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VARIATIONAL SOURCE CONDITIONS FOR INVERSE ROBIN AND FLUX PROBLEMS BY PARTIAL MEASUREMENTS

De-Han Chen and Daijun Jiang

School of Mathematics and Statistics, Central China Normal University
Wuhan 430079, China

Irwin Yousept

Universität Duisburg-Essen, Fakultät für Mathematik, Thea-Leymann-Str. 9
D-45127 Essen, Germany

Jun Zou*

Department of Mathematics, The Chinese University of Hong Kong
Shatin, Hong Kong, China

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1. **Introduction.** We are concerned in this work with the convergence of the Tikhonov regularization for two inverse problems modelled by the following elliptic equation with mixed boundary conditions:

$$\begin{cases} -\nabla\cdot(p(x)\nabla u(x)) = f(x),\ x\in\Omega, \\ p(x)\dfrac{\partial u}{\partial n}(x) = q(x),\ x\in\Gamma_1, \\ p(x)\dfrac{\partial u}{\partial n}(x)+\kappa(x)u(x) = u_a(x),\ x\in\Gamma_2, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^d$ $(d=2,3)$ is a bounded and connected domain with a $C^{1,1}$-boundary $\partial\Omega$ consisting of two disjointed parts Γ_1 and Γ_2, i.e., $\partial\Omega=\Gamma_1\cup\Gamma_2$ and $\Gamma_1\cap\Gamma_2=\emptyset$ (e.g. Fig. 1). Furthermore, the functions $p:\Omega\to\mathbb{R}$, $f:\Omega\to\mathbb{R}$, and $u_a:\Gamma_2\to\mathbb{R}$ denote, respectively, the diffusivity coefficient, the source strength, and the ambient temperature. Assuming that all these three data are available, our analysis focuses on the reconstruction of the Robin coefficient $\kappa:\Gamma_2\to\mathbb{R}$ and the boundary flux $q:\Gamma_1\to\mathbb{R}$ separately, as specified below.

Inverse Robin Problem (IRP): Suppose that p, f, u_a and q are all known, and let the open connected part Γ_a of Γ_1 be accessible (e.g., through some preset

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* Corresponding author: Jun Zou.

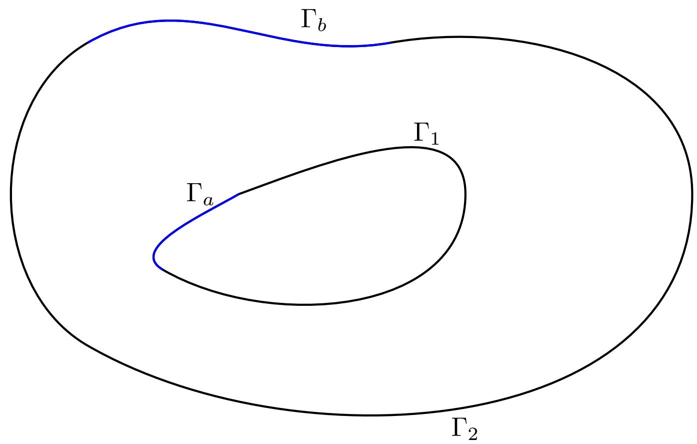

Figure 1. An example of Ω in $\mathbb{R}^2$ and Γ_a (resp. Γ_b) is an open part of Γ_1 (resp. Γ_2)

device). Recover the unknown Robin coefficient κ on the inaccessible boundary Γ_2 from the noisy measurement data z^δ of u on the accessible partial boundary Γ_a.

Inverse flux Problem (IFP): Suppose that p, f, u_a and κ are all known, and let the open connected part Γ_b of Γ_2 be accessible (e.g., through some preset device). Recover the unknown heat flux q on the inaccessible boundary Γ_1 from the noisy measurement data z^δ of u on the accessible partial boundary Γ_b.

In heat transfer, Robin boundary condition characterizes the convective heat conduction between the conducting body and the ambient environment [47]. Therefore, it is of immense practical interest in thermal problems [37], e.g. the thermal analysis of rocket nozzles and gas-turbine blades, the safety analysis of nuclear reactor elements for the loss-of-coolant accident, the thermal protection of space shuttles, the analysis of quenching processes and the development of thermally high-performance materials and composites. However, it is very difficult to acquire the exact value of the Robin coefficient. Engineers thus seek to estimate them from the accessible boundary measurements [36], which naturally gives rise to the inverse Robin problems. On the other hand, the inverse flux problems find numerous important applications in diffusive, thermal and heat transfer problems, including the real-time monitoring in steel industry [2] and the visualization by liquid crystal thermography [30]. Since it is difficult to obtain an accurate measurement on the inaccessible boundary, such as the interior boundary of nuclear reactors and steel furnaces, engineers attempt to reconstruct the flux from the measurements on the accessible boundary. Both the inverse Robin and flux problems are severly ill-posed and have attracted a lot of attention. In particular, some theoretical and numerical analysis have been studied under the framework of Tikhonov regularization (see e.g. [27, 28, 34, 49]).

The classical convergence rate was studied as early as in 1989 [14] for general inverse problems. The essence of this classical regularization theory lies in its source condition which involves the adjoint operator to the Fréchet derivative of the forward map and requires the existence of a small source function in certain sense. This smallness condition is usually hard to be verified for inverse problems of PDEs, which motivated many further studies of the convergence rates of the solutions generated by Tikhonov regularization for the inverse problems in elliptic and parabolic equations. A new effort was made in [15] for an inverse conductivity problem in a parabolic system to relax the restrictive requirements in the classical convergence theory [14]. A much simpler source condition was presented, which involved only the forward map and does not require the smallness of the source function, but the proposed source conditions can be verified only in the one spatial dimension. Convergence rates of the Tikhonov regularizations were further studied in [19] for the identification of conductivity and radiativity respectively in elliptic systems, where parameters to be identified were assumed to be known over all the boundaries. Then the source conditions can be relaxed in [15] and the source conditions can be verified in general dimensional spaces for elliptic systems. But the parameters to be identified may not be accessible over the entire boundary in most applications. The authors derived recently in [9] some reasonable convergence rates of the Tikhonov regularizations for elliptic and parabolic inverse radiativity problems in the framework of the variational source condition. Even if the radiativities may not be known on the boundary, the variational source conditions can still be rigorously verified, and the convergence results also revealed the explicit relation between the regularity of the radiativities and the convergence rates. On the other hand, an inverse conductivity problem by using the boundary data was considered in [31] for a nonlinear elliptic equation. The convergence rate of the Tikhonov regularization was derived under some source conditions similar to the ones proposed in [15]. But to verify these source conditions, it requires sufficient smoothness of the conductivity (belong to H^4), sufficient smoothness of the trace of the forward solution on the accessible boundary and the full knowledge about the conductivity on the accessible boundary. A novel convergence theory was proposed in [25] for general nonlinear inverse operator equation under a special source condition and a strong nonlinearity condition. An inverse elliptic Robin problem by the boundary data was taken to be an application for the novel theory in [25], but the special source condition and strong nonlinearity condition may not be easy to verify in general dimensional spaces. We emphasize that all the above results were achieved under the full data in the whole domain (see, e.g., [9, 15, 19]) or on the entire boundary (see, e.g., [31, 25]).

It is important to note that we may only be able to collect measurements on a part of the physical boundary or domain in most applications. We are not aware of any results in the literature about the convergence rates of Tikhonov regularizations for inverse elliptic problems when only the partial data are available. This is a major motivation of this work. To this end, we shall study the convergence of Tikhonov regularizations for the considered two inverse problems (IRP) and (IFP) in the framework of the variational source condition (VSC). VSC was initiated by Hofmann et al. [20], and its extensions were developed independently in [4], [16] and [17]. In comparison with the classical source condition, VSC does not involve the Fréchet differentiability of the forward operator, and the resulting convergence rates for the regularized solutions follow immediately from VSC under an appropriate parameter choice rule (see, e.g., [21]). We refer the reader to [22, 24, 23] for more details about the connections between VSC and classical source conditions.

Similarly to the classical source conditions, the verification of VSCs has proved to be still a nontrivial and highly technical mathematical task, and it is often highly problem dependent. VSC was verified for some abstract linear inverse problems with ℓ^p penalties [3, 6, 8], in particular, for elastic-net regularizations in [8]. The main techniques used there are the operator theory and a delicate construction of index functions. For inverse problems of PDEs, the validity of VSC was established in [22, 24] for inverse scattering problems, using the conditional stability estimates via geometrical optics solutions. Recently, the validity of VSC was shown for the ill-posed backward nonlinear Maxwell's equations in [10], by means of the semigroup theory and extrapolation of Hilbert spaces. The authors of this work verified VSC and obtained a Hölder type convergence rate in [7, 9, 11] for the Tikhonov regularized solutions of the inverse elliptic and parabolic radiativity problems. It is important to emphasize that all the above results were established under the full measurement data.

In this work, we shall first derive some logarithmic type stability estimates for the proposed inverse problems and then propose some new variational source conditions in order to achieve some desirable convergence rates of the Tikhonov regularizations for the inverse problems. We are aware of a novel methodology that was developed recently in [46], which enables us to derive the convergence rates directly from some conditional stability estimates without VSCs. However, it is still unclear how to apply the approach to some concrete inverse problems of PDEs and how to derive the required conditional stability estimates.

There are several important novelties in this work. Firstly, we are able to rigorously verify some VSC under a newly established logarithmic-type stability and the sufficient conditions depend mainly on the regularity of the true Robin coefficient or distributed flux. We shall also construct two counterexamples to illustrate that the optimal stabilities for both the inverse Robin and flux problems are of logarithmic type (See Remark 3.3). The second novelty is that the convergence rates can be achieved when measurement data are available over an arbitrary small accessible partial boundary Γ_a (or Γ_b). As the third novelty, our results reveal the relation between the regularity of the Robin coefficient (resp. flux) and the corresponding convergence rate.

The paper is organized as follows. In Section 2, we introduce our general assumptions and present the mathematical formulations of (IRP) and (IFP). In Section 3, the logarithmic type stability estimates are derived for both (IRP) and (IFP). Section 4 is devoted to verifying VSC and establishing convergence raters for (IRP) and (IFP). Some concluding remarks are given in Section 5.

2. **Preliminaries.**

Given a linear operator $T : X \to X$ on a complex Banach space X, let $\mathrm{Ker}(T)$, $D(T)$, $\rho(T)$ and $\sigma(T)$ stand for the kernel, domain, resolvent, and spectrum of T. A linear operator $T : D(T) \subset X \to X$ is called closed if its graph $\{(x, Tx),\ x \in D(T)\}$ is closed in $X \times X$. Furthermore, the adjoint of a densely defined operator $T : D(T) \subset X \to X$ is denoted by $T^* : D(T^*) \subset X \to X$. We call $T : D(T) \subset X \to X$ symmetric if $Tx = T^*x$ holds true for all $x \in D(T)$, i.e., $(Tx, y)_X = (x, Ty)_X$ for all $x, y \in D(T)$. If a symmetric operator T satisfies that $D(T) = D(T^*)$, then T is said to be self-adjoint.

For $1 \leq i, j \leq d$ and a sufficiently regular Sobolev function $u : \mathbb{R}^d \to \mathbb{R}$, we write $\partial_i u := \partial u/\partial x_i$, $\nabla u = (\partial_1 u, \ldots, \partial_d u)$, and $\partial_{i,j} u := \partial^2 u/\partial x_i \partial x_j$. Given the Hessian matrix u'' of a function u, we write $u''(x, y) := \sum_{i,j=1}^d \partial_{i,j} u x_i y_j$ with the vectors $x = (x_1, \cdots, x_d)$ and $y = (y_1, \cdots, y_d) \in \mathbb{C}^d$.

For any $s \in \mathbb{R}$, we define the fractional Sobolev space

$$H^s(\mathbb{R}^d) := \{u \in \mathcal{S}(\mathbb{R}^d)' \mid \|u\|^2_{H^s(\mathbb{R}^d)} := \int_{\mathbb{R}^d} (1+|\xi|^2)^s |(\mathcal{F}u)(\xi)|^2 d\xi < +\infty\},$$

where $\mathcal{F} : \mathcal{S}(\mathbb{R}^d)' \to \mathcal{S}(\mathbb{R}^d)'$ is the Fourier transform, and $\mathcal{S}(\mathbb{R}^d)'$ denotes the tempted distribution space (see, e.g., [35, 48, 50]). For a bounded domain $U \subset \mathbb{R}^d$ with a Lipschitz boundary ∂U, the space $H^s(U)$ with a possibly non-integer exponent $s \geq 0$ is defined as the space of all complex-valued functions $v \in L^2(U)$ satisfying $V_{|U} = v$ for some $V \in H^s(\mathbb{R}^d)$, endowed with the norm

$$\|v\|_{s,U} := \inf_{\substack{V_{|U}=v \\ V \in H^s(\mathbb{R}^d)}} \|V\|_{H^s(\mathbb{R}^d)}.$$

For every $s \in [0,\infty)$, we denote by $\lfloor s \rfloor \in [0, s]$ the largest integer less or equal to s. If s is a non-negative integer, then $H^s(U)$ coincides with the classical Sobolev space. For a compact, d-dimensional $C^{k,\kappa}$-manifold M with an integer $k \geq 0$ and $\kappa \in \{0,1\}$, we can define the Sobolev space $H^s(M)$ on M for all $0 \leq s \leq k+\kappa$ via partitions of unity and Sobolev spaces $H^s(\mathbb{R}^d)$ (see, e.g., [48]). In particular, for a bounded domain U of class $C^{k,\kappa}$, its boundary ∂U is a compact $C^{k,\kappa}$ manifold and then the Sobolev space $H^s(\partial U)$ is defined as in [48]. Moreover, in the case of $s \in (0,\infty)$ with $s = \lfloor s \rfloor + \sigma$ and $0 < \sigma < 1$, the norm of $\|u\|_{H^s(\partial U)}$ is also equivalent to

$$\left(\sum_{|\alpha| \leq \lfloor s \rfloor} \|D^\alpha u\|^2_{L^2(\partial U)} + \sum_{|\alpha| \leq \lfloor s \rfloor} \iint_{\partial U \times \partial U} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n-1+2\sigma}} dS_x dS_y \right)^{\frac{1}{2}}.$$

By $H^{-s}(M)$ we denote the dual of $H^s(M)$ with respect to the inner product in $L^2(M)$. We also write $H^0(M) = L^2(M)$ and denote its norm and scalar product by $\|\cdot\|_M$ and $(\cdot,\cdot)_M$ respectively. By a standard argument (as used in [35, Theorem 7.7]), for every $-(k+\kappa) \leq s_1 < s_2 \leq k+\kappa$, one has for all $\theta \in [0,1]$ and $s = s_1(1-\theta) + s_2\theta$ that

$$[H^{s_1}(M), H^{s_2}(M)]_\theta = H^s(M), \tag{2.1}$$

with equivalent norms (See. e.g. [42, Proposition 2.3.11 and 2.4.3]). In particular, one has

$$\|u\|_{s,M} \leq C\|u\|^{1-s}_{s_1,M} \|u\|^s_{s_2,M} \quad \forall\, u \in H^{s_2}(M) \tag{2.2}$$

for some $C > 0$, depending only on M, s_0, s_1 and θ. We would like to mention that (2.2) also holds for $M = U$ and $0 \leq s_1, s_2 \leq k+\kappa$.

We then recall some estimates on the pointwise product operator $(u,v) \to uv$ in the fractional Sobolev spaces. In view of [44, Corollary 2.1], [50, p. 49] and [43, Theorem 4], we have the following results with the help of partition of unity together with a local mapping.

Lemma 2.1. *Let M be a d-dimensional, $C^{1,1}$-compact manifold.*

(i) *Assume that s and t are two real numbers such that $0 < r, s < \frac{d}{2}$, then for $t := r + s - \frac{d}{2}$, $u \in H^s(M)$ and $v \in H^r(M)$, one has $uv \in H^t(M)$ and the following inequality holds*

$$\|uv\|_{t,M} \leq C\|u\|_{s,M}\|v\|_{r,M}$$

with a constant $C > 0$ independent of u and v.

(ii) *If* $2 > s > \frac{d}{2}$, *then*

$$\|uv\|_{s,M} \le C\|u\|_{s,M}\|v\|_{s,M}, \quad \forall\, u, v \in H^s(M),$$

with a constant $C > 0$ *independent of* u *and* v.

(iii) *If* $F : \mathbb{C} \to \mathbb{C}$ *is a globally Lipschitz function, then* $F(u) \in H^1(M)$ *when* $u \in H^1(M)$ *and* $\|F(u)\|_{H^1(M)} \le C\|u\|_{H^1(M)}$ *for a positive constant* $C > 0$, *independent of* u.

Let us now formulate the regularity assumptions on the fixed data involved in (1.1).

(H1) Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded and connected $C^{1,1}$-domain. There exist $(d-1)$-dimensional compact $C^{1,1}$-manifolds Γ_1, Γ_2 such that $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\partial\Omega = \Gamma_1 \cup \Gamma_2$.

(H2) We assume that $p \in C^{0,1}(\overline{\Omega})$ with

$$p_{min} := \min_{x\in\overline{\Omega}} p(x) > 0, \tag{2.3}$$

$f \in L^2(\Omega)$, and $u_a \in H^{1/2}(\Gamma_2)$.

In our study below, the measurements are only taken on a part Γ_p of the whole boundary $\partial\Omega$. Since it may not necessarily be a compact manifold itself, we need to impose some further requirements for the definitions of the Sobolev spaces on the boundary. More precisely, we say that a tuple (Γ_p, Ω) is of $C^{1,1}$ provided that for each $x \in \overline{\Omega}$, there exists an open neighborhood N_x in $\mathbb{R}^n$ and $C^{1,1}$-diffeomorphisms $\Phi_x : N_x \to \mathbb{R}^n$ such that $V_x := \Phi_x(N_x)$ satisfies

$$\begin{aligned} \Phi_x(N_x \cap \Omega) &= V_x \cap \mathbb{R}^n_+, \\ \Phi_x(N_x \cap \partial\Omega) &= V_x \cap \partial\mathbb{R}^n_+, \\ \Phi_x(N_x \cap \Gamma_p) &= V_x \cap \{\partial\mathbb{R}^{n-1}_+ \times \{0\}\}, \\ \Phi_x(N_x \cap \partial\Gamma_p) &= V_x \cap \{\mathbb{R}^{n-2} \times \{(0,0)\}\}, \end{aligned}$$

where $\mathbb{R}^n_+ := \{(x_1, x_2, \cdots, x_n); x_n > 0\}$ for $n \ge 1$ (see e.g. [41, 40]). If a tuple (Γ_p, Ω) is of $C^{1,1}$, for each $s \ge 0$, we define $H^s(\Gamma_p)$ to be the space of all complex-valued functions $v \in L^2(\partial\Omega)$ satisfying $V_{|\Gamma_p} = v$ for some $V \in H^s(\partial\Omega)$, endowed with the norm

$$\|v\|_{s,\Gamma_p} := \inf_{\substack{V_{|U}=v \\ V\in H^s(\partial\Omega)}} \|V\|_{H^s(\partial\Omega)}. \tag{2.4}$$

On the other hand, we set $H^s_0(\Gamma_p)$ to be the subspace of $H^s(\Gamma_p)$ such that for each $v \in H^s_0(\Gamma_p)$, its trivial extension $\widetilde{v} = v\chi_{\Gamma_p}$ belongs to $H^s(\partial\Omega)$. The dual space of $H^s_0(\Gamma_p)$ is denoted by $H^{-s}(\Gamma_p)$ (See e.g. [39, Subsection 1.2.1.4]). Then, the trace mapping $\gamma_p : H^s(\Omega) \to H^{s-\frac{1}{2}}(\Gamma_p)$ is continuous when $s > \frac{1}{2}$ ([39, Theorem 1.37]). From (2.2) and (2.4) it follows that the following interpolation result is valid:

$$\|u\|_{1/2,\Gamma_p} \le C_p\|u\|^{1/2}_{1,\Gamma_p}\|u\|^{1/2}_{0,\Gamma_p} \quad \forall u \in H^1(\Gamma_p), \tag{2.5}$$

where C_p is independent of u (see e.g. [41]).

In this paper, we shall use the following admissible set for the Robin coefficient data.

$$\mathcal{U} := \{\kappa \in L^\infty(\Gamma_2) \mid \underline{\kappa} \le \kappa(x) \le \bar{\kappa} \quad \text{for a.e. } x \in \Gamma_2\} \tag{2.6}$$

with fixed constants $0 < \underline{\kappa} < \bar{\kappa} < \infty$.

The following well-possedness results follow from [35, Theorem 7.4] and the compact perturbation theorem of Fredholm operators, whose proof is beyond the scope of this paper and is included in the Appendix.

Lemma 2.2. *Suppose* **(H1)**-**(H2)** *hold. Then, for all* $q \in L^2(\Gamma_1)$ *and* $\kappa \in \mathcal{U}$, *the system* (1.1) *admits a unique weak solution* $u \in H^{3/2}(\Omega)$ *satisfying*

$$\|u\|_{H^{3/2}(\Omega)} \le C(\|f\|_{0,\Omega} + \|q\|_{0,\Gamma_1} + \|u_a\|_{0,\Gamma_2})$$

with a constant $C > 0$, *independent of* u, f, q, u_a, *and* $\kappa \in \mathcal{U}$.

Let us now recall a stability estimate result, which plays a profound role in our analysis. In the following, for a $C^{1,1}$-tuple (Γ_p, Ω), we denote by $H^1_{co}(\Omega \cup \Gamma_p)$ the closed subspace of $H^1(\Omega)$ containing all functions with compact support in $\Omega \cup \Gamma_p$.

Lemma 2.3 ([1, Theorem 1.9])**.** *Let the conditions of Lemma 2.2 be satisfied. Furthermore, suppose that* (Γ_p, Ω) *is a* $C^{1,1}$*-couple and* $u \in H^1(\Omega)$ *satisfying*

$$\int_\Omega p\nabla u \cdot \nabla v dx = \langle p\partial_n u, v\rangle_{H^{-1/2}(\Gamma_p)\times H^{1/2}(\Gamma_p)} \quad \forall\, v \in H^1_{co}(\Omega \cup \Gamma_p),$$

and

$$\|u\|_{1,\Omega} \le M \tag{2.7}$$

for some positive constant M. *Then, there exist* $\mu^* \in (0,1)$ *and* $C > 0$, *independent of* M *and* u, *such that*

$$\|u\|_{L^2(\Omega)} \le C(M + \|u\|_{1/2,\Gamma_p} + \|p\partial_n u\|_{-1/2,\Gamma_p}) \frac{1}{\log^{\mu^*}\left(\frac{M + \|u\|_{1/2,\Gamma_p} + \|p\partial_n u\|_{-1/2,\Gamma_p}}{\|u\|_{1/2,\Gamma_p} + \|p\partial_n u\|_{-1/2,\Gamma_p}}\right)}. \tag{2.8}$$

Remark 2.4. F. John and L.E. Payne [29, 38] initiated the general study of stabilities for ill-posed elliptic Cauchy problems, i.e., assuming a solution u of the Cauchy problem is a priori known to be bounded over Ω, and the Cauchy data are prescribed on some portion Γ of the boundary $\partial\Omega$, one wishes to estimate the solution u in an interior domain or the whole domain Ω. The former is called stability estimates in the interior, while the later is called global stability estimates. For linear elliptic equations with analytic coefficients and classical solutions, F. John [29] established Hölder continuity for stability estimates in the interior and logarithmic global stability estimates. In [38], the author relaxed the restriction on coefficients and established similar estimates for strong solutions of elliptic equations, while in [1] we can find the same estimates for merely weak solution of elliptic equations. Moreover, the classical example dated back to Hadamard shows that for the stability estimate in the interior we cannot expect anything better than a Hölder rate [1, 18], whereas for the global stability the optimal rate will be of logarithmic type (see, e.g., [1]).

2.1. **Mathematical formulation for** (IRP)**.** We first introduce an assumption for the inverse Robin problem.

(HR) Assume that $q \in L^2(\Gamma_1)$, and $\kappa^\dagger \in \mathcal{U}$ denotes the true Robin coefficient of (1.1). Furthermore, let $\Gamma_a \subset \Gamma_1$ be a connected open part of Γ_1 such that (Γ_a, Ω) is a $C^{1,1}$-tuple.

Let us now assume that **(H1)-(H2)** and **(HR)** are satisfied. In view of Lemma 2.2, the elliptic problem (1.1) admits for every $\kappa \in \mathcal{U}$ a unique weak solution $u = u(\kappa) \in H^{3/2}(\Omega)$ satisfying the variational form:

(2.9)
$$\int_\Omega p\nabla u(\kappa)\cdot\nabla v dx + \int_{\Gamma_2} \kappa u(\kappa) v dS = \int_\Omega f v dx + \int_{\Gamma_1} q v dS + \int_{\Gamma_2} u_a v dS \quad \forall v \in H^1(\Omega).$$

Introducing the noisy data $u^\delta \in L^2(\Gamma_a)$ resulting from the partial measurement of the true solution $u(\kappa^\dagger)$ on $\Gamma_a \subset \Gamma_1$ satisfying (with a noise level $\delta > 0$)

$$\|u^\delta - u(\kappa^\dagger)\|_{0,\Gamma_a} \le \delta, \tag{2.10}$$

the inverse Robin coefficient problem of our interest reads as follows:

$$\text{Find } \kappa \in \mathcal{U} \text{ s.t.} \quad u(\kappa) = u^\delta. \tag{IRP}$$

To deal with the ill-posedness in (IRP), we consider the Tikhonov regularization method:

$$\min_{\kappa\in\mathcal{U}} \frac{1}{2}\|u(\kappa) - u^\delta\|^2_{0,\Gamma_a} + \frac{\alpha}{2}\|\kappa - \kappa^*\|^2_{0,\Gamma_2}, \tag{2.11}$$

where $\alpha > 0$ is a regularization parameter and $\kappa^* \in L^\infty(\Gamma_2)$ is an a priori estimate of the true parameter $\kappa^\dagger$. Note that κ^* does not necessarily lie in the admissible set $\mathcal{U}$. By the well-known arguments (cf. [28, Lemma 3.1]), the Tikhonov regularization problem (2.11) admits a solution, which we refer to as the regularized solution to the inverse Robin problem (IRP).

Lemma 2.5. *Under* **(H1)-(H2)** *and* **(HR)**, (2.11) *admits a global optimal solution* $\kappa^\delta_\alpha \in \mathcal{U}$.

2.2. **Mathematical formulation for (IFP).** In the case of the inverse flux problem, the Robin coefficient $\kappa \in \mathcal{U}$ is supposed to be the known data, and our goal is to reconstruct the flux parameter $q \in L^2(\Gamma_1)$. In other words, in place of **(HR)**, we now pose the following assumption:

(HF) Suppose that $\kappa \in \mathcal{U}$, and $q^\dagger \in L^2(\Gamma_1)$ denotes the true boundary flux of (1.1). Furthermore, let $\Gamma_b \subset \Gamma_2$ be a connected open part of Γ_2 such that (Γ_b, Ω) is a $C^{1,1}$ couple.

As in (IRP), to emphasize the solution dependence on the heat flux q, the unique weak solution to (1.1) is denoted by $u = u(q) \in H^1(\Omega)$ satisfying the variational formulation:

$$\int_\Omega p\nabla u(q)\cdot\nabla v dx + \int_{\Gamma_2} k u(q) v dS = \int_\Omega f v dx + \int_{\Gamma_1} q v dS + \int_{\Gamma_2} u_a v dS. \tag{2.12}$$

Introducing the noisy data $z^\delta \in L^2(\Gamma_b)$ resulting from the partial measurement of the true solution $u(q^\dagger)$ on $\Gamma_b \subset \Gamma_2$ satisfying

$$\|z^\delta - u(q^\dagger)\|_{0,\Gamma_b} \le \delta, \tag{2.13}$$

we focus on the following inverse flux problem:

$$\text{Find } q \in L^2(\Gamma_1) \text{ s.t.} \quad u(q) = z^\delta. \tag{IFP}$$

Then, the Tikhonov regularization method for (IRP) reads as:

$$\min_{q\in L^2(\Gamma_1)} \frac{1}{2}\|u(q) - z^\delta\|^2_{0,\Gamma_b} + \frac{\alpha}{2}\|q\|^2_{0,\Gamma_1}, \tag{2.14}$$

where $\alpha > 0$ is the regularization parameter. The well-known arguments (cf. [49]) yield the existence of the solutions to (2.14).

Lemma 2.6. *Under* **(H1)**-**(H2)** *and* **(HF)**, (2.14) *admits a global optimal solution* $q_\alpha^\delta \in L^2(\Gamma_1)$.

3. Conditional stability estimates.

3.1. Stability estimate for (IRP).

Theorem 3.1. *Let* (**H1**), (**H2**), *and* (**HR**) *hold. Suppose that* $|u(\kappa^\dagger)| \geq c_0$ *a.e. on* Γ_2 *for some positive constant* $c_0 > 0$, *then for every* $\epsilon \in (0, 1/2)$, *there exists a positive constant* C *such that*

$$\|\kappa - \kappa^\dagger\|_{-\frac{1}{2}-\epsilon,\Gamma_2} \leq C \frac{1}{\log^{\mu^*/3}\left(\frac{1}{\|u(\kappa)-u(\kappa^\dagger)\|_{0,\Gamma_a}} + 1\right)} \quad \forall \kappa \in \mathcal{U} \tag{3.1}$$

with $\mu^* \in (0, 1)$ *is the same as in Lemma 2.3.*

Proof. Let $\epsilon \in (0, 1/2)$. The claim is trivial for $\kappa = \kappa^\dagger$. Therefore, let $\kappa^\dagger \neq \kappa \in \mathcal{U}$. We split the proof into two steps.

Step 1. We first prove that there exists a constant $C > 0$, independent of $\kappa \in \mathcal{U}$, such that

$$\|\kappa - \kappa^\dagger\|_{-\frac{1}{2}-\epsilon,\Gamma_2} \leq C\|u(\kappa) - u(\kappa^\dagger)\|_{1,\Omega}. \tag{3.2}$$

In view of the definition of the variational form (2.9) for $u(\kappa)$, we have that for any $v \in H^1(\Omega)$,

$$\int_\Omega p\nabla u(\kappa) \cdot \nabla v dx + \int_{\Gamma_2} \kappa u(\kappa) v dS = \int_\Omega p\nabla u(\kappa^\dagger) \cdot \nabla v dx + \int_{\Gamma_2} \kappa^\dagger u(\kappa^\dagger) v dS,$$

which ensures that

$$\int_\Omega p\nabla(u(\kappa) - u(\kappa^\dagger)) \cdot \nabla v dx + \int_{\Gamma_2} \kappa(u(\kappa) - u(\kappa^\dagger)) v dS = \int_{\Gamma_2} (\kappa^\dagger - \kappa) u(\kappa^\dagger) v dS. \tag{3.3}$$

Hence, using the Cauchy-Schwarz inequality and the trace theorem, we get

$$\begin{aligned}\left|\int_{\Gamma_2} (\kappa^\dagger - \kappa) u(\kappa^\dagger) v dS\right| &\leq \bar{\kappa}\|u(\kappa) - u(\kappa^\dagger)\|_{0,\Gamma_2}\|v\|_{0,\Gamma_2} + \|p\|_{C(\overline{\Omega})}\|u(\kappa) - u(\kappa^\dagger)\|_{1,\Omega}\|v\|_{1,\Omega} \\ &\leq C\|u(\kappa) - u(\kappa^\dagger)\|_{1,\Omega}\|v\|_{1,\Omega}.\end{aligned} \tag{3.4}$$

As $u(\kappa^\dagger) \in H^{\frac{3}{2}}(\Omega)$ (by Lemma 2.2), we obtain by trace theorem that $u(\kappa^\dagger) \in H^1(\Gamma_2)$. Based on the condition that $|u(\kappa^\dagger)| \geq c_0 > 0$ a.e. on Γ_2, Lemma 2.1 (iii) implies that $1/u(\kappa^\dagger) \in H^1(\Gamma_2)$. Let $\varphi \in H^{1/2+\epsilon}(\Gamma_2)$ be arbitrarily fixed. In the case of $d = 2$, Lemma 2.1 (ii) implies that $\varphi/u(\kappa^\dagger) \in H^{1/2+\epsilon}(\Gamma_2)$ with

$$\|\varphi/u(\kappa^\dagger)\|_{1/2+\epsilon,\Gamma_2} \leq C\|\varphi\|_{1/2+\epsilon,\Gamma_2}\|1/u(\kappa^\dagger)\|_{1/2+\epsilon,\Gamma_2} \tag{3.5}$$

with a constant $C > 0$ independent of φ and $u(\kappa^\dagger)$. As the trace mapping $H^{1+\epsilon}(\Omega) \to H^{1/2+\epsilon}(\Gamma_2)$ is surjective with some continuous right inverse (see. e.g. [35]), we can find some $v_\varphi \in H^{1+\epsilon}(\Omega)$ such that $v_\varphi|_{\Gamma_2} = \varphi/u(\kappa^\dagger)$ and

$$\|v_\varphi\|_{1+\epsilon,\Omega} \leq C\|\varphi/u(\kappa^\dagger)\|_{1/2+\epsilon,\Gamma_2} \leq C\|\varphi\|_{1/2+\epsilon,\Gamma_2}\|1/u(\kappa^\dagger)\|_{1/2+\epsilon,\Gamma_2}. \tag{3.6}$$

For this reason, we can deduce by choosing $v = v_\varphi$ in (3.4) and using (3.6) that

$$\begin{aligned}\left|\int_{\Gamma_2}(\kappa^\dagger-\kappa)\varphi dS\right| &\le C\|u(\kappa)-u(\kappa^\dagger)\|_{1,\Omega}\|v_\varphi\|_{1,\Omega} \le C\|u(\kappa)-u(\kappa^\dagger)\|_{1,\Omega}\|v_\varphi\|_{1+\epsilon,\Omega}\\ &\le C\|u(\kappa)-u(\kappa^\dagger)\|_{1,\Omega}\|\varphi\|_{1/2+\epsilon,\Gamma_2}\|1/u(\kappa^\dagger)\|_{1/2+\epsilon,\Gamma_2}. \end{aligned} \tag{3.7}$$

As $\varphi \in H^{1/2+\epsilon}(\Gamma_2)$ was chosen arbitrarily, the estimate (3.7) yields the desired (3.2) for the case of $d = 2$.

Let us now prove (3.2) for the three-dimensional case, namely $d = 3$. In this case, Lemma 2.1 (i) implies that

$$\|\varphi/u(\kappa^\dagger)\|_{1/2+\epsilon/2,\Gamma_2} \le C\|\varphi\|_{1/2+\epsilon,\Gamma_2}\|1/u(\kappa^\dagger)\|_{1-\epsilon/2,\Gamma_2} \tag{3.8}$$

with a constant $C > 0$ independent of φ and $u(\kappa^\dagger)$. Again, as the trace mapping $H^{1+\epsilon/2}(\Omega) \to H^{1/2+\epsilon/2}(\Gamma_2)$ is also surjective with some continuous right inverse, we can find some $v_\varphi \in H^{1+\epsilon/2}(\Omega)$ such that $v_\varphi|_{\Gamma_2} = \varphi/u(\kappa^\dagger)$ and

$$\|v_\varphi\|_{1+\epsilon/2,\Omega} \le C\|\varphi/u(\kappa^\dagger)\|_{1/2+\epsilon/2,\Gamma_2} \le C\|\varphi\|_{1/2+\epsilon,\Gamma_2}\|1/u(\kappa^\dagger)\|_{1-\epsilon/2,\Gamma_2}. \tag{3.9}$$

Therefore, applying (3.9) to (3.4) with $v = v_\varphi$ in (3.4), it follows that

$$\begin{aligned}\left|\int_{\Gamma_2}(\kappa^\dagger-\kappa)\varphi dS\right| &\le C\|u(\kappa)-u(\kappa^\dagger)\|_{1,\Omega}\|v_\varphi\|_{1+\epsilon/2,\Omega}\\ &\le C\|u(\kappa)-u(\kappa^\dagger)\|_{1,\Omega}\|\varphi\|_{1/2+\epsilon,\Gamma_2}\|1/u(\kappa^\dagger)\|_{1-\epsilon/2,\Gamma_2},\end{aligned}$$

which concludes (3.2) for the case $d = 3$.

Step 2. Thanks to (3.2), we can prove (3.1) by estimating $\|u(\kappa)-u(\kappa^\dagger)\|_{1,\Omega}$ in terms of $\|u(\kappa)-u(\kappa^\dagger)\|_{0,\Gamma_a}$. In view of the definition of $u(\kappa)$, the function $w \doteq u(\kappa)-u(\kappa^\dagger)$ satisfies the following elliptic system

$$\begin{cases} -\nabla\cdot(p\nabla w) &= 0 \quad \text{in } \Omega,\\ p\dfrac{\partial w}{\partial n} &= 0 \quad \text{on } \Gamma_1,\\ p\dfrac{\partial w}{\partial n}+\kappa w &= (\kappa^\dagger-\kappa)u(\kappa^\dagger) \quad \text{on } \Gamma_2.\end{cases} \tag{3.10}$$

Then, an combination of Lemma 2.2 and the trace theorem implies that

$$\|u(\kappa)-u(\kappa^\dagger)\|_{\frac{3}{2},\Omega} \le C\|(\kappa^\dagger-\kappa)u(\kappa^\dagger)\|_{0,\Gamma_2} \le C2\bar{\kappa}\|u(\kappa^\dagger)\|_{1,\Omega} \le M \quad \forall\,\kappa\in\mathcal{U}, \tag{3.11}$$

where M is a sufficiently large constant that will be fixed later. Applying the interpolation result (2.2) with $M = \Omega$, $s_0 = 0$, $s_1 = 3/2$ and $s = 1$ and (3.11), it follows that there exists a constant $C > 0$ independent of $\kappa \in \mathcal{U}$ such that

$$\|u(\kappa)-u(\kappa^\dagger)\|_{1,\Omega} \le C\|u(\kappa)-u(\kappa^\dagger)\|_{3/2,\Omega}^{2/3}\|u(\kappa)-u(\kappa^\dagger)\|_{0,\Omega}^{1/3} \le CM^{2/3}\|u(\kappa)-u(\kappa^\dagger)\|_{0,\Omega}^{1/3}. \tag{3.12}$$

or this reason along with (3.2), we can conclude that

$$\|\kappa-\kappa^\dagger\|_{-\frac{1}{2}-\epsilon,\Gamma_2} \le C\|u(\kappa)-u(\kappa^\dagger)\|_{1,\Omega} \le CM^{2/3}\|u(\kappa)-u(\kappa^\dagger)\|_{0,\Omega}^{1/3} \quad \forall \kappa\in\mathcal{U}. \tag{3.13}$$

Hence, it remains to estimate $\|u(\kappa)-u(\kappa^\dagger)\|_{0,\Omega}$ in terms of $\|u(\kappa)-u(\kappa^\dagger)\|_{1/2,\Gamma_a}$. Thanks to (3.11), we may apply Lemma 2.3 with $\Gamma_p = \Gamma_a \subset \Gamma_1$ to (3.10), and obtain that

$$\|u(\kappa)-u(\kappa^\dagger)\|_{0,\Omega} \le C(M+\|u(\kappa)-u(\kappa^\dagger)\|_{1/2,\Gamma_a})\frac{1}{\log^{\mu^*}\left(\frac{M+\|u(\kappa)-u(\kappa^\dagger)\|_{1/2,\Gamma_a}}{\|u(\kappa)-u(\kappa^\dagger)\|_{1/2,\Gamma_a}}\right)}. \tag{3.14}$$

On the other hand, by the interpolation result (2.5) and the continuity of the trace mapping $H^{3/2}(\Omega) \to H^1(\Gamma_a)$, it holds that

$$\|u(\kappa) - u(\kappa^\dagger)\|_{1/2,\Gamma_a} \leq C\|u(\kappa) - u(\kappa^\dagger)\|_{1,\Gamma_a}^{1/2}\|u(\kappa) - u(\kappa^\dagger)\|_{0,\Gamma_a}^{1/2}$$

$$\leq \quad C\|u(\kappa) - u(\kappa^\dagger)\|_{\frac{3}{2},\Omega}^{1/2}\|u(\kappa) - u(\kappa^\dagger)\|_{0,\Gamma_a}^{1/2} \underbrace{\leq}_{(3.11)} CM^{1/2}\|u(\kappa) - u(\kappa^\dagger)\|_{0,\Gamma_a}^{1/2} \tag{3.15}$$

with a constant $C > 0$ independent of $\kappa \in \mathcal{U}$. As the mapping $x \mapsto (M + x)/x$ is monotonically decreasing in $(0, \infty)$, we deduce from (3.15) that

$$\frac{M + \|u(\kappa) - u(\kappa^\dagger)\|_{1/2,\Gamma_a}}{\|u(\kappa) - u(\kappa^\dagger)\|_{1/2,\Gamma_a}} \geq \frac{M^{1/2}/C + \|u(\kappa) - u(\kappa^\dagger)\|_{0,\Gamma_a}^{1/2}}{\|u(\kappa) - u(\kappa^\dagger)\|_{0,\Gamma_a}^{1/2}} \geq \frac{(1 + \|u(\kappa) - u(\kappa^\dagger)\|_{0,\Gamma_a})^{1/2}}{\|u(\kappa) - u(\kappa^\dagger)\|_{0,\Gamma_a}^{1/2}}$$

by choosing M large enough. Applying the above inequality to (3.14), it follows that

$$\begin{aligned}\|u(\kappa) - u(\kappa^\dagger)\|_{0,\Omega} \quad &\leq \quad 2^{\mu^*}C(M + \|u(\kappa) - u(\kappa^\dagger)\|_{1/2,\Gamma_a})\frac{1}{\log^{\mu^*}\left(\frac{1}{\|u(\kappa)-u(\kappa^\dagger)\|_{0,\Gamma_a}} + 1\right)} \\ &\underbrace{\leq}_{(3.11)} \quad C\frac{M}{\log^{\mu^*}\left(\frac{1}{\|u(\kappa)-u(\kappa^\dagger)\|_{0,\Gamma_a}} + 1\right)}.\end{aligned}$$

Concluding from (3.2), (3.13), and the above estimate, we conclude the desired claim. □

Remark 3.2. We add a remark about the condition on the lower bound of the forward solution in Theorem 3.1. If f is non-negative a.e. in Ω, q is non-negative a.e. on Γ_1, and u_a is nonnegative a.e. on Γ_2, then $u(\kappa^\dagger) \geq 0$ a.e. in Ω by [33, Theorem 2]. If, in addition, Ω is convex and smooth, $f \in L^\infty(\Omega)$, $u(\kappa^\dagger) \in L^\infty(\Omega)$ and $u(\kappa^\dagger) \not\equiv 0$, then $u(\kappa^\dagger)$ is Hölder continuous, and there exists $C_0 > 0$ such that $u(\kappa^\dagger) \geq C_0$ in $\overline{\Omega}$ (see, e.g., [33, Theorem 4].).

At this point, let us underline that the established logarithmic-type estimate (3.1) is readily sharp and cannot be improved by Hölder- or Lipschitz-type estimate. To demonstrate this fact, we consider the following counterexample extending the one proposed by [12] to our inverse problem (IRP).

Let $B_1(0)$ (resp. $B_{1/2}(0)$) denote the open ball in $\mathbb{R}^2$ of radius 1 (resp. 1/2) centered at 0. We set $\Omega = B_1(0)\backslash B_{1/2}(0) \subset \mathbb{R}^2$, $\Gamma_1 = \partial B_1(0)$, and $\Gamma_2 = \partial B_{1/2}(0)$ and consider the following elliptic problem:

$$\begin{cases} \triangle u &= 0 \quad \text{in} \quad \Omega\,, \\ \frac{\partial u}{\partial n} + \kappa u &= 0 \quad \text{on} \quad \Gamma_2\,, \\ \frac{\partial u}{\partial n} &= 1 \quad \text{on} \quad \Gamma_1. \end{cases} \tag{3.16}$$

We construct the special solutions for κ and u by means of the separation of variables in polar coordinates. More precisely, we make use of the coordinate transformation $(x, y) = (r\cos\theta, r\sin\theta)$ in (3.16), leading to

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0. \tag{3.17}$$

It is straightforward to justify that the following function

$$u(r, \theta) = 10 + \ln r \tag{3.18}$$

fulfills (3.17) and the boundary condition $\frac{\partial u}{\partial n} = \frac{\partial u}{\partial r} = 1$ on Γ_1. Furthermore, by specifying

$$\kappa \equiv \frac{2}{10 - \ln 2}, \tag{3.19}$$

it holds that

$$\frac{\partial u}{\partial n} + \kappa u = -\frac{1}{r} + \kappa(10 + \ln r) = -2 + \kappa(10 - \ln 2) = 0 \quad \text{on} \quad \Gamma_2 = \partial B_{1/2}(0)\,.$$

In conclusion, we see that (3.18) and (3.19) are the special solutions of (3.16). In an analogous way, we can justify that

$$u^n = 10 + \ln r + \epsilon_n(r^n + r^{-n})\sin n\theta \quad \text{and} \quad \kappa^n = \frac{2 + n^{-1}(2^{-2n+1} - 2)\sin n\theta}{10 - \ln 2 + n^{-2}(2^{-2n} + 1)\sin n\theta},$$

with $\epsilon_n := 2^{-n}n^{-2}$, are the solutions to (3.16). Due to our construction, we can compute

$$\|u - u^n\|_{0,\Gamma_1} = \epsilon_n \left(\int_0^{2\pi} \sin^2(n\theta)d\theta\right)^{1/2} = 2^{-n}n^{-3}\pi^{1/2} \tag{3.20}$$

and

$$\begin{aligned}
\|u - u^n\|_{1,\Omega} &= \left(\int_{1/2}^1\int_0^{2\pi}[(u-u^n)^2 + (\partial_r u - \partial_r u^n)^2 + r^{-2}(\partial_\theta u - \partial_\theta u^n)^2]rdrd\theta\right)^{1/2}\\
&= \epsilon_n\Big(\int_{1/2}^1\int_0^{2\pi}[(r^n + r^{-n})^2\sin^2(n\theta) + n^2(r^{n-1} - r^{-n-1})^2\sin^2(n\theta)\\
&\quad + n^2(r^{n-1} + r^{-n-1})^2\cos^2(n\theta)]rdrd\theta\Big)^{1/2}\\
&\sim n^{-3/2}. \qquad (3.21)
\end{aligned}$$

Next, we claim that there exists a constant $C > 0$, independent of n, such that

$$C^{-1}\|u - u^n\|_{1,\Omega} \le \|\kappa - \kappa^n\|_{-1/2,\Gamma_2} \le C\|u - u^n\|_{1,\Omega}. \tag{3.22}$$

Indeed, we first observe that u is constant over Γ_2 and obtain by similar arguments for (3.2) (see (3.4)) that

$$\|\kappa - \kappa^n\|_{-\frac{1}{2},\Gamma_2} \le C\|u - u^n\|_{1,\Omega} \tag{3.23}$$

with a constant $C > 0$ independent of n. On the other hand, similar to (3.3), we have

$$\int_\Omega \nabla(u^n - u)\cdot\nabla v dx + \int_{\Gamma_2}\kappa^n(u^n - u)v dS = \int_{\Gamma_2}(\kappa - \kappa^n)uv dS \quad \forall v \in H^1(\Omega). \tag{3.24}$$

By choosing $v = u^n - u$ in (3.24) and using the generalized Poincare's inequality (cf. [48, Theorem 7.7]) along with the trace theorem, it follows that there exists a constant $C > 0$, independent of u^n, u, κ^n and κ, such that

$$\begin{aligned}
\|u^n - u\|_{1,\Omega}^2 &\le C\|\kappa^n - \kappa\|_{-1/2,\Gamma_2}\|u^n - u\|_{1/2,\Gamma_2}\\
&\le C\|\kappa^n - \kappa\|_{-1/2,\Gamma_2}\|u^n - u\|_{1,\Omega},
\end{aligned}$$

which, together with (3.23), implies the desired claim (3.22). it follows from (3.21)-(3.22) that

$$\|\kappa - \kappa^n\|_{-\frac{1}{2},\Gamma_2} \sim \|u - u^n\|_{1,\Omega} \sim n^{-3/2}. \tag{3.25}$$

In conclusion, (3.20) implies that the error $\|u-u^n\|_{0,\Gamma_1}$ decays exponentially, while (3.25) ensures that $\|\kappa-\kappa^n\|_{-\frac{1}{2},\Gamma_2}$ decays only polynomially. Therefore, the following Hölder-type estimate

$$\|\kappa-\kappa^n\|_{-\frac{1}{2},\Gamma_2} \le C\|u-u^n\|_{0,\Gamma_1}^{\alpha}$$

can not be satisfied for any fixed $C>0$ and $\alpha>0$.

Remark 3.3. The above example demonstrates that one can only expect logarithmic-type stabilities for (IRP). However, if the whole boundary data is given, i.e., $\Gamma_a = \partial\Omega$, then we can get a Hölder-type stability estimate instead of logarithmic-type. We obtain by the classical well-posendess theory that

$$\|u(\kappa)-u(\kappa^\dagger)\|_{1,\Omega} \le C\|u(\kappa)-u(\kappa^\dagger)\|_{1/2,\partial\Omega} \quad \forall\, \kappa\in\mathcal{U}$$

with a constant $C>0$ independent of $\kappa\in\mathcal{U}$. This, together with (3.2), implies that

$$\|\kappa-\kappa^\dagger\|_{-\frac{1}{2}-\epsilon,\Gamma_2} \le C\|u(\kappa)-u(\kappa^\dagger)\|_{1/2,\partial\Omega} \quad \forall\, \kappa\in\mathcal{U}. \tag{3.26}$$

On the other hand, an interplay of (3.11) and the continuity of trace mapping implies that

$$\|u(\kappa)-u(\kappa^\dagger)\|_{1,\partial\Omega} \le C\|u(\kappa)-u(\kappa^\dagger)\|_{3/2,\Omega} \le CM \quad \forall\, \kappa\in\mathcal{U}. \tag{3.27}$$

Then we conclude from (3.26)-(3.27), and the well-known interpolation inequality (see (2.5)) that

$$\|\kappa-\kappa^\dagger\|_{-\frac{1}{2}-\epsilon,\Gamma_2} \le C\|u(\kappa)-u(\kappa^\dagger)\|_{1/2,\partial\Omega} \le CM^{1/2}\|u(\kappa)-u(\kappa^\dagger)\|_{0,\partial\Omega}^{1/2} \quad \forall\, \kappa\in\mathcal{U}. \tag{3.28}$$

3.2. **Local stability estimate for (IFP).** The goal of this section is to derive a logarithmic type stability estimate for the inverse flux problem (IFP). Similar to the previous stability estimate (3.1), our analysis makes use of Lemma 2.3 and the following admissible set for the heat flux (with $r>0$):

$$\mathcal{B}_r := \{q\in L^2(\Gamma_1) \mid \|q-q^\dagger\|_{0,\Gamma_1} \le r\}\,.$$

Theorem 3.4. *Let* (**H1**)*,* (**H2**)*, and* (**HF**) *hold. For every $r>0$, there exists a constant $C(r)>0$ such that*

$$\|q-q^\dagger\|_{-\frac{1}{2},\Gamma_1} \le C(r)\frac{1}{\log^{\mu^*/3}\left(\frac{1}{\|u(q)-u(q^\dagger)\|_{0,\Gamma_b}}+1\right)} \quad \forall\, q\in\mathcal{B}_r, \tag{3.29}$$

where $\mu^*\in(0,1)$ is the same as in Lemma 2.3.

Proof. Let $r>0$ and $q\in\mathcal{B}_r$. In view of the definition (2.12), it holds for all $v\in H^1(\Omega)$ that

$$\begin{aligned}&\int_\Omega p\nabla u(q)\cdot\nabla v dx + \int_{\Gamma_2}\kappa u(q) v dS - \int_{\Gamma_1} q v dS\\ &= \int_\Omega p\nabla u(q^\dagger)\cdot\nabla v dx + \int_{\Gamma_2}\kappa u(q^\dagger) v dS - \int_{\Gamma_1} q^\dagger v dS.\end{aligned}$$

As the trace mapping $H^1(\Omega)\to H^{1/2}(\Gamma_2)$ is continuous, the identity above implies

$$\begin{aligned}|\int_{\Gamma_1}(q-q^\dagger) v dS| &\le \|p\|_{C(\overline{\Omega})}\|\nabla(u(q)-u(q^\dagger))\|_{0,\Omega}\|\nabla v\|_{0,\Omega} + \bar{\kappa}\|u(q)-u(q^\dagger)\|_{0,\Gamma_2}\|v\|_{0,\Gamma_2}\\ &\le C\|u(q)-u(q^\dagger)\|_{1,\Omega}\|v\|_{1,\Omega}.\end{aligned} \tag{3.30}$$

Since the trace mapping $H^1(\Omega) \to H^{1/2}(\Gamma_1)$ is surjective with continuous right inverse (see. e.g. [35]), then for any given $\varphi \in H^{1/2}(\Gamma_1)$, there exists a function $v_\varphi \in H^1(\Omega)$ such that $v_\varphi \mid_{\Gamma_1} = \varphi$ and $\|v_\varphi\|_{1,\Omega} \le C\|\varphi\|_{1/2,\Gamma_1}$. For this reason, it follows from (3.30) that

$$|\int_{\Gamma_1} (q - q^\dagger)\varphi dS| \le C\|u(q) - u(q^\dagger)\|_{1,\Omega}\|\varphi\|_{1/2,\Gamma_1} \quad \forall\, \varphi \in H^{1/2}(\Gamma_1),$$

which yields

$$\|q - q^\dagger\|_{-\frac{1}{2},\Gamma_1} \le C\|u(q) - u(q^\dagger)\|_{1,\Omega}. \tag{3.31}$$

On the other hand, it is readily checked that $u(q) - u(q^\dagger)$ satisfies the following elliptic system

$$\begin{cases} -\nabla \cdot (p\nabla(u(q) - u(q^\dagger))) = 0 & \text{in} \quad \Omega, \\ p\dfrac{\partial(u(q) - u(q^\dagger))}{\partial n} = q - q^\dagger & \text{on} \quad \Gamma_1, \\ p\dfrac{\partial(u(q) - u(q^\dagger))}{\partial n} = -\kappa(u(q) - u(q^\dagger)) & \text{on} \quad \Gamma_2. \end{cases} \tag{3.32}$$

Then, Lemma 2.2 implies the existence of a positive constant $\widehat{C}(r)$ such that

$$\max\{\|u(q) - u(q^\dagger)\|_{1,\Omega}, \|u(q) - u(q^\dagger)\|_{3/2,\Omega}\} \le \widehat{C}(r) \qquad \forall q \in \mathcal{B}_r. \tag{3.33}$$

Hence, using (3.33), (3.31), and the interpolation result (2.2) with $M = \Omega$ $s_0 = 0$, $s_1 = 3/2$ and $s = 1$, we get

$$\begin{aligned} \|q - q^\dagger\|_{-\frac{1}{2},\Gamma_1} \le& C\|u(q) - u(q^\dagger)\|_{1,\Omega} \le C\|u(q) - u(q^\dagger)\|_{3/2,\Omega}^{2/3}\|u(q) - u(q^\dagger)\|_{0,\Omega}^{1/3} \\ \le& C\widehat{C}(r)^{2/3}\|u(q) - u(q^\dagger)\|_{0,\Omega}^{1/3}. \end{aligned} \tag{3.34}$$

Furthermore, applying Lemma 2.3 with $\Gamma_p = \Gamma_b \subset \Gamma_2$ to (3.32) yields that

$$\begin{aligned} \|u(q) - u^\dagger\|_{0,\Omega} \le& C(\widehat{C}(r) + \|u(q) - u(q^\dagger)\|_{1/2,\Gamma_b} + \|k(u(q) - u(q^\dagger))\|_{-1/2,\Gamma_b}) \\ & \times \frac{1}{\log^{\mu^*}\left(\frac{\widehat{C}(r)+\|u(q)-u(q^\dagger)\|_{1/2,\Gamma_b}+\|k(u(q)-u(q^\dagger))\|_{-1/2,\Gamma_b}}{\|u(q)-u(q^\dagger)\|_{1/2,\Gamma_b}+\|k(u(q)-u(q^\dagger))\|_{-1/2,\Gamma_b}}\right)}. \end{aligned} \tag{3.35}$$

In view of (2.5), (3.33), the continuity of the trace mapping $H^1(\Omega) \to L^2(\Gamma_b)$, and the embeddings $L^2(\Gamma_b) \hookrightarrow H^{-1/2}(\Gamma_b)$ and $H^{3/2}(\Omega) \hookrightarrow H^1(\Omega)$, we obtain

$$\begin{aligned} & \|u(q) - u(q^\dagger)\|_{1/2,\Gamma_b} + \|k(u(q) - u(q^\dagger))\|_{-1/2,\Gamma_b} \\ \le\;& C(\|u(q) - u(q^\dagger)\|_{1,\Gamma_b}^{1/2}\|u(q) - u(q^\dagger)\|_{0,\Gamma_b}^{1/2} + \|k(u(q) - u(q^\dagger))\|_{0,\Gamma_b}) \\ \le\;& C(\|u(q) - u(q^\dagger)\|_{3/2,\Omega}^{1/2}\|u(q) - u(q^\dagger)\|_{0,\Gamma_b}^{1/2} + \|u(q) - u(q^\dagger)\|_{0,\Gamma_b}^{1/2} \cdot \|u(q) - u(q^\dagger)\|_{0,\Gamma_b}^{1/2}) \\ \le\;& C\widehat{C}(r)^{1/2}\|u(q) - u(q^\dagger)\|_{0,\Gamma_b}^{1/2}. \end{aligned} \tag{3.36}$$

Then, the similar arguments as used in the proof of Theorem 3.1 along with (3.35) and (3.36) ensure that

$$\|u(q) - u(q^\dagger)\|_{1,\Omega} \le C(r) \frac{1}{\log^{\mu^*/3}\left(\frac{1+\|u(q)-u(q^\dagger)\|_{0,\Gamma_b}}{\|u(q)-u(q^\dagger)\|_{0,\Gamma_b}}\right)}, \tag{3.37}$$

which, together with (3.34), completes the proof. □

We emphasize that the established logarithmic-type estimate (3.29) is sharp and can not be improved, to get Hölder- or Lipschitz-type estimates. To demonstrate this fact, we construct a counterexample to our inverse problem (IFP).

We denote by $B_2(0)$ the open ball in $\mathbb{R}^2$ of radius 2 centered at 0. We set $\Omega = B_2(0)\backslash B_1(0) \subset \mathbb{R}^2$, $\Gamma_2 = \partial B_1(0)$, and $\Gamma_1 = \partial B_2(0)$ and consider the following elliptic system

$$\begin{cases} \triangle u &= 0 \quad \text{in} \quad \Omega\,, \\ \frac{\partial u}{\partial n} + u &= 0 \quad \text{on} \quad \Gamma_2\,, \\ \frac{\partial u}{\partial n} &= q \quad \text{on} \quad \Gamma_1\,. \end{cases} \tag{3.38}$$

We will construct special solutions by the separation of variables in polar coordinates. Obviously, the system (3.38) admits trivial solutions $u \equiv 0$ and $q \equiv 0$. By a similar argument as in Subsection, we justify that for all $n \ge 1$,

$$u^n := \epsilon_n((n+2)r^{n+1} + nr^{-n-1})\sin(n+1)\theta \quad \text{and} \quad q^n := \epsilon_n(n+1)[(n+2)2^n - n2^{-n-2}]\sin(n+1)\theta$$

where $\epsilon_n = 2^{-n}n^{-2}$, are the solutions of (3.38). Since Γ_1 is a circle, we recall that the norm in $H^{-1/2}(\Gamma_1)$ can be equivalently computed as follows (See [45, page 25]):

$$\|f\|^2_{-1/2,\Gamma_1} = \sum_{m=-\infty}^{+\infty} (1+m^2)^{-1} \left| \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-im\theta} d\theta \right|^2 \quad \forall f \in H^{-1/2}(\Gamma_1). \tag{3.39}$$

From this formula and the result that

$$\left|\frac{1}{2\pi}\int_0^{2\pi} \sin(n+1)\theta e^{-im\theta} d\theta\right| = \begin{cases} 1/2, & m = \pm(n+1), \\ 0, & m \ne \pm(n+1), \end{cases}$$

it follows that

$$\begin{aligned} &\|q_n\|_{-\frac{1}{2},\Gamma_1} \\ &= \epsilon_n(n+1)[(n+2)2^n - n2^{-n-2}] \left(\sum_{m=-\infty}^{+\infty} (1+m^2)^{-1} \left| \frac{1}{2\pi}\int_0^{2\pi} \sin(n+1)\theta e^{-im\theta} d\theta \right|^2 \right)^{1/2} \\ &= \epsilon_n(n+1)[(n+2)2^n - n2^{-n-2}] \cdot \frac{1}{\sqrt{n^2+2n+2}} \\ &\sim n^{-\frac{1}{2}} \text{ as } \quad n \to +\infty. \end{aligned} \tag{3.40}$$

In an analogous way, we obtain that

$$\|u^n\|_{0,\Gamma_2} = 2\epsilon_n(n+1)\frac{1}{\sqrt{n^2+2n+2}} \sim 2^{-n}n^{-1} \text{ as } \quad n \to +\infty.$$

In conclusion, the error $\|u^n - u\|_{0,\Gamma_2} = \|u^n\|_{0,\Gamma_2}$ decays exponentially, while $\|q_n - q\|_{-\frac{1}{2},\Gamma_1} = \|q^n\|_{-\frac{1}{2},\Gamma_1}$ decays only polynomially. Therefore there exists no constants $C > 0$ and $\alpha > 0$ such that

$$\|q_n - q\|_{-\frac{1}{2},\Gamma_1} \le C\|u^n - u\|^{\alpha}_{0,\Gamma_2} \quad \forall\, n \in \mathbb{N}.$$

4. Convergence rates analysis.

4.1. Convergence rate for (IRP).
Following the general principle of variational source conditions (VSCs) [20] for inverse problems, we propose the following VSC for the Tikhonov regularization (2.11) associated with (IFP):

$$\begin{aligned}&\frac{\|\kappa-\kappa^\dagger\|_{0,\Gamma_2}^2}{4}\\ &\leq\frac{\|\kappa-\kappa^*\|_{0,\Gamma_2}^2}{2}-\frac{\|\kappa^\dagger-\kappa^*\|_{0,\Gamma_2}^2}{2}+\frac{C}{\log^\mu\left(\frac{1}{\|u(\kappa)-u(\kappa^\dagger)\|_{0,\Gamma_a}}+1\right)}\quad\forall\kappa\in\mathcal{U},\end{aligned}\tag{4.1}$$

where $\mu>0$ will be selected later.

Lemma 4.1. *Let* (**H1**), (**H2**) *and* (**HR**) *hold. Let $\mu^*\in(0,1)$ be the same as in Lemma 2.3. Assume that $|u(\kappa^\dagger)|\geq c_0$ a.e. on Γ_2 for some positive constant $c_0>0$, and $\kappa^\dagger-\kappa^*\in H^\beta(\Gamma_2)$ with $\beta>0$. Then, (4.1) holds true for $\mu\in(0,\mu^*]$ satisfying*

$$\begin{cases}\mu=\mu^*/3 & \text{if } \beta>\frac{1}{2},\\ \mu<\frac{4\beta\mu^*}{3+6\beta} & \text{if } \beta\in(0,\frac{1}{2}].\end{cases}$$

Proof. By the parallelogram law in Hilbert spaces, (4.1) is equivalent to

$$(\kappa^\dagger-\kappa^*,\kappa^\dagger-\kappa)_{\Gamma_2}\leq\frac{1}{4}\|\kappa-\kappa^\dagger\|_{0,\Gamma_2}^2+C\frac{1}{\log^\mu\left(\frac{1}{\|u(\kappa)-u(\kappa^\dagger)\|_{0,\Gamma_a}}+1\right)}\quad\forall\kappa\in\mathcal{U}.\tag{4.2}$$

Trivially, (4.2) holds true for $\kappa^\dagger-\kappa^*=0$. In the sequel, we therefore consider the case $\kappa^\dagger\neq\kappa^*$. In this case, if $\beta>1/2$, Theorem 3.1 yields that

$$\begin{aligned}|(\kappa^\dagger-\kappa^*,\kappa^\dagger-\kappa)_{\Gamma_2}|&\leq\|\kappa^\dagger-\kappa^*\|_{\beta,\Gamma_2}\|\kappa^\dagger-\kappa\|_{-\beta,\Gamma_2}\\ \leq\quad &C\|\kappa^\dagger-\kappa^*\|_{\beta,\Gamma_2}\frac{1}{\log^{\mu^*/3}\left(\frac{1}{\|u(\kappa)-u(\kappa^\dagger)\|_{0,\Gamma_a}}+1\right)}\quad\forall\kappa\in\mathcal{U},\end{aligned}$$

which verifies (4.2) with $\mu=\mu^*/3$.

Next, let us consider the case $\beta\in(0,1/2]$ and $\epsilon\in(0,1/2)$. Using (2.2) with $s_0=-1/2-\epsilon$, $s_0=0$, and $\theta=\frac{1+2\epsilon-2\beta}{1+2\epsilon}$, we obtain that

$$\begin{aligned}|(\kappa^\dagger-\kappa^*,\kappa^\dagger-\kappa)_{\Gamma_2}|\leq&\|\kappa^\dagger-\kappa^*\|_{\beta,\Gamma_2}\|\kappa^\dagger-\kappa\|_{-\beta,\Gamma_2}\\ &\leq C\|\kappa^\dagger-\kappa^*\|_{\beta,\Gamma_2}\|\kappa^\dagger-\kappa\|_{-1/2-\epsilon,\Gamma_2}^{\frac{2\beta}{1+2\epsilon}}\|\kappa^\dagger-\kappa\|_{0,\Gamma_2}^{\frac{1+2\epsilon-2\beta}{1+2\epsilon}}.\end{aligned}$$

Applying Theorem 3.1 to the inequality above, it follows that

$$|(\kappa^\dagger-\kappa^*,\kappa^\dagger-\kappa)_{\Gamma_2}|\leq C\|\kappa^\dagger-\kappa^*\|_{\beta,\Gamma_2}\frac{1}{\log^{\frac{2\beta\mu^*}{3(1+2\epsilon)}}\left(\frac{1}{\|u(\kappa)-u(\kappa^\dagger)\|_{0,\Gamma_a}}+1\right)}\|\kappa^\dagger-\kappa\|_{0,\Gamma_2}^{\frac{1+2\epsilon-2\beta}{1+2\epsilon}}.$$

Then, making the use of Young's inequality

$$ab\leq\frac{c^p}{p}a^p+\frac{1}{qc^q}b^q$$

with $p=\frac{2(1+2\epsilon)}{1+2\epsilon+2\beta}$, $q=\frac{2(1+2\epsilon)}{1+2\epsilon-2\beta}$ and a suitable parameter $c>0$, we conclude that

$$|(\kappa^\dagger-\kappa^*,\kappa^\dagger-\kappa)_{\Gamma_2}|\leq\frac{1}{4}\|\kappa^\dagger-\kappa\|_{0,\Gamma_2}^2+C\frac{1}{\log^{\frac{4\beta\mu^*}{3(1+2\epsilon+2\beta)}}\left(\frac{1}{\|u(\kappa)-u(\kappa^\dagger)\|_{0,\Gamma_a}}+1\right)}.$$

Since $\epsilon\in(0,1/2)$ is arbitrary, this completes the proof. □

Lemma 4.2. *For each fixed $\mu \in (0, 1/3)$, the following function is concave:*

$$\psi(\delta) := \log^{-\mu}(\frac{1}{\delta} + 1) \quad \forall\, \delta \in (0, \infty)\,.$$

Proof. It suffices to show that $\psi''(\delta) < 0$ for all $\delta > 0$. An direct computation yields

$$\psi''(\delta) = \frac{\mu}{(\delta^2+\delta)^2 \log^{(\mu+1)}(\frac{1}{\delta}+1)}[(\mu+1)\log^{-1}(\frac{1}{\delta}+1) - 1 - 2\delta]. \tag{4.3}$$

By applying Jensen's inequality to the convex function $x \mapsto 1/x$, we can obtain that

$$\log(\frac{1}{\delta}+1) = \log(1+\delta) - \log\delta = \int_{\delta}^{\delta+1} \frac{1}{x} dx \geq \frac{1}{\int_{\delta}^{\delta+1} x dx} = \frac{1}{\frac{2\delta+1}{2}},$$

which, along with the fact that $0 < \mu \leq 1/3$, implies

$$(\mu+1)\log^{-1}(\frac{1}{\delta}+1) - 1 - 2\delta \leq (\mu+1)(\delta+\frac{1}{2}) - 2(\delta+\frac{1}{2}) = (\mu-1)(\delta+\frac{1}{2}) < 0 \quad \forall \delta > 0 \tag{4.4}$$

The desired claim follows now from (4.3) and (4.4). □

In view of Lemmas 4.1-4.2, we readily derive the following convergence result [21, Theorem 1].

Theorem 4.3. *Let* (**H1**), (**H2**) *and* (**HR**) *hold. Suppose that that $|u(\kappa^\dagger)| \geq c_0$ a.e. on Γ_2 for some positive constant $c_0 > 0$, and $\kappa^\dagger - \kappa^* \in H^\beta(\Gamma_2)$ with $\beta > 0$. Then, under the parameter choice $\beta(\delta) = \delta^2 \log^\mu(\frac{1}{\delta}+1)$, with μ as in Lemma 4.1, it holds that*

$$\|u(\kappa_\alpha^\delta) - u(\kappa^\dagger)\|_{0,\Gamma_a} = O(\delta) \ \ \text{as } \delta \to 0 \tag{4.5}$$

and

$$\|\kappa_\alpha^\delta - \kappa^\dagger\|_{0,\Gamma_2} = O(\log^{-\mu/2}(\frac{1}{\delta})) \ \ \text{as } \delta \to 0, \tag{4.6}$$

4.2. **Convergence rate for (IFP).** In this subsection, we propose the following VSC for the Tikhonov regularization (2.14) associated with (IFP):

$$\begin{aligned}&\frac{\|q-q^\dagger\|_{0,\Gamma_1}^2}{4} \\ \leq &\frac{\|q\|_{0,\Gamma_1}^2}{2} - \frac{\|q^\dagger\|_{0,\Gamma_1}^2}{2} + C(r)\frac{1}{\log^\mu\left(\frac{1}{\|u(q)-u(q^\dagger)\|_{0,\Gamma_b}}+1\right)} \quad \forall\, r > 0,\, q \in \mathcal{B}_r,\end{aligned} \tag{4.7}$$

with a constant $C(r) > 0$ depending on r, and a constant $\mu > 0$ to be specified later. Then the following results follow from the same arguments as that for Lemma 4.1 but using Theorem 3.4.

Lemma 4.4. *Let* (**H1**), (**H2**), *and* (**HF**) *hold, and $\mu^* \in (0,1)$ be the same as in Lemma 2.3. Assume that $q^\dagger \in H^\beta(\Gamma_1)$ with some $\beta > 0$, then (4.7) holds true with μ satisfying*

$$\begin{cases} \mu = \frac{\mu^*}{3} & \text{if } \beta > \frac{1}{2}, \\ \mu = \frac{4\beta\mu^*}{3+6\beta} & \text{if } \beta \in (0, \frac{1}{2}]\,. \end{cases}$$

With the aid of the proposed VSC (4.7), we are ready to establish the following results about the convergence rate, whose proof follows the same reasoning as that for Theorem 4.3 with some necessary modifications.

Theorem 4.5. *Let* (**H1**), (**H2**), *and* (**HF**) *be satisfied. Suppose that* $q^\dagger \in H^\beta(\Gamma_2)$ *holds for some* $\beta > 0$. *Then, under the parameter choice* $\alpha(\delta) = \delta^2 \log^\mu(\frac{1}{\delta} + 1)$, *the following convergence properties*

$$\|u(q_\alpha^\delta) - u^\dagger\|_{0,\Gamma_b} = O(\delta) \ \text{as } \delta \to 0 \quad \text{and} \quad \|q_\alpha^\delta - q^\dagger\|_{0,\Gamma_1} = O(\log^{-\mu/2}(\frac{1}{\delta})) \ \text{as } \delta \to 0$$

hold with μ *being the same as in Theorem 4.4.*

Proof. Let $\delta_{\max} > 0$ be fixed. From the definition of q_α^δ it follows that

$$\frac{1}{2}\|u(q_\alpha^\delta) - z^\delta\|_{0,\Gamma_b}^2 + \frac{\alpha}{2}\|q_\alpha^\delta\|_{0,\Gamma_1}^2 \le \frac{1}{2}\|u(q^\dagger) - z^\delta\|_{0,\Gamma_b}^2 + \frac{\alpha}{2}\|q^\dagger|_{0,\Gamma_1}^2 \le \frac{\delta^2}{2} + \frac{\alpha}{2}\|q^\dagger\|_{0,\Gamma_1}^2,$$

which ensures that $\|q_\alpha^\delta\|_{0,\Gamma_1}^2 \le \delta^2/\alpha + \|q^\dagger|_{0,\Gamma_1}^2 \le r^* = \log^{-\mu}(\frac{1}{\delta_{\max}} + 1) + \|q^\dagger|_{0,\Gamma_1}^2$ for all $0 < \delta \le \delta_{\max}$. Therefore, we can obtain the desired estimates along the lines of the proof of Theorem 4.3, by applying VSC (4.7) with $r = r^*$. □

5. Concluding remarks.

In this paper, we have established some logarithmic type stability estimates for both the inverse Robin and flux problems, which are then applied to help us rigorously verify the variational source conditions in general dimensional spaces. We have also obtained the logarithmic type convergence rates and presented two counterexamples to show that the logarithmic type convergence rates are optimal.

There are some potential studies related to this work that can be conducted in the future. Firstly, we may expect similar results for inverse parabolic Robin and flux problems. The most crucial step for this extension is to establish some global stability estimates for the weak solution of the Cauchy problem associated with the parabolic system. Secondly, it is very interesting but also challenging to study the explicit convergence rates of Tikhonov regularization for inverse elliptic conductivity, diffusivity problems and inverse problems arising from electromagnetic applications (see e.g. [32]), when only partial interior measurement data is available, e.g., collected from a subregion inside the physical domain.

6. Appendix.

We present a proof of Lemma 2.2 in this appendix. Following [35], we define ρ to be a positive and $C^{1,1}(\overline{\Omega})$ function, vanishing on $\partial\Omega$ of the same order of $d(x, \partial\Omega)$, such that

$$\lim_{x \to x_0} \frac{\rho(x)}{d(x, \partial\Omega)} \neq 0.$$

Since $d(x, \partial\Omega)$ is $C^{1,1}$ in a neigborhood of Γ ([5, Theorem 2.1.]), we can define ρ to a strictly positive function in $C^{1,1}(\Omega)$ that equals to $d(x, \partial\Omega)$ in a certain neighborhood of the boundary $\partial\Omega$ (See, e.g., [13]). For $k = 0, 1, 2$, we set $X^k(\Omega) := \{u \in L^2(\Omega); \rho^{|\alpha|} D^\alpha u \in L^2(\Omega),\ |\alpha| \le k\}$, endowed with the inner product

$$(u, v)_{X^k} = \sum_{|\alpha| \le k} (\rho^{|\alpha|} D^\alpha u, \rho^{|\alpha|} D^\alpha v)_{L^2(\Omega)}.$$

Then, for $s = k + \theta$ with an integer $k = 0, 1$ and $0 < \theta < 1$, we define

$$X^s(\Omega) := [X^k(\Omega), X^{k+1}(\Omega)]_\theta,$$

and $X^{-s}(\Omega)$ to be the dual space of $X^s(\Omega)$. Let us mention that the embedding $X^t(\Omega) \hookrightarrow X^s(\Omega)$ is continuous for all $t \ge s$ (See [35, Section 6.3]). Let us write

$$Lu := \nabla \cdot p \nabla u \quad \forall u \in L^2(\Omega)$$

in the sense of the distribution, i.e.,

$$\langle Lu, \varphi\rangle_{H^{-2}(\Omega),H_0^2(\Omega)} := \int_\Omega u\nabla\cdot p\nabla\varphi dx \quad \forall\varphi\in H^2(\Omega).$$

Then, following the concept in [35], we can define a subspace of $H^s(\Omega)$ as follows:

$$D_L^s(\Omega) := \{u\in H^s(\Omega) \mid Lu\in X^{s-2}(\Omega)\}, \tag{6.1}$$

which is a Hilbert space with the norm $\|u\|_{X_L^s} := \|u\|_{s,\Omega} + \|Lu\|_{X^{s-2}(\Omega)}$, and is continuously embedded into $H^s(\Omega)$ for $s\geq 0$. These spaces are useful in characterizing the elliptic boundary value problems with general boundary conditions. In particular, the boundary conditions are allowed to be in the sense of distributions, and the normal derivative is well-defined for functions in $D_L^{3/2}(\Omega)$.

Theorem 6.1 ([35, Theorem 7.4]). *Suppose* **(H1)**-**(H2)** *hold, and set the normal derivatives*

$$B_1 u = p\frac{\partial u}{\partial n} \quad \text{on } \Gamma_1 \quad \text{and} \quad B_2 u = p\frac{\partial u}{\partial n} + u \quad \text{on } \Gamma_2,$$

and sdefine

$$A: D_L^{3/2}(\Omega)\to V := (X^{-1/2}(\Omega), L^2(\Gamma_1), L^2(\Gamma_2)) \quad \text{with} \quad Au := (Lu, B_1u, B_2u).$$

Then, the operator $A: D_L^{3/2}(\Omega)\to V$ is an algebra and topological homomorphism.

Proof of Lemma 2.2. We first prove the uniqueness of the solution to system (1.1). Suppose that u_1 and u_2 are two corresponding solutions to (1.1), it follows immediately that

$$\int_\Omega p\nabla(u_1-u_2)\cdot\nabla v dx + \int_{\Gamma_2}\kappa(u_1-u_2)v dS = 0 \quad \forall v\in H^1(\Omega). \tag{6.2}$$

By selecting $v=u_1-u_2$ in (6.2), one can derive $\|\nabla(u_1-u_2)\|_{L^2(\Omega)^d} = 0 = \|u_1-u_2\|_{L^2(\Gamma_2)}$. Then the norm equivalence of $\|u_1-u_2\|^2_{H^1(\Omega)}$ and $(\|\nabla(u_1-u_2)\|^2_{L^2(\Omega)^d} + \|u_1-u_2\|^2_{L^2(\Gamma_2)})$ implies that $\|u_1-u_2\|^2_{H^1(\Omega)}=0$. Hence, we have $u_1=u_2$ a.e. in Ω.

Then let us prove the existence and stability result. To this end, we first define the operator

$$C_\kappa: H^1(\Omega)\to V, \quad C_\kappa u := (0,0,(\kappa-1)\gamma u), \tag{6.3}$$

where $\gamma: H^1(\Omega)\to H^{1/2}(\Gamma_2)$ denotes the trace operator. Obviously, C_κ is compact. In view of this, Theorem 6.1, and the injectivity of the $A+C_\kappa$, it follows from the compact perturbation of the Fredholm operator ([48, Theorem 12.8]) that $A+C_\kappa: D_L^{3/2}(\Omega)\to V$ is also an algebra and topological homomorphism. Therefore, given any $(f,q,u_a)\in V$, the inverse $u:=(A+C_\kappa)^{-1}(f,q,u_a)$ exists, and it is exactly the weak solution u of system (1.1), which proves the existence.

Now, let us prove the stability. As it is readily checked that

$$u = A^{-1}((f,q,u_a) - C_\kappa u)$$

satisfies

$$\|u\|_{D_L^{3/2}(\Omega)} \leq \|A^{-1}\|_{V\to D_L^{3/2}(\Omega)}(\|(f,q,u_a)\|_V + \|C_\kappa u\|_V) \tag{6.4}$$

In view of the weak formulation (2.9), we obtain that

$$\int_\Omega p|\nabla u|^2 dx + \int_{\Gamma_2}\kappa|u|^2 dS = \int_\Omega fu(\kappa)dx + \int_{\Gamma_1} qudS + \int_{\Gamma_2} u_a u dS. \tag{6.5}$$

By the Hölder's inequality and Poincáre's inequalities, it follows that

$$\|u\|_{H^1(\Omega)} \leq C(\|f\|_{0,\Omega} + \|q\|_{0,\Gamma_1} + \|u_a\|_{0,\Gamma_2}) \tag{6.6}$$

with a constant $C > 0$, independent of f, q, u_a, and $\kappa \in \mathcal{U}$. Since the operator $C_\kappa : H^1(\Omega) \to L^2(\Gamma_2)$ is continuous with $\|C_\kappa\|_{H^1(\Omega)\to V} \leq (\overline{\kappa}+1)\|\gamma\|_{H^1(\Omega)\to L^2(\Omega)}$, it follows from (6.3) and (6.6) that

$$\|C_\kappa u\|_V \leq C(\|f\|_{0,\Omega} + \|q\|_{0,\Gamma_1} + \|u_a\|_{0,\Gamma_2}). \tag{6.7}$$

Concluding from (6.4), (6.7), and the invertibility of $A : D_L^{3/2}(\Omega) \to V$, we obtain

$$\|u\|_{D_L^{3/2}(\Omega)} \leq C(\|f\|_{0,\Omega} + \|f\|_{X^{-1/2}(\Omega)} + \|q\|_{0,\Gamma_1} + \|u_a\|_{0,\Gamma_2})$$

with a constant $C > 0$, independent of f, q, u_a, and $\kappa \in \mathcal{U}$. As the embedding $D_L^{3/2}(\Omega) \hookrightarrow H^{3/2}(\Omega)$ and $L^2(\Omega) \hookrightarrow X^{-1/2}(\Omega)$ are continuous, we completes the proof. □

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E-mail address: dehan.chen@uni-due.de
E-mail address: jiangdaijun@mail.ccnu.edu.cn
E-mail address: irwin.yousept@uni-due.de
E-mail address: zou@math.cuhk.edu.hk