as

$$\int_\zeta (\operatorname{Curl} S)^\top \,\mathrm{d}\,\lambda = \int_\zeta \operatorname{Grad} w\,\mathrm{d}\,\lambda = 0, \qquad \int_\zeta T\,\mathrm{d}\,\lambda = \int_\zeta \operatorname{Grad} v\,\mathrm{d}\,\lambda = 0,$$

completing the proof. $\square$

Note that for $l,j=1,\dots,p$ and $k=1,\dots,6$ and for the curves $\zeta_{x_{l,0},x_{l,1}}\subset\zeta_l$ with the chosen starting points $x_{l,0}\in\Upsilon_{l,0}$ and respective endpoints $x_{l,1}\in\Upsilon_{l,1}$ we can compute[11] by Lemma B.23

$$\begin{aligned}
\mathbb{R}^3\ni a_l(\Theta_{j,k}) &:= \int_{\zeta_l}(\operatorname{Curl}\Theta_{j,k})^\top\,\mathrm{d}\,\lambda = \int_{\zeta_{x_{l,0},x_{l,1}}}(\operatorname{Curl}\operatorname{symGrad}\theta_{j,k})^\top\,\mathrm{d}\,\lambda\\
&= \frac{1}{2}\operatorname{curl}\theta_{j,k}(x_{l,1}) - \frac{1}{2}\underbrace{\operatorname{curl}\theta_{j,k}(x_{l,0})}_{=0}\\
&= \frac{1}{2}\delta_{l,j}\operatorname{curl}\widehat{r}_k(x_{l,1}) = \delta_{l,j}\begin{cases} e^k, & \text{if } k=1,2,3,\\ 0, & \text{if } k=4,5,6,\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{R}^3\ni b_l(\Theta_{j,k}) &:= \int_{\zeta_l}\Theta_{j,k}\,\mathrm{d}\,\lambda + \int_{\zeta_l}\operatorname{spn}\big((\operatorname{Curl}\Theta_{j,k})^\top(y)\,\mathrm{d}\,\lambda_y\big)(x_{l,1}-y)\\
&= \int_{\zeta_{x_{l,0},x_{l,1}}}\operatorname{symGrad}\theta_{j,k}\,\mathrm{d}\,\lambda\\
&\qquad + \int_{\zeta_{x_{l,0},x_{l,1}}}\operatorname{spn}\big((\operatorname{Curl}\operatorname{symGrad}\theta_{j,k})^\top(y)\,\mathrm{d}\,\lambda_y\big)(x_{l,1}-y)\\
&= \int_{\zeta_{x_{l,0},x_{l,1}}}\Big(\operatorname{symGrad}\theta_{j,k}(y)\\
&\qquad + \int_{\zeta_{x_{l,0},y}}\operatorname{spn}\big((\operatorname{Curl}\operatorname{symGrad}\theta_{j,k})^\top\,\mathrm{d}\,\lambda\big)\Big)\,\mathrm{d}\,\lambda_y\\
&= \theta_{j,k}(x_{l,1})\underbrace{-\theta_{j,k}(x_{l,0}) - \frac{1}{2}\operatorname{curl}\theta_{j,k}(x_{l,0})\times(x_{l,1}-x_{l,0})}_{=0}\\
&= \delta_{l,j}\widehat{r}_k(x_{l,1}) = \delta_{l,j}\begin{cases} e^k\times x_{l,1}, & \text{if } k=1,2,3,\\ e^{k-3}, & \text{if } k=4,5,6.\end{cases}
\end{aligned}$$

Thus, we have functionals $\beta_{l,\ell}$ for $l=1,\dots,p$ and $\ell=1,\dots,6$ given by

$$\beta_{l,\ell}(\Theta_{j,k}) := \begin{cases}\big\langle a_l(\Theta_{j,k}),e^\ell\big\rangle, & \text{if } \ell=1,2,3,\\ \big\langle b_l(\Theta_{j,k}),e^{\ell-3}\big\rangle, & \text{if } \ell=4,5,6,\end{cases}\qquad j=1,\dots,p,\quad k=1,\dots,6.$$

Then for $l,j=1,\dots,p$ and for $\ell=1,2,3$

$$\beta_{l,\ell}(\Theta_{j,k}) = \big\langle a_l(\Theta_{j,k}),e^\ell\big\rangle = \delta_{l,j}\begin{cases}\langle e^k,e^\ell\rangle=\delta_{\ell,k}, & \text{if } k=1,2,3,\\ \langle 0,e^\ell\rangle = 0, & \text{if } k=4,5,6,\end{cases}$$

i.e.,

$$\beta_{l,\ell}(\Theta_{j,k}) = \delta_{l,j}\delta_{\ell,k},\qquad k=0,\dots,6,$$

[11]Note that $\operatorname{curl}\widehat{r}_k = 2e^k$ for $k=1,2,3$, since, e.g.,

$$\begin{aligned}
\operatorname{curl}\widehat{r}_1(x) &= \operatorname{curl}(e^1\times x) = \operatorname{curl}(x_2\,e^1\times e^2 + x_3\,e^1\times e^3) = \operatorname{curl}(x_2e^3 - x_3e^2)\\
&= \operatorname{grad}(x_2)\times e^3 - \operatorname{grad}(x_3)\times e^2 = e^2\times e^3 - e^3\times e^2 = 2e^1.
\end{aligned}$$

and for $\ell = 4, 5, 6$

$$\beta_{l,\ell}(\Theta_{j,k}) = \big\langle b_l(\Theta_{j,k}), e^{\ell-3}\big\rangle = \delta_{l,j}\begin{cases}\langle e^k \times x_{l,1}, e^{\ell-3}\rangle = \langle e^{\ell-3} \times e^k, x_{l,1}\rangle, & \text{if } k = 1, 2, 3,\\ \langle e^{k-3}, e^{\ell-3}\rangle = \delta_{\ell,k}, & \text{if } k = 4, 5, 6,\end{cases}$$

i.e.,

$$\beta_{l,\ell}(\Theta_{j,k}) = \delta_{l,j}\delta_{\ell,k} + \delta_{l,j}(\delta_{1,k} + \delta_{2,k} + \delta_{3,k})(x_{l,1})_{\widehat{\ell-3,k}}, \qquad k = 0, \dots, 6,$$

where

$$(x_{l,1})_{\widehat{\ell-3,k}} := \langle e^{\ell-3} \times e^k, x_{l,1}\rangle = \langle e^{\ell-3} \times e^k, e^i\rangle (x_{l,1})_i = \pm (x_{l,1})_i$$

for the even resp. odd permutation $(\ell - 3, k, i)$ of $(1, 2, 3)$ and

$$(x_{l,1})_{\widehat{\ell-3,k}} := 0$$

for all other ℓ and k. In particular, $(x_{l,1})_{\widehat{\ell-3,k}} = 0$ if $\ell - 3 = k$ or $\ell = 1, 2, 3$ or $k = 4, 5, 6$. Therefore, we have for $l, j = 1, \dots, p$ and $k, \ell = 1, \dots, 6$

$$\begin{aligned}\beta_{l,\ell}(\Theta_{j,k}) &= \delta_{l,j}\delta_{\ell,k} + \delta_{l,j}(x_{l,1})_{\widehat{\ell-3,k}}\\ &= \delta_{l,j}\delta_{\ell,k} + \delta_{l,j}(\delta_{\ell,4} + \delta_{\ell,5} + \delta_{\ell,6})(\delta_{1,k} + \delta_{2,k} + \delta_{3,k})(1 - \delta_{\ell-3,k})(x_{l,1})_{\widehat{\ell-3,k}}.\end{aligned} \tag{59}$$

Let Assumption 2 be satisfied. For the elasticity complex, simliar to (3), (4), and (40), (46), (53) we have the orthogonal decompositions

$$\begin{aligned} L^{2,3\times 3}_{\mathbb{S}}(\Omega) &= \operatorname{ran}(\operatorname{symGrad}, \Omega) \oplus_{L^{2,3\times3}_{\mathbb{S}}(\Omega)} \ker(\mathring{\operatorname{Div}}_{\mathbb{S}}, \Omega),\\ \ker(\operatorname{CurlCurl}^\top_{\mathbb{S}}, \Omega) &= \operatorname{ran}(\operatorname{symGrad}, \Omega) \oplus_{L^{2,3\times3}_{\mathbb{S}}(\Omega)} \mathcal{H}^{\mathsf{ela}}_{N,\mathbb{S}}(\Omega).\end{aligned} \tag{60}$$

Remark B.24. *It holds* $\operatorname{dom}(\operatorname{symGrad}, \Omega) = H^{1,3}(\Omega)$ *by* [22, Lemma 3.2]. *Moreover, the range in* (60) *is closed by the Poincaré type estimate*

$$\exists\, c > 0 \quad \forall\, \phi \in H^{1,3}(\Omega) \cap \mathsf{RM}_{\mathsf{pw}}^{\perp_{L^{2,3}(\Omega)}} \qquad |\phi|_{L^{2,3}(\Omega)} \le c |\operatorname{symGrad} \phi|_{L^{2,3\times3}(\Omega)},$$

which is implied by Rellich's selection theorem and [22, Lemma 3.2] *as Assumption 2 holds.*

Let us denote in (60) the orthogonal projector onto $\ker(\mathring{\operatorname{Div}}_{\mathbb{S}}, \Omega)$ resp. $\mathcal{H}^{\mathsf{ela}}_{N,\mathbb{S}}(\Omega)$ by π. By Lemma B.22 there exists some $\psi_{j,k} \in H^{1,3}(\Omega)$ such that

$$\mathcal{H}^{\mathsf{ela}}_{N,\mathbb{S}}(\Omega) \ni \pi\Theta_{j,k} = \Theta_{j,k} - \operatorname{symGrad} \psi_{j,k}, \quad (\Theta_{j,k} - \operatorname{symGrad}\psi_{j,k})\big|_{\Omega_F} = \operatorname{symGrad}(\theta_{j,k} - \psi_{j,k}).$$

As $\mathcal{H}^{\mathsf{ela}}_{N,\mathbb{S}}(\Omega) \subset C^{\infty,3\times3}(\Omega)$, cf. (25), we conclude by $\pi\Theta_{j,k}, \Theta_{j,k} \in C^{\infty,3\times3}(\Omega)$ that also $\operatorname{symGrad}\psi_{j,k} \in C^{\infty,3\times3}(\Omega)$ and hence $\psi_{j,k} \in C^{\infty,3}(\Omega)$. Hence all path integrals over the closed curves ζ_l are well defined. Furthermore, we observe by Lemma B.23

$$\begin{aligned} a_l(\operatorname{symGrad}\psi_{j,k}) &= \int_{\zeta_l} (\operatorname{Curl}\operatorname{symGrad}\psi_{j,k})^\top \,\mathrm{d}\,\lambda\\ &= \frac{1}{2}\big(\operatorname{curl}\psi_{j,k}(x_{l,1}) - \operatorname{curl}\psi_{j,k}(x_{l,1})\big) = 0,\end{aligned}$$

and

$$\begin{aligned} b_l(\operatorname{symGrad}\psi_{j,k}) = \int_{\zeta_l} &\operatorname{symGrad}\psi_{j,k} \,\mathrm{d}\,\lambda\\ &+ \int_{\zeta_l} \operatorname{spn}\big((\operatorname{Curl}\operatorname{symGrad}\psi_{j,k})^\top(y)\,\mathrm{d}\,\lambda_y\big)(x_{l,1} - y)\end{aligned}$$

$$
\begin{aligned}
&= \int_{\zeta_{x_{l,1},x_{l,1}}} \Big(\operatorname{symGrad} \psi_{j,k}(y) \\
&\qquad + \int_{\zeta_{x_{l,1},y}} \operatorname{spn}\big((\operatorname{Curl} \operatorname{symGrad} \psi_{j,k})^\top \operatorname{d}\lambda \big) \Big) \operatorname{d}\lambda_y \\
&= \psi_{j,k}(x_{l,1}) - \psi_{j,k}(x_{l,1}) - \frac{1}{2} \operatorname{curl} \psi_{j,k}(x_{l,1}) \times (x_{l,1} - x_{l,1}) = 0.
\end{aligned}
$$

Therefore, by (59)

$$
\begin{aligned}
\beta_{l,\ell}(\pi\Theta_{j,k}) &= \beta_{l,\ell}(\Theta_{j,k}) - \underbrace{\beta_{l,\ell}(\operatorname{symGrad}\psi_{j,k})}_{=0} = \delta_{l,j}\delta_{\ell,k} + \delta_{l,j}(x_{l,1})_{\widehat{\ell-3,k}} \\
&= \delta_{l,j}\delta_{\ell,k} + \delta_{l,j}(\delta_{\ell,4} + \delta_{\ell,5} + \delta_{\ell,6})(\delta_{1,k} + \delta_{2,k} + \delta_{3,k})(1 - \delta_{\ell-3,k})(x_{l,1})_{\widehat{\ell-3,k}}
\end{aligned}
\tag{61}
$$

for all $l, j = 1, \dots, p$ and all $\ell, k = 1, \dots, 6$. We shall show that

$$
\mathcal{B}_N^{\mathsf{ela}} := \{\pi\Theta_{j,k}\}_{\substack{j=1,\dots,p,\\ k=1,\dots,6}} \subset \mathcal{H}_{N,\mathbb{S}}^{\mathsf{ela}}(\Omega)
\tag{62}
$$

defines a basis of $\mathcal{H}_{N,\mathbb{S}}^{\mathsf{ela}}(\Omega)$.

Note that $\psi_{j,k} \in H^{1,3}(\Omega) \cap \mathsf{RM}_{\mathsf{pw}}^{\perp_{L^{2,3}(\Omega)}}$ can be found by the standard variational formulation

$$
\forall\, \phi \in H^{1,3}(\Omega) \qquad \langle \operatorname{symGrad}\psi_{j,k}, \operatorname{symGrad}\phi \rangle_{L^{2,3\times3}(\Omega)} = \langle \Theta_{j,k}, \operatorname{symGrad}\phi \rangle_{L^{2,3\times3}(\Omega)},
$$

i.e., $\psi_{j,k} = \Delta_{\mathbb{S}}^{-1} \operatorname{Div}_{\mathbb{S}} \Theta_{j,k}$. Therefore,

$$
\pi\Theta_{j,k} = \Theta_{j,k} - \operatorname{symGrad}\psi_{j,k} = (1 - \operatorname{symGrad}\Delta_{\mathbb{S}}^{-1}\operatorname{Div}_{\mathbb{S}})\Theta_{j,k}.
$$

Let us also mention that $\psi_{j,k}$ solves in classical terms the Neumann elasticity problem

$$
\begin{aligned}
-\Delta_{\mathbb{S}}\psi_{j,k} &= -\operatorname{Div}_{\mathbb{S}}\Theta_{j,k} && \text{in } \Omega, \\
(\operatorname{Grad}\psi_{j,k})\nu &= \Theta_{j,k}\nu && \text{on } \Gamma, \\
\int_{\Omega_l} (\psi_{j,k})_\ell &= 0 && \text{for } l = 1, \dots, n, \quad \ell = 1,2,3, \\
\int_{\Omega_l} \big(x \times \psi_{j,k}(x)\big)_\ell \operatorname{d}\lambda_x &= 0 && \text{for } l = 1, \dots, n, \quad \ell = 1,2,3,
\end{aligned}
\tag{63}
$$

which is uniquely solvable.

Lemma B.25. *Let Assumption 2 as well as Assumption 3 be satisfied. Then it holds* $\mathcal{H}_{N,\mathbb{S}}^{\mathsf{ela}}(\Omega) = \operatorname{lin}\mathcal{B}_N^{\mathsf{ela}}$.

Proof. Let $H \in \mathcal{H}_{N,\mathbb{S}}^{\mathsf{ela}}(\Omega) = \ker(\mathring{\operatorname{Div}}_{\mathbb{S}}, \Omega) \cap \ker(\operatorname{Curl}\operatorname{Curl}_{\mathbb{S}}^\top, \Omega) \subset C_{\mathbb{S}}^{\infty,3\times3}(\Omega)$, cf. (25). With the above introduced functionals a_l and b_l we recall

$$
\begin{aligned}
\mathbb{R}^3 \ni a_l(H) &= \int_{\zeta_l} (\operatorname{Curl} H)^\top \operatorname{d}\lambda, \\
\mathbb{R}^3 \ni b_l(H) &:= \int_{\zeta_l} H \operatorname{d}\lambda + \int_{\zeta_l} \operatorname{spn}\big((\operatorname{Curl} H)^\top(y) \operatorname{d}\lambda_y \big)(x_{l,1} - y),
\end{aligned}
$$

and define for $l = 1, \dots, p$ the numbers

$$
\begin{aligned}
\gamma_{l,\ell} &:= \gamma_{l,\ell}(H) := \big\langle a_l(H), e^\ell \big\rangle = \beta_{l,\ell}(H), && \ell = 1,2,3, \\
\gamma_{l,\ell} &:= \gamma_{l,\ell}(H) := \Big\langle b_l(H) - \sum_{k=1}^3 \beta_{l,k}(H) e^k \times x_{l,1}, e^{\ell-3} \Big\rangle
\end{aligned}
$$

$$= \beta_{l,\ell}(H) - \sum_{k=1}^{3} \beta_{l,k}(H)(x_{l,1})_{\widehat{\ell-3,k}}, \qquad \ell = 4,5,6,$$

where we recall $(x_{l,1})_{\widehat{\ell-3,k}} = (\delta_{\ell,4} + \delta_{\ell,5} + \delta_{\ell,6})(\delta_{1,k} + \delta_{2,k} + \delta_{3,k})(1 - \delta_{\ell-3,k})(x_{l,1})_{\widehat{\ell-3,k}}$ by definition, cf. (59), (61). We shall show that

$$\mathcal{H}^{\mathsf{ela}}_{N,\mathbb{S}}(\Omega) \ni \widehat{H} := H - \sum_{j=1}^{p}\sum_{k=1}^{6} \gamma_{j,k}\pi\Theta_{j,k} = 0 \quad \text{in } \Omega.$$

Similar to the proofs of Lemma B.3, Lemma B.12, and Lemma B.19, the aim is to prove that there exists $v \in H^{1,3}(\Omega)$ such that $\operatorname{symGrad} v = \widehat{H}$, since then

$$|\widehat{H}|^2_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)} = \langle \operatorname{symGrad} v, \widehat{H}\rangle_{L^{2,3\times 3}_{\mathbb{S}}(\Omega)} = 0.$$

By (61) we observe for $l = 1,\dots,p$ and for $\ell = 1,2,3$

$$\Big(\int_{\zeta_l} (\operatorname{Curl}\widehat{H})^\top \,\mathrm{d}\,\lambda\Big)_\ell = \underbrace{\big(a_l(\widehat{H})\big)_\ell}_{=\beta_{l,\ell}(\widehat{H})} = \underbrace{\beta_{l,\ell}(H)}_{=\gamma_{l,\ell}} - \sum_{j=1}^{p}\sum_{k=1}^{6} \gamma_{j,k} \underbrace{\beta_{l,\ell}(\pi\Theta_{j,k})}_{=\delta_{l,j}\delta_{\ell,k}} = 0,$$

and thus by Assumption 3 (A.1) for any closed piecewise C^1-curve ζ in Ω

$$\int_{\zeta} (\operatorname{Curl}\widehat{H})^\top \,\mathrm{d}\,\lambda = 0. \tag{64}$$

Recall the connected components $\Omega_1,\dots,\Omega_n$ of Ω. For $1 \le k \le n$ let some $x_0 \in \Omega_k$ be fixed. By (64) and $\operatorname{Curl}(\operatorname{Curl}\widehat{H})^\top = \operatorname{CurlCurl}^\top_{\mathbb{S}} \widehat{H} = 0$, cf. Lemma B.23, the vector field $w : \Omega \to \mathbb{R}^3$ and the tensor field $T : \Omega \to \mathbb{R}^{3\times 3}$ given by

$$w(x) := \int_{\zeta(x_0,x)} (\operatorname{Curl}\widehat{H})^\top \,\mathrm{d}\,\lambda, \quad T := \widehat{H} + \operatorname{spn} w, \qquad x \in \Omega_k,$$

where $\zeta(x_0,x)$ is any piecewise C^1-curve connecting x_0 with x, are well defined, i.e., independent of the respective curve $\zeta(x_0,x)$, and belong to $C^{\infty,3}(\Omega_k)$ and $C^{\infty,3\times 3}(\Omega_k)$, respectively. Moreover, $\operatorname{Grad} w = (\operatorname{Curl}\widehat{H})^\top$ and by Lemma B.8

$$\begin{aligned}\operatorname{Curl} T &= \operatorname{Curl}\widehat{H} + \operatorname{Curl}\operatorname{spn} w = \operatorname{Curl}\widehat{H} + (\operatorname{div} w)\operatorname{Id} - (\operatorname{Grad} w)^\top\\ &= (\operatorname{tr}\operatorname{Grad} w)\operatorname{Id} = (\operatorname{tr}\operatorname{Curl}\widehat{H})\operatorname{Id} = 0,\end{aligned}$$

as $\operatorname{skw}\widehat{H} = 0$. Note that for $\zeta_{x_{l,0},x_{l,1}} \subset \zeta_l \subset \Omega_k$ we have with $c := w(x_{l,1}) \in \mathbb{R}^3$

$$w(x) = \underbrace{w(x) - w(x_{l,1})}_{=\int_{\zeta_{x_{l,1},x}} \operatorname{Grad} w \,\mathrm{d}\,\lambda} + c = \int_{\zeta_{x_{l,1},x}} (\operatorname{Curl}\widehat{H})^\top \,\mathrm{d}\,\lambda + c, \qquad x \in \zeta_l,$$

and

$$\int_{\zeta_l} (\operatorname{spn} c)\,\mathrm{d}\,\lambda = (\operatorname{spn} c)\int_{\zeta_l} \operatorname{Id}\,\mathrm{d}\,\lambda = (\operatorname{spn} c)\int_{\zeta_l} \operatorname{Grad} x \,\mathrm{d}\,\lambda_x = 0.$$

Moreover, the closed curve ζ_l may be considered as the closed curve $\zeta_{x_{l,1},x_{l,1}}$ with circulation 1 along ζ_l. By Lemma B.23 and by the definition of b_l we have for $l = 1,\dots,p$

$$\int_{\zeta_l} T \,\mathrm{d}\,\lambda = \int_{\zeta_l} \widehat{H}\,\mathrm{d}\,\lambda + \int_{\zeta_l} (\operatorname{spn} w)\,\mathrm{d}\,\lambda$$

$$= \int_{\zeta_l} \widehat{H} \,\mathrm{d}\,\lambda + \int_{\zeta_{x_{l,1},x_{l,1}}} \operatorname{spn}\Big(\int_{\zeta_{x_{l,1},y}} (\operatorname{Curl}\widehat{H})^\top \,\mathrm{d}\,\lambda\Big) \,\mathrm{d}\,\lambda_y$$
$$= \int_{\zeta_l} \widehat{H} \,\mathrm{d}\,\lambda + \int_{\zeta_l} \operatorname{spn}\big((\operatorname{Curl}\widehat{H})^\top(y) \,\mathrm{d}\,\lambda_y\big)(x_{l,1} - y) = b_l(\widehat{H}).$$

Hence, for $\ell = 4, 5, 6$ we get by (61)

$$\Big(\int_{\zeta_l} T \,\mathrm{d}\,\lambda\Big)_{\ell-3} = \Big\langle \int_{\zeta_l} T \,\mathrm{d}\,\lambda, e^{\ell-3}\Big\rangle = \langle b_l(\widehat{H}), e^{\ell-3}\rangle = \beta_{l,\ell}(\widehat{H})$$
$$= \beta_{l,\ell}(H) - \sum_{j=1}^{p}\sum_{k=1}^{6} \gamma_{j,k} \underbrace{\beta_{l,\ell}(\pi\Theta_{j,k})}_{=\delta_{l,j}\delta_{\ell,k}+\delta_{l,j}(x_{l,1})_{\widehat{\ell-3,k}}} = \beta_{l,\ell}(H) - \gamma_{l,\ell} - \sum_{k=1}^{3} \underbrace{\gamma_{l,k}}_{=\beta_{l,k}(H)} (x_{l,1})_{\widehat{\ell-3,k}} = 0.$$

Therefore, $\int_{\zeta_l} T \,\mathrm{d}\,\lambda = 0$ and thus by Assumption 3 (A.1) for any closed piecewise C^1-curve ζ in Ω

$$\int_{\zeta} T \,\mathrm{d}\,\lambda = 0. \tag{65}$$

By (65), cf. Lemma B.23, the vector field $v : \Omega \to \mathbb{R}^3$ given by

$$v(x) := \int_{\zeta_{x_0,x}} T \,\mathrm{d}\,\lambda, \qquad x \in \Omega_k,$$

where $\zeta(x_0, x)$ is any piecewise C^1-curve connecting x_0 with x, is well defined, i.e., independent of the respective curve $\zeta(x_0, x)$. Moreover, v belongs to $C^{\infty,3}(\Omega_k)$ and satisfies $\operatorname{Grad} v = T \in C^{\infty,3\times3}(\Omega_k)$ as well as

$$\operatorname{symGrad} v = \operatorname{sym} T = \operatorname{sym} \widehat{H} = \widehat{H} \in C^{\infty,3\times3}(\Omega_k) \cap L^{2,3\times3}_{\mathbb{S}}(\Omega_k).$$

Similar to the end of the proof of Lemma B.3, elliptic regularity and, e.g., [14, Theorem 2.6 (1)] or [1, Theorem 3.2 (2)] show that $v \in C^{\infty,3}(\Omega_k)$ with $\operatorname{symGrad} v \in L^{2,3\times3}_{\mathbb{S}}(\Omega_k)$ implies $v \in H^{1,3}(\Omega_k)$ and thus $v \in H^{1,3}(\Omega)$, completing the proof. Let us note that $v \in H^{1,3}(\Omega)$ implies also $T \in L^{2,3\times3}(\Omega)$ and hence $w \in L^{2,3}(\Omega)$. □

Lemma B.26. *Let Assumption 2 and Assumption 3 be satisfied. Then $\mathcal{B}^{\mathsf{ela}}_N$ is linear independent.*

Proof. Let $\sum_{j=1}^{p}\sum_{k=1}^{6} \gamma_{j,k}\pi\Theta_{j,k} = 0$, $\gamma_{j,k} \in \mathbb{R}$. (61) implies for $l = 1, \dots, p$

$$0 = \sum_{j=1}^{p}\sum_{k=1}^{6} \gamma_{j,k}\beta_{l,\ell}(\pi\Theta_{j,k}) = \gamma_{l,\ell}, \qquad \ell = 1, 2, 3,$$
$$0 = \sum_{j=1}^{p}\sum_{k=1}^{6} \gamma_{j,k}\beta_{l,\ell}(\pi\Theta_{j,k}) = \gamma_{l,\ell} + \sum_{k=1}^{3} \gamma_{l,k}(x_{l,1})_{\widehat{\ell-3,k}} = \gamma_{l,\ell}, \qquad \ell = 4, 5, 6,$$

finishing the proof. □

Theorem B.27. *Let Assumption 2 as well as Assumption 3 be satisfied. Then it holds $\dim \mathcal{H}^{\mathsf{ela}}_{N,\mathbb{S}}(\Omega) = 6p$ and a basis of $\mathcal{H}^{\mathsf{ela}}_{N,\mathbb{S}}(\Omega)$ is given by (62).*

Proof. Use Lemma B.25 and Lemma B.26. □

Fakultät für Mathematik, Universität Duisburg-Essen, Campus Essen, Germany
Email address, Dirk Pauly: dirk.pauly@uni-due.de

Department of Mathematics and Statistics, University of Strathclyde, Livingstone Tower, 26 Richmond Street, Glasgow G1 1XH, Scotland
Email address, Marcus Waurick: marcus.waurick@strath.ac.uk