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# NUMERICAL ANALYSIS FOR MAXWELL OBSTACLE PROBLEMS IN ELECTRIC SHIELDING*

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**Abstract.** This paper proposes and examines a finite element method (FEM) for a Maxwell obstacle problem in electric shielding. The model is given by a coupled system comprising the Faraday equation and an evolutionary variational inequality (VI) of Ampère–Maxwell-type. Based on the leapfrog (Yee) time-stepping and the Nédélec edge elements, we set up a fully discrete FEM where the obstacle is discretized in such a way that no additional nonlinear solver is required for the computation of the discrete VI. While the $L^2$-stability is achieved for the discrete solutions and the associated difference quotients, the scheme only guarantees the $L^1$-stability for the discrete magnetic curl field in the obstacle region. The lack of the global $L^2$-stability for the magnetic curl field is justified by the low regularity issue in Maxwell obstacle problems and turns to be the main challenge in the convergence analysis. Our convergence proof consists of two main stages. First, exploiting the $L^1$-stability in the obstacle region, we derive a convergence result towards a weaker system involving smooth feasible test functions. In the second step, we recover the original system by enlarging the feasible test function set through a specific constraint preserving mollification process in the spirit of Ern and Guermond [*Comput. Methods Appl. Math.*, 16 (2016), pp. 51–75]. This paper is closed by three-dimensional numerical results of the proposed FEM confirming the theoretical convergence result and, in particular, the Faraday shielding effect.

**Key words.** Maxwell obstacle problems, electric shielding, FEM, leapfrog time-stepping, stability, convergence, constraint preserving mollification

**AMS subject classifications.** 65N30, 35L85, 78M30

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## 1. Introduction.

Electric shielding is a physical process of blocking or canceling external electric fields through obstacles made by conductive materials. This physical phenomenon was discovered in 1836 by Faraday, who experimentally verified that a conductive enclosure is able to eliminate the effect of an external electric field by charge cancelation on the boundary and leaving a zero field inside the cavity. Such an effect is also known under the term *Faraday cage*. See Figure 1 for a simple experiment of a Faraday cage conducted at our mathematical faculty. From the mathematical point of view (see Duvaut and Lions [7]), electric shielding falls into the class of obstacle problems: In the free region, the electromagnetic waves satisfy the Maxwell equations, whereas pointwise constraints are applied to the electric field in the shielded area. This leads to a nonstandard coupled system consisting of the Faraday equation and an evolutionary variational inequality (VI) of Ampère–Maxwell-type. Faraday cage effects can also be treated by homogenization techniques (see [16, 3] for electrostatic Faraday cage models), which are, however, not the focus of this paper.

We refer to [24] for a recent well posedness result for Maxwell obstacle problems, which turn out to be more difficult to handle than Maxwell VI of the second kind with $L^1$-nonlinearities emerging in type-II superconductivity [23, 19, 22, 20]. In particular, solutions to Maxwell obstacle problems suffer from low regularity as we shall address shortly.

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FIG. 1. *Electric field measurement with and without Faraday shielding.*

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz polyhedral domain representing the hold-all domain. Inside this domain, we consider a Lipschitz polyhedral domain $\omega$ satisfying $\overline{\omega} \subset \Omega$.

The subset $\omega$ stands for the obstacle region representing the area shielded by a closed conductive enclosure. Thus, a pointwise constraint is applied to the electric field in $\omega$ leading to the following feasible electric set:

$$\boldsymbol{K} \coloneqq \{\boldsymbol{e} \in \boldsymbol{L}^2(\Omega) \mid |\boldsymbol{e}(x)| \leq d \quad \text{for a.e. } x \in \omega\}$$

for some fixed upper bound $d \in [0, \infty)$. Then, given initial data $(\boldsymbol{E}_0, \boldsymbol{H}_0) \in (\boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})) \times \boldsymbol{H}(\mathbf{curl})$ and a source field $\boldsymbol{f} \in C^{0,1}([0,T], \boldsymbol{L}^2(\Omega))$, the electric obstacle problem we focus on reads as

$$\text{(P)} \qquad \begin{cases} \displaystyle\int_\Omega \epsilon \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t)) + \sigma\boldsymbol{E}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t)) - \boldsymbol{H}(t)\cdot\mathbf{curl}(\boldsymbol{v}-\boldsymbol{E}(t))\,\mathrm{d}x \\ \displaystyle\geq \int_\Omega \boldsymbol{f}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t))\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{K}\cap\boldsymbol{H}_0(\mathbf{curl}) \text{ for a.e. } t \in (0,T), \\ \displaystyle\mu\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}(t) + \mathbf{curl}\,\boldsymbol{E}(t) = \mathbf{0} \quad \text{for a.e. } t\in(0,T), \\ (\boldsymbol{E},\boldsymbol{H}) \in W^{1,\infty}((0,T),\boldsymbol{L}^2(\Omega)\times\boldsymbol{L}^2(\Omega)) \cap L^\infty((0,T),\boldsymbol{H}_0(\mathbf{curl})\times\boldsymbol{L}^2(\Omega)), \\ \boldsymbol{E}(t)\in\boldsymbol{K} \text{ for all } t\in[0,T] \text{ and } (\boldsymbol{E},\boldsymbol{H})(0) = (\boldsymbol{E}_0,\boldsymbol{H}_0). \end{cases}$$

The existence of a unique solution $(\boldsymbol{E}, \boldsymbol{H})$ to (P) follows from Theorems 1 and 2 in [24]. As shown in [24, Theorem 1], in the free region $\Omega \setminus \overline{\omega}$, the unique solution of (P) satisfies the Ampère–Maxwell equation and the local magnetic $L^2$-regularity property $\mathbf{curl}\,\boldsymbol{H} \in L^\infty((0,T), \boldsymbol{L}^2(\Omega\setminus\overline{\omega}))$. However, in the obstacle region $\omega$, the $L^2$-regularity of $\mathbf{curl}\,\boldsymbol{H}$ is not a priori guaranteed.

This paper aims to construct and analyze an efficient finite element method (FEM) for (P). We are not aware of any previous work on the numerical analysis of (P). Note that the implicit Euler time-stepping does not seem to be a suitable choice for (P) as it results in a second-order elliptic curl-curl VI at every time step. Therefore, at every time step, one requires a nonsmooth method such as the semismooth Newton method for solving the VI problem, leading to highly inefficient computational efforts in the case of fine time discretization. For this reason, our discretization does not employ the implicit Euler method. To describe our method, let us begin by introducing a partition of the time interval $[0,T]$ as follows: Given $N \in \mathbb{N}$, we set

$$\tau \coloneqq T/N, \quad 0 = t_0 < t_{\frac{1}{2}} < t_1 < \cdots < t_{N-\frac{1}{2}} < t_N = T, \quad t_n \coloneqq n\tau \quad \forall n \in \{0,\ldots,N\},$$

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and intermediate time steps

$$t_{n-\frac{1}{2}} := \frac{t_n + t_{n-1}}{2} = t_n - \frac{\tau}{2} \quad \forall n \in \{1, \dots, N\}.$$

Motivated by the leapfrog (Yee) time-stepping [21] (cf. Li [11], Li, Wang Waters, and Machorro [12], Cohen and Monk [6], and Monk [17]), we consider the following:

- the Ampère–Maxwell variational inequality in (P) at the intermediate time steps $t_{n-\frac{1}{2}}$;
- the Faraday equation in (P) at the time steps $t_n$;

and make use of the central difference approximations

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t_{n-\frac{1}{2}}) \approx \frac{\boldsymbol{E}(t_n) - \boldsymbol{E}(t_{n-1})}{\tau}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}(t_n) \approx \frac{\boldsymbol{H}(t_{n+\frac{1}{2}}) - \boldsymbol{H}(t_{n-\frac{1}{2}})}{\tau},$$

and mean value approximations

$$\boldsymbol{E}(t_{n-\frac{1}{2}}) \approx \frac{\boldsymbol{E}(t_n) + \boldsymbol{E}(t_{n-1})}{2}.$$

Then, invoking the piecewise constant finite element space $\mathbf{DG}_h$ (see (2.2)) and the lowest-order Nédélec finite element space $\mathbf{ND}_h$ (see (2.1)) for the spatial discretization of the electric and magnetic fields, respectively, we arrive at

$$(\mathrm{LF}_{N,h}) \qquad \begin{cases} \displaystyle\int_\Omega \epsilon\delta\boldsymbol{E}_h^n \cdot (\boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}}) + \sigma\boldsymbol{E}_h^{n-\frac{1}{2}} \cdot (\boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}}) \\ \qquad - \mathbf{curl}\,\boldsymbol{H}_h^{n-\frac{1}{2}} \cdot (\boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}})\,\mathrm{d}x \\ \geq \displaystyle\int_\Omega \boldsymbol{f}_h^{n-\frac{1}{2}} \cdot (\boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}})\,\mathrm{d}x \quad \forall \boldsymbol{v}_h \in \boldsymbol{K} \cap \mathbf{DG}_h \quad \forall n \in \{1, \dots, N\} \\ \displaystyle\int_\Omega \mu\delta\boldsymbol{H}_h^{n+\frac{1}{2}} \cdot \boldsymbol{w}_h + \boldsymbol{E}_h^n \cdot \mathbf{curl}\,\boldsymbol{w}_h\,\mathrm{d}x = \mathbf{0} \\ \forall \boldsymbol{w}_h \in \mathbf{ND}_h \quad \forall n \in \{1, \dots, N\}, \end{cases}$$

where

$$(1.1) \quad \delta\boldsymbol{E}_h^n := \frac{\boldsymbol{E}_h^n - \boldsymbol{E}_h^{n-1}}{\tau}, \quad \delta\boldsymbol{H}_h^{n+\frac{1}{2}} := \frac{\boldsymbol{H}_h^{n+\frac{1}{2}} - \boldsymbol{H}_h^{n-\frac{1}{2}}}{\tau}, \quad \boldsymbol{E}_h^{n-\frac{1}{2}} := \frac{\boldsymbol{E}_h^n + \boldsymbol{E}_h^{n-1}}{2}$$

for all $n \in \{1, \dots, N\}$. Furthermore, $\boldsymbol{E}_h^0 \in \mathbf{DG}_h$, $\boldsymbol{H}_h^{\frac{1}{2}} \in \mathbf{ND}_h$, and $\boldsymbol{f}_h^{n-\frac{1}{2}} \in \mathbf{DG}_h$ are given proper FE approximations specified as in (2.8). Now, to complete the discrete scheme, we have to properly include the obstacle structure $\boldsymbol{K}$ in the discrete system. We propose applying the pointwise electric constraint at *the intermediate time steps* $t_{n-\frac{1}{2}}$ (instead of at the time steps $t_n$), i.e.,

$$(1.2) \qquad \boldsymbol{E}_h^{n-\frac{1}{2}} \in \boldsymbol{K} \cap \mathbf{DG}_h \quad \forall n \in \{1, \dots, N\}.$$

The choice (1.2) is of paramount importance to obtain an efficiently computable explicit formula for the discrete electric field (Theorem 2.2). Thus, differently from the implicit Euler method, the numerical realization of our discretization does not require an additional nonlinear solver for solving the underlying VI. Altogether, utilizing

$$(1.3) \qquad \delta\boldsymbol{E}_h^n = \frac{2}{\tau}\left(\boldsymbol{E}_h^{n-\frac{1}{2}} - \boldsymbol{E}_h^{n-1}\right) \quad \forall n \in \{1, \dots, N\}$$

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and (1.1)–(1.2) in ($\mathrm{LF}_{N,h}$), we finally end up with the following fully discrete FEM:

$$(\mathrm{P}_{N,h})\qquad \begin{cases} \text{Find } \{(\boldsymbol{E}_h^{n-\frac{1}{2}}, \boldsymbol{H}_h^{n+\frac{1}{2}})\}_{n=1}^N \subset (\boldsymbol{K} \cap \mathbf{DG}_h) \times \mathbf{ND}_h \text{ such that} \\ \displaystyle\int_\Omega \left(\frac{2\epsilon}{\tau} + \sigma\right) \boldsymbol{E}_h^{n-\frac{1}{2}} \cdot (\boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}})\,\mathrm{d}x \\ \displaystyle\geq \int_\Omega \left(\boldsymbol{f}_h^{n-\frac{1}{2}} + \mathbf{curl}\,\boldsymbol{H}_h^{n-\frac{1}{2}} + \frac{2\epsilon}{\tau}\boldsymbol{E}_h^{n-1}\right) \cdot (\boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}})\,\mathrm{d}x \\ \forall \boldsymbol{v}_h \in \boldsymbol{K} \cap \mathbf{DG}_h \quad \forall n \in \{1, \ldots, N\}, \\ \boldsymbol{E}_h^n = 2\boldsymbol{E}_h^{n-\frac{1}{2}} - \boldsymbol{E}_h^{n-1}, \\ \displaystyle\int_\Omega \frac{\mu}{\tau}\boldsymbol{H}_h^{n+\frac{1}{2}} \cdot \boldsymbol{w}_h + \boldsymbol{E}_h^n \cdot \mathbf{curl}\,\boldsymbol{w}_h\,\mathrm{d}x = \int_\Omega \frac{\mu}{\tau}\boldsymbol{H}_h^{n-\frac{1}{2}} \cdot \boldsymbol{w}_h\,\mathrm{d}x \\ \forall \boldsymbol{w}_h \in \mathbf{ND}_h \quad \forall n \in \{1, \ldots, N\}. \end{cases}$$

This paper analyzes the proposed FEM ($\mathrm{P}_{N,h}$) and delivers three main novelties: well posedness, stability, and convergence. The well posedness is obtained by the celebrated result [13] due to our particular choice (1.2), which leads to a computable explicit formula for the (exact) discrete electric field (see Theorem 2.2). The stability analysis relies on an additional $\boldsymbol{H}^1(\Omega)$-regularity assumption for the initial electric field $\boldsymbol{E}_0$. Along with a linear CFL-condition (3.2), it allows us to prove $L^2$-stability for the discrete solutions and the associated difference quotients (1.1) (see Proposition 3.3 and Corollary 3.5). Based on Proposition 3.3, our analysis reveals local $L^2$-stability for $\{\mathbf{curl}\,\boldsymbol{H}_h^{n+1/2}\}$ in the free region $\Omega \setminus \overline{\omega}$, while only $L^1$-stability for $\{\mathbf{curl}\,\boldsymbol{H}_h^{n+1/2}\}$ is achieved in the obstacle region $\omega$ (see Proposition 3.6). This result is somehow justified by the low regularity issue in (P) pointed out earlier: In the free region $\Omega \setminus \overline{\omega}$, we have $\mathbf{curl}\,\boldsymbol{H} \in L^\infty((0,T), \boldsymbol{L}^2(\Omega \setminus \overline{\omega}))$, but there is no a priori knowledge on the $L^2$-regularity of $\mathbf{curl}\,\boldsymbol{H}$ in the obstacle region $\omega$. The lack of the global $L^2$-stability for the rotation field makes the convergence analysis of ($\mathrm{P}_{N,h}$) rather challenging. Our strategy to prove a convergence result (Theorem 4.3) comprises two main stages. First, exploiting the $L^2$-stability estimates (Proposition 3.3 and Corollary 3.5) and the $L^1$-stability result (Proposition 3.6), we derive a convergence result for ($\mathrm{P}_{N,h}$) towards a weaker system (4.28) involving smooth test functions $\boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{\mathcal{C}}_0^\infty(\Omega)$. The second step is to recover the original system (P) from (4.28) by enlarging the feasible smooth test function set to $\boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})$. We realize this part through a mollification process, which requires us to modify the recent result by Ern and Guermond [8] to constraint preserving mollification (see section 4.1).

**1.1. Preliminaries.** Given a real Hilbert space $H$, we denote by $(\cdot,\cdot)_H$ and $\|\cdot\|_H$ its scalar product and induced norm, respectively. In the case of $H = \mathbb{R}^d$, we simply write a dot and $|\cdot|$ for the Euclidean scalar product and norm. Discussing problems of Maxwell-type, there naturally arise function spaces of $\mathbb{R}^3$-valued functions. We will therefore use a bold typeface to indicate them. Let $\boldsymbol{L}^2(\Omega)$ denote the space of all (equivalence classes of) $\mathbb{R}^3$-valued Lebesgue square-integrable functions. Moreover, we introduce the Hilbert space

$$\boldsymbol{H}(\mathbf{curl}) \coloneqq \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega) \mid \mathbf{curl}\,\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)\}$$

endowed with its natural graph norm. Here the **curl**-operator is to be understood in the sense of distributions. Furthermore, let $\boldsymbol{\mathcal{C}}_0^\infty(\Omega)$ denote the space of infinitely differentiable $\mathbb{R}^3$-valued functions with compact support in $\Omega$. The space $\boldsymbol{H}_0(\mathbf{curl})$

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stands for the closure of $\boldsymbol{\mathcal{C}}_0^\infty(\Omega)$ with respect to the $\boldsymbol{H}(\mathbf{curl})$-topology. We recall that $\boldsymbol{H}_0(\mathbf{curl})$ admits the useful characterization

$$\boldsymbol{H}_0(\mathbf{curl}) = \{\boldsymbol{z} \in \boldsymbol{H}(\mathbf{curl}) \mid (\boldsymbol{z}, \mathbf{curl}\,\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} = (\mathbf{curl}\,\boldsymbol{z}, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)}\ \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl})\}. \tag{1.4}$$

See, for instance, [23, Appendix A] for a proof of (1.4). For a given uniformly positive function $\alpha \in L^\infty(\Omega)$, that is, there exists a constant $\underline{\alpha} > 0$ such that

$$\alpha(x) \geq \underline{\alpha} \quad \text{for a.e. } x \in \Omega,$$

we denote by $\boldsymbol{L}^2_\alpha(\Omega)$ the vector space $\boldsymbol{L}^2(\Omega)$ equipped with the weighted scalar product $(\alpha\cdot,\cdot)_{\boldsymbol{L}^2(\Omega)}$. Finally, $C > 0$ represents a generic constant which is independent of $N$ and the spatial discretization parameter $h > 0$. However, this constant may depend on other quantities, including the model parameters, the domain, and the time interval. We close this section by presenting the basic assumption for our analysis.

*Assumption* 1.1. *There exist a family of Lipschitz polyhedral domains* $\{\Omega_j\}_{j=1}^{j_0}$ *in* $\Omega$ *and a subfamily* $\{\Omega_j^\omega\}_{j=1}^{l_0} \subset \{\Omega_j\}_{j=1}^{j_0}$ *such that*

$$\Omega_i \cap \Omega_j = \emptyset \quad \forall i \neq j \in \{1, \ldots, j_0\}, \quad \overline{\Omega} = \bigcup_{j=1}^{j_0} \overline{\Omega}_j, \quad \overline{\omega} = \bigcup_{j=1}^{l_0} \overline{\Omega_j^\omega}.$$

*All material parameters are assumed to be piecewise constants, i.e., there exist real constants* $c_j^\epsilon, c_j^\mu > 0$ *and* $c_j^\sigma \geq 0$ *such that*

$$\epsilon(x) = c_j^\epsilon, \quad \mu(x) = c_j^\mu, \quad \sigma(x) = c_j^\sigma \quad \textit{for a.e. } x \in \Omega_j \textit{ and every } j \in \{1, \ldots, j_0\}.$$

*Furthermore, we denote the lower bounds for* $\epsilon$ *and* $\mu$*, respectively, by* $\underline{\epsilon}, \underline{\mu} \in (0, \infty)$*, i.e.,* $\epsilon(x) \geq \underline{\epsilon}$ *and* $\mu(x) \geq \underline{\mu}$ *hold for a.e.* $x \in \Omega$.

**2. Well posedness.** In all what follows, let Assumption 1.1 be satisfied. Let $\{\mathcal{T}_h\}_{h>0}$ denote a quasi-uniform family of triangulations of $\Omega$ with $h > 0$ standing for the largest diameter of $T \in \mathcal{T}_h$. The triangulation is chosen such that for every $h > 0$, there exists a subfamily $\mathcal{T}_h^\omega$ of $\mathcal{T}_h$ with the property $\bigcup_{T \in \mathcal{T}_h^\omega} T = \overline{\omega}$, and that $\epsilon, \mu$ and $\sigma$ are constant in every $T \in \mathcal{T}_h$. We denote the Nédélec finite element space of the first family [18] (cf. [2]) by

$$\mathbf{ND}_h \coloneqq \{\boldsymbol{v}_h \in \boldsymbol{H}(\mathbf{curl}) \mid \boldsymbol{v}_{h|_T} = a_T + b_T \times \cdot \ \text{ for some } a_T, b_T \in \mathbb{R}^3\ \forall T \in \mathcal{T}_h\}, \tag{2.1}$$

and the piecewise constant finite element space by

$$\mathbf{DG}_h \coloneqq \{\boldsymbol{w}_h \in \boldsymbol{L}^2(\Omega) \mid \boldsymbol{w}_{h|_T} = a_T \ \text{ for some } a_T \in \mathbb{R}^3\ \forall T \in \mathcal{T}_h\}. \tag{2.2}$$

Let us now introduce the standard $\boldsymbol{L}^2(\Omega)$-orthogonal projector onto the space $\mathbf{DG}_h$ by $\boldsymbol{Q}_h \colon \boldsymbol{L}^2(\Omega) \to \mathbf{DG}_h$ defined by

$$\boldsymbol{Q}_h \boldsymbol{v} = \sum_{T \in \mathcal{T}_h} \chi_T \frac{1}{|T|} \int_T \boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{L}^2(\Omega), \tag{2.3}$$

where $\chi_T : \mathbb{R}^3 \to \{0, 1\}$ denotes the characteristic function of $T$. For every $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$, it is well known that $\boldsymbol{Q}_h \boldsymbol{v} \to \boldsymbol{v}$ in $\boldsymbol{L}^2(\Omega)$ as $h \to 0$. Particularly, if $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$, we obtain convergence with a linear rate, i.e.,

$$\|\boldsymbol{Q}_h \boldsymbol{v} - \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \leq Ch \|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \quad \forall h > 0 \quad \forall \boldsymbol{v} \in \boldsymbol{H}^1(\Omega). \tag{2.4}$$

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Further important properties of $\boldsymbol{Q}_h \colon \boldsymbol{L}^2(\Omega) \to \mathbf{DG}_h$ are summarized in the following lemma.

LEMMA 2.1. *Let Assumption* 1.1 *hold. Then,* $\boldsymbol{Q}_h \colon \boldsymbol{L}^2(\Omega) \to \mathbf{DG}_h$ *satisfies*

$$\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \quad \Rightarrow \quad \|\boldsymbol{Q}_h \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \le \|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \quad \forall h > 0, \tag{2.5}$$

$$\boldsymbol{v} \in \boldsymbol{K} \quad \Rightarrow \quad \boldsymbol{Q}_h \boldsymbol{v} \in \boldsymbol{K} \cap \mathbf{DG}_h \quad \forall h > 0, \tag{2.6}$$

$$\boldsymbol{v} \in \boldsymbol{\mathcal{C}}^{0,1}(\overline{\Omega}) \quad \Rightarrow \quad \|\boldsymbol{Q}_h \boldsymbol{v} - \boldsymbol{v}\|_{\boldsymbol{L}^\infty(\Omega)} \le \mathrm{Lip}(\boldsymbol{v}) h \quad \forall h > 0, \tag{2.7}$$

*where* $\mathrm{Lip}(\boldsymbol{v}) > 0$ *denotes the Lipschitz constant of* $\boldsymbol{v} \in \boldsymbol{\mathcal{C}}^{0,1}(\overline{\Omega})$.

*Proof.* The first property (2.5) is an immediate consequence of the definition. Suppose that $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$ satisfies $|\boldsymbol{v}(x)| \le d$ for a.e. $x \in \omega$. Then, for a.e. $y \in \omega$, it follows that

$$|\boldsymbol{Q}_h \boldsymbol{v}(y)| = \left| \sum_{T \in \mathcal{T}_h} \chi_T(y) \frac{1}{|T|} \int_T \boldsymbol{v} \,\mathrm{d}x \right| \le \sum_{T \in \mathcal{T}_h^\omega} \chi_T(y) \frac{1}{|T|} \int_T |\boldsymbol{v}| \,\mathrm{d}x \le d \sum_{T \in \mathcal{T}_h^\omega} \chi_T(y) = d.$$

In conclusion, (2.6) is valid. Now, suppose that $\boldsymbol{v} \in \boldsymbol{\mathcal{C}}^{0,1}(\overline{\Omega})$. Then,

$$\begin{aligned} \|\boldsymbol{Q}_h \boldsymbol{v} - \boldsymbol{v}\|_{\boldsymbol{L}^\infty(\Omega)} &= \operatorname*{ess\,sup}_{y \in \Omega} \left| \sum_{T \in \mathcal{T}_h} \chi_T(y) \frac{1}{|T|} \int_T \boldsymbol{v}(x) \,\mathrm{d}x - \boldsymbol{v}(y) \right| \\ &\le \operatorname*{ess\,sup}_{y \in \Omega} \sum_{T \in \mathcal{T}_h} \chi_T(y) \frac{1}{|T|} \int_T |\boldsymbol{v}(x) - \boldsymbol{v}(y)| \,\mathrm{d}x \le \mathrm{Lip}(\boldsymbol{v}) h. \end{aligned}$$

This completes the proof. ∎

Let us now state the initial discrete values and right-hand side data involved in the scheme $(\mathrm{P}_{N,h})$:

$$\boldsymbol{f}_h^{n-\frac{1}{2}} \coloneqq \boldsymbol{Q}_h \boldsymbol{f}(t_{n-\frac{1}{2}}), \quad \boldsymbol{E}_h^0 = \boldsymbol{Q}_h \boldsymbol{E}_0, \quad \boldsymbol{H}_h^{\frac{1}{2}} = \boldsymbol{\Pi}_h \boldsymbol{H}_0 \quad \forall n \in \{1, \ldots, N\} \; \forall h > 0, \tag{2.8}$$

where $\boldsymbol{\Pi}_h \colon \boldsymbol{H}(\mathbf{curl}) \to \mathbf{ND}_h$ denotes the classical Hilbert projection.

THEOREM 2.2. *Let Assumption* 1.1 *hold. Then, for every* $N \in \mathbb{N}$ *and* $h > 0$, *the fully discrete problem* $(\mathrm{P}_{N,h})$ *admits a unique solution* $\{(\boldsymbol{E}_h^{n-\frac{1}{2}}, \boldsymbol{H}_h^{n+\frac{1}{2}})\}_{n=1}^N \subset (\boldsymbol{K} \cap \mathbf{DG}_h) \times \mathbf{ND}_h$. *In particular,* $\boldsymbol{E}_h^{n-\frac{1}{2}}$ *explicitly comes as*

$$\boldsymbol{E}_h^{n-\frac{1}{2}} = \begin{cases} \dfrac{d \boldsymbol{g}_h^{n-\frac{1}{2}}}{|\boldsymbol{g}_h^{n-\frac{1}{2}}|} & \text{on } \mathcal{M}_h^{n-\frac{1}{2}}, \\ \left( \dfrac{2\epsilon}{\tau} + \sigma \right)^{-1} \boldsymbol{g}_h^{n-\frac{1}{2}} & \text{on } \Omega \setminus \mathcal{M}_h^{n-\frac{1}{2}} \end{cases} \tag{2.9}$$

*with right-hand sides and strict superlevel sets*

$$\begin{aligned} \boldsymbol{g}_h^{n-\frac{1}{2}} &\coloneqq \boldsymbol{f}_h^{n-\frac{1}{2}} + \mathbf{curl}\, \boldsymbol{H}_h^{n-\frac{1}{2}} + \frac{2\epsilon}{\tau} \boldsymbol{E}_h^{n-1}, \\ \mathcal{M}_h^{n-\frac{1}{2}} &\coloneqq \left\{ x \in \omega \;\middle|\; \left( \frac{2\epsilon}{\tau} + \sigma \right)^{-1} |\boldsymbol{g}_h^{n-\frac{1}{2}}(x)| > d \right\}. \end{aligned}$$

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*Proof.* Let $n \in \{1, \dots, N\}$ be arbitrarily fixed. We assume that $(\boldsymbol{E}_h^{n-1}, \boldsymbol{H}_h^{n-\frac{1}{2}})$ is already computed in agreement with $(\mathrm{P}_{N,h})$. By virtue of [13], we obtain the existence of a unique solution $\boldsymbol{E}_h^{n-\frac{1}{2}} \in \boldsymbol{K} \cap \mathbf{DG}_h$ to the $\boldsymbol{L}^2(\Omega)$-elliptic variational inequality

$$\int_\Omega \left(\frac{2\epsilon}{\tau} + \sigma\right) \boldsymbol{E}_h^{n-\frac{1}{2}} \cdot (\boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}})\,\mathrm{d}x \geq \int_\Omega \boldsymbol{g}_h^{n-\frac{1}{2}} \cdot (\boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}})\,\mathrm{d}x \quad \forall \boldsymbol{v}_h \in \boldsymbol{K} \cap \mathbf{DG}_h\,. \tag{2.10}$$

The discrete magnetic field $\boldsymbol{H}_h^{n+\frac{1}{2}} \in \mathbf{ND}_h$ is then obtained by the Lax–Milgram lemma since $\|\cdot\|_{\boldsymbol{L}^2(\Omega)}$ and $\|\cdot\|_{\boldsymbol{H}(\mathbf{curl})}$ are equivalent norms in the finite-dimensional space $\mathbf{ND}_h$. Let us now verify the explicit formula (2.9). Let $\boldsymbol{v}_h \in \boldsymbol{K} \cap \mathbf{DG}_h$. First, it holds that

$$\int_{\mathcal{M}_h^{n-\frac{1}{2}}} \left| \frac{d\left(\frac{2\epsilon}{\tau} + \sigma\right)}{|\boldsymbol{g}_h^{n-\frac{1}{2}}|} - 1 \right| \boldsymbol{g}_h^{n-\frac{1}{2}} \cdot \left(\boldsymbol{v}_h - \frac{d\boldsymbol{g}_h^{n-\frac{1}{2}}}{|\boldsymbol{g}_h^{n-\frac{1}{2}}|}\right) \mathrm{d}x \tag{2.11}$$
$$= \int_{\mathcal{M}_h^{n-\frac{1}{2}}} \left| \frac{d\left(\frac{2\epsilon}{\tau} + \sigma\right)}{|\boldsymbol{g}_h^{n-\frac{1}{2}}|} - 1 \right| \underbrace{\boldsymbol{g}_h^{n-\frac{1}{2}} \cdot \boldsymbol{v}_h}_{\leq d|\boldsymbol{g}_h^{n-\frac{1}{2}}|} \mathrm{d}x - \int_{\mathcal{M}_h^{n-\frac{1}{2}}} \left| \frac{d\left(\frac{2\epsilon}{\tau} + \sigma\right)}{|\boldsymbol{g}_h^{n-\frac{1}{2}}|} - 1 \right| d|\boldsymbol{g}_h^{n-\frac{1}{2}}|\,\mathrm{d}x \leq 0.$$

Since

$$\frac{d\left(\frac{2\epsilon}{\tau} + \sigma\right)}{|\boldsymbol{g}_h^{n-\frac{1}{2}}|} - 1 = \frac{d}{\left(\frac{2\epsilon}{\tau} + \sigma\right)^{-1} |\boldsymbol{g}_h^{n-\frac{1}{2}}|} - 1 < 0 \quad \text{on } \mathcal{M}_h^{n-\frac{1}{2}},$$

multiplying (2.11) by a sign implies

$$\int_{\mathcal{M}_h^{n-\frac{1}{2}}} \left( \frac{d\left(\frac{2\epsilon}{\tau} + \sigma\right)}{|\boldsymbol{g}_h^{n-\frac{1}{2}}|} - 1 \right) \boldsymbol{g}_h^{n-\frac{1}{2}} \cdot \left(\boldsymbol{v}_h - \frac{d\boldsymbol{g}_h^{n-\frac{1}{2}}}{|\boldsymbol{g}_h^{n-\frac{1}{2}}|}\right) \mathrm{d}x \geq 0. \tag{2.12}$$

Now, rearrangement in (2.12) yields

$$\int_{\mathcal{M}_h^{n-\frac{1}{2}}} \left(\frac{2\epsilon}{\tau} + \sigma\right) \frac{d\boldsymbol{g}_h^{n-\frac{1}{2}}}{|\boldsymbol{g}_h^{n-\frac{1}{2}}|} \cdot \left(\boldsymbol{v}_h - \frac{d\boldsymbol{g}_h^{n-\frac{1}{2}}}{|\boldsymbol{g}_h^{n-\frac{1}{2}}|}\right) \mathrm{d}x \geq \int_{\mathcal{M}_h^{n-\frac{1}{2}}} \boldsymbol{g}_h^{n-\frac{1}{2}} \cdot \left(\boldsymbol{v}_h - \frac{d\boldsymbol{g}_h^{n-\frac{1}{2}}}{|\boldsymbol{g}_h^{n-\frac{1}{2}}|}\right) \mathrm{d}x. \tag{2.13}$$

By construction, for the set $\Omega \setminus \mathcal{M}_h^{n-\frac{1}{2}}$ there is nothing to show. As a conclusion, $\boldsymbol{E}_h^{n-\frac{1}{2}}$ as stated in (2.9) is the unique solution to (2.10). ∎

**3. Stability.** From the classical inverse estimate for finite-dimensional subspaces of $H^1(\Omega)$ (see [1]) and the continuous embedding $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{H}(\mathbf{curl})$, we obtain an inverse estimate for the space $\mathbf{ND}_h$. To be specific, there exists a constant $C_{\mathrm{inv}} > 0$ such that

$$\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{C_{\mathrm{inv}}}{h} \|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{v} \in \mathbf{ND}_h\,. \tag{3.1}$$

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*Assumption* 3.1.

(i) *The linear CFL-condition*

$$\tau \leq \frac{1}{2c_\nu C_{\text{inv}}} h \tag{3.2}$$

*holds true. Here,* $c_\nu := 1/\sqrt{\underline{\epsilon\mu}}$ *denotes the uniform lower bound for the wave propagation speed in* $\Omega$.

(i) *The initial electromagnetic field* $(\boldsymbol{E}_0, \boldsymbol{H}_0) \in (\boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})) \times \boldsymbol{H}(\mathbf{curl})$ *is assumed to additionally satisfy* $\boldsymbol{E}_0 \in \boldsymbol{H}^1(\Omega)$.

Both the CFL-condition (3.2) and $\boldsymbol{E}_0 \in \boldsymbol{H}^1(\Omega)$ serve as the fundamentals for our stability analysis. In view of (2.4), the corresponding discrete initial value $\boldsymbol{E}_h^0 = \boldsymbol{Q}_h \boldsymbol{E}_0$ satisfies

$$\|\boldsymbol{E}_h^0 - \boldsymbol{E}_0\|_{\boldsymbol{L}^2(\Omega)} \leq Ch \|\boldsymbol{E}_0\|_{\boldsymbol{H}^1(\Omega)} \quad \forall h > 0. \tag{3.3}$$

In what follows, we mainly take advantage of the structure $(\text{LF}_{N,h})$, which is by the construction automatically satisfied by the unique solution to $(\text{P}_{N,h})$.

Lemma 3.2. *Let Assumptions* 1.1 *and* 3.1 *hold. Then, there exists a constant* $C > 0$ *such that for all* $h > 0$ *and* $N \in \mathbb{N}$ *the unique solution to* $(\text{P}_{N,h})$ *satisfies*

$$\|\delta \boldsymbol{E}_h^1\|_{\boldsymbol{L}^2(\Omega)} + \|\delta \boldsymbol{H}_h^{\frac{3}{2}}\|_{\boldsymbol{L}^2(\Omega)} \leq C. \tag{3.4}$$

*Proof.* Let $h > 0$ and $N \in \mathbb{N}$ be arbitrarily fixed. We start by setting $\boldsymbol{v}_h = \mathbf{0}$ in $(\text{LF}_{N,h})$ to obtain that

$$\int_\Omega \epsilon \delta \boldsymbol{E}_h^1 \cdot \boldsymbol{E}_h^{\frac{1}{2}} + \sigma \boldsymbol{E}_h^{\frac{1}{2}} \cdot \boldsymbol{E}_h^{\frac{1}{2}} - \mathbf{curl}\, \boldsymbol{H}_h^{\frac{1}{2}} \cdot \boldsymbol{E}_h^{\frac{1}{2}} \,\mathrm{d}x \leq \int_\Omega \boldsymbol{f}_h^{\frac{1}{2}} \cdot \boldsymbol{E}_h^{\frac{1}{2}} \,\mathrm{d}x. \tag{3.5}$$

Multiplying the above inequality by $\tau$, applying (1.3), and using that $\sigma$ is nonnegative,

$$\int_\Omega 2\epsilon (\boldsymbol{E}_h^{\frac{1}{2}} - \boldsymbol{E}_h^0) \cdot \boldsymbol{E}_h^{\frac{1}{2}} - \tau\, \mathbf{curl}\, \boldsymbol{H}_h^{\frac{1}{2}} \cdot \boldsymbol{E}_h^{\frac{1}{2}} \,\mathrm{d}x \leq \int_\Omega \tau \boldsymbol{f}_h^{\frac{1}{2}} \cdot \boldsymbol{E}_h^{\frac{1}{2}} \,\mathrm{d}x,$$

from which we deduce that

$$2\underline{\epsilon} \|\boldsymbol{E}_h^{\frac{1}{2}}\|^2_{\boldsymbol{L}^2(\Omega)} \leq \|\tau \boldsymbol{f}_h^{\frac{1}{2}} + 2\epsilon \boldsymbol{E}_h^0 + \tau\, \mathbf{curl}\, \boldsymbol{H}_h^{\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{E}_h^{\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)}. \tag{3.6}$$

Now, by construction of $\boldsymbol{f}_h^{\frac{1}{2}}, \boldsymbol{H}_h^{\frac{1}{2}}$, and $\boldsymbol{E}_h^0$, the first norm on the right-hand side of (3.6) is uniformly bounded. As a consequence, it follows that

$$\|\boldsymbol{E}_h^{\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} \leq C \tag{3.7}$$

with $C > 0$, independent of $N$ and $h$. Let us mention that according to Assumption 3.1 along with Lemma 2.1 and (2.8), the field $\boldsymbol{E}_h^0$ is admissible, i.e., $\boldsymbol{E}_h^0 \in \boldsymbol{K} \cap \mathbf{DG}_h$. Thus, we may set $\boldsymbol{v}_h = \boldsymbol{E}_h^0$ in $(\text{LF}_{N,h})$ to conclude that $\delta \boldsymbol{E}_h^1$ admits a uniform bound in $\boldsymbol{L}^2(\Omega)$. This way we receive after multiplication with $-\frac{2}{\tau}$ together with applying (1.3) that

$$\int_\Omega \epsilon \delta \boldsymbol{E}_h^1 \cdot \delta \boldsymbol{E}_h^1 + \sigma \boldsymbol{E}_h^{\frac{1}{2}} \cdot \delta \boldsymbol{E}_h^1 - \mathbf{curl}\, \boldsymbol{H}_h^{\frac{1}{2}} \cdot \delta \boldsymbol{E}_h^1 \,\mathrm{d}x \leq \int_\Omega \boldsymbol{f}_h^{\frac{1}{2}} \cdot \delta \boldsymbol{E}_h^1 \,\mathrm{d}x,$$

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and therefore

$$\underline{\epsilon}\|\delta \boldsymbol{E}_h^1\|^2_{\boldsymbol{L}^2(\Omega)} \le \|\boldsymbol{f}_h^{\frac{1}{2}} - \sigma \boldsymbol{E}_h^{\frac{1}{2}} + \mathbf{curl}\, \boldsymbol{H}_h^{\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)}\|\delta \boldsymbol{E}_h^1\|_{\boldsymbol{L}^2(\Omega)}.$$

Utilizing (3.7) then leads to

$$\|\delta \boldsymbol{E}_h^1\|^2_{\boldsymbol{L}^2(\Omega)} \le C \tag{3.8}$$

with $C > 0$, independent of $N$ and $h$. Let us now prove that $\delta \boldsymbol{H}_h^{\frac{3}{2}}$ admits a uniform bound in $\boldsymbol{L}^2(\Omega)$. We start by setting $\boldsymbol{w}_h = \delta \boldsymbol{H}_h^{\frac{3}{2}}$ in $(\mathrm{LF}_{N,h})$ to derive

$$\int_\Omega \mu \delta \boldsymbol{H}_h^{\frac{3}{2}} \cdot \delta \boldsymbol{H}_h^{\frac{3}{2}} + \boldsymbol{E}_h^1 \cdot \mathbf{curl}\, \delta \boldsymbol{H}_h^{\frac{3}{2}} \,\mathrm{d}x = \mathbf{0}. \tag{3.9}$$

To complete the proof, we employ (3.9) to estimate

$$\begin{aligned}
&\underline{\mu}\|\delta \boldsymbol{H}_h^{\frac{3}{2}}\|^2_{\boldsymbol{L}^2(\Omega)} \le \left|\int_\Omega \boldsymbol{E}_h^1 \cdot \mathbf{curl}\, \delta \boldsymbol{H}_h^{\frac{3}{2}} \,\mathrm{d}x\right| \\
\underbrace{\le}_{(1.4)}\;& \left|\int_\Omega (\boldsymbol{E}_h^1 - \boldsymbol{E}_h^0) \cdot \mathbf{curl}\, \delta \boldsymbol{H}_h^{\frac{3}{2}} \,\mathrm{d}x\right| + \left|\int_\Omega (\boldsymbol{E}_h^0 - \boldsymbol{E}_0) \cdot \mathbf{curl}\, \delta \boldsymbol{H}_h^{\frac{3}{2}} \,\mathrm{d}x\right| \\
&+ \left|\int_\Omega \mathbf{curl}\, \boldsymbol{E}_0 \cdot \delta \boldsymbol{H}_h^{\frac{3}{2}} \,\mathrm{d}x\right| \\
\le\;& \tau\|\delta \boldsymbol{E}_h^1\|_{\boldsymbol{L}^2(\Omega)}\|\,\mathbf{curl}\, \delta \boldsymbol{H}_h^{\frac{3}{2}}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{E}_h^0 - \boldsymbol{E}_0\|_{\boldsymbol{L}^2(\Omega)}\|\,\mathbf{curl}\, \delta \boldsymbol{H}_h^{\frac{3}{2}}\|_{\boldsymbol{L}^2(\Omega)} \\
&+ \|\,\mathbf{curl}\, \boldsymbol{E}_0\|_{\boldsymbol{L}^2(\Omega)}\|\delta \boldsymbol{H}_h^{\frac{3}{2}}\|_{\boldsymbol{L}^2(\Omega)} \\
\underbrace{\le}_{(3.1)}\;& \left(\tau\|\delta \boldsymbol{E}_h^1\|_{\boldsymbol{L}^2(\Omega)}\frac{C_{\mathrm{inv}}}{h} + \|\boldsymbol{E}_h^0 - \boldsymbol{E}_0\|_{\boldsymbol{L}^2(\Omega)}\frac{C_{\mathrm{inv}}}{h} + \|\,\mathbf{curl}\, \boldsymbol{E}_0\|_{\boldsymbol{L}^2(\Omega)}\right)\|\delta \boldsymbol{H}_h^{\frac{3}{2}}\|_{\boldsymbol{L}^2(\Omega)} \\
\underbrace{\le}_{(3.2),(3.3)}\;& \left(\frac{\sqrt{\underline{\epsilon}}\sqrt{\underline{\mu}}}{2}\|\delta \boldsymbol{E}_h^1\|_{\boldsymbol{L}^2(\Omega)} + C C_{\mathrm{inv}}\|\boldsymbol{E}_0\|_{\boldsymbol{H}^1(\Omega)} + \|\,\mathbf{curl}\, \boldsymbol{E}_0\|_{\boldsymbol{L}^2(\Omega)}\right)\|\delta \boldsymbol{H}_h^{\frac{3}{2}}\|_{\boldsymbol{L}^2(\Omega)} \\
\underbrace{\le}_{(3.8)}\;& C\|\delta \boldsymbol{H}_h^{\frac{3}{2}}\|_{\boldsymbol{L}^2(\Omega)}
\end{aligned}$$

with a constant $C > 0$, independent of $N$ and $h$. ∎

Proposition 3.3. *Let Assumptions* 1.1 *and* 3.1 *be satisfied. Then, there exists a constant $C > 0$ such that for every $N \in \mathbb{N}$ with $N \ge 2$ and $h > 0$ the unique solution to* $(\mathrm{P}_{N,h})$ *satisfies*

$$\max_{n\in\{2,\dots,N\}} \left[\|\delta \boldsymbol{E}_h^n\|_{\boldsymbol{L}^2(\Omega)} + \|\delta \boldsymbol{H}_h^{n-\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)}\right] \le C. \tag{3.10}$$

*Proof.* Let $N \in \mathbb{N}$ with $N \ge 2$ and $h > 0$ be arbitrarily fixed. We choose $n_0 \in \{2, \dots, N\}$ and $n \in \{2, \dots, n_0\}$. Let us first note that it holds that

$$\boldsymbol{E}_h^{n-\frac{3}{2}} - \boldsymbol{E}_h^{n-\frac{1}{2}} = \frac{\boldsymbol{E}_h^{n-1} + \boldsymbol{E}_h^{n-2}}{2} - \frac{\boldsymbol{E}_h^{n} + \boldsymbol{E}_h^{n-1}}{2} = -\frac{\tau}{2}(\delta \boldsymbol{E}_h^n + \delta \boldsymbol{E}_h^{n-1}). \tag{3.11}$$

By construction, both the fields $\boldsymbol{E}_h^{n-\frac{1}{2}}$ and $\boldsymbol{E}_h^{n-\frac{3}{2}}$ are admissible, i.e., they belong to $\boldsymbol{K} \cap \mathbf{DG}_h$. Hence we are able to test with $\boldsymbol{E}_h^{n-\frac{3}{2}}$ (resp., with $\boldsymbol{E}_h^{n-\frac{1}{2}}$) in the $n$th

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inequality of $(\mathrm{LF}_{N,h})$ (resp., the $(n-1)$th inequality of $(\mathrm{LF}_{N,h})$) and thus obtain by multiplication with $-\frac{2}{\tau}$ together with (3.11) that

$$\begin{aligned}&\int_\Omega \epsilon\delta \boldsymbol{E}_h^n \cdot (\delta \boldsymbol{E}_h^n + \delta \boldsymbol{E}_h^{n-1}) + \sigma \boldsymbol{E}_h^{n-\frac{1}{2}} \cdot (\delta \boldsymbol{E}_h^n + \delta \boldsymbol{E}_h^{n-1}) \\ &\quad - \mathbf{curl}\, \boldsymbol{H}_h^{n-\frac{1}{2}} \cdot (\delta \boldsymbol{E}_h^n + \delta \boldsymbol{E}_h^{n-1})\,\mathrm{d}x \le \int_\Omega \boldsymbol{f}_h^{n-\frac{1}{2}} \cdot (\delta \boldsymbol{E}_h^n + \delta \boldsymbol{E}_h^{n-1})\,\mathrm{d}x\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}&-\int_\Omega \epsilon\delta \boldsymbol{E}_h^{n-1} \cdot (\delta \boldsymbol{E}_h^n + \delta \boldsymbol{E}_h^{n-1}) + \sigma \boldsymbol{E}_h^{n-\frac{3}{2}} \cdot (\delta \boldsymbol{E}_h^n + \delta \boldsymbol{E}_h^{n-1}) \\ &\quad - \mathbf{curl}\, \boldsymbol{H}_h^{n-\frac{3}{2}} \cdot (\delta \boldsymbol{E}_h^n + \delta \boldsymbol{E}_h^{n-1})\,\mathrm{d}x \le -\int_\Omega \boldsymbol{f}_h^{n-\frac{3}{2}} \cdot (\delta \boldsymbol{E}_h^n + \delta \boldsymbol{E}_h^{n-1})\,\mathrm{d}x.\end{aligned} \tag{3.13}$$

Adding together the inequalities (3.12) and (3.13) as well as using $\sigma$ being nonnegative yields

$$\begin{aligned}&\int_\Omega \epsilon(\delta \boldsymbol{E}_h^n - \delta \boldsymbol{E}_h^{n-1}) \cdot (\delta \boldsymbol{E}_h^n + \delta \boldsymbol{E}_h^{n-1}) - \mathbf{curl}(\boldsymbol{H}_h^{n-\frac{1}{2}} - \boldsymbol{H}_h^{n-\frac{3}{2}}) \cdot (\delta \boldsymbol{E}_h^n + \delta \boldsymbol{E}_h^{n-1})\,\mathrm{d}x \\ &\qquad \le \int_\Omega (\boldsymbol{f}_h^{n-\frac{1}{2}} - \boldsymbol{f}_h^{n-\frac{3}{2}}) \cdot (\delta \boldsymbol{E}_h^n + \delta \boldsymbol{E}_h^{n-1})\,\mathrm{d}x.\end{aligned} \tag{3.14}$$

We sum up the inequality (3.14) over $\{2, \dots, n_0\}$:

$$\begin{aligned}&\sum_{n=2}^{n_0} \int_\Omega \epsilon(\delta \boldsymbol{E}_h^n - \delta \boldsymbol{E}_h^{n-1}) \cdot (\delta \boldsymbol{E}_h^n + \delta \boldsymbol{E}_h^{n-1})\,\mathrm{d}x \\ &\quad - \sum_{n=2}^{n_0} \int_\Omega \mathbf{curl}(\boldsymbol{H}_h^{n-\frac{1}{2}} - \boldsymbol{H}_h^{n-\frac{3}{2}}) \cdot (\delta \boldsymbol{E}_h^n + \delta \boldsymbol{E}_h^{n-1})\,\mathrm{d}x \\ &\le \sum_{n=2}^{n_0} \int_\Omega (\boldsymbol{f}_h^{n-\frac{1}{2}} - \boldsymbol{f}_h^{n-\frac{3}{2}}) \cdot (\delta \boldsymbol{E}_h^n + \delta \boldsymbol{E}_h^{n-1})\,\mathrm{d}x.\end{aligned} \tag{3.15}$$

For the left-hand side of (3.15), we have

$$\begin{aligned}\sum_{n=2}^{n_0} \int_\Omega \epsilon(\delta \boldsymbol{E}_h^n - \delta \boldsymbol{E}_h^{n-1}) \cdot (\delta \boldsymbol{E}_h^n + \delta \boldsymbol{E}_h^{n-1})\,\mathrm{d}x &= \sum_{n=2}^{n_0} \|\delta \boldsymbol{E}_h^n\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} - \|\delta \boldsymbol{E}_h^{n-1}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} \\ &= \|\delta \boldsymbol{E}_h^{n_0}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} - \|\delta \boldsymbol{E}_h^1\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}\end{aligned} \tag{3.16}$$

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and

$$
\begin{aligned}
&-\sum_{n=2}^{n_0}\int_\Omega \mathbf{curl}(\boldsymbol{H}_h^{n-\frac{1}{2}}-\boldsymbol{H}_h^{n-\frac{3}{2}})\cdot(\delta\boldsymbol{E}_h^n+\delta\boldsymbol{E}_h^{n-1})\,\mathrm{d}x\\
&=-\tau\sum_{n=2}^{n_0}\int_\Omega \mathbf{curl}\,\delta\boldsymbol{H}_h^{n-\frac{1}{2}}\cdot(\delta\boldsymbol{E}_h^n+\delta\boldsymbol{E}_h^{n-1})\,\mathrm{d}x\\
&=-\tau\sum_{n=2}^{n_0-1}\int_\Omega \mathbf{curl}\,\delta\boldsymbol{H}_h^{n-\frac{1}{2}}\cdot\delta\boldsymbol{E}_h^n\,\mathrm{d}x-\tau\int_\Omega \mathbf{curl}\,\delta\boldsymbol{H}_h^{n_0-\frac{1}{2}}\cdot\delta\boldsymbol{E}_h^{n_0}\,\mathrm{d}x\\
&\quad-\tau\sum_{n=3}^{n_0}\int_\Omega \mathbf{curl}\,\delta\boldsymbol{H}_h^{n-\frac{1}{2}}\cdot\delta\boldsymbol{E}_h^{n-1}\,\mathrm{d}x-\tau\int_\Omega \mathbf{curl}\,\delta\boldsymbol{H}_h^{\frac{3}{2}}\cdot\delta\boldsymbol{E}_h^{1}\,\mathrm{d}x\\
&=-\tau\sum_{n=2}^{n_0-1}\int_\Omega \mathbf{curl}(\delta\boldsymbol{H}_h^{n+\frac{1}{2}}+\delta\boldsymbol{H}_h^{n-\frac{1}{2}})\cdot\delta\boldsymbol{E}_h^n\,\mathrm{d}x\\
&\quad-\tau\int_\Omega \mathbf{curl}\,\delta\boldsymbol{H}_h^{n_0-\frac{1}{2}}\cdot\delta\boldsymbol{E}_h^{n_0}\,\mathrm{d}x-\tau\int_\Omega \mathbf{curl}\,\delta\boldsymbol{H}_h^{\frac{3}{2}}\cdot\delta\boldsymbol{E}_h^{1}\,\mathrm{d}x.
\end{aligned}\tag{3.17}
$$

Now, by the definition of the discrete difference quotients, the first summand on the right-hand side of (3.17) can be rewritten as

$$
\begin{aligned}
&-\tau\sum_{n=2}^{n_0-1}\int_\Omega \mathbf{curl}(\delta\boldsymbol{H}_h^{n+\frac{1}{2}}+\delta\boldsymbol{H}_h^{n-\frac{1}{2}})\cdot\delta\boldsymbol{E}_h^n\,\mathrm{d}x\\
&=-\sum_{n=2}^{n_0-1}\int_\Omega \mathbf{curl}(\delta\boldsymbol{H}_h^{n+\frac{1}{2}}+\delta\boldsymbol{H}_h^{n-\frac{1}{2}})\cdot\boldsymbol{E}_h^n\,\mathrm{d}x\\
&\quad+\sum_{n=2}^{n_0-1}\int_\Omega \mathbf{curl}(\delta\boldsymbol{H}_h^{n+\frac{1}{2}}+\delta\boldsymbol{H}_h^{n-\frac{1}{2}})\cdot\boldsymbol{E}_h^{n-1}\,\mathrm{d}x=:R.
\end{aligned}\tag{3.18}
$$

Testing with $\boldsymbol{w}_h=\delta\boldsymbol{H}_h^{n+\frac{1}{2}}+\delta\boldsymbol{H}_h^{n-\frac{1}{2}}$ in the $n$th (resp., the $(n-1)$th) equality of $(\mathrm{LF}_{N,h})$, we continue with

$$
\begin{aligned}
R&=\sum_{n=2}^{n_0-1}\int_\Omega \mu\delta\boldsymbol{H}_h^{n+\frac{1}{2}}\cdot(\delta\boldsymbol{H}_h^{n+\frac{1}{2}}+\delta\boldsymbol{H}_h^{n-\frac{1}{2}})\,\mathrm{d}x\\
&\quad-\sum_{n=2}^{n_0-1}\int_\Omega \mu\delta\boldsymbol{H}_h^{n-\frac{1}{2}}\cdot(\delta\boldsymbol{H}_h^{n+\frac{1}{2}}+\delta\boldsymbol{H}_h^{n-\frac{1}{2}})\,\mathrm{d}x\\
&=\sum_{n=2}^{n_0-1}\int_\Omega \mu(\delta\boldsymbol{H}_h^{n+\frac{1}{2}}-\delta\boldsymbol{H}_h^{n-\frac{1}{2}})\cdot(\delta\boldsymbol{H}_h^{n+\frac{1}{2}}+\delta\boldsymbol{H}_h^{n-\frac{1}{2}})\,\mathrm{d}x\\
&=\sum_{n=2}^{n_0-1}\|\delta\boldsymbol{H}_h^{n+\frac{1}{2}}\|^2_{\boldsymbol{L}^2_\mu(\Omega)}-\|\delta\boldsymbol{H}_h^{n-\frac{1}{2}}\|^2_{\boldsymbol{L}^2_\mu(\Omega)}=\|\delta\boldsymbol{H}_h^{n_0-\frac{1}{2}}\|^2_{\boldsymbol{L}^2_\mu(\Omega)}-\|\delta\boldsymbol{H}_h^{\frac{3}{2}}\|^2_{\boldsymbol{L}^2_\mu(\Omega)}.
\end{aligned}\tag{3.19}
$$

Let us now consider the inverse estimate (3.1) and the imposed CFL-condition (3.2)

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to obtain that

$$\begin{aligned}&\tau\int_\Omega \mathbf{curl}\,\delta \boldsymbol{H}_h^{n_0-\frac{1}{2}}\cdot\delta\boldsymbol{E}_h^{n_0}\,\mathrm{d}x \le \tau\|\mathbf{curl}\,\delta\boldsymbol{H}_h^{n_0-\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)}\|\delta\boldsymbol{E}_h^{n_0}\|_{\boldsymbol{L}^2(\Omega)}\\ &\le \frac{\tau}{h}\frac{C_{\mathrm{inv}}}{\sqrt{\underline{\epsilon}}\sqrt{\underline{\mu}}}\|\delta\boldsymbol{H}_h^{n_0-\frac{1}{2}}\|_{\boldsymbol{L}^2_\mu(\Omega)}\|\delta\boldsymbol{E}_h^{n_0}\|_{\boldsymbol{L}^2_\epsilon(\Omega)} \le \frac{1}{2}\|\delta\boldsymbol{H}_h^{n_0-\frac{1}{2}}\|_{\boldsymbol{L}^2_\mu(\Omega)}\|\delta\boldsymbol{E}_h^{n_0}\|_{\boldsymbol{L}^2_\epsilon(\Omega)}\\ &\le \frac{1}{4}\|\delta\boldsymbol{H}_h^{n_0-\frac{1}{2}}\|^2_{\boldsymbol{L}^2_\mu(\Omega)} + \frac{1}{4}\|\delta\boldsymbol{E}_h^{n_0}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}.\end{aligned}\tag{3.20}$$

By an analogous argumentation, we infer that

$$\tau\int_\Omega \mathbf{curl}\,\delta\boldsymbol{H}_h^{\frac{3}{2}}\cdot\delta\boldsymbol{E}_h^1\,\mathrm{d}x \le \frac{1}{4}\|\delta\boldsymbol{H}_h^{\frac{3}{2}}\|^2_{\boldsymbol{L}^2_\mu(\Omega)} + \frac{1}{4}\|\delta\boldsymbol{E}_h^1\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}.\tag{3.21}$$

Finally let us estimate the right-hand side of (3.15) as follows:

$$\begin{aligned}&\sum_{n=2}^{n_0}\int_\Omega(\boldsymbol{f}_h^{n-\frac{1}{2}}-\boldsymbol{f}_h^{n-\frac{3}{2}})\cdot(\delta\boldsymbol{E}_h^n+\delta\boldsymbol{E}_h^{n-1})\,\mathrm{d}x\\ &\le \sum_{n=2}^{n_0}\|\boldsymbol{f}_h^{n-\frac{1}{2}}-\boldsymbol{f}_h^{n-\frac{3}{2}}\|_{\boldsymbol{L}^2(\Omega)}\|\delta\boldsymbol{E}_h^n+\delta\boldsymbol{E}_h^{n-1}\|_{\boldsymbol{L}^2(\Omega)}\\ &\le \sum_{n=2}^{n_0}\frac{4N}{\underline{\epsilon}}\|\boldsymbol{f}_h^{n-\frac{1}{2}}-\boldsymbol{f}_h^{n-\frac{3}{2}}\|^2_{\boldsymbol{L}^2(\Omega)} + \sum_{n=2}^{n_0}\frac{\underline{\epsilon}}{16N}\left(\|\delta\boldsymbol{E}_h^n\|_{\boldsymbol{L}^2(\Omega)}+\|\delta\boldsymbol{E}_h^{n-1}\|_{\boldsymbol{L}^2(\Omega)}\right)^2\\ &\underbrace{\le}_{(2.5)} \frac{4L^2T^2}{\underline{\epsilon}} + \sum_{n=2}^{n_0}\frac{\underline{\epsilon}}{8N}\left(\|\delta\boldsymbol{E}_h^n\|^2_{\boldsymbol{L}^2(\Omega)}+\|\delta\boldsymbol{E}_h^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)}\right)\\ &\le \frac{4L^2T^2}{\underline{\epsilon}} + \sum_{n=1}^{n_0}\frac{1}{4N}\|\delta\boldsymbol{E}_h^n\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}\\ &\le \frac{4L^2T^2}{\underline{\epsilon}} + \frac{1}{4}\|\delta\boldsymbol{E}_h^{n_0}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} + \sum_{n=1}^{n_0-1}\frac{1}{4N}\|\delta\boldsymbol{E}_h^n\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)},\end{aligned}\tag{3.22}$$

where $L>0$ denotes the Lipschitz constant of $\boldsymbol{f}\in C^{0,1}([0,T],\boldsymbol{L}^2(\Omega))$. Applying (3.16)–(3.22) to (3.15) now yields

$$\begin{aligned}&\frac{1}{2}\|\delta\boldsymbol{E}_h^{n_0}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} + \frac{3}{4}\|\delta\boldsymbol{H}_h^{n_0-\frac{1}{2}}\|^2_{\boldsymbol{L}^2_\mu(\Omega)}\\ &\qquad\le \frac{4L^2T^2}{\underline{\epsilon}} + \frac{5}{4}\|\delta\boldsymbol{E}_h^1\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} + \frac{5}{4}\|\delta\boldsymbol{H}_h^{\frac{3}{2}}\|^2_{\boldsymbol{L}^2_\mu(\Omega)} + \sum_{n=1}^{n_0-1}\frac{1}{4N}\|\delta\boldsymbol{E}_h^n\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}.\end{aligned}\tag{3.23}$$

The discrete version of the Gronwall lemma (see, for example, [5]) together with Lemma 3.2 then leads to

$$\|\delta\boldsymbol{E}_h^{n_0}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} + \|\delta\boldsymbol{H}_h^{n_0-\frac{1}{2}}\|^2_{\boldsymbol{L}^2_\mu(\Omega)} \le C\exp\left(\sum_{n=1}^{n_0-1}\frac{1}{N}\right) \le C\exp(1)$$

or, equivalently,

$$\|\delta\boldsymbol{E}_h^{n_0}\|_{\boldsymbol{L}^2(\Omega)} + \|\delta\boldsymbol{H}_h^{n_0-\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} \le C$$

with $C>0$, independent of $N$ and $h$. This completes the proof. ∎

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*Remark* 3.4. We underline that (3.10) does not guarantee the stability of the term $\|\delta \boldsymbol{H}_h^{N+\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)}$. The stability of this term can be obtained by performing one additional step $N+1$ in $(\mathrm{LF}_{N,h})$ under an appropriate choice for $\boldsymbol{f}_h^{N+\frac{1}{2}}$. However, as we shall see in the upcoming section, the estimate (3.10) is readily sufficient for proving the convergence of the proposed scheme $(\mathrm{LF}_{N,h})$, i.e., without performing one additional step $N+1$ in $(\mathrm{LF}_{N,h})$.

COROLLARY 3.5. *Let Assumptions* 1.1 *and* 3.1 *be satisfied. Then, there exists a constant $C>0$ such that for every $N \in \mathbb{N}$ with $N \geq 2$ and $h>0$ the unique solution to* $(\mathrm{P}_{N,h})$ *satisfies*

$$\max_{n\in\{1,\dots,N\}} \left[\|\boldsymbol{E}_h^n\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{H}_h^{n-\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)}\right] \leq C. \tag{3.24}$$

*Proof.* Using the reversed triangle inequality, it follows by the definition of the difference quotients (1.1) together with Proposition 3.3 and Lemma 3.2 that

$$\frac{1}{\tau}\left(\|\boldsymbol{E}_h^n\|_{\boldsymbol{L}^2(\Omega)} - \|\boldsymbol{E}_h^{n-1}\|_{\boldsymbol{L}^2(\Omega)}\right) \leq \|\delta \boldsymbol{E}_h^n\|_{\boldsymbol{L}^2(\Omega)} \leq C,$$

which implies

$$\|\boldsymbol{E}_h^n\|_{\boldsymbol{L}^2(\Omega)} \leq \tau C + \|\boldsymbol{E}_h^{n-1}\|_{\boldsymbol{L}^2(\Omega)} \quad \forall n \in \{1,\dots,N\}.$$

Using the same argumentation for $\|\boldsymbol{H}_h^{n-\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)}$, we derive iteratively that

$$\begin{aligned}\|\boldsymbol{E}_h^n\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{H}_h^{n-\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} &\leq n\tau C + \|\boldsymbol{E}_h^0\|_{\boldsymbol{L}^2(\Omega)} + (n-1)\tau C + \|\boldsymbol{H}_h^{\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} \\ &\leq 2TC + \|\boldsymbol{E}_h^0\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{H}_h^{\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} \leq C \quad \forall n \in \{1,\dots,N\}\end{aligned} \tag{3.25}$$

with $C>0$, independent of $N$ and $h$. ∎

PROPOSITION 3.6. *Let Assumptions* 1.1 *and* 3.1 *be satisfied. Then, there exists a constant $C>0$ such that for every $N \in \mathbb{N}$ with $N \geq 2$ and $h>0$ the unique solution to* $(\mathrm{P}_{N,h})$ *satisfies*

$$\max_{n\in\{1,\dots,N-1\}} \left[\|\,\mathbf{curl}\,\boldsymbol{H}_h^{n-\frac{1}{2}}\|_{\boldsymbol{L}^1(\omega)} + \|\,\mathbf{curl}\,\boldsymbol{H}_h^{n-\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega\setminus\omega)}\right] \leq C. \tag{3.26}$$

*Proof.* Let $n \in \{1,\dots,N-1\}$ be arbitrarily fixed. We define

$$\begin{aligned}\boldsymbol{z}_h^{n-\frac{1}{2}}(x) &:= \begin{cases} \dfrac{d\,\mathbf{curl}\,\boldsymbol{H}_h^{n-\frac{1}{2}}(x)}{|\,\mathbf{curl}\,\boldsymbol{H}_h^{n-\frac{1}{2}}(x)|} & \text{if } \mathbf{curl}\,\boldsymbol{H}_h^{n-\frac{1}{2}}(x) \neq 0, \\ \mathbf{0} & \text{if } \mathbf{curl}\,\boldsymbol{H}_h^{n-\frac{1}{2}}(x) = 0, \end{cases} \\ \boldsymbol{z}_{h,\omega}^{n-\frac{1}{2}} &:= \begin{cases} \boldsymbol{z}_h^{n-\frac{1}{2}} & \text{on } \omega, \\ \boldsymbol{E}_h^{n-\frac{1}{2}} & \text{on } \Omega\setminus\omega. \end{cases}\end{aligned} \tag{3.27}$$

Obviously, $\boldsymbol{z}_{h,\omega}^{n-\frac{1}{2}}$ is an element of the set $\boldsymbol{K}\cap\mathbf{DG}_h$, and we can therefore set $\boldsymbol{v}_h = \boldsymbol{z}_{h,\omega}^{n-\frac{1}{2}}$ in $(\mathrm{LF}_{N,h})$ to obtain

$$\begin{aligned}&\int_\omega \epsilon\delta\boldsymbol{E}_h^n \cdot (\boldsymbol{z}_h^{n-\frac{1}{2}} - \boldsymbol{E}_h^{n-\frac{1}{2}}) + \sigma\boldsymbol{E}_h^{n-\frac{1}{2}} \cdot (\boldsymbol{z}_h^{n-\frac{1}{2}} - \boldsymbol{E}_h^{n-\frac{1}{2}})\,\mathrm{d}x \\ &\qquad - \int_\omega \mathbf{curl}\,\boldsymbol{H}_h^{n-\frac{1}{2}} \cdot (\boldsymbol{z}_h^{n-\frac{1}{2}} - \boldsymbol{E}_h^{n-\frac{1}{2}})\,\mathrm{d}x \geq \int_\omega \boldsymbol{f}_h^{n-\frac{1}{2}} \cdot (\boldsymbol{z}_h^{n-\frac{1}{2}} - \boldsymbol{E}_h^{n-\frac{1}{2}})\,\mathrm{d}x.\end{aligned} \tag{3.28}$$

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Altogether, in view of Proposition 3.3 and Corollary 3.5, we obtain that

$$
\begin{aligned}
&d\|\operatorname{\mathbf{curl}} \boldsymbol{H}_h^{n-\frac{1}{2}}\|_{\boldsymbol{L}^1(\omega)} \underbrace{=}_{(3.27)} \int_\omega \operatorname{\mathbf{curl}} \boldsymbol{H}_h^{n-\frac{1}{2}} \cdot \boldsymbol{z}_h^{n-\frac{1}{2}} \,\mathrm{d}x \\
\underbrace{\leq}_{(3.28)} &\int_\omega \left(\epsilon\delta \boldsymbol{E}_h^n + \sigma \boldsymbol{E}_h^{n-\frac{1}{2}} - \boldsymbol{f}_h^{n-\frac{1}{2}}\right) \cdot (\boldsymbol{z}_h^{n-\frac{1}{2}} - \boldsymbol{E}_h^{n-\frac{1}{2}}) \,\mathrm{d}x + \int_\omega \operatorname{\mathbf{curl}} \boldsymbol{H}_h^{n-\frac{1}{2}} \cdot \boldsymbol{E}_h^{n-\frac{1}{2}} \,\mathrm{d}x \\
\leq \;& C + \int_\omega \operatorname{\mathbf{curl}} \boldsymbol{H}_h^{n-\frac{1}{2}} \cdot \boldsymbol{E}_h^{n-\frac{1}{2}} \,\mathrm{d}x \\
= \;& C + \frac{1}{2}\int_\omega \operatorname{\mathbf{curl}} \boldsymbol{H}_h^{n-\frac{1}{2}} \cdot \boldsymbol{E}_h^{n} \,\mathrm{d}x + \frac{1}{2}\int_\omega \operatorname{\mathbf{curl}} \boldsymbol{H}_h^{n-\frac{1}{2}} \cdot \boldsymbol{E}_h^{n-1} \,\mathrm{d}x \\
\underbrace{=}_{(\mathrm{LF}_{N,h})} &C - \frac{1}{2}\int_\omega \mu\delta \boldsymbol{H}_h^{n+\frac{1}{2}} \cdot \boldsymbol{H}_h^{n-\frac{1}{2}} \,\mathrm{d}x - \frac{1}{2}\int_\omega \mu\delta \boldsymbol{H}_h^{n-\frac{1}{2}} \cdot \boldsymbol{H}_h^{n-\frac{1}{2}} \,\mathrm{d}x \leq C.
\end{aligned}
$$

To obtain a bound for the term $\|\operatorname{\mathbf{curl}} \boldsymbol{H}_h^{n-\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega\setminus\omega)}$, we define

$$
\boldsymbol{z}_{h,\Omega\setminus\omega}^{n-\frac{1}{2}} := \begin{cases} \boldsymbol{E}_h^{n-\frac{1}{2}} & \text{on } \omega, \\ \operatorname{\mathbf{curl}} \boldsymbol{H}_h^{n-\frac{1}{2}} + \boldsymbol{E}_h^{n-\frac{1}{2}} & \text{on } \Omega\setminus\omega. \end{cases}
$$

Then, $\boldsymbol{z}_{h,\Omega\setminus\omega}^{n-\frac{1}{2}}$ is also an element of the set $\boldsymbol{K}\cap\mathbf{DG}_h$, and so using it as a test function in $(\mathrm{LF}_{N,h})$ leads to

$$
\begin{aligned}
\|\operatorname{\mathbf{curl}} \boldsymbol{H}_h^{n-\frac{1}{2}}\|^2_{\boldsymbol{L}^2(\Omega\setminus\omega)} &\leq \int_{\Omega\setminus\omega} \left(\epsilon\delta \boldsymbol{E}_h^n + \sigma \boldsymbol{E}_h^{n-\frac{1}{2}} - \boldsymbol{f}_h^{n-\frac{1}{2}}\right) \cdot \operatorname{\mathbf{curl}} \boldsymbol{H}_h^{n-\frac{1}{2}} \,\mathrm{d}x \\
&\leq C\|\operatorname{\mathbf{curl}} \boldsymbol{H}_h^{n-\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega\setminus\omega)},
\end{aligned}
$$

where we have used again Proposition 3.3 and Corollary 3.5 for the last inequality. This completes the proof. ∎

**4. Convergence.** Given $N \in \mathbb{N}$ with $N \geq 2$ and $h > 0$, we consider the solution $\{(\boldsymbol{E}_h^{n-\frac{1}{2}}, \boldsymbol{H}_h^{n+\frac{1}{2}})\}_{n=1}^N$ to $(\mathrm{P}_{N,h})$. Invoking those finite element solutions, we set up linear and piecewise constant interpolations

$$
\begin{aligned}
\boldsymbol{E}_{N,h}, \overline{\boldsymbol{E}}_{N,h}, \widehat{\boldsymbol{f}}_{N,h} &\colon [0,T] \to \mathbf{DG}_h, \\
\widehat{\boldsymbol{E}}_{N,h} &\colon [0,T] \to \boldsymbol{K}\cap\mathbf{DG}_h, \\
\boldsymbol{H}_{N,h}, \widehat{\boldsymbol{H}}_{N,h} &\colon [0,T] \to \mathbf{ND}_h,
\end{aligned}
$$

which, for $t \in [0,T]$, are defined by

$$
\boldsymbol{E}_{N,h}(t) = \begin{cases} \boldsymbol{E}_h^0 & \text{if } t = 0, \\ \boldsymbol{E}_h^{n-1} + (t - t_{n-1})\delta \boldsymbol{E}_h^n & \text{if } t \in (t_{n-1}, t_n], \end{cases} \tag{4.1}
$$

$$
\boldsymbol{H}_{N,h}(t) = \begin{cases} \boldsymbol{H}_h^{\frac{1}{2}} & \text{if } t = 0, \\ \boldsymbol{H}_h^{n-\frac{1}{2}} + (t - t_{n-1})\delta \boldsymbol{H}_h^{n+\frac{1}{2}} & \text{if } t \in (t_{n-1}, t_n] \text{ for } n \in \{1, \ldots, N-1\}, \\ \boldsymbol{H}_h^{N-\frac{1}{2}} & \text{if } t \in (t_{n-1}, t_n] \text{ for } n = N, \end{cases}
$$

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and

$$\widehat{\boldsymbol{E}}_{N,h}(t) = \begin{cases} \boldsymbol{E}_h^0 & \text{if } t = 0, \\ \boldsymbol{E}_h^{n-\frac{1}{2}} & \text{if } t \in (t_{n-1}, t_n], \end{cases}$$

$$\widehat{\boldsymbol{H}}_{N,h}(t) = \begin{cases} \boldsymbol{H}_h^{\frac{1}{2}} & \text{if } t = 0, \\ \boldsymbol{H}_h^{n-\frac{1}{2}} & \text{if } t \in (t_{n-1}, t_n] \text{ for } n \in \{1, \dots, N-1\}, \\ \boldsymbol{H}_h^{N-\frac{3}{2}} & \text{if } t \in (t_{n-1}, t_n] \text{ for } n = N, \end{cases} \tag{4.2}$$

$$\overline{\boldsymbol{E}}_{N,h}(t) = \begin{cases} \boldsymbol{E}_h^0 & \text{if } t = 0, \\ \boldsymbol{E}_h^n & \text{if } t \in (t_{n-1}, t_n], \end{cases}$$

$$\widehat{\boldsymbol{f}}_{N,h}(t) = \begin{cases} \boldsymbol{f}_h^{\frac{1}{2}} & \text{if } t = 0, \\ \boldsymbol{f}_h^{n-\frac{1}{2}} & \text{if } t \in (t_{n-1}, t_n]. \end{cases}$$

Note that, since the pointwise electric constraint is applied at the intermediate time steps $t_{n-\frac{1}{2}}$ instead of at the time steps $t_n$, only the range of the piecewise constant interpolation $\widehat{\boldsymbol{E}}_{N,h}$ is contained in the obstacle set $\boldsymbol{K}$.

Now, by the above construction and in view of $(\mathrm{LF}_{N,h})$ as well as (1.2) it then follows that

$$(\tilde{\mathrm{P}}_{N,h}) \quad \begin{cases} \displaystyle\int_\Omega \epsilon \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}_{N,h}(t) \cdot (\boldsymbol{v}_h - \widehat{\boldsymbol{E}}_{N,h}(t)) + \sigma \widehat{\boldsymbol{E}}_{N,h}(t) \cdot (\boldsymbol{v}_h - \widehat{\boldsymbol{E}}_{N,h}(t)) \\ \qquad - \mathbf{curl}\, \widehat{\boldsymbol{H}}_{N,h}(t) \cdot (\boldsymbol{v}_h - \widehat{\boldsymbol{E}}_{N,h}(t))\, \mathrm{d}x \\ \geq \displaystyle\int_\Omega \widehat{\boldsymbol{f}}_{N,h}(t) \cdot (\boldsymbol{v}_h - \widehat{\boldsymbol{E}}_{N,h}(t))\, \mathrm{d}x \quad \forall \boldsymbol{v}_h \in \boldsymbol{K} \cap \mathbf{DG}_h \quad \forall t \in \left(0, T - \frac{T}{N}\right], \\ \displaystyle\int_\Omega \mu \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}_{N,h}(t) \cdot \boldsymbol{w}_h + \overline{\boldsymbol{E}}_{N,h}(t) \cdot \mathbf{curl}\, \boldsymbol{w}_h \, \mathrm{d}x = \boldsymbol{0} \\ \forall \boldsymbol{w}_h \in \mathbf{ND}_h \quad \forall t \in \left(0, T - \frac{T}{N}\right], \\ \widehat{\boldsymbol{E}}_{N,h}(t) \in \boldsymbol{K} \cap \mathbf{DG}_h \quad \forall t \in [0, T]. \end{cases}$$

The convergence analysis of $(\tilde{\mathrm{P}}_{N,h})$ turns out to be challenging due to the lack of $L^\infty((0,T), \boldsymbol{L}^2(\omega))$-boundedness of $\mathbf{curl}\, \widehat{\boldsymbol{H}}_{N,h}$. Provided the worse boundedness in $L^\infty((0,T), \boldsymbol{L}^1(\omega))$ (see Proposition 3.6), our first step consists of bypassing the missing boundedness by exploiting $\boldsymbol{Q}_h \boldsymbol{v}$ for functions $\boldsymbol{v} \in \boldsymbol{\mathcal{C}}_0^\infty(\Omega)$. In this way, we are able to derive a convergence result towards a solution of a time integrated version of the variational inequality in (P) with test functions $\boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{\mathcal{C}}_0^\infty(\Omega)$. The final step is to enlarge the test function set to $\boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})$, which requires the construction of a constraint preserving mollification operator.

**4.1. Constraint preserving mollification.** Recently, Ern and Guermond [8] established novel mollification operators with pivotal commuting and convergence properties (cf. also [4, 9]). Their construction is based on the use of a transversal vector field [10] along with a cut-off strategy in a careful combination with mollification techniques. Our goal is to extend [8] to constraint preserving mollification operators in the sense that the mollification of a function in $\boldsymbol{K}$ lies as well in $\boldsymbol{K}$. The extension is mainly complicated due to the fact that there is no a priori knowledge of how the obstacle region $\omega$ behaves under the expansion as in [8]. We tackle this issue by

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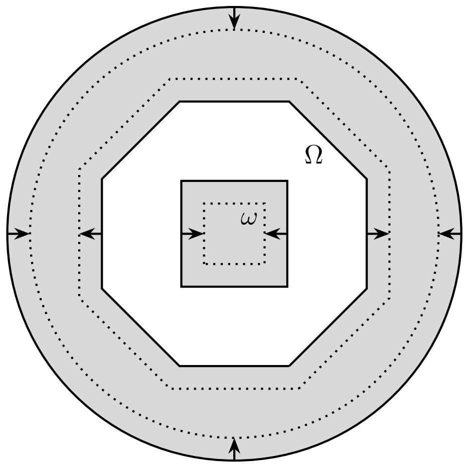


FIG. 2. *Schematic drawing of $\mathcal{O}$ (gray) and its inwardly transversal vector field.*

modifying the mollification operator in [8] with a certain scaling and by choosing the transversal vector field in a way that the obstacle set boundary $\partial\omega$ is transported inwardly (cf. Figure 2). Given $\boldsymbol{v} \in \boldsymbol{L}^1(\Omega)$ we denote its zero-extension to the whole space $\mathbb{R}^3$ by $\widetilde{\boldsymbol{v}} \in \boldsymbol{L}^1(\mathbb{R}^3)$. Furthermore, let

$$\rho\colon \mathbb{R}^3 \to \mathbb{R}, \quad \rho(x) = \begin{cases} \eta \exp\left(-\dfrac{1}{1-|x|^2}\right) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

where $\eta > 0$ is chosen such that

$$\int_{\mathbb{R}^3} \rho(x)\,\mathrm{d}x = \int_{B(0,1)} \rho(x)\,\mathrm{d}x = 1. \tag{4.3}$$

At first, since $\Omega$ is bounded, there exist some $x_\Omega \in \mathbb{R}^3$ and a radius $r_\Omega > 0$ such that $\overline{\Omega} \subset B(x_\Omega, r_\Omega)$. Then

$$\mathcal{O} \coloneqq B(x_\Omega, r_\Omega) \setminus \big(\overline{\Omega} \setminus \omega\big) = \big(B(x_\Omega, r_\Omega) \setminus \overline{\Omega}\big) \cup \omega$$

represents a bounded and open set with Lipschitz boundary. Therefore, as shown in [10, Corollary 2.13], the set $\mathcal{O}$ (of locally finite perimeter) admits a continuous inwardly globally transversal vector field, i.e., there exist a vector field $\widehat{\boldsymbol{k}} \in \boldsymbol{\mathcal{C}}(\partial\mathcal{O})$ and a real number $\kappa > 0$ with the property $\widehat{\boldsymbol{k}}(x) \cdot \boldsymbol{n}(x) \leq -\kappa$ for a.e. $x \in \partial\mathcal{O}$. Here, $\boldsymbol{n}$ denotes the unit normal vector field pointing outward on $\mathcal{O}$. Now, by the piecewise smoothness of $\partial\mathcal{O}$ in combination with [25, Lemma 5.9.5] and [15, Remark 15.1], the measure-theoretic boundary of $\mathcal{O}$ defined by

$$\partial_\star\mathcal{O} \coloneqq \left\{x \in \partial\mathcal{O} \mid \limsup_{r\to 0^+} r^{-3} \min\left\{|\mathcal{O} \cap B(x,r)|\,, |\mathcal{O}^{\mathrm{c}} \cap B(x,r)|\right\} > 0\right\}$$

coincides with $\partial\mathcal{O}$ up to a set of (surface-)measure zero, as a result of which we can deduce that

$$|\partial_\star\mathcal{O} \cap B(x,r)| = |\partial\mathcal{O} \cap B(x,r)| > 0 \quad \forall x \in \partial\mathcal{O} \quad \forall r > 0. \tag{4.4}$$

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Note that in (4.4) with $|\cdot|$ we refer to the two-dimensional Lebesgue measure. Together with the boundary $\partial\mathcal{O}$ being compact, we are able to apply [10, Proposition 2.3 (iv)] which implies that there exists a vector field $\boldsymbol{k} \in \boldsymbol{\mathcal{C}}^\infty(\mathbb{R}^3)$ whose restriction to $\partial\mathcal{O}$ is inwardly globally transversal with $|\boldsymbol{k}(y)| = 1$ for every $y \in \partial\mathcal{O}$. By the use of this special vector field, for every $\delta > 0$ we introduce the mapping

$$\boldsymbol{\theta}_\delta \colon \mathbb{R}^3 \to \mathbb{R}^3, \quad y \mapsto y + \delta\boldsymbol{k}(y). \tag{4.5}$$

LEMMA 4.1. *There exist* $\delta_0 > 0$ *and* $\zeta > 0$ *such that*

$$\boldsymbol{\theta}_\delta(\mathcal{O}) + B(0, \delta\zeta) \subset \mathcal{O} \quad \forall \delta \in (0, \delta_0).$$

*Proof.* As shown in the proof of [10, Proposition 4.15], there exists some $\delta_{0,1} > 0$ such that

$$\partial\boldsymbol{\theta}_\delta(\mathcal{O}) = \{y + \delta\boldsymbol{k}(y) \mid y \in \partial\mathcal{O}\} \quad \forall \delta \in (0, \delta_{0,1}).$$

As obtained from the proof of Lemma 4.16 in [10], there exists some $\delta_{0,2} > 0$ such that

$$H \colon \partial\mathcal{O} \times (-\delta_{0,2}, \delta_{0,2}) \to \mathbb{R}^3, \quad (x, \delta) \mapsto y + \delta\boldsymbol{k}(y)$$

is a bi-Lipschitz mapping. In particular, with $L_H$ denoting the Lipschitz constant of $H$, it holds that

$$|H(y,\delta) - H(z,\rho)| \geq \frac{1}{L_H}|(y,\delta) - (z,\rho)| \quad \forall (y,\delta), (z,\rho) \in \partial\mathcal{O} \times (-\delta_{0,2}, \delta_{0,2}).$$

Now let $\delta_0 \coloneqq \min\{\delta_{0,1}, \delta_{0,2}\}$ and $\delta \in (0, \delta_0)$ be arbitrarily fixed. Given $y \in \partial\mathcal{O}$, $y + \delta\boldsymbol{k}(y) \in \partial\boldsymbol{\theta}_\delta(\mathcal{O})$, and $z \in \partial\mathcal{O}$, we have

$$|y+\delta\boldsymbol{k}(y)-z| = |H(y,\delta)-H(z,0)| \geq \frac{1}{L_H}|(y,\delta)-(z,0)| = \frac{1}{L_H}\sqrt{|y-z|^2 + \delta^2} \geq \frac{1}{L_H}\delta.$$

Therefore, $\operatorname{dist}(\partial\boldsymbol{\theta}_\delta(\mathcal{O}), \partial\mathcal{O}) \geq \frac{1}{L_H}\delta$ holds for every $\delta \in (0, \delta_0)$. Now, [10, Proposition 4.15] yields that $\overline{\boldsymbol{\theta}_\delta(\mathcal{O})} \subset \mathcal{O}$, and therefore the claim follows for $\zeta \coloneqq \frac{1}{L_H}$ and $\delta_0$ as above. ∎

By $\mathbb{K}_\delta$ we denote the Jacobian mapping $\mathrm{D}\boldsymbol{\theta}_\delta \colon \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$. It is well known (see [8, pp. 59–60]) that there exists a constant $c_\theta > 0$ such that for $\delta > 0$ it holds that

$$\sup_{y \in \Omega} \|\mathbb{K}_\delta(y) - \mathbb{I}\|_{\mathbb{R}^{3\times 3}} \leq c_\theta\delta. \tag{4.6}$$

We now introduce the following mollification operator:

$$\boldsymbol{\mathcal{K}}_\delta \colon \boldsymbol{L}^1(\Omega) \to \boldsymbol{L}^1(\Omega), \quad \boldsymbol{v} \mapsto \frac{1}{1 + c_\theta\delta}\int_{B(0,1)} \rho(x)\mathbb{K}_\delta^{\mathrm{T}}(\cdot)\widetilde{\boldsymbol{v}}(\boldsymbol{\theta}_\delta(\cdot) + \delta\zeta x)\,\mathrm{d}x, \tag{4.7}$$

where $\widetilde{\boldsymbol{v}} \in \boldsymbol{L}^1(\mathbb{R}^3)$ is the zero-extension of $\boldsymbol{v} \in \boldsymbol{L}^1(\Omega)$. In the following theorem, we prove the main constraint preserving property of the mollification (4.7) relying on the use of the following positive constants:

$$c_{\boldsymbol{k}} \coloneqq \max_{p \in \overline{\omega}} |\boldsymbol{k}(p)| \quad \text{and} \quad \lambda \coloneqq \operatorname{dist}(\omega, \mathbb{R}^3 \setminus \Omega).$$

Note that $\lambda > 0$ holds true due to $\overline{\omega} \subset \Omega$.

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THEOREM 4.2. *For $\delta \in (0, \delta_0)$, it holds that*

$$\begin{aligned} \boldsymbol{v} \in \boldsymbol{L}^1(\Omega) \quad &\Rightarrow \quad \boldsymbol{\mathcal{K}}_\delta \boldsymbol{v} \in \boldsymbol{\mathcal{C}}_0^\infty(\Omega), \\ \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}) \quad &\Rightarrow \quad \lim_{\delta \to 0} \|\boldsymbol{K}_\delta \boldsymbol{v} - \boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl})} = 0. \end{aligned} \tag{4.8}$$

*If* $0 < \delta < \min\left\{\delta_0, \frac{\lambda}{c_{\boldsymbol{k}} + \zeta}\right\}$, *then* $\boldsymbol{\mathcal{K}}_\delta$ *satisfies*

$$\boldsymbol{v} \in \boldsymbol{K} \quad \Rightarrow \quad \boldsymbol{\mathcal{K}}_\delta \boldsymbol{v} \in \boldsymbol{K}. \tag{4.9}$$

*Proof.* The vector field $\boldsymbol{k}$ is particularly inwardly (globally) transversal for the boundary $\partial(B(x_\Omega, r_\Omega) \setminus \overline{\Omega})$. The proof for (4.8) therefore follows the same arguments as in [8, Lemma 4.1] and [8, Theorem 4.4] together with the fact that $\frac{1}{1+c_\theta \delta} \to 1$ for $\delta \to 0$. Now let $0 < \delta < \min\left\{\delta_0, \frac{\lambda}{c_{\boldsymbol{k}} + \zeta}\right\}$. Due to Lemma 4.1 we know that

$$\boldsymbol{\theta}_\delta(\omega) + B(0, \delta\zeta) \subset \mathcal{O} = \left(B(x_\Omega, r_\Omega) \setminus \overline{\Omega}\right) \cup \omega. \tag{4.10}$$

Let us now prove that (4.10) can be refined as

$$\boldsymbol{\theta}_\delta(\omega) + B(0, \delta\zeta) \subset \omega. \tag{4.11}$$

To this aim, we assume the contrary: There exist $y \in \omega$ and $x \in B(0, \delta\zeta)$ such that

$$\boldsymbol{\theta}_\delta(y) + x \in B(x_\Omega, r_\Omega) \setminus \overline{\Omega}. \tag{4.12}$$

Then (4.12) leads to a contradiction as follows:

$$\begin{aligned} \lambda = \; & \operatorname{dist}(\omega, \mathbb{R}^3 \setminus \Omega) \leq \operatorname{dist}(\omega, \boldsymbol{\theta}_\delta(y) + x) = \inf_{z \in \omega} |\boldsymbol{\theta}_\delta(y) + x - z| \leq |\boldsymbol{\theta}_\delta(y) + x - y| \\ \underbrace{=}_{(4.5)} \; & |\delta \boldsymbol{k}(y) + x| \leq \delta \max_{p \in \overline{\omega}} |\boldsymbol{k}(p)| + \delta\zeta = \delta(c_{\boldsymbol{k}} + \zeta) < \lambda, \end{aligned}$$

where the last inequality follows from our particular choice of $\delta$. This concludes (4.11). Now let $\boldsymbol{v} \in \boldsymbol{K}$ be given. In view of (4.7) and (4.11), it holds for a.e. $y \in \omega$ that

$$\begin{aligned} |\boldsymbol{K}_\delta \boldsymbol{v}(y)| &= \left| \frac{1}{1 + c_\theta \delta} \int_{B(0,1)} \rho(x) \mathbb{K}_\delta^{\mathrm{T}}(y) \widetilde{\boldsymbol{v}}(\boldsymbol{\theta}_\delta(y) + \delta\zeta x) \, \mathrm{d}x \right| \\ \leq \; & \frac{1}{1 + c_\theta \delta} \int_{B(0,1)} \left| \rho(x) \left(\mathbb{K}_\delta^{\mathrm{T}}(y) - \mathbb{I}\right) \widetilde{\boldsymbol{v}}(\boldsymbol{\theta}_\delta(y) + \delta\zeta x) \right| \, \mathrm{d}x \\ & + \frac{1}{1 + c_\theta \delta} \int_{B(0,1)} |\rho(x) \widetilde{\boldsymbol{v}}(\boldsymbol{\theta}_\delta(y) + \delta\zeta x)| \, \mathrm{d}x \\ \underbrace{=}_{(4.11)} \; & \frac{1}{1 + c_\theta \delta} \int_{B(0,1)} \left| \rho(x) \left(\mathbb{K}_\delta^{\mathrm{T}}(y) - \mathbb{I}\right) \boldsymbol{v}(\underbrace{\boldsymbol{\theta}_\delta(y) + \delta\zeta x}_{\in \omega}) \right| \, \mathrm{d}x \\ & + \frac{1}{1 + c_\theta \delta} \int_{B(0,1)} \left| \rho(x) \boldsymbol{v}(\underbrace{\boldsymbol{\theta}_\delta(y) + \delta\zeta x}_{\in \omega}) \right| \, \mathrm{d}x \\ \underbrace{\leq}_{\boldsymbol{v} \in \boldsymbol{K}} \; & \frac{d}{1 + c_\theta \delta} \int_{B(0,1)} \rho(x) \left\|\mathbb{K}_\delta^{\mathrm{T}}(y) - \mathbb{I}\right\|_{\mathbb{R}^{3 \times 3}} \, \mathrm{d}x + \frac{d}{1 + c_\theta \delta} \int_{B(0,1)} \rho(x) \, \mathrm{d}x \\ \underbrace{\leq}_{(4.3),(4.6)} \; & d \left( \frac{c_\theta \delta}{1 + c_\theta \delta} + \frac{1}{1 + c_\theta \delta} \right) = d. \end{aligned}$$

In conclusion, (4.9) is valid. ∎

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**4.2. Convergence result.** In the following, let $N = N(h) \in \mathbb{N}$ denote a natural number depending on $h > 0$ with the property $N(h) \to \infty$ as $h \to 0$ maintaining the linear CFL-condition (3.2).

THEOREM 4.3. *Let Assumptions* 1.1 *and* 3.1 *hold. Then*

$$(\boldsymbol{E}_{N,h}, \boldsymbol{H}_{N,h}) \overset{*}{\rightharpoonup} (\boldsymbol{E}, \boldsymbol{H}) \quad \textit{weakly-* in } L^\infty((0,T), \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \textit{ as } h \to 0,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{E}_{N,h}, \boldsymbol{H}_{N,h}) \overset{*}{\rightharpoonup} \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{E}, \boldsymbol{H}) \quad \textit{weakly-* in } L^\infty((0,T), \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \textit{ as } h \to 0,$$

*where* $(\boldsymbol{E}, \boldsymbol{H})$ *is the unique solution to* (P). *If, additionally,*

$$\boldsymbol{H} \in L^1((0,T), \boldsymbol{H}(\mathbf{curl})), \quad \{\mathbf{curl}\, \widehat{\boldsymbol{H}}_{N,h}\}_{h>0} \textit{ bounded in } L^p((0,T), \boldsymbol{L}^2(\omega)) \tag{4.13}$$

*for some* $p > 1$, *then*

$$(\boldsymbol{E}_{N,h}, \boldsymbol{H}_{N,h}) \to (\boldsymbol{E}, \boldsymbol{H}) \quad \textit{in } \mathcal{C}([0,T], \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \textit{ as } h \to 0.$$

*Proof.* The proof is divided into four parts.

*Step* 1*: Preparation.* Proposition 3.3, Corollary 3.5, and Proposition 3.6 yield the existence of subsequences, denoted w.l.o.g. by the same symbol, such that

$$\begin{aligned}
(\boldsymbol{E}_{N,h}, \boldsymbol{H}_{N,h}) &\overset{*}{\rightharpoonup} (\boldsymbol{E}, \boldsymbol{H}) \quad \text{weakly-* in } L^\infty((0,T), \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \text{ as } h \to 0,\\
(\widehat{\boldsymbol{E}}_{N,h}, \widehat{\boldsymbol{H}}_{N,h}) &\overset{*}{\rightharpoonup} (\widehat{\boldsymbol{E}}, \widehat{\boldsymbol{H}}) \quad \text{weakly-* in } L^\infty((0,T), \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \text{ as } h \to 0,\\
\overline{\boldsymbol{E}}_{N,h} &\overset{*}{\rightharpoonup} \overline{\boldsymbol{E}} \quad \text{weakly-* in } L^\infty((0,T), \boldsymbol{L}^2(\Omega)) \text{ as } h \to 0,\\
\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{E}_{N,h}, \boldsymbol{H}_{N,h}) &\overset{*}{\rightharpoonup} \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{E}, \boldsymbol{H}) \quad \text{weakly-* in } L^\infty((0,T), \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \text{ as } h \to 0
\end{aligned} \tag{4.14}$$

for some $(\boldsymbol{E}, \boldsymbol{H}) \in W^{1,\infty}((0,T), \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega))$, $(\widehat{\boldsymbol{E}}, \widehat{\boldsymbol{H}}) \in L^\infty((0,T), \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega))$, and $\overline{\boldsymbol{E}} \in L^\infty((0,T), \boldsymbol{L}^2(\Omega))$. The constructions (4.1) and (4.2) imply

$$\begin{aligned}
\|\boldsymbol{E}_{N,h}(t) - \overline{\boldsymbol{E}}_{N,h}(t)\|_{\boldsymbol{L}^2(\Omega)} &\le \max_{n \in \{1,\dots,N\}} \tau \|\delta \boldsymbol{E}_h^n\|_{\boldsymbol{L}^2(\Omega)} \le \frac{TC}{N} \quad \forall t \in [0,T],\\
\|\overline{\boldsymbol{E}}_{N,h}(t) - \widehat{\boldsymbol{E}}_{N,h}(t)\|_{\boldsymbol{L}^2(\Omega)} &\le \max_{n \in \{1,\dots,N\}} \frac{\tau}{2} \|\delta \boldsymbol{E}_h^n\|_{\boldsymbol{L}^2(\Omega)} \le \frac{TC}{2N} \quad \forall t \in [0,T],\\
\|\boldsymbol{H}_{N,h}(t) - \widehat{\boldsymbol{H}}_{N,h}(t)\|_{\boldsymbol{L}^2(\Omega)} &\le \max_{n \in \{1,\dots,N-1\}} \tau \|\delta \boldsymbol{H}_h^{n+\frac{1}{2}}\|_{\boldsymbol{L}^2(\Omega)} \le \frac{TC}{N} \quad \forall t \in [0,T],
\end{aligned} \tag{4.15}$$

from which we conclude that $\boldsymbol{E} = \widehat{\boldsymbol{E}} = \overline{\boldsymbol{E}}$ and $\boldsymbol{H} = \widehat{\boldsymbol{H}}$. By standard arguments, the first and last convergence properties in (4.14) lead to the following pointwise weak convergence:

$$(\boldsymbol{E}_{N,h}, \boldsymbol{H}_{N,h})(t) \rightharpoonup (\boldsymbol{E}, \boldsymbol{H})(t) \quad \text{weakly in } \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega) \text{ as } h \to 0 \ \forall t \in [0,T]. \tag{4.16}$$

Let us now verify the Faraday law for the weak limit $(\boldsymbol{E}, \boldsymbol{H})$. Given $\boldsymbol{w} \in \boldsymbol{H}(\mathbf{curl})$, there exists a sequence $\{\boldsymbol{w}_h\}_{h>0} \subset \boldsymbol{H}(\mathbf{curl})$ with $\boldsymbol{w}_h \in \mathbf{ND}_h$ for all $h > 0$ such that $\boldsymbol{w}_h \to \boldsymbol{w}$ in $\boldsymbol{H}(\mathbf{curl})$ as $h \to 0$. Using this converging sequence and (4.14), we deduce

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that

$$\int_0^T \left(\left(\mu \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}(t), \boldsymbol{w}\right)_{\boldsymbol{L}^2(\Omega)} + (\boldsymbol{E}(t), \mathbf{curl}\,\boldsymbol{w})_{\boldsymbol{L}^2(\Omega)}\right)\phi(t)\,\mathrm{d}t$$
$$\underbrace{=}_{(4.14)} \lim_{h\to 0}\int_0^T \left(\left(\mu \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}_{N,h}(t), \boldsymbol{w}_h\right)_{\boldsymbol{L}^2(\Omega)} + \left(\overline{\boldsymbol{E}}_{N,h}(t), \mathbf{curl}\,\boldsymbol{w}_h\right)_{\boldsymbol{L}^2(\Omega)}\right)\phi(t)\,\mathrm{d}t$$
$$\underbrace{=}_{(\tilde{\mathrm{P}}_{N,h})} \lim_{h\to 0}\int_{T-\frac{T}{N}}^T \left(\left(\mu \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}_{N,h}(t), \boldsymbol{w}_h\right)_{\boldsymbol{L}^2(\Omega)} + \left(\overline{\boldsymbol{E}}_{N,h}(t), \mathbf{curl}\,\boldsymbol{w}_h\right)_{\boldsymbol{L}^2(\Omega)}\right)\phi(t)\,\mathrm{d}t = 0$$

for all $\phi \in \mathcal{C}_0^\infty(0,T)$, where the last equality holds true since the integrand is uniformly bounded in time (Proposition 3.3 and Corollary 3.5). Thus, by the fundamental theorem of variational calculus and since $\boldsymbol{w} \in \boldsymbol{H}(\mathbf{curl})$ was chosen arbitrarily, it follows from the above identity and (1.4) that

$$\boldsymbol{E}(t) \in \boldsymbol{H}_0(\mathbf{curl}) \quad \text{with} \quad \mathbf{curl}\,\boldsymbol{E}(t) = -\mu\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}(t) \quad \text{for a.e. } t \in (0,T), \tag{4.17}$$

which particularly implies that

$$\boldsymbol{E} \in W^{1,\infty}((0,T), \boldsymbol{L}^2(\Omega)) \cap L^\infty((0,T), \boldsymbol{H}_0(\mathbf{curl})). \tag{4.18}$$

Since $(\boldsymbol{E}_{N,h}, \boldsymbol{H}_{N,h})(0) = (\boldsymbol{E}_h^0, \boldsymbol{H}_h^{\frac{1}{2}}) \to (\boldsymbol{E}_0, \boldsymbol{H}_0)$ holds, we obtain, thanks to (4.16), that

$$(\boldsymbol{E}, \boldsymbol{H})(0) = (\boldsymbol{E}_0, \boldsymbol{H}_0). \tag{4.19}$$

By the construction (4.2), it holds that $\widehat{\boldsymbol{E}}_{N,h}(t) \in \boldsymbol{K} \cap \mathbf{DG}_h$ for all $t \in [0,T]$, and so (4.15) and (4.16) imply that

$$\boldsymbol{E}(t) \in \boldsymbol{K} \quad \forall t \in [0,T] \tag{4.20}$$

since $\boldsymbol{K}$ is weakly closed in $\boldsymbol{L}^2(\Omega)$.

*Step 2: Derivation of the weak system* (4.28) *for* $(\boldsymbol{E}, \boldsymbol{H})$. Let $\boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{\mathcal{C}}_0^\infty(\Omega)$ be arbitrarily fixed. In view of (2.6), $\boldsymbol{Q}_h\boldsymbol{v} \in \boldsymbol{K} \cap \mathbf{DG}_h$ such that we may insert $\boldsymbol{v}_h = \boldsymbol{Q}_h\boldsymbol{v}$ in $(\tilde{\mathrm{P}}_{N,h})$ to deduce that

$$\int_0^T (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{E}(t))_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t \tag{4.21}$$
$$\underbrace{=}_{(2.5),(2.7),(4.14)} \limsup_{h\to 0}\int_0^{T-\frac{T}{N}} (\widehat{\boldsymbol{f}}_{N,h}(t), \boldsymbol{Q}_h\boldsymbol{v} - \widehat{\boldsymbol{E}}_{N,h}(t))_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t$$
$$\underbrace{\leq}_{(\tilde{\mathrm{P}}_{N,h}),(4.14)} \int_0^T \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t), \boldsymbol{v}\right)_{\boldsymbol{L}^2_\epsilon(\Omega)}\mathrm{d}t - \liminf_{h\to 0}\int_0^{T-\frac{T}{N}} \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}_{N,h}(t), \widehat{\boldsymbol{E}}_{N,h}(t)\right)_{\boldsymbol{L}^2_\epsilon(\Omega)}\mathrm{d}t$$
$$+ \int_0^T (\sigma\boldsymbol{E}(t), \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t - \liminf_{h\to 0}\int_0^{T-\frac{T}{N}} \left(\sigma\widehat{\boldsymbol{E}}_{N,h}(t), \widehat{\boldsymbol{E}}_{N,h}(t)\right)_{\boldsymbol{L}^2(\Omega)}\mathrm{d}t$$
$$- \liminf_{h\to 0}\int_0^{T-\frac{T}{N}} \left(\mathbf{curl}\,\widehat{\boldsymbol{H}}_{N,h}(t), \boldsymbol{Q}_h\boldsymbol{v}\right)_{\boldsymbol{L}^2(\Omega)}\mathrm{d}t$$
$$+ \limsup_{h\to 0}\int_0^{T-\frac{T}{N}} \left(\mathbf{curl}\,\widehat{\boldsymbol{H}}_{N,h}(t), \widehat{\boldsymbol{E}}_{N,h}(t)\right)_{\boldsymbol{L}^2(\Omega)}\mathrm{d}t.$$

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Let us now proceed by estimating the individual parts appearing in the right-hand side of (4.21). At first, for $\boldsymbol{w} \in \boldsymbol{L}^2(\Omega)$, we estimate

$$\begin{aligned}
&\left(\boldsymbol{E}_{N,h}\left(T-\frac{T}{N}\right)-\boldsymbol{E}(T),\boldsymbol{w}\right)_{\boldsymbol{L}^2(\Omega)}\\
&=\left(\boldsymbol{E}_{N,h}\left(T-\frac{T}{N}\right)-\boldsymbol{E}_{N,h}(T),\boldsymbol{w}\right)_{\boldsymbol{L}^2(\Omega)}+\left(\boldsymbol{E}_{N,h}(T)-\boldsymbol{E}(T),\boldsymbol{w}\right)_{\boldsymbol{L}^2(\Omega)}\\
&=-\int_{T-\frac{T}{N}}^{T}\left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}_{N,h}(t),\boldsymbol{w}\right)_{\boldsymbol{L}^2(\Omega)}\mathrm{d}t+\left(\boldsymbol{E}_{N,h}(T)-\boldsymbol{E}(T),\boldsymbol{w}\right)_{\boldsymbol{L}^2(\Omega)}\to 0 \text{ as } h\to 0,
\end{aligned}$$

where we have used the boundedness of $\{\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}_{N,h}\}$ and (4.16) for the above convergence. Using the same argumentation for the discrete magnetic fields, we obtain that

$$(\boldsymbol{E}_{N,h},\boldsymbol{H}_{N,h})\left(T-\frac{T}{N}\right)\rightharpoonup(\boldsymbol{E}(T),\boldsymbol{H}(T))\quad\text{weakly in } \boldsymbol{L}^2(\Omega)\times\boldsymbol{L}^2(\Omega), \tag{4.22}$$

as $h\to 0$. Now, by the weak sequential lower semicontinuity of the squared norm, we infer that

$$\begin{aligned}
&\liminf_{h\to 0}\int_0^{T-\frac{T}{N}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}_{N,h}(t),\boldsymbol{E}_{N,h}(t)\right)_{\boldsymbol{L}^2_\epsilon(\Omega)}\mathrm{d}t\\
&=\liminf_{h\to 0}\frac{1}{2}\left(\left\|\boldsymbol{E}_{N,h}\left(T-\frac{T}{N}\right)\right\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}-\|\boldsymbol{E}_h^0\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}\right)\\
&\underbrace{\geq}_{(4.22)}\frac{1}{2}\left(\|\boldsymbol{E}(T)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}-\|\boldsymbol{E}_0\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}\right)=\int_0^T\left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t),\boldsymbol{E}(t)\right)_{\boldsymbol{L}^2_\epsilon(\Omega)}\mathrm{d}t,
\end{aligned}\tag{4.23}$$

and therefore

$$\begin{aligned}
&\liminf_{h\to 0}\int_0^{T-\frac{T}{N}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}_{N,h}(t),\widehat{\boldsymbol{E}}_{N,h}(t)\right)_{\boldsymbol{L}^2_\epsilon(\Omega)}\mathrm{d}t\\
&=\liminf_{h\to 0}\int_0^{T-\frac{T}{N}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}_{N,h}(t),\widehat{\boldsymbol{E}}_{N,h}(t)-\boldsymbol{E}_{N,h}(t)\right)_{\boldsymbol{L}^2_\epsilon(\Omega)}\mathrm{d}t\\
&\quad+\liminf_{h\to 0}\int_0^{T-\frac{T}{N}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}_{N,h}(t),\boldsymbol{E}_{N,h}(t)\right)_{\boldsymbol{L}^2_\epsilon(\Omega)}\mathrm{d}t\\
&\underbrace{\geq}_{(4.15),(4.23)}\int_0^T\left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t),\boldsymbol{E}(t)\right)_{\boldsymbol{L}^2_\epsilon(\Omega)}\mathrm{d}t.
\end{aligned}\tag{4.24}$$

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Furthermore,

$$
\begin{aligned}
(4.25)\qquad & \liminf_{h\to 0}\int_0^{T-\frac{T}{N}} \left(\sigma\widehat{\boldsymbol{E}}_{N,h}(t),\widehat{\boldsymbol{E}}_{N,h}(t)\right)_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t\\
&= \liminf_{h\to 0}\int_0^{T-\frac{T}{N}} \left(\sigma(\widehat{\boldsymbol{E}}_{N,h}(t)-\boldsymbol{E}(t)),\widehat{\boldsymbol{E}}_{N,h}(t)-\boldsymbol{E}(t)\right)_{\boldsymbol{L}^2(\Omega)}\\
&\qquad + \left(\sigma(\widehat{\boldsymbol{E}}_{N,h}(t)-\boldsymbol{E}(t)),\boldsymbol{E}(t)\right)_{\boldsymbol{L}^2(\Omega)} + \left(\sigma\boldsymbol{E}(t),\widehat{\boldsymbol{E}}_{N,h}(t)\right)_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t\\
&\geq \liminf_{h\to 0}\int_0^{T-\frac{T}{N}} \left(\sigma(\widehat{\boldsymbol{E}}_{N,h}(t)-\boldsymbol{E}(t)),\boldsymbol{E}(t)\right)_{\boldsymbol{L}^2(\Omega)} + \left(\sigma\boldsymbol{E}(t),\widehat{\boldsymbol{E}}_{N,h}(t)\right)_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t\\
&\underbrace{=}_{(4.14)} \int_0^T (\sigma\boldsymbol{E}(t),\boldsymbol{E}(t))_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t.
\end{aligned}
$$

By the same argumentation as in (4.23) and (4.24), we infer that

$$
\liminf_{h\to 0}\int_0^{T-\frac{T}{N}} \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}_{N,h}(t),\widehat{\boldsymbol{H}}_{N,h}(t)\right)_{\boldsymbol{L}^2_\mu(\Omega)}\,\mathrm{d}t \geq \int_0^T \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}(t),\boldsymbol{H}(t)\right)_{\boldsymbol{L}^2_\mu(\Omega)}\,\mathrm{d}t,
$$

from which it follows that

$$
\begin{aligned}
&\limsup_{h\to 0}\int_0^{T-\frac{T}{N}} \left(\mathbf{curl}\,\widehat{\boldsymbol{H}}_{N,h}(t),\widehat{\boldsymbol{E}}_{N,h}(t)\right)_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t\\
&\underbrace{=}_{(4.15)} \limsup_{h\to 0}\int_0^{T-\frac{T}{N}} \left(\mathbf{curl}\,\widehat{\boldsymbol{H}}_{N,h}(t),\overline{\boldsymbol{E}}_{N,h}(t)\right)_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t\\
(4.26)\qquad &\underbrace{=}_{(\tilde{\mathrm{P}}_{N,h})} -\liminf_{h\to 0}\int_0^{T-\frac{T}{N}} \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}_{N,h}(t),\widehat{\boldsymbol{H}}_{N,h}(t)\right)_{\boldsymbol{L}^2_\mu(\Omega)}\,\mathrm{d}t\\
&\leq -\int_0^T \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}(t),\boldsymbol{H}(t)\right)_{\boldsymbol{L}^2_\mu(\Omega)}\,\mathrm{d}t \underbrace{=}_{(4.17)} \int_0^T (\mathbf{curl}\,\boldsymbol{E}(t),\boldsymbol{H}(t))_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t.
\end{aligned}
$$

Due to Proposition 3.6 and Lemma 2.1, it holds that

$$
\begin{aligned}
&\left|\int_0^{T-\frac{T}{N}} \left(\mathbf{curl}\,\widehat{\boldsymbol{H}}_{N,h}(t),\boldsymbol{Q}_h\boldsymbol{v}-\boldsymbol{v}\right)_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t\right|\\
&\qquad\qquad \leq \|\,\mathbf{curl}\,\widehat{\boldsymbol{H}}_{N,h}\|_{L^1((0,T),\boldsymbol{L}^1(\Omega))}\|\boldsymbol{Q}_h\boldsymbol{v}-\boldsymbol{v}\|_{\boldsymbol{L}^\infty(\Omega)} \to 0,
\end{aligned}
$$

and consequently,

$$
\begin{aligned}
&\liminf_{h\to 0}\int_0^{T-\frac{T}{N}} \left(\mathbf{curl}\,\widehat{\boldsymbol{H}}_{N,h}(t),\boldsymbol{Q}_h\boldsymbol{v}\right)_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t\\
&\geq \liminf_{h\to 0}\int_0^{T-\frac{T}{N}} \left(\mathbf{curl}\,\widehat{\boldsymbol{H}}_{N,h}(t),\boldsymbol{Q}_h\boldsymbol{v}-\boldsymbol{v}\right)_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t\\
(4.27)\qquad &+ \int_0^{T-\frac{T}{N}} \left(\mathbf{curl}\,\widehat{\boldsymbol{H}}_{N,h}(t),\boldsymbol{v}\right)_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t\\
&= \liminf_{h\to 0}\int_0^{T-\frac{T}{N}} \left(\widehat{\boldsymbol{H}}_{N,h}(t),\mathbf{curl}\,\boldsymbol{v}\right)_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t \underbrace{=}_{(4.14)} \int_0^T (\boldsymbol{H}(t),\mathbf{curl}\,\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t.
\end{aligned}
$$

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Applying (4.24)–(4.27) to (4.21) and taking (4.17)–(4.20) into account, we conclude that the weak-star limit $(\boldsymbol{E}, \boldsymbol{H})$ satisfies

$$
(4.28)\quad \begin{cases} \displaystyle\int_0^T \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t), \boldsymbol{v} - \boldsymbol{E}(t)\right)_{\boldsymbol{L}^2_\epsilon(\Omega)} + (\sigma \boldsymbol{E}(t), \boldsymbol{v} - \boldsymbol{E}(t))_{\boldsymbol{L}^2(\Omega)} \\ \qquad - (\boldsymbol{H}(t), \mathbf{curl}(\boldsymbol{v} - \boldsymbol{E}(t)))_{\boldsymbol{L}^2(\Omega)}\, \mathrm{d}t \\ \displaystyle\geq \int_0^T (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{E}(t))_{\boldsymbol{L}^2(\Omega)}\, \mathrm{d}t \quad \forall \boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{\mathcal{C}}_0^\infty(\Omega), \\ \displaystyle\mu \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}(t) + \mathbf{curl}\, \boldsymbol{E}(t) = \boldsymbol{0} \quad \text{for a.e. } t \in (0, T), \\ (\boldsymbol{E}, \boldsymbol{H}) \in W^{1,\infty}((0,T), \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \cap L^\infty((0,T), \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{L}^2(\Omega)), \\ \boldsymbol{E}(t) \in \boldsymbol{K} \text{ for all } t \in [0, T] \text{ and } (\boldsymbol{E}, \boldsymbol{H})(0) = (\boldsymbol{E}_0, \boldsymbol{H}_0). \end{cases}
$$

*Step* 3*:* (4.28) $\Rightarrow$ (P) *through a mollification process.* Let $\boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})$ and $0 < \delta < \min\left\{\delta_0, \frac{\lambda}{c_{\boldsymbol{k}} + \zeta}\right\}$. In view of Theorem 4.2, $\boldsymbol{\mathcal{K}}_\delta \boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{\mathcal{C}}_0^\infty(\Omega)$ is a feasible test function for (4.28). For this reason,

$$
(4.29)\quad \begin{aligned} &\int_0^T (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{E}(t))_{\boldsymbol{L}^2(\Omega)}\, \mathrm{d}t \underbrace{=}_{(4.8)} \lim_{\delta \to 0} \int_0^T (\boldsymbol{f}(t), \boldsymbol{\mathcal{K}}_\delta \boldsymbol{v} - \boldsymbol{E}(t))_{\boldsymbol{L}^2(\Omega)}\, \mathrm{d}t \\ &\underbrace{\leq}_{(4.28)} \lim_{\delta \to 0} \int_0^T \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t), \boldsymbol{\mathcal{K}}_\delta \boldsymbol{v} - \boldsymbol{E}(t)\right)_{\boldsymbol{L}^2_\epsilon(\Omega)} + (\sigma \boldsymbol{E}(t), \boldsymbol{\mathcal{K}}_\delta \boldsymbol{v} - \boldsymbol{E}(t))_{\boldsymbol{L}^2(\Omega)} \\ &\qquad - (\boldsymbol{H}(t), \mathbf{curl}(\boldsymbol{\mathcal{K}}_\delta \boldsymbol{v} - \boldsymbol{E}(t)))_{\boldsymbol{L}^2(\Omega)}\, \mathrm{d}t \\ &\underbrace{=}_{(4.8)} \int_0^T \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t), \boldsymbol{v} - \boldsymbol{E}(t)\right)_{\boldsymbol{L}^2_\epsilon(\Omega)} + (\sigma \boldsymbol{E}(t), \boldsymbol{v} - \boldsymbol{E}(t))_{\boldsymbol{L}^2(\Omega)} \\ &\qquad - (\boldsymbol{H}(t), \mathbf{curl}(\boldsymbol{v} - \boldsymbol{E}(t)))_{\boldsymbol{L}^2(\Omega)}\, \mathrm{d}t. \end{aligned}
$$

Since simple (in time) functions with values in $\boldsymbol{H}_0(\mathbf{curl})$ are dense in the space $L^2((0,T), \boldsymbol{H}_0(\mathbf{curl}))$, it follows that

$$
(4.30)\quad \begin{aligned} &\int_0^T \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t), \boldsymbol{v}(t) - \boldsymbol{E}(t)\right)_{\boldsymbol{L}^2_\epsilon(\Omega)} + (\sigma \boldsymbol{E}(t), \boldsymbol{v}(t) - \boldsymbol{E}(t))_{\boldsymbol{L}^2(\Omega)} \\ &\qquad - (\boldsymbol{H}(t), \mathbf{curl}(\boldsymbol{v}(t) - \boldsymbol{E}(t)))_{\boldsymbol{L}^2(\Omega)}\, \mathrm{d}t \\ &\geq \int_0^T (\boldsymbol{f}(t), \boldsymbol{v}(t) - \boldsymbol{E}(t))_{\boldsymbol{L}^2(\Omega)}\, \mathrm{d}t \quad \forall \boldsymbol{v} \in L^2((0,T), \boldsymbol{H}_0(\mathbf{curl})) \\ &\qquad\qquad \text{with } \boldsymbol{v}(t) \in \boldsymbol{K} \text{ for a.e. } t \in (0,T). \end{aligned}
$$

Finally, to prove that the Ampère–Maxwell VI in (P) is satisfied, let us assume the contrary: there exist $\boldsymbol{z} \in \boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl})$, and $M \subset (0,T)$ with $|M| > 0$, s.t.

$$
\begin{aligned} &\int_\Omega \epsilon \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t) \cdot (\boldsymbol{z} - \boldsymbol{E}(t))\, \mathrm{d}x + \int_\Omega \sigma \boldsymbol{E}(t) \cdot (\boldsymbol{z} - \boldsymbol{E}(t)) - \boldsymbol{H}(t) \cdot \mathbf{curl}(\boldsymbol{z} - \boldsymbol{E}(t))\, \mathrm{d}x \\ &\qquad < \int_\Omega \boldsymbol{f}(t) \cdot (\boldsymbol{z} - \boldsymbol{E}(t))\, \mathrm{d}x \quad \text{for a.e. } t \in M, \end{aligned}
$$



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which implies

$$\int_M \int_\Omega \epsilon \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t)\cdot(\boldsymbol{z}-\boldsymbol{E}(t))\,\mathrm{d}x + \int_\Omega \sigma \boldsymbol{E}(t)\cdot(\boldsymbol{z}-\boldsymbol{E}(t)) - \boldsymbol{H}(t)\cdot \mathbf{curl}(\boldsymbol{z}-\boldsymbol{E}(t))\,\mathrm{d}x\,\mathrm{d}t$$
$$< \int_M \int_\Omega \boldsymbol{f}(t)\cdot(\boldsymbol{z}-\boldsymbol{E}(t))\,\mathrm{d}x\,\mathrm{d}t. \tag{4.31}$$

Inserting $\boldsymbol{v} := \chi_M \boldsymbol{z} + \chi_{(0,T)\setminus M}\boldsymbol{E}$ into (4.30) immediately contradicts (4.31). In conclusion, we have shown that $(\boldsymbol{E},\boldsymbol{H})$ is the unique solution to (P).

*Step 4: Uniform convergence.* Suppose that $\boldsymbol{H} \in L^1((0,T),\boldsymbol{H}(\mathbf{curl}))$ and that the sequence $\{\mathbf{curl}\,\widehat{\boldsymbol{H}}_{N,h}\}_{h>0}$ is bounded in $L^p((0,T),\boldsymbol{L}^2(\omega))$ for some $p>1$. Let $\boldsymbol{v} \in \boldsymbol{K}$. As shown in Theorem 4.2, it holds that $\boldsymbol{\mathcal{K}}_\delta \boldsymbol{v} \in \boldsymbol{K}$. Now, [8, Theorem 4.4] additionally reveals that $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$ is sufficient to obtain $\|\boldsymbol{\mathcal{K}}_\delta \boldsymbol{v} - \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \to 0$ as $\delta \to 0$. Thus, together with $\boldsymbol{H} \in L^1((0,T),\boldsymbol{H}(\mathbf{curl}))$, we obtain

$$\int_\Omega \left(\epsilon \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t) + \sigma\boldsymbol{E}(t) - \mathbf{curl}\,\boldsymbol{H}(t)\right)\cdot(\boldsymbol{v}-\boldsymbol{E}(t))\,\mathrm{d}x \\ \geq \int_\Omega \boldsymbol{f}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t))\,\mathrm{d}x \quad \forall \boldsymbol{v}\in\boldsymbol{K} \text{ for a.e. } t\in(0,T). \tag{4.32}$$

Now, $(\tilde{\mathrm{P}}_{N,h})$ implies that

$$\int_\Omega \left(\epsilon \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}_{N,h}(t) + \sigma\widehat{\boldsymbol{E}}_{N,h}(t) - \mathbf{curl}\,\widehat{\boldsymbol{H}}_{N,h}(t)\right)\cdot(\boldsymbol{v}_h-\widehat{\boldsymbol{E}}_{N,h}(t))\,\mathrm{d}x \\ \geq \int_\Omega \widehat{\boldsymbol{f}}_{N,h}(t)\cdot(\boldsymbol{v}_h-\widehat{\boldsymbol{E}}_{N,h}(t))\,\mathrm{d}x \quad \forall \boldsymbol{v}_h\in\boldsymbol{K}\cap\mathbf{DG}_h \quad \forall t\in\left(0,T-\frac{T}{N}\right]. \tag{4.33}$$

For a.e. $t\in(0,T-T/N]$, the inequalities (4.32) and (4.33) allow for testing with $\boldsymbol{v} = \widehat{\boldsymbol{E}}_{N,h}(t)$ and $\boldsymbol{v}_h = \boldsymbol{Q}_h\boldsymbol{E}(t)$. Let $\rho\in(0,T)$ be arbitrarily fixed. Adding the resulting inequalities and integrating over the time interval $(0,\rho)$ then yields the estimate

$$\begin{aligned}
&\int_0^\rho \int_\Omega \epsilon\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{E}_{N,h}(t)-\boldsymbol{E}(t))\cdot(\widehat{\boldsymbol{E}}_{N,h}(t)-\boldsymbol{E}(t))\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^\rho \int_\Omega \epsilon\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}_{N,h}(t)\cdot(\boldsymbol{E}(t)-\boldsymbol{Q}_h\boldsymbol{E}(t))\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^\rho \int_\Omega \sigma(\widehat{\boldsymbol{E}}_{N,h}(t)-\boldsymbol{E}(t))\cdot(\widehat{\boldsymbol{E}}_{N,h}(t)-\boldsymbol{E}(t))\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^\rho \int_\Omega \sigma\widehat{\boldsymbol{E}}_{N,h}(t)\cdot(\boldsymbol{E}(t)-\boldsymbol{Q}_h\boldsymbol{E}(t))\,\mathrm{d}x\,\mathrm{d}t \\
&- \int_0^\rho \int_\Omega \mathbf{curl}(\widehat{\boldsymbol{H}}_{N,h}(t)-\boldsymbol{H}(t))\cdot(\widehat{\boldsymbol{E}}_{N,h}(t)-\boldsymbol{E}(t))\,\mathrm{d}x\,\mathrm{d}t \\
&- \int_0^\rho \int_\Omega \mathbf{curl}\,\widehat{\boldsymbol{H}}_{N,h}(t)\cdot(\boldsymbol{E}(t)-\boldsymbol{Q}_h\boldsymbol{E}(t))\,\mathrm{d}x\,\mathrm{d}t \\
\leq &\int_0^\rho \int_\Omega (\widehat{\boldsymbol{f}}_{N,h}(t)-\boldsymbol{f}(t))\cdot(\widehat{\boldsymbol{E}}_{N,h}(t)-\boldsymbol{E}(t))\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^\rho \int_\Omega \widehat{\boldsymbol{f}}_{N,h}(t)\cdot(\boldsymbol{E}(t)-\boldsymbol{Q}_h\boldsymbol{E}(t))\,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{4.34}$$

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for sufficiently small $h > 0$. The first term on the left-hand side of the last inequality satisfies

$$
\begin{aligned}
&\limsup_{h\to 0} \int_0^\rho \int_\Omega \epsilon \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{E}_{N,h}(t) - \boldsymbol{E}(t)) \cdot (\widehat{\boldsymbol{E}}_{N,h}(t) - \boldsymbol{E}(t))\,\mathrm{d}x\,\mathrm{d}t \\
&= \limsup_{h\to 0} \Bigg( \int_0^\rho \int_\Omega \epsilon \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{E}_{N,h}(t) - \boldsymbol{E}(t)) \cdot (\widehat{\boldsymbol{E}}_{N,h}(t) - \boldsymbol{E}_{N,h}(t))\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \frac{1}{2}\|\boldsymbol{E}_{N,h}(\rho) - \boldsymbol{E}(\rho)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} - \frac{1}{2}\|\boldsymbol{E}_h^0 - \boldsymbol{E}_0\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} \Bigg) \\
&\underbrace{=}_{(4.15)} \limsup_{h\to 0} \|\boldsymbol{E}_{N,h}(\rho) - \boldsymbol{E}(\rho)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}.
\end{aligned} \tag{4.35}
$$

The remaining pointwise norm can be extracted as follows:

$$
\begin{aligned}
&\limsup_{h\to 0} -\int_0^\rho \int_\Omega \mathbf{curl}(\widehat{\boldsymbol{H}}_{N,h}(t) - \boldsymbol{H}(t)) \cdot (\widehat{\boldsymbol{E}}_{N,h}(t) - \boldsymbol{E}(t))\,\mathrm{d}x\,\mathrm{d}t \\
\underbrace{=}_{(4.13),(4.15)} &\limsup_{h\to 0} -\int_0^\rho \int_\Omega \mathbf{curl}(\widehat{\boldsymbol{H}}_{N,h}(t) - \boldsymbol{H}(t)) \cdot (\overline{\boldsymbol{E}}_{N,h}(t) - \boldsymbol{E}(t))\,\mathrm{d}x\,\mathrm{d}t \\
= &\limsup_{h\to 0} -\int_0^\rho \int_\Omega \mathbf{curl}(\widehat{\boldsymbol{H}}_{N,h}(t) - \boldsymbol{\Pi}_h\boldsymbol{H}(t)) \cdot (\overline{\boldsymbol{E}}_{N,h}(t) - \boldsymbol{E}(t))\,\mathrm{d}x\,\mathrm{d}t \\
\underbrace{=}_{(1.4),(\mathrm{P}),(\tilde{\mathrm{P}}_{N,h})} &\limsup_{h\to 0} \int_0^\rho \int_\Omega \mu \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{H}_{N,h}(t) - \boldsymbol{H}(t)) \cdot (\widehat{\boldsymbol{H}}_{N,h}(t) - \boldsymbol{\Pi}_h\boldsymbol{H}(t))\,\mathrm{d}x\,\mathrm{d}t \\
= &\limsup_{h\to 0} \int_0^\rho \int_\Omega \mu \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{H}_{N,h}(t) - \boldsymbol{H}(t)) \cdot (\widehat{\boldsymbol{H}}_{N,h}(t) - \boldsymbol{H}(t))\,\mathrm{d}x\,\mathrm{d}t \\
= &\limsup_{h\to 0} \|\boldsymbol{H}_{N,h}(\rho) - \boldsymbol{H}(\rho)\|^2_{\boldsymbol{L}^2_\mu(\Omega)},
\end{aligned} \tag{4.36}
$$

where the same argument as in (4.35) was used for the last equality. Let us recall that, due to Lemma 3.2, Proposition 3.3, and Corollary 3.5, there exists a constant $C > 0$, independent of $h$, such that

$$
\|(\boldsymbol{E}_{N,h}, \boldsymbol{H}_{N,h})\|_{W^{1,\infty}((0,T),\boldsymbol{L}^2(\Omega)\times\boldsymbol{L}^2(\Omega))} + \|\widehat{\boldsymbol{E}}_{N,h}\|_{L^\infty((0,T),\boldsymbol{L}^2(\Omega))} \le C \quad \forall h > 0. \tag{4.37}
$$

On the other hand, by the convergence property of $\boldsymbol{Q}_h$ along with the fact that $\boldsymbol{E} \in L^\infty((0,T), \boldsymbol{L}^2(\Omega))$ and Lemma 2.1, the Lebesgue dominated convergence theorem implies that

$$
\|\boldsymbol{E} - \boldsymbol{Q}_h\boldsymbol{E}\|_{L^s((0,T),\boldsymbol{L}^2(\Omega))} \to 0 \quad \text{as } h \to 0 \quad \forall 1 \le s < \infty. \tag{4.38}
$$

Indeed, given any $1 \le s < \infty$, the necessary bound for the Lebesgue dominated convergence theorem is obtained as follows:

$$
\begin{aligned}
\|\boldsymbol{E}(t) - \boldsymbol{Q}_h\boldsymbol{E}(t)\|^s_{\boldsymbol{L}^2(\Omega)} &\le \left(\|\boldsymbol{E}(t)\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{Q}_h\boldsymbol{E}(t)\|_{\boldsymbol{L}^2(\Omega)}\right)^s \\
&\underbrace{\le}_{(2.5)} 2^s\|\boldsymbol{E}(t)\|^s_{\boldsymbol{L}^2(\Omega)} \qquad \text{for a.e. } t \in (0,T)
\end{aligned}
$$

for the right-hand side being of class $L^\infty(0,T) \hookrightarrow L^1(0,T)$. Thus, by (4.37) and the nonnegativity of $\sigma$, applying (4.35)–(4.38) to (4.34), using the assumed boundedness of

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$\{\mathbf{curl}\,\widehat{\boldsymbol{H}}_{N,h}\}_{h>0}$ in $L^p((0,T),\boldsymbol{L}^2(\omega))$ with $p>1$ together with the shown boundedness in $L^\infty((0,T),\boldsymbol{L}^2(\Omega\setminus\omega))$ from Proposition 3.6, we find that

$$\lim_{h\to 0}\|(\boldsymbol{E}_{N,h},\boldsymbol{H}_{N,h})(\rho)-(\boldsymbol{E},\boldsymbol{H})(\rho)\|_{\boldsymbol{L}^2(\Omega)\times\boldsymbol{L}^2(\Omega)}\to 0 \quad \text{as } h\to 0 \quad \forall\rho\in(0,T).$$

Finally, utilizing once again the boundedness in (4.37), the proof is finished by the use of Arzelà–Ascoli's theorem. ☐

## 5. Numerical tests.

To close this paper, we carry out numerical tests of the proposed FEM ($\mathrm{P}_{N,h}$). Motivated by our real experiment (see Figure 1), we consider a numerical simulation with $\Omega=(-1,1)^3$, $T=1$, $\epsilon,\mu=1$, $\sigma=0$, $(\boldsymbol{E}_0,\boldsymbol{H}_0)=(\mathbf{0},\mathbf{0})$, and

$$\boldsymbol{f}\colon[0,1]\times\Omega\to\mathbb{R}^3,\quad \boldsymbol{f}(t,\cdot)=(0,2+10t,0).$$

The obstacle set is defined by

$$\begin{aligned}\boldsymbol{K}&=\{\boldsymbol{v}\in\boldsymbol{L}^2(\Omega)\mid|\boldsymbol{v}(x)|\le 0.05 \text{ for a.e. } x\in\omega\},\\ \omega&=(-0.25,0.25)\times(-0.5,0.5)^2.\end{aligned}\tag{5.1}$$

Note that $d=0.05$ for the electric obstacle in $\boldsymbol{K}$ is just an arbitrary choice. We may as well set $d=0$ or any nonnegative real number for the upper bound $d$.

As stated in the introduction, thanks to Theorem 2.2, there is no need to invoke an additional nonlinear solver for the computation of the VI in ($\mathrm{P}_{N,h}$). Its exact solution is given by (2.9), which makes the numerical realization of ($\mathrm{P}_{N,h}$) particularly efficient and superior to the implicit Euler time-stepping. As to numerical precision, we went with 320 time-steps and roughly 1.800.000 degrees of freedom (DoF) for $\mathbf{ND}_h$ as well as roughly 4.700.000 DoF for $\mathbf{DG}_h$. Our computations were solely done on the open-source platform FEniCS [14], and ParaView was used for visualization purposes. Figure 3 depicts two computed electric fields at the final time step $t=T$. The left figure depicts the computed electric field of the classical Maxwell equations in the absence of an electric obstacle, i.e., $\boldsymbol{K}=\boldsymbol{L}^2(\Omega)$, whereas the second one is the computed solution based on ($\mathrm{P}_{N,h}$) with the given obstacle (5.1). See also Figure 4 for the evolution of the electric field at $t=1/4,1/2,3/4,1$. Evidently, our numerical method is able to confirm the Faraday shielding effect in the obstacle region $\omega$.

Finally, to test the convergence behavior of ($\mathrm{P}_{N,h}$), we fix the above-mentioned computed solution as a reference solution $(\boldsymbol{E}_{\mathrm{ref}},\boldsymbol{H}_{\mathrm{ref}})$ since the true solution is unknown. Thereafter, we consider four different numerical solutions at coarser grids (maintaining a linear CFL-condition) and compute their error to the reference solution based on

$$\begin{aligned}\mathrm{Err}_{N,h}(\boldsymbol{E})&\coloneqq\max_{n\in\{0,\dots,N\}}\|\boldsymbol{E}_{N,h}(t_n)-\boldsymbol{E}_{\mathrm{ref}}(t_n)\|_{\boldsymbol{L}^2(\Omega)},\\ \mathrm{Err}_{N,h}(\boldsymbol{H})&\coloneqq\max_{n\in\{0,\dots,N\}}\|\boldsymbol{H}_{N,h}(t_n)-\boldsymbol{H}_{\mathrm{ref}}(t_n)\|_{\boldsymbol{L}^2(\Omega)}.\end{aligned}$$

We should point out that, based on our numerical tests, the error quantities $\mathrm{Err}_{N,h}(\boldsymbol{E})$ and $\mathrm{Err}_{N,h}(\boldsymbol{H})$ coincide with the corresponding errors at the final time $t_N=T$, i.e., $\mathrm{Err}_{N,h}(\boldsymbol{E})=\|\boldsymbol{E}_{N,h}(T)-\boldsymbol{E}_{\mathrm{ref}}(T)\|_{\boldsymbol{L}^2(\Omega)}$ and $\mathrm{Err}_{N,h}(\boldsymbol{H})=\|\boldsymbol{H}_{N,h}(T)-\boldsymbol{H}_{\mathrm{ref}}(T)\|_{\boldsymbol{L}^2(\Omega)}$. From Table 1, we clearly monitor a convergence behavior as the discretization becomes finer and finer, which serves as well as a numerical evidence of our convergence result (Theorem 4.3).

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TABLE 1
*Convergence behavior of the scheme.*

| $N$ | $5 \cdot 2^2$ | $5 \cdot 2^3$ | $5 \cdot 2^4$ | $5 \cdot 2^5$ | $5 \cdot 2^6$ |
|---|---|---|---|---|---|
| $h$ | $1/2^2$ | $1/2^3$ | $1/2^4$ | $1/2^5$ | $1/2^6$ |
| DoF($\mathbf{DG}_h$) | 1.152 | 9.216 | 31.024 | 589.824 | 4.718.592 |
| DoF($\mathbf{ND}_h$) | 604 | 4.184 | 73.728 | 238.688 | 1.872.064 |
| $\mathrm{Err}_{N,h}(\boldsymbol{E})$ | 3.2415 | 1.2647 | 0.9207 | 0.5267 | – |
| $\mathrm{Err}_{N,h}(\boldsymbol{H})$ | 3.1408 | 1.4920 | 0.8186 | 0.4352 | – |

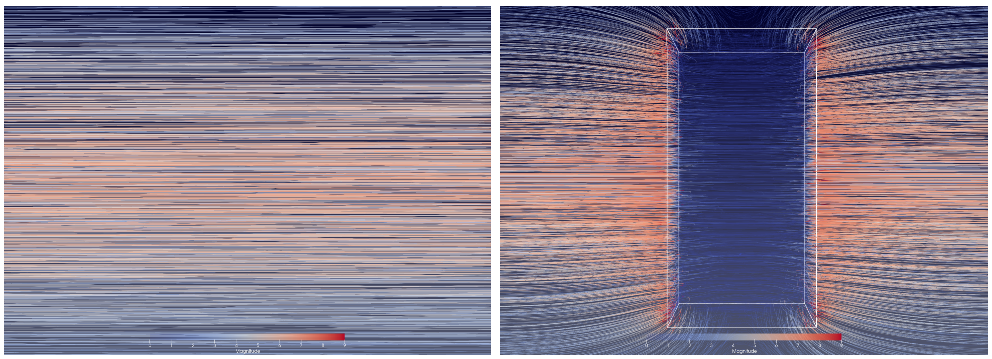

FIG. 3. *Electric field without obstacle (left) and with obstacle (right).*

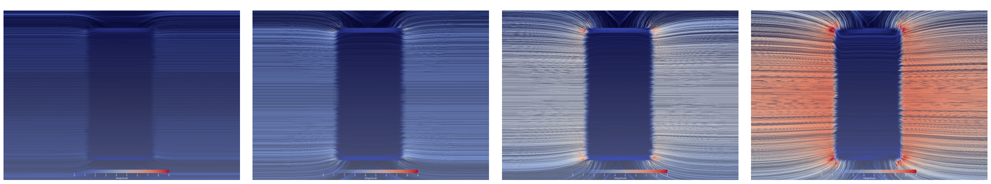

FIG. 4. *Evolution of the shielded electric field (two-dimensional slice) at* $t = 1/4, 1/2, 3/4, 1$.

## REFERENCES


[1] M. Ainsworth and J. T. Oden, *A Posteriori Error Estimation in Finite Element Analysis*, Pure Appl. Math. (N. Y.), Wiley-Interscience [John Wiley & Sons], New York, 2000, https://doi.org/10.1002/9781118032824.

[2] A. Bossavit, *Computational Electromagnetism*, Electromagnetism, Academic Press, San Diego, 1998.

[3] S. J. Chapman, D. P. Hewett, and L. N. Trefethen, *Mathematics of the Faraday cage*, SIAM Rev., 57 (2015), pp. 398–417, https://doi.org/10.1137/140984452.

[4] S. H. Christiansen and R. Winther, *Smoothed projections in finite element exterior calculus*, Math. Comp., 77 (2008), pp. 813–829, https://doi.org/10.1090/S0025-5718-07-02081-9.

[5] D. S. Clark, *Short proof of a discrete Gronwall inequality*, Discrete Appl. Math., 16 (1987), pp. 279–281, https://doi.org/10.1016/0166-218X(87)90064-3.

[6] G. Cohen and P. Monk, *Gauss point mass lumping schemes for Maxwell's equations*, Numer. Methods Partial Differential Equations, 14 (1998), pp. 63–88, https://doi.org/10.1002/(SICI)1098-2426(199801)14:1<63::AID-NUM4>3.3.CO;2-O.


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[7] G. Duvaut and J.-L. Lions, *Inequalities in Mechanics and Physics*, Grundlehren der Mathematischen Wissenschaften, 219, Springer-Verlag, Berlin, New York, 1976, translated from the French by C. W. John.
[8] A. Ern and J.-L. Guermond, *Mollification in strongly Lipschitz domains with application to continuous and discrete de Rham complexes*, Comput. Methods Appl. Math., 16 (2016), pp. 51–75, https://doi.org/10.1515/cmam-2015-0034.
[9] A. Ern and J.-L. Guermond, *Abstract nonconforming error estimates and application to boundary penalty methods for diffusion equations and time-harmonic Maxwell's equations*, Comput. Methods Appl. Math., 18 (2018), pp. 451–475, https://doi.org/10.1515/cmam-2017-0058.
[10] S. Hofmann, M. Mitrea, and M. Taylor, *Geometric and transformational properties of Lipschitz domains, Semmes-Kenig-Toro domains, and other classes of finite perimeter domains*, J. Geom. Anal., 17 (2007), pp. 593–647, https://doi.org/10.1007/BF02937431.
[11] J. Li, *Unified analysis of leap-frog methods for solving time-domain Maxwell's equations in dispersive media*, J. Sci. Comput., 47 (2011), pp. 1–26.
[12] J. Li, J. Wang Waters, and E. A. Machorro, *An implicit leap-frog discontinuous Galerkin method for the time-domain Maxwell's equations in metamaterials*, Comput. Methods Appl. Mech. Engrg., 223/224 (2012), pp. 43–54, https://doi.org/10.1016/j.cma.2012.02.016.
[13] J.-L. Lions and G. Stampacchia, *Variational inequalities*, Comm. Pure Appl. Math., 20 (1967), pp. 493–519, https://doi.org/10.1002/cpa.3160200302.
[14] A. Logg, K.-A. Mardal, and G. N. Wells, eds., *Automated Solution of Differential Equations by the Finite Element Method*, Lect. Notes Comput. Sci. Eng. 84, Springer, Berlin, Heidelberg, 2012, https://doi.org/10.1007/978-3-642-23099-8.
[15] F. Maggi, *Sets of Finite Perimeter and Geometric Variational Problems*, An Introduction to Geometric Measure Theory, Cambridge Stud. Adv. Math. 135, Cambridge University Press, Cambridge, UK, 2012, https://doi.org/10.1017/CBO9781139108133.
[16] P. A. Martin, *On acoustic and electric Faraday cages*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 470 (2014), 2014034.
[17] P. Monk, *A mixed method for approximating Maxwell's equations*, SIAM J. Numer. Anal., 28 (1991), pp. 1610–1634, https://doi.org/10.1137/0728081.
[18] J.-C. Nédélec, *Mixed finite elements in* $\mathbf{R}^3$, Numer. Math., 35 (1980), pp. 315–341.
[19] M. Winckler and I. Yousept, *Fully discrete scheme for Bean's critical-state model with temperature effects in superconductivity*, SIAM J. Numer. Anal., 57 (2019), pp. 2685–2706, https://doi.org/10.1137/18M1231407.
[20] M. Winckler, I. Yousept, and J. Zou, *Adaptive edge element approximation for* $\mathbf{H}(\mathbf{curl})$ *elliptic variational inequalities of second kind*, SIAM J. Numer. Anal., 58 (2020), pp. 1941–1964, https://doi.org/10.1137/19M1281320.
[21] K. Yee, *Numerical solution of initial boundary value problems involving Maxwell's equations in isotropic media*, IEEE Trans. Antennas Propag., 14 (1966), pp. 302–307.
[22] I. Yousept, *Hyperbolic Maxwell variational inequalities for Bean's critical-state model in type-II superconductivity*, SIAM J. Numer. Anal., 55 (2017), pp. 2444–2464, https://doi.org/10.1137/16M1091939.
[23] I. Yousept, *Hyperbolic Maxwell variational inequalities of the second kind*, ESAIM Control Optim. Calc. Var., 26 (2020), 34, https://doi.org/10.1051/cocv/2019015.
[24] I. Yousept, *Well-posedness theory for electromagnetic obstacle problems*, J. Differential Equations, 269 (2020), pp. 8855–8881, https://doi.org/10.1016/j.jde.2020.05.009.
[25] W. P. Ziemer, *Weakly Differentiable Functions*, Sobolev Spaces and Functions of Bounded Variation, Grad. Texts in Math. 120, Springer-Verlag, New York 1989, https://doi.org/10.1007/978-1-4612-1015-3.

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