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VARIATIONAL SOURCE CONDITIONS IN L^p-SPACES*

DE-HAN CHEN† AND IRWIN YOUSEPT‡

Abstract. We propose and analyze variational source conditions (VSC) for the Tikhonov regularization method with L^p-penalties applied to an ill-posed operator equation in a Banach space. Our analysis is built on the celebrated Littlewood–Paley theory and the concept of (Rademacher) R-boundedness. With these two analytical principles, we validate the proposed VSC under a conditional stability estimate in terms of a dual Triebel–Lizorkin-type norm. In the final part of the paper, the developed theory is applied to an inverse elliptic problem with measure data for the reconstruction of possibly unbounded diffusion coefficients. By means of VSC, convergence rates for the associated Tikhonov regularization with L^p-penalties are obtained.

Key words. variational source conditions, Littlewood–Paley theory, R-boundedness, Tikhonov regularization with L^p-penalties, ill-posed operator equation, convergence rates, PDE with measure data, reconstruction of unbounded coefficients

AMS subject classifications. 42B25, 42B35, 43A15

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1. Introduction.

Let us consider an ill-posed operator equation of the type

$$T(x) = y \quad \text{in } Y, \tag{1.1}$$

where Y is a Banach space, and $T : D(T) \to Y$ is a weakly sequentially continuous mapping with a closed and convex subset $D(T) \subset L^p(\Omega, \mu)$ for some $1 < p < +\infty$ and σ-finite measure (Ω, μ). We underline that the Lebesgue space $L^p(\Omega, \mu)$ is real, but the Banach space Y is allowed to be complex or real. Moreover, the right-hand side y lies in the range of T. The operator equation (1.1) is supposed to be *locally ill-posed* with a specific characterization through the stability estimate (3.7) comprising a dual Triebel–Lizorkin-type norm. The local degree of the ill-posedness is exactly described by the involved dual norm in (3.7) (Remark 3.4(ii)). To construct a stable approximation to the ill-posed problem (1.1), we employ the celebrated Tikhonov regularization method taking into account the noisy data $y^\delta \in Y$ under the deterministic noise model: $\|y - y^\delta\|_Y \leq \delta$. More precisely, for a given $\alpha > 0$, the solution of (1.1) is approximated by a minimizer of

$$\min_{x \in D(T)} T_\alpha^\delta(x) := \frac{1}{\ell}\|T(x) - y^\delta\|_Y^\ell + \frac{\alpha}{p}\|x - x^*\|_p^{\hat{p}} \tag{1.2}$$

for a fixed constant $\ell > 1$, $\hat{p} := \max\{p, 2\}$, and a fixed a priori guess x^* of x. In view of the presupposed conditions on $T : D(T) \to Y$ and $D(T) \subset L^p(\Omega, \mu)$, the

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†School of Mathematics & Statistics, Central China Normal University, Wuhan 430079, China (chen.dehan@uni-due.de).
‡Corresponding author. Universität Duisburg-Essen, Fakultät für Mathematik, Thea-Leymann-Str. 9, D-45127 Essen, Germany (irwin.yousept@uni-due.de).

existence and plain convergence for the Tikhonov regularization method (1.2) follow by standard arguments (see [14, 25, 46]).

In general, a convergence rate for (1.2) is guaranteed under a smoothness assumption on the true solution, well-known as the so-called source condition (cf. [14, 15, 25, 33, 34]). However, classical source conditions are rather restrictive since they require the Fréchet differentiability of the operator T and further properties on its first-order derivative (see [8, 15, 16, 18, 33, 34, 40, 45]). These restrictions make the classical source condition less appealing for the convergence analysis of (1.2). Our focus is therefore set on the concept of *variational source condition* (VSC) introduced originally by Hofmann et al. [25] in the case of a linear index function. Convergence rates based on VSC for a general index function were shown independently in [6, 19, 22]. In contrast to the classical source condition, VSC is applicable to a wider class of inverse problems with possibly nonsmooth forward operators. More importantly, convergence rates can be deduced from VSC in a straightforward manner (cf. Hofmann and Mathé [27]) without any additional nonlinearity condition such as tangential cone condition. We refer to Hohage and Weidling [31] for a general characterization of VSC in Hilbert spaces. See also [10, 30, 53] regarding VSC for inverse problems governed by partial differential equations (PDEs). All these results were derived by means of the spectral theory for self-adjoint operators in Hilbert spaces.

Although the study of VSC was initiated in the Banach space setting, general sufficient conditions for VSC in Banach spaces are somewhat restrictive (see [21, 46]), compared with those for the Hilbertian case, which are mainly related to conditional stability estimates and smoothness of the true solution. Such methodologies have been applied to various inverse problems governed by PDEs in the Hilbertian setting (see [9, 10, 30, 31, 53]). More recently, less restrictive sufficient conditions for VSC in Besov spaces were proposed by Hohage and others [29, 54] using a new characterization of subgradient smoothness. Their results lead to optimal convergence rates for the Tikhonov regularization method with wavelet Besov-norm penalties. However, we note that $L^p(\Omega, \mu)$ for $p \neq 2$ is not a Besov space, and therefore [29, 54] are not directly applicable to (1.1)–(1.2).

Striving to fill this gap, our paper develops novel sufficient criteria for VSC in the L^p-setting based on a sophisticated application of the Littlewood–Paley (LP) decomposition and the concept of the (Rademacher) $\mathcal{R}$-boundedness. The LP theory is a systematic method to understand various properties of functions by decomposing them in infinite dydic sums with frequency localized components. On the other hand, the concept of $\mathcal{R}$-boundedness was initially introduced to study multiplier theorems for vector-valued functions [7]. These two mathematical concepts are of central significance in the vector-valued harmonic analysis and its profound application to PDEs (cf. [7, 32]). For the sake of completeness, we provide some basics and standard results concerning the LP decomposition and $\mathcal{R}$-boundedness in sections 2.2 and 2.3. Invoking these two analytical tools, we prove our main result (Theorem 3.3) on the sufficient criteria for VSC in the L^p-setting, leading to convergence rates for the Tikhonov regularization method (1.2). The proposed sufficient conditions consist of the existence of a LP decomposition for the (complex) space $L^q(\Omega, \mu; \mathbb{C})$, $q := \frac{p}{p-1}$, together with the previously mentioned conditional stability estimate (3.7) and a regularity assumption for the true solution in terms of a Triebel–Lizorkin-type norm.

The final part of this paper focuses on an inverse reconstruction problem of possibly unbounded diffusion L^p-coefficients in elliptic equations with measure data. Such problems are mainly motivated from geological or medical applications involving dirac measures as source terms. They include acoustic monopoles in full waveform inversion and electrostatic phenomena with a current dipole source in electroencephalography. We analyze the mathematical property of the corresponding forward operator and prove the existence and plain convergence of the corresponding regularized solution (Theorem 4.5). Finally, we transfer our abstract theoretical finding to this specific inverse problem and verify its requirements (see Theorem 4.6 and Lemmas 4.13 and 4.14), leading to convergence rates for the associated Tikhonov regularization method (Corollary 4.8).

2. Preliminaries. We begin by recalling some terminologies and notations used in what follows. Let X, Y be complex or real Banach spaces. The space of all linear and bounded operators from X to Y is denoted by $B(X,Y) = \{A : X \to Y \text{ is linear and bounded}\}$, endowed with the operator norm $\|A\|_{B(X,Y)} := \sup_{\|x\|_X=1} \|Ax\|_Y$. If $X = Y$, then we simply write $B(X)$ for $B(X,X)$. The notation X^* stands for the dual space of X. A linear operator $A : D(A) \subset X \to X$ is called closed if its graph $\{(x, Ax),\ x \in D(A)\}$ is closed in $X \times X$. If $A : D(A) \subset X \to X$ is a linear and closed operator, then

$$\rho(A) := \{\lambda \in \mathbb{C} \mid \lambda \mathrm{id} - A : D(A) \to X \text{ is bijective and } (\lambda \mathrm{id} - A)^{-1} \in B(X)\}$$

and

$$\sigma(A) := \mathbb{C} \setminus \rho(A)$$

denote respectively the resolvent set and spectrum of A. For every $\lambda \in \rho(A)$, the operator $R(\lambda, A) := (\lambda \mathrm{id} - A)^{-1} \in B(X)$ is referred to as the resolvent operator of A.

If (Ω, μ) is a σ-finite measure and $1 \leq p < +\infty$, then $L^p(\Omega, \mu)$ (resp., $L^p(\Omega, \mu; \mathbb{C})$) denotes the space of all equivalence classes of real-valued (resp., complex-valued) μ-measurable and p-integrable functions with the corresponding norm $\|f\|_p := (\int_\Omega |f|^p d\mu)^{\frac{1}{p}}$. If $\Omega \subset \mathbb{R}^n$ is a measurable and μ is the Lebesgue measure, then we simply write $L^p(\Omega)$ (resp., $L^q(\Omega; \mathbb{C})$) for $L^p(\Omega, \mu)$ (resp., $L^q(\Omega, \mu; \mathbb{C})$). For $f, g \in L^1(\mathbb{R}^n; \mathbb{C})$, $f \star g$ denotes the convolution of f and g. Moreover, let $\langle \cdot, \cdot \rangle_{p,q} := \int_\Omega f \overline{g} d\mu$ stand for the duality product between $f \in L^p(\Omega, \mu; \mathbb{C})$ and $g \in L^q(\Omega, \mu; \mathbb{C})$ for $\frac{1}{p} + \frac{1}{q} = 1$.

Finally, for nonnegative real numbers a, b, we write $a \lesssim b$ if $a \leq Cb$ holds true for a positive constant $C > 0$ independent of a and b. If $a \lesssim b$ and $b \lesssim a$, we then write $a \cong b$.

2.1. Sobolev spaces. For every $-\infty < s < \infty$ and $p \geq 1$, we define the (classical) fractional Sobolev space

$$H^s_p(\mathbb{R}^n; \mathbb{C}) := \{u \in \mathcal{S}(\mathbb{R}^n; \mathbb{C})' \mid \|u\|_{H^s_p(\mathbb{R}^n;\mathbb{C})} := \|\mathcal{F}^{-1}[(1+|\cdot|^2)^{\frac{s}{2}} |(\mathcal{F}u)]\|_{L^p(\mathbb{R}^n;\mathbb{C})} < +\infty\},$$

where $\mathcal{S}(\mathbb{R}^n; \mathbb{C})'$ denotes the tempered distribution space and $\mathcal{F} : \mathcal{S}(\mathbb{R}^n; \mathbb{C})' \to \mathcal{S}(\mathbb{R}^n; \mathbb{C})'$ is the Fourier transform (see, e.g., [56]). For a bounded open set $U \subset \mathbb{R}^n$ with a Lipschitz boundary ∂U, the space $H^s_p(U; \mathbb{C})$ with a possibly noninteger exponent $s \geq 0$ is defined as the space of all complex-valued functions $v \in L^p(U; \mathbb{C})$ satisfying $V_{|U} = v$ for some $V \in H^s_p(\mathbb{R}^n; \mathbb{C})$, endowed with the norm

$$\|v\|_{H^s_p(U;\mathbb{C})} := \inf_{\substack{V_{|U}=v \\ V \in H^s_p(\mathbb{R}^n;\mathbb{C})}} \|V\|_{H^s_p(\mathbb{R}^n;\mathbb{C})}.$$

Furthermore, the real counterpart to $H^s_p(U; \mathbb{C})$ is simply denoted by $H^s_p(U)$.

PROPOSITION 2.1 ([51, Theorems 2.4.2 and 4.10.1] and [56, Theorem 1.36]). *Let Ω be a bounded open set in $\mathbb{R}^n$ with a Lipschitz boundary and $1 < \tau < +\infty$.*

(i) *If $\tau < n$, then for any $1 \leq s \leq \frac{\tau n}{n-\tau}$, the embedding $H^1_\tau(\Omega;\mathbb{C}) \hookrightarrow L^s(\Omega;\mathbb{C})$ is continuous. It is compact if $s < \frac{\tau n}{n-\tau}$.*

(ii) *If $\tau \geq n$, then for any $1 \leq s < +\infty$, the embedding $H^1_\tau(\Omega;\mathbb{C}) \hookrightarrow L^s(\Omega;\mathbb{C})$ is compact.*

(iii) *Let $0 \leq s_1, s_2 < +\infty$ and $1 \leq \tau_1, \tau_2 \leq +\infty$. Furthermore, let $\rho \in (0,1)$ and*

$$s := (1-\rho)s_1 + \rho s_2, \quad \frac{1}{\tau} := \frac{1-\rho}{\tau_1} + \frac{\rho}{\tau_2}.$$

Then, there exists a constant $C > 0$ such that

$$\|u\|_{H^s_\tau(\Omega;\mathbb{C})} \leq C\|u\|^{1-\rho}_{H^{s_1}_{\tau_1}(\Omega;\mathbb{C})}\|u\|^{\rho}_{H^{s_2}_{\tau_2}(\Omega;\mathbb{C})} \quad \forall\, u \in H^{s_1}_{\tau_1}(\Omega;\mathbb{C}) \cap H^{s_2}_{\tau_2}(\Omega;\mathbb{C}). \tag{2.1}$$

In the following, we also summarize the well-known composition rule and product estimates for Sobolev functions (cf. [49, Chapter 2, Propositions 1.1 and 6.1]).

PROPOSITION 2.2 (composition rule and product estimates). *Let Ω be a bounded open set in $\mathbb{R}^n$ with a Lipschitz boundary.*

(i) *Let $1 \leq \tau < +\infty$. If $F : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz and satisfies $F(0) = 0$, then $F(u) \in H^1_\tau(\Omega)$ holds true for all $u \in H^1_\tau(\Omega)$.*

(ii) *For all $1 < \tau, \tau_1, \tau_2 < +\infty$ satisfying $\frac{1}{\tau} = \frac{1}{\tau_1} + \frac{1}{\tau_2}$, there exists a constant $C > 0$ such that*

$$\|uv\|_{H^1_\tau(\Omega;\mathbb{C})} \leq C\|u\|_{H^1_{\tau_1}(\Omega;\mathbb{C})}\|v\|_{H^1_{\tau_2}(\Omega;\mathbb{C})}$$

holds true for all $u \in H^1_{\tau_1}(\Omega;\mathbb{C})$ and $v \in H^1_{\tau_2}(\Omega;\mathbb{C})$.

2.2. Littlewood–Paley decomposition. In its simplest manifestation, the LP decomposition is a method to understand various properties of functions by decomposing them into an infinite dydic sum of frequency localized components. A prominent example for an LP decomposition can be found in the classical theory of harmonic analysis as follows: Let $1 < q < +\infty$ and $s \geq 0$. Then, every $f \in H^s_q(\mathbb{R}^n;\mathbb{C})$ can be decomposed into

$$f = \sum_{j=0}^{\infty} f \star \check{\varphi}_j \quad \text{and} \quad \|f\|_{H^s_q(\mathbb{R}^n;\mathbb{C})} \cong \left\| \left(\sum_{j=0}^{\infty} 2^{2js}|f \star \check{\varphi}_j|^2 \right)^{\frac{1}{2}} \right\|_q, \tag{2.2}$$

where $\{\varphi_j\}_{j=0}^\infty$ is a family of compactly supported smooth functions satisfying $\operatorname{supp}(\varphi_0) \subset \{\xi \mid |\xi| \leq 2\}$, $\operatorname{supp}(\varphi_1) \subset \{\xi \mid 1 \leq |\xi| \leq 4\}$, $\varphi_j(\cdot) := \varphi_1(\cdot 2^{1-j})$ for $j \geq 2$, and $\sum_{j=0}^\infty \varphi(\xi) = 1$ for all $\xi \in \mathbb{R}^n$. Furthermore, $\check{\varphi}_j$ denotes the inverse Fourier transformation of φ_j (cf. [48, section 4.1]). Motivated by (2.2) and following [35], we introduce the following definition.

DEFINITION 2.3. *Let (Ω, μ) be a σ-finite measure and $1 < q < +\infty$. We say that $L^q(\Omega,\mu;\mathbb{C})$ admits an LP decomposition if there is a family of uniformly bounded and pairwisely commutative linear operators $\{P_j\}_{j=0}^\infty \subset B(L^q(\Omega,\mu;\mathbb{C}))$ satisfying the following conditions:*

(i) *The partition of identity:*

$$z = \sum_{j=0}^{\infty} P_j z \quad \forall\, z \in L^q(\Omega,\mu;\mathbb{C}). \tag{2.3}$$

(ii) *Almost orthogonality:*

$$P_j P_k z = 0 \quad \forall\, z \in L^q(\Omega,\mu;\mathbb{C}) \quad \forall j,k \in \mathbb{N}\cup\{0\} \text{ with } |j-k| \geq 2. \tag{2.4}$$

(iii) *Norm equivalence: there exists a constant $c^* \geq 1$ such that*

$$\frac{1}{c^*}\|z\|_q \leq \left\| \left(\sum_{j=0}^{\infty} |P_j z|^2 \right)^{\frac{1}{2}} \right\|_q \leq c^* \|z\|_q \quad \forall z \in L^q(\Omega,\mu;\mathbb{C}). \tag{2.5}$$

Remark 2.4. The third condition in Definition 2.3 implies that $\{P_j\}_{j=0}^{\infty}$ is uniformly bounded in $B(L^q(\Omega,\mu;\mathbb{C}))$. Therefore, we may remove the uniform boundedness assumption on $\{P_j\}_{j=0}^{\infty}$ in the definition. From the partition of identity and the almost orthogonality, it follows that

$$P_j(P_j + P_{j-1} + P_{j+1}) = P_j \quad \forall j \geq 1. \tag{2.6}$$

Note that (2.2) gives a classical example of an LP decomposition on $L^q(\mathbb{R}^n;\mathbb{C})$. Also, if $q = 2$ and $\{e_j\}_{j=0}^{\infty}$ is an orthonormal basis of $L^2(\Omega,\mu;\mathbb{C})$, then the family of operators $\mathcal{P} = \{P_j\}_{j=0}^{\infty}$ with $P_j z := (z, e_j)_{L^2(\Omega,\mu;\mathbb{C})} z$ is an LP decomposition on $L^2(\mathbb{R}^n;\mathbb{C})$.

With the help of the LP decomposition and inspired by the classical Triebel–Lizorkin spaces, if $L^q(\Omega,\mu)$ admits an LP decomposition $\mathcal{P} = \{P_j\}_{j=0}^{\infty} \subset B(L^q(\Omega,\mu;\mathbb{C}))$, then the space

$$F_q^s(\mathcal{P}) := \left\{ z \in L^q(\Omega,\mu;\mathbb{C}) \mid \|z\|_{F_q^s(\mathcal{P})} := \left\| \left(\sum_{j=0}^{\infty} 2^{2js} |P_j z|^2 \right)^{\frac{1}{2}} \right\|_q < +\infty \right\} \quad \forall s \geq 0 \tag{2.7}$$

defines a Banach space. Obviously, $F_q^0(\mathcal{P}) = L^q(\Omega,\mu;\mathbb{C})$ holds true with norm equivalence. According to Definition 2.3, $F_q^s(\mathcal{P})$ is a dense subspace of $L^q(\Omega,\mu;\mathbb{C})$, and the embedding $F_q^s(\mathcal{P}) \hookrightarrow L^q(\Omega,\mu;\mathbb{C})$ is continuous.

2.3. $\mathcal{R}$-boundedness.

DEFINITION 2.5. *Let (Ω,μ) be a σ-finite measure and $1 < q < +\infty$. A subset $\mathcal{T} \subset B(L^q(\Omega,\mu;\mathbb{C}))$ is called $\mathcal{R}$-bounded if there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$, $T_1,\ldots,T_n \in \mathcal{T}$ and $z_1,\ldots,z_n \in L^q(\Omega,\mu;\mathbb{C})$, the following inequality holds:*

$$\left\| \left(\sum_{k=1}^{n} |T_k z_k|^2 \right)^{\frac{1}{2}} \right\|_q \leq C \left\| \left(\sum_{k=1}^{n} |z_k|^2 \right)^{\frac{1}{2}} \right\|_q. \tag{2.8}$$

The infimum of all such constants $C > 0$ is called the $\mathcal{R}$-bound of $\mathcal{T}$ and is denoted by $\mathcal{R}(\mathcal{T})$.

Remark 2.6. The notion of $\mathcal{R}$-boundedness can also be defined by using Rademacher functions, and Definition 2.5 is also referred to as ℓ^2-boundedness (cf. [32]) or $\mathcal{R}^2$-boundedness (cf. [37]). By Khintchine's inequality, these two definitions are equivalent in $L^q(\Omega,\mu;\mathbb{C})$ (see [43, Remark 4.1.3] or [32, Proposition 6.3.3]). Since our work only focuses on $L^q(\Omega,\mu;\mathbb{C})$ and considers (2.8), we choose the terminology "$\mathcal{R}$-boundedness." We note that for the case $q = 2$, the $\mathcal{R}$-boundedness of $\mathcal{T}$ is equivalent to the uniform boundedness of $\mathcal{T}$ (cf. [43, Remark 4.1.3]).

We recall some elementary properties regarding to $\mathcal{R}$-boundedness.

PROPOSITION 2.7 (cf. [32, Example 8.1.7 and Proposition 8.1.19] and [37, Propositions 2.9 and 2.10]). *Let (Ω, μ) be a σ-finite measure and $1 < q < +\infty$. Then, the following claims hold true:*

(i) *Every singleton $\{T\}$ in $B(L^q(\Omega, \mu; \mathbb{C}))$ is $\mathcal{R}$-bounded with*

$$\mathcal{R}(\{T\}) \leq C_G \|T\|_{B(L^q(\Omega,\mu;\mathbb{C}))},$$

where $C_G > 0$ denotes the Grothendieck's constant. In particular, $\mathcal{R}(\{\mathrm{id}\}) = 1$.

(ii) *If $\mathcal{T}, \mathcal{S} \subset B(L^q(\Omega, \mu; \mathbb{C}))$ are $\mathcal{R}$-bounded subsets, then both $\mathcal{T} + \mathcal{S}$ and $\mathcal{T} \cup \mathcal{S}$ are $\mathcal{R}$-bounded with*

$$\mathcal{R}(\mathcal{T} + \mathcal{S}) \leq \mathcal{R}(\mathcal{T}) + \mathcal{R}(\mathcal{S}) \quad \textit{and} \quad \mathcal{R}(\mathcal{T} \cup \mathcal{S}) \leq \mathcal{R}(\mathcal{T}) + \mathcal{R}(\mathcal{S}).$$

Let us mention that the exact value of Grothendieck's constant is still an open problem, and it is known that $\frac{\pi}{2} \leq C_G \leq \frac{\pi}{2\ln(1+\sqrt{2})}$ (cf. [36]). A direct consequence of Proposition 2.7 is summarized in the following corollary.

COROLLARY 2.8. *If a subset $\mathcal{T} \subset B(L^q(\Omega, \mu; \mathbb{C}))$ is finite, then it is $\mathcal{R}$-bounded.*

2.4. Existence of LP decompositions via sectorial operators. In this section, we recall the notion of the sectorial operator and discuss some LP decomposition for $L^q(\Omega, \mu; \mathbb{C})$ with the help of sectorial operators. In the following, let X be a complex Banach space. For $\omega \in (0, \pi)$, let $\Sigma_\omega := \{z \in \mathbb{C}\backslash\{0\} \mid |\arg z| < \omega\}$ denote the symmetric sector around the positive axis of aperture angle 2ω.

DEFINITION 2.9 ([24, 35]). *Let $\omega \in (0, \pi)$. A linear and closed operator $A : D(A) \subset X \to X$ is called ω-sectorial if the following conditions hold:*

(i) *the spectrum $\sigma(A)$ is contained in $\overline{\Sigma_\omega}$;*

(ii) *$R(A)$ is dense in X;*

(iii) *for all $\theta \in (\omega, \pi)$ $\exists\, C_\theta > 0$ for all $\lambda \in \mathbb{C}\backslash\overline{\Sigma_\theta} : \|\lambda R(\lambda, A)\| \leq C_\theta$.*

We say that A is 0-sectorial operator if A is ω-sectorial for all $\omega \in (0, \pi)$.

Note that (ii) and (iii) imply that every ω-sectorial operator is injective (cf. [24]). For every $\theta \in (0, \pi)$, we denote by $H^\infty(\Sigma_\theta; \mathbb{C})$ the space of all bounded holomorphic functions on Σ_θ, which is a Banach algebra with the norm $\|f\|_{\infty,\theta} := \sup_{z\in\Sigma_\theta} |f(z)|$. Moreover, we introduce the subspace $H_0^\infty(\Sigma_\theta; \mathbb{C}) := \{f \in H^\infty(\Sigma_\theta; \mathbb{C}) \mid \exists\, C, \epsilon > 0 \text{ such that } |f(z)| \leq C\frac{|z|^\epsilon}{(1+|z|)^\epsilon}\}$. Then, for an ω-sectorial operator A and a function $f \in H_0^\infty(\Sigma_\theta; \mathbb{C})$ with $\theta \in (\omega, \pi)$, one can define a linear and bounded operator

$$G_A(f) : X \to X, \quad G_A(f) := \frac{1}{2\pi i} \int_\Gamma f(\lambda) R(\lambda, A) d\lambda, \tag{2.9}$$

where Γ is the boundary of the sector Σ_σ with $\sigma \in (\omega, \theta)$, oriented counterclockwise. Note that by the Cauchy integral formula for vector-valued holomorphic functions, the above integral has the same value for all $\sigma \in (\omega, \theta)$. Therefore, the definition (2.9) is independent of the choice of Γ. If there exists a constant $C > 0$ such that

$$\|G_A(f)\|_{B(X)} \leq C\|f\|_{\infty,\theta} \quad \forall f \in H_0^\infty(\Sigma_\theta; \mathbb{C}),$$

then we say that A has a bounded $H_0^\infty(\Sigma_\theta; \mathbb{C})$-calculus. In this case, the Cauchy integral formula (2.9) can be extended to a bounded homomorphism $H^\infty(\Sigma_\theta; \mathbb{C}) \to$

$B(X)$, $f \mapsto G_A(f)$. For any $\alpha \geq 0$, we can choose an integer n strictly larger than α such that the function $f_\alpha(z) := z^\alpha(1+z)^{-n}$ belongs to $H_0^\infty(\Sigma_\theta;\mathbb{C})$, and so the operator

$$A^\alpha : D(A^\alpha) \subset X \to X, \quad A^\alpha := (1+A)^n G_A(f_\alpha)$$

defines a linear and closed operator (cf. [24, Lemma A.1.3]) with the effective domain $D(A^\alpha) := \{x \in X \mid G_A(f_\alpha)x \in D(A^n)\}$. In particular, $D(A^\alpha)$ equipped with the graph norm

$$\|A^\alpha \cdot\|_X + \|\cdot\|_X \tag{2.10}$$

defines a Banach space. Clearly, $D(A^0) = X$ and $D(A^1) = D(A)$.

Let $(A, D(A))$ be a 0-sectorial operator and $\eta > 0$. If there is a constant $C > 0$ such that for all $\omega \in (0,\pi)$,

$$\|G_A(f)\|_{B(X)} \leq \frac{C}{\omega^\eta}\|f\|_{\infty,\omega} \quad \forall\, f \in H^\infty(\Sigma_\omega),$$

then we say that A has a (bounded) $\mathcal{M}^\eta$-calculus (see, e.g., [11, Theorem 4.10] and [35]). Another equivalent definition of $\mathcal{M}^\eta$-calculus can be found in [35] (see [11, Theorem 4.10] for the proof).

Under the existence of a 0-sectorial operator with $\mathcal{M}^\eta$-calculus for some $\eta > 0$, the following key lemma guarantees the existence of an LP decomposition for $L^q(\Omega,\mu;\mathbb{C})$.

Lemma 2.10 ([35, Theorems 4.1 and 4.5]). *Let (Ω,μ) be a σ-finite measure. If $X = L^q(\Omega,\mu;\mathbb{C})$ for some $1 < q < +\infty$, and there exists a 0-sectorial operator $A : D(A) \subset X \to X$ with $\mathcal{M}^\eta$-calculus for some $\eta > 0$, then X admits an LP decomposition $\mathcal{P} = \{P_j\}_{j=0}^\infty$ such that*

$$F_q^s(\mathcal{P}) = D(A^s) \quad \forall\, s \geq 0, \tag{2.11}$$

where $F_q^s(\mathcal{P})$ is defined as in (2.7).

Example 2.11. Let Ω be a bounded domain of $\mathbb{R}^n$ $(n \geq 2)$ with a $C^{1,1}$-boundary and $X = L^q(\Omega;\mathbb{C})$ for $1 < q < +\infty$.

(i) Dirichlet boundary condition. If we define $Au := -\Delta u$ for all $u \in D(A)$ with $D(A) := H^2(\Omega;\mathbb{C}) \cap \mathring{H}^1(\Omega;\mathbb{C})$, which corresponds to Dirichlet boundary condition, then $A : D(A) \subset L^2(\Omega;\mathbb{C}) \to L^2(\Omega;\mathbb{C})$ is a self-adjoint operator with $0 \in \rho(A)$ and $-A$ generates a strongly continuous semigroup $\{e^{-At}\}_{t\geq 0}$, whose kernel $\{p_t\}_{t\in(0,+\infty)}$ satisfies the following Gaussian upper bound estimate:

$$|p_t(x,y)| \lesssim \frac{1}{t^{\frac{n}{2}}}\exp\left(-c\frac{|x-y|^2}{t}\right) \quad \forall (t,x,y) \in (0,+\infty)\times\Omega\times\Omega \tag{2.12}$$

for some $c > 0$ (see, e.g., [42, Theorem 6.10] and [42, Chapter 7]). We can extend $\{e^{-At}\}_{t\geq 0}$ to a strongly continuous semigroup on $L^q(\Omega;\mathbb{C})$, whose generator is denoted by $-A_q : D(A_q) \subset L^q(\Omega;\mathbb{C}) \to L^q(\Omega;\mathbb{C})$. Then, $A_q : D(A_q) \subset L^q(\Omega;\mathbb{C}) \to L^q(\Omega;\mathbb{C})$ is a 0-sectorial operator with bounded $\mathcal{M}^\eta$-calculus for $\eta > \lfloor\frac{n}{2}\rfloor + 1$ (see, e.g., [35, Lemma 6.1] and [42, Theorem 7.23]), where $\lfloor c\rfloor$ denotes the largest integer smaller than c. Therefore, according to Lemma 2.10, $L^q(\Omega;\mathbb{C})$ admits an LP decomposition $\mathcal{P}_D$ and

$$F_q^\theta(\mathcal{P}_D) = D(A_q^\theta) = \begin{cases} H_q^{2\theta}(\Omega;\mathbb{C}), & 0 \leq \theta < \frac{1}{2q}, \\ \{H_q^{2\theta}(\Omega;\mathbb{C}) \mid \gamma u = 0\}, & 1 \geq \theta > \frac{1}{2q} \text{ and } \theta \neq \frac{q+1}{2q}, \end{cases}$$

where we have used the characterization of $D(A_q^\theta)$ from [56, Theorem 16.15].

(ii) Neumann boundary condition. Let us now consider $Au := -\Delta u + u$ for $u \in D(A)$, where $D(A) := \{H^2(\Omega;\mathbb{C}) \mid \frac{\partial u}{\partial n} = 0 \,\text{on}\, \partial\Omega\}$. Then, $A : D(A) \subset L^2(\Omega;\mathbb{C}) \to L^2(\Omega;\mathbb{C})$ is also a self-adjoint operator with $0 \in \rho(A)$ and hence $-A$ generates a strongly continuous semigroup $(e^{-At})_{t\geq 0}$. Its kernel also satisfies the classical Gaussian upper estimate (2.12) ([42, Theorem 6.10] and [42, Chapter 7]). As in the first case, $(e^{-At})_{t\geq 0}$ can be extended to a strongly continuous semigroup on $L^q(\Omega;\mathbb{C})$ with the generator denoted by $-A_q : D(A_q) \subset L^q(\Omega;\mathbb{C}) \to L^q(\Omega;\mathbb{C})$. Again, thanks to [35, Lemma 6.1] and [42, Theorem 7.23], $A_q : D(A_q) \subset L^q(\Omega;\mathbb{C}) \to L^q(\Omega;\mathbb{C})$ is a 0-sectorial operator over $L^q(\Omega;\mathbb{C})$ with bounded $\mathcal{M}^\eta$-calculus for $\eta > \lfloor \frac{n}{2} \rfloor + 1$. Therefore, by Lemma 2.10, $L^q(\Omega;\mathbb{C})$ admits an LP decomposition $\mathcal{P}_N$ such that

$$F_q^\theta(\mathcal{P}_N) = D(A_q^\theta) = \begin{cases} H_q^{2\theta}(\Omega;\mathbb{C}), & 0 \leq \theta < \frac{q+1}{2q}, \\ \{H_q^{2\theta}(\Omega;\mathbb{C}) \mid \frac{\partial u}{\partial n} = 0\}, & 1 \geq \theta > \frac{q+1}{2q}, \end{cases} \tag{2.13}$$

where we have used the characterization of $D(A_q^\theta)$ from [56, Theorem 16.11].

Further examples for sectorial operators with bounded $\mathcal{M}^\eta$-calculus can be found in [35, Lemma 6.1].

3. Sufficient conditions for VSC in $L^p(\Omega,\mu)$.

In what follows, let (Ω,μ) be a σ-finite measure, $1 < p < +\infty$, $q := \frac{p}{p-1}$, and $\hat{p} := \max\{p,2\}$. It is well-known that the real Lebesgue space $L^p(\Omega,\mu)$ is $\hat{p}$-uniformly convex (see, e.g., [55]), and there exists a constant $c_p > 0$ such that

$$\|w+y\|_p^{\hat{p}} \geq \|w\|_p^{\hat{p}} + \hat{p}\langle y, J_{\hat{p}}(w)\rangle_{p,q} + c_p\|y\|_p^{\hat{p}} \quad \forall w,y \in L^p(\Omega,\mu), \tag{3.1}$$

where $J_{\hat{p}} : L^p(\Omega,\mu) \to L^q(\Omega,\mu)$ denotes the generalized duality map (cf. [55]) satisfying

$$\langle w, J_{\hat{p}}(w)\rangle_{p,q} = \|w\|_p^{\hat{p}} \quad \text{and} \quad \|J_{\hat{p}}(w)\|_q = \|w\|_p^{\hat{p}-1}. \tag{3.2}$$

Given an x^*-minimum norm solution $x^\dagger \in D(T) \subset L^p(\Omega,\mu)$ to the ill-posed operator equation (1.1), i.e.,

$$\|x^\dagger - x^*\|_p = \min\{\|x - x^*\|_p \mid x \in D(T) \,\text{such that}\, T(x) = y\},$$

our goal is to find a constant $\beta \in (0, c_p)$ and a concave index function $\Psi : (0,+\infty) \to (0,+\infty)$ such that the VSC

$$\langle x^\dagger - x, J_{\hat{p}}(x^\dagger - x^*)\rangle_{p,q} \leq \frac{c_p - \beta}{\hat{p}}\|x - x^\dagger\|_p^{\hat{p}} + \Psi(\|T(x) - T(x^\dagger)\|_Y) \quad \forall\, x \in D(T) \tag{3.3}$$

holds true. Note that a function $\Psi : (0,+\infty) \to (0,+\infty)$ is called an index function if it is continuous, is strictly increasing, and satisfies the limit condition $\lim_{\delta\to 0^+} \Psi(\delta) = 0$.

Remark 3.1. Inserting $y = x - x^\dagger$ and $w = x^\dagger - x^*$ in (3.1), we immediately obtain that

$$\langle x^\dagger - x, J_{\hat{p}}(x^\dagger - x^*)\rangle_{p,q} \geq \frac{1}{\hat{p}}\left(\|x^\dagger - x^*\|_p^{\hat{p}} - \|x - x^*\|_p^{\hat{p}} + c_p\|x - x^\dagger\|_p^{\hat{p}}\right) \quad \forall x \in D(T).$$

Therefore, (3.3) implies that

$$\frac{\beta}{\hat{p}}\|x^\dagger - x\|_p^{\hat{p}} \leq \frac{1}{\hat{p}}\|x - x^*\|_p^{\hat{p}} - \frac{1}{\hat{p}}\|x^\dagger - x^*\|_p^{\hat{p}} + \Psi(\|T(x) - T(x^\dagger)\|_Y) \quad \forall x \in D(T). \tag{3.4}$$

VSC of the type (3.4) has been proposed in [27, 46]. Thus, as (3.3) implies (3.4), the following convergence rate result follows directly from [27, Theorem 1] and [46, Theorem 4.13]).

COROLLARY 3.2. *Suppose that VSC* (3.3) *holds true for some* $\beta \in (0, c_p)$ *and concave index function* $\Psi : (0, +\infty) \to (0, +\infty)$. *If the regularization parameter in* (1.2) *is chosen as* $\alpha(\delta) := \frac{\delta^\ell}{\Psi(\delta)}$, *then every solution* $x^\delta_{\alpha(\delta)} \in D(T)$ *to* (1.2) *satisfies*

$$\|x^\delta_{\alpha(\delta)} - x^\dagger\|_p^{\hat{p}} = O(\Psi(\delta)) \quad \text{as } \delta \to 0^+. \tag{3.5}$$

Let us now state our main assumption on the existence of an LP decomposition for the dual space of $L^p(\Omega, \mu; \mathbb{C})$:

(H0) $L^q(\Omega, \mu; \mathbb{C})$ admits an LP decomposition $\mathcal{P} = \{P_j\}_{j=0}^\infty \subset B(L^q(\Omega, \mu; \mathbb{C}))$ in the sense of Definition 2.3.

If (H0) holds, then for every $\theta \geq 0$, we can construct a Banach space $F_q^\theta := F_q^\theta(\mathcal{P})$ by (2.7). Since the embedding $F_q^\theta \hookrightarrow L^q(\Omega, \mu; \mathbb{C})$ is dense and continuous, the embedding $L^p(\Omega, \mu; \mathbb{C}) \hookrightarrow (F_q^\theta)^*$ is continuous, and therefore

$$|\langle f, g\rangle_{p,q}| \leq \|f\|_{(F_q^\theta)^*} \|g\|_{F_q^\theta} \quad \forall\, (f, g) \in L^p(\Omega, \mu; \mathbb{C}) \times F_q^\theta. \tag{3.6}$$

THEOREM 3.3. *Let* (Ω, μ) *be a* σ*-finite measure,* $1 < p < +\infty$, *and* $q = \frac{p}{p-1}$ *satisfying* (H0). *Suppose that there exist a concave index function* $\Psi_0 : (0, +\infty) \to (0, +\infty)$ *and a constant* $\theta \geq 0$ *such that*

$$\|x^\dagger - x\|_{(F_q^\theta)^*} \lesssim \Psi_0(\|T(x^\dagger) - T(x)\|_Y) \quad \forall\, x \in D(T). \tag{3.7}$$

Moreover, assume that $f^\dagger := J_{\hat{p}}(x^\dagger - x^*)$ *is nonzero and belongs to* $F_q^{s\theta}$ *for some* $0 < s \leq 1$. *Then, VSC* (3.3) *holds true for* $\beta = \frac{c_p}{2}$ *and a concave index function* $\Psi : (0, +\infty) \to (0, +\infty)$, *defined by*

$$\Psi(\delta) := \begin{cases} C\|f^\dagger\|_{F_q^\theta} \Psi_0(\delta) & \text{if } s = 1, \\ C \inf\limits_{\lambda \geq 1} \left[\dfrac{1}{2^{\lambda \hat{q} s}} \|f^\dagger\|^{\hat{q}}_{F_q^{s\theta}} + 2^{(\lambda+1)(1-s)} \|f^\dagger\|_{F_q^{s\theta}} \Psi_0(\delta) \right] & \text{if } s \in (0,1), \end{cases} \tag{3.8}$$

for all sufficiently large $C > 0$ *and* $\hat{q} := \min\{q, 2\}$. *Furthermore, the index function* (3.8) *satisfies*

$$\Psi(\delta) \lesssim \Psi_0(\delta)^{\frac{\hat{q}s}{1+(\hat{q}-1)s}} \tag{3.9}$$

for all sufficiently small $\delta > 0$.

Remark 3.4.

(i) If $f^\dagger$ is zero, then $x^* = x^\dagger$, i.e., the a priori guess x^* is exactly the true solution $x^\dagger$. In this case, VSC (3.3) holds true for all $\beta \in (0, c_p)$ and all index functions Ψ.

(ii) The condition (3.7) characterizes the local ill-posedness of the forward operator $T : L^p(\Omega, \mu) \supset D(T) \to Y$ at $x^\dagger$. As the topology of $(F_q^\theta)^*$ becomes coarser for growing θ, i.e., $(F_q^{\theta_1})^* \subset (F_q^{\theta_2})^*$ holds for any $0 \leq \theta_1 < \theta_2$, the ill-posedness grows if θ becomes larger. On the other hand, if $\theta = 0$, then $(F_q^0)^* = L^p(\Omega, \mu; \mathbb{C})$, and (3.7) implies the local well-posedness at $x^\dagger$ in the following sense:

$$\{x_n\}_{n\in\mathbb{N}} \subset D(T) \quad \text{and} \quad \lim_{n\to\infty} T(x_n) = T(x^\dagger) \text{ in } Y \implies \lim_{n\to\infty} x_n = x^\dagger \text{ in } L^p(\Omega, \mu).$$

(iii) The existence of a concave index function Ψ_0 satisfying (3.7) can be obtained by conditional stability estimates, including Hölder/Lipschitz-type estimates and logarithmic type estimates, for the corresponding inverse problem (1.1) related to the forward operator $T: L^p(\Omega,\mu) \supset D(T) \to Y$. The claim for the case of $\theta = 0$ can be found in [46, Theorem 4.26]. In this case, the assumption (H0) is not required, and (3.3) holds for all $\beta \in (0, c_p)$ and $\Psi = C\|f^\dagger\|_{F^\theta_q}\Psi_0$ for all sufficiently large $C > 0$.

(iv) We underline that the regularity condition $f^\dagger := J_{\hat{p}}(x^\dagger - x^*) \in F^{s\theta}_q$ is not a source condition. This regularity requirement along with (H0) and the stability estimate (3.7) yield that VSC (3.3) is satisfied for the index function (3.8). No other smoothness conditions are needed.

Proof. If $s = 1$ or $\theta = 0$, then (3.6) and (3.7) imply that

$$\langle x^\dagger - x, J_{\hat{p}}(x^\dagger - x^*)\rangle_{p,q} \le \|x^\dagger - x\|_{(F^\theta_q)^*}\|f^\dagger\|_{F^\theta_q} \le \|f^\dagger\|_{F^\theta_q}\Psi_0(\|T(x^\dagger) - T(x)\|_Y) \tag{3.10}$$

holds true for all $x \in D(T)$. Therefore, if $s = 1$ or $\theta = 0$, VSC (3.3) is satisfied for all $\beta \in (0, c_p)$ and $\Psi(\delta) = \|f^\dagger\|_{F^\theta_q}\Psi_0(\delta)$.

We now prove the claim for $0 < s < 1$ and $\theta > 0$. To this aim, let $x \in D(T)$ be arbitrarily fixed. For any fixed $\lambda \ge 1$, we introduce

$$\mathscr{P}_\lambda z := \sum_{k=0}^{\lfloor\lambda\rfloor} P_k z \quad \forall z \in L^q(\Omega,\mu) \quad \text{and} \quad \mathscr{Q}_\lambda := I - \mathscr{P}_\lambda,$$

where we recall that $\lfloor\lambda\rfloor \in \mathbb{N}$ denotes the largest integer satisfying $\lfloor\lambda\rfloor \le \lambda$. Then,

$$\langle x^\dagger - x, f^\dagger\rangle_{p,q} = \langle x^\dagger - x, \mathscr{Q}_\lambda f^\dagger\rangle_{p,q} + \langle x^\dagger - x, \mathscr{P}_\lambda f^\dagger\rangle_{p,q} =: \mathbf{I_1} + \mathbf{I_2}. \tag{3.11}$$

Let us first derive a proper estimate for $\mathbf{I_1}$. Since $\hat{p} = \max\{2, p\}$ and $\hat{q} = \min\{q, 2\} = \frac{\hat{p}}{\hat{p}-1}$, Young's inequality implies that

$$\mathbf{I_1} \le \|x^\dagger - x\|_p\|\mathscr{Q}_\lambda f^\dagger\|_q \le \frac{c_p}{2\hat{p}}\|x^\dagger - x\|_p^{\hat{p}} + \frac{1}{\hat{q}}\left(\frac{2}{c_p}\right)^{\hat{q}-1}\|\mathscr{Q}_\lambda f^\dagger\|_q^{\hat{q}}. \tag{3.12}$$

Next, in view of the almost orthogonality (2.4) and the partition of identity (2.3), it holds for all $z \in L^q(\Omega,\mu)$ that

$$P_j\mathscr{Q}_\lambda z = P_j\sum_{k=\lfloor\lambda\rfloor+1}^{\infty} P_k z = \begin{cases} P_j z, \quad j \ge \lfloor\lambda\rfloor + 2, \\ P_{\lfloor\lambda\rfloor+1}(P_{\lfloor\lambda\rfloor+1} + P_{\lfloor\lambda\rfloor+2})z, \quad j = \lfloor\lambda\rfloor + 1, \\ P_{\lfloor\lambda\rfloor}P_{\lfloor\lambda\rfloor+1}z, \quad j = \lfloor\lambda\rfloor, \\ 0, \quad j \le \lfloor\lambda\rfloor - 1. \end{cases} \tag{3.13}$$

By (2.5), (3.13), and the fact that $\{P_j\}_{j=0}^\infty$ is pairwisely commutative, we obtain that

(3.14)

$$\begin{aligned}\frac{1}{c^*}\|\mathscr{Q}_\lambda f^\dagger\|_q &\le \left\| \left(\sum_{j=0}^\infty |P_j\mathscr{Q}_\lambda f^\dagger|^2\right)^{\frac{1}{2}} \right\|_q \\ &= \left\| \left(|P_{\lfloor\lambda\rfloor+1}P_{\lfloor\lambda\rfloor}f^\dagger|^2 + |\left(P_{\lfloor\lambda\rfloor+1} + P_{\lfloor\lambda\rfloor+2}\right)P_{\lfloor\lambda\rfloor+1}f^\dagger|^2 + \sum_{j=\lfloor\lambda\rfloor+2}^\infty |P_j f^\dagger|^2 \right)^{\frac{1}{2}} \right\|_q.\end{aligned}$$

From Proposition 2.7, it follows that the finite set $\{P_{\lfloor\lambda\rfloor+1}, P_{\lfloor\lambda\rfloor+1}+P_{\lfloor\lambda\rfloor+2}, \mathrm{id}\}$ is $\mathcal{R}$-bounded with

$$\begin{aligned}\mathcal{R}(\{P_{\lfloor\lambda\rfloor+1}, P_{\lfloor\lambda\rfloor+1}+P_{\lfloor\lambda\rfloor+2}, I\}) &\leq \mathcal{R}(\{P_{\lfloor\lambda\rfloor+1}\}) + \mathcal{R}(\{P_{\lfloor\lambda\rfloor+1}+P_{\lfloor\lambda\rfloor+2}\}) + \mathcal{R}(\{\mathrm{id}\})\\ &\leq C_R := 1 + 3C_G \sup_{j\geq 0} \|P_j\|_{B(L^q(\Omega,\mu;\mathbb{C}))}.\end{aligned}$$

Let now $N \in \mathbb{N}$ be arbitrarily fixed with $N > \lfloor\lambda\rfloor$. According to the definition of the $\mathcal{R}$-boundedness (see Definition 2.5), by choosing

$$n := N - \lfloor\lambda\rfloor + 1, \quad T_1 := P_{\lfloor\lambda\rfloor+1}, \quad T_2 := P_{\lfloor\lambda\rfloor+1} + P_{\lfloor\lambda\rfloor+2}, \quad T_k := \mathrm{id} \quad \forall k = 3, \dots, n,$$

and $z_k := P_{\lfloor\lambda\rfloor+k-1} f^\dagger$ for all $k = 1, \dots n$ in (2.8), we obtain

$$\begin{aligned}&\left\| \left(|P_{\lfloor\lambda\rfloor+1} P_{\lfloor\lambda\rfloor} f^\dagger|^2 + |(P_{\lfloor\lambda\rfloor+1} + P_{\lfloor\lambda\rfloor+2}) P_{\lfloor\lambda\rfloor+1} f^\dagger|^2 + \sum_{j=\lfloor\lambda\rfloor+2}^{N} |P_j f^\dagger|^2 \right)^{\frac{1}{2}} \right\|_q\\ &\qquad \leq C_R \left\| \left(\sum_{j=\lfloor\lambda\rfloor}^{N} |P_j f^\dagger|^2 \right)^{\frac{1}{2}} \right\|_q .\end{aligned}$$

Since N was chosen arbitrarily, it follows that

(3.15)
$$\begin{aligned}&\left\| \left(|P_{\lfloor\lambda\rfloor+1} P_{\lfloor\lambda\rfloor} f^\dagger|^2 + |\left(P_{\lfloor\lambda\rfloor+1} + P_{\lfloor\lambda\rfloor+2}\right) P_{\lfloor\lambda\rfloor+1} f^\dagger|^2 + \sum_{j=\lfloor\lambda\rfloor+2}^{\infty} |P_j f^\dagger|^2 \right)^{\frac{1}{2}} \right\|_q\\ &\quad \leq C_R \left\| \left(\sum_{j=\lfloor\lambda\rfloor}^{\infty} |P_j f^\dagger|^2 \right)^{\frac{1}{2}} \right\|_q \leq \frac{C_R}{2^{(\lambda-1)s\theta}} \left\| \left(\sum_{j=0}^{\infty} 2^{2js\theta} |P_j f^\dagger|^2 \right)^{\frac{1}{2}} \right\|_q = \frac{2^{s\theta} C_R}{2^{\lambda s\theta}} \|f^\dagger\|_{F_q^{s\theta}},\end{aligned}$$

where we have used the definition (2.7) for the last identity. Combining (3.12) and (3.14)–(3.15) results in

$$\mathbf{I_1} \leq \frac{c_p}{2\hat{p}} \|x^\dagger - x\|_p^{\hat{p}} + \frac{C}{2^{\lambda\hat{q}s\theta}} \|f^\dagger\|_{F_q^{s\theta}}^{\hat{q}} \quad \forall\, x \in D(T) \tag{3.16}$$

for some $C > 0$, depending only on c^*, c_p, $\hat{q}$, s, θ, and C_R.

Next, we estimate the second term $\mathbf{I_2}$ by applying (3.6) and (3.7) to (3.11):

$$\mathbf{I_2} = \langle x^\dagger - x, \mathscr{P}_\lambda f^\dagger \rangle_{p,q} \lesssim \|\mathscr{P}_\lambda f^\dagger\|_{F_q^\theta} \Psi_0(\|T(x^\dagger) - T(x)\|_Y). \tag{3.17}$$

Let us now derive an appropriate upper bound for $\|\mathscr{P}_\lambda f^\dagger\|_{F_q^\theta}$. Similar to (3.13), invoking the almost orthogonality (2.4) and the partition of identity (2.3), we deduce that

$$P_j \mathscr{P}_\lambda z = P_j \sum_{k=0}^{\lfloor\lambda\rfloor} P_k z = \begin{cases} 0, & j \geq \lfloor\lambda\rfloor + 2,\\ P_{\lfloor\lambda\rfloor+1} P_{\lfloor\lambda\rfloor} z, & j = \lfloor\lambda\rfloor + 1,\\ P_{\lfloor\lambda\rfloor}(P_{\lfloor\lambda\rfloor} + P_{\lfloor\lambda\rfloor-1}) z, & j = \lfloor\lambda\rfloor,\\ P_j z, & j \leq \lfloor\lambda\rfloor - 1, \end{cases} \tag{3.18}$$

holds true for all $z \in L^q(\Omega,\mu)$. Since the finite set $\{P_{\lfloor\lambda\rfloor}, P_{\lfloor\lambda\rfloor}+P_{\lfloor\lambda\rfloor-1}, \mathrm{id}\}$ is $\mathcal{R}$-bounded with $\mathcal{R}(\{P_{\lfloor\lambda\rfloor}, P_{\lfloor\lambda\rfloor}+P_{\lfloor\lambda\rfloor-1}, \mathrm{id}\}) \le 1+3C_G \sup_{j\ge 0}\|P_j\|_{B(L^q(\Omega,\mu;\mathbb{C}))} = C_R$, using (3.18) and analogous arguments for (3.15), we infer that

(3.19)

$$
\begin{aligned}
&\|\mathscr{P}_\lambda f^\dagger\|_{F^\theta_q} \\
&\quad = \left\| \left(\sum_{j=0}^{\infty} 2^{2j\theta} |P_j \mathscr{P}_\lambda f^\dagger|^2 \right)^{\frac{1}{2}} \right\|_q \\
&\quad = \left\| \left(\sum_{j=0}^{\lfloor\lambda\rfloor-1} 2^{2j\theta} |P_j f^\dagger|^2 + 2^{2\lfloor\lambda\rfloor\theta} \left| \left(P_{\lfloor\lambda\rfloor}+P_{\lfloor\lambda\rfloor-1}\right) P_{\lfloor\lambda\rfloor} f^\dagger\right|^2 + 2^{2(\lfloor\lambda\rfloor+1)\theta} |P_{\lfloor\lambda\rfloor} P_{\lfloor\lambda\rfloor+1} f^\dagger|^2 \right)^{\frac{1}{2}} \right\|_q \\
&\quad \le C_R \left\| \left(\sum_{j=0}^{\lfloor\lambda\rfloor+1} 2^{2j\theta} |P_j f^\dagger|^2 \right)^{\frac{1}{2}} \right\|_q \le C_R 2^{(\lambda+1)\theta(1-s)} \left\| \left(\sum_{j=0}^{\lfloor\lambda\rfloor+1} 2^{2sj\theta} |P_j f^\dagger|^2 \right)^{\frac{1}{2}} \right\|_q \\
&\quad \le C_R 2^{(\lambda+1)\theta(1-s)} \|f^\dagger\|_{F^{s\theta}_q}.
\end{aligned}
$$

Applying (3.19) to (3.17) leads to

$$
\mathbf{I_2} \lesssim 2^{(\lambda+1)\theta(1-s)} \|f^\dagger\|_{F^{s\theta}_q} \Psi_0(\|T(x^\dagger)-T(x)\|_Y). \tag{3.20}
$$

Finally, combining (3.11), (3.16), and (3.20) together, we arrive at

$$
\begin{aligned}
\langle x^\dagger - x, f^\dagger\rangle_{p,q} \le{}& \frac{c_p}{2\hat{p}} \|x^\dagger - x\|_p^{\hat{p}} \\
&+ C \inf_{\lambda\ge 1} \left(\frac{1}{2^{\lambda\hat{q}s\theta}} \|f^\dagger\|^{\hat{q}}_{F^{s\theta}_q} + 2^{(\lambda+1)\theta(1-s)} \|f^\dagger\|_{F^{s\theta}_q} \Psi_0(\|T(x^\dagger)-T(x)\|_Y) \right) \ \forall\, x \in D(T)
\end{aligned}
$$

for all sufficiently large $C>0$, independent of x. The function $\Psi : (0,\infty) \to (0,\infty)$ defined by

$$
\Psi(\delta) := C \inf_{\lambda\ge 1} \left(\frac{1}{2^{\lambda\hat{q}s\theta}} \|f^\dagger\|^{\hat{q}}_{F^{s\theta}_q} + 2^{(\lambda+1)\theta(1-s)} \|f^\dagger\|_{F^{s\theta}_q} \Psi_0(\delta) \right) \tag{3.21}
$$

is concave, continuous, and strictly increasing (cf. the proof of [10, Theorem 4.3]). In conclusion, VSC (3.3) holds true for $\beta = \frac{c_p}{2}$ and the concave index function (3.21) for all sufficiently large $C>0$.

Eventually, since $s, \hat{q}, \theta$ are fixed and $\lim_{\delta\to 0}\Psi_0(\delta) = 0$, if δ is small enough, there exists $\lambda_0 \ge 1$ such that $\frac{1}{2^{\lambda_0\theta}} = \Psi_0(\delta)^{\frac{1}{1+(\hat{q}-1)s}}$, which implies that $(\frac{1}{2^{\lambda_0\theta}})^{\hat{q}s} = \Psi_0(\delta)^{\frac{\hat{q}s}{1+(\hat{q}-1)s}}$ and $(\frac{1}{2^{\lambda_0\theta}})^{s-1}\Psi_0(\delta) = \Psi_0(\delta)^{\frac{s-1}{1+(\hat{q}-1)s}}\Psi_0(\delta) = \Psi_0(\delta)^{\frac{\hat{q}s}{1+(\hat{q}-1)s}}$. Therefore, if δ is small enough, (3.21) yields that

$$
\Psi(\delta) \lesssim \frac{1}{2^{\lambda_0\hat{q}s\theta}} \|f^\dagger\|^{\hat{q}}_{F^{s\theta}_q} + 2^{(\lambda_0+1)\theta(1-s)} \|f^\dagger\|_{F^{s\theta}_q}\Psi_0(\delta) = (\|f^\dagger\|^{\hat{q}}_{F^s_q} + 2^{\theta(1-s)}\|f^\dagger\|_{F^s_q})\Psi_0(\delta)^{\frac{\hat{q}s}{1+(\hat{q}-1)s}}.
$$

This completes the proof. ∎

Let us close this section by presenting an exemplary application of Theorem 3.3 with an optimal convergence rate. We consider $p=q=2$ and an unbounded, self-adjoint, and strictly positive operator $A : D(A) \subset L^2(\Omega,\mu;\mathbb{C}) \to L^2(\Omega,\mu;\mathbb{C})$. By the functional calculus for self-adjoint operator (see, e.g., [35, Lemma 6.1(2)]), $A : D(A) \subset L^2(\Omega, \mu; \mathbb{C}) \to L^2(\Omega, \mu; \mathbb{C})$ is a 0-sectorial operator with $\mathcal{M}^\eta$-calculus for some $\eta > 0$. Therefore, in view of Lemma 2.10, $L^2(\Omega, \mu; \mathbb{C})$ admits an LP decomposition $\mathcal{P} = \{P_j\}_{j=0}^\infty$ such that

$$F_2^\theta(\mathcal{P}) = D(A^\theta) \quad \forall \theta \geq 0, \tag{3.22}$$

and its dual space $F_2^\theta(\mathcal{P})^*$ coincides with $D(A^{-\theta})$. We suppose that the forward operator $T : D(T) \subset L^2(\Omega, \mu) \to L^2(\Omega, \mu)$ is linear, and there exists $\theta \geq 0$ such that

$$\|x\|_{D(A^{-\theta})} \lesssim \|Tx\|_{L^2(\Omega,\mu)} \quad \forall x \in D(T).$$

Now, if $x^* = 0$ and $x^\dagger \in D(A^{s\theta})$ for some $0 < s \leq 1$, then Theorem 3.3 yields that VSC (3.3) holds true for the index function

$$\Psi(\delta) := C\delta^{\frac{2s}{1+s}}$$

for any sufficiently large $C > 0$. Eventually, Corollary 3.2 yields the convergence rate

$$\|x^\delta_{\alpha(\delta)} - x^\dagger\|_{L^2(\Omega,\mu)} = O(\delta^{\frac{s}{1+s}}) \quad \text{as } \delta \to 0^+ \tag{3.23}$$

for the Tikhonov regularization method (1.2) with $\ell = 2$ and the parameter choice $\alpha(\delta) := C^{-1}\delta^{\frac{2}{s+1}}$. It is well-known that the convergence rate (3.23) is optimal (see [41] or [50, Theorem 1.1.]).

4. Parameter identification of elliptic equations with measure data in the L^p-setting.

Throughout this section, let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded $C^{1,1}$ domain, and let $\kappa \in L^\infty(\Omega)$ be a real-valued function satisfying

$$0 < \lambda_0 \leq \kappa(x) \leq \Lambda \text{ for a.e. } x \in \Omega, \tag{4.1}$$

with two positive real constants $\lambda_0 < \Lambda$. We consider the inverse problem of reconstructing the possibly unbounded diffusion coefficient $a : \Omega \to \mathbb{R}$ of the following elliptic equation:

$$\begin{cases} \nabla(\kappa\nabla u) + au = \mu_\Omega & \text{in}\, \Omega, \\ \kappa\frac{\partial u}{\partial \nu} = \mu_\Gamma & \text{on}\, \Gamma := \partial\Omega, \end{cases} \tag{4.2}$$

where μ_Ω and μ_Γ are regular signed Borel measures on Ω and Γ.

DEFINITION 4.1. *Let $\mu_\Omega + \mu_\Gamma =: \mu_{\overline{\Omega}} \in C(\overline{\Omega})^*$ be a regular signed Borel measure on $\overline{\Omega}$. A function $u \in H_1^1(\Omega)$ is said to be a weak solution of* (4.2) *if $au \in L^1(\Omega)$ and*

$$\int_\Omega \kappa\nabla u \cdot \nabla\varphi + au\varphi dx = \int_{\overline{\Omega}} \varphi d\mu_{\overline{\Omega}} \quad \forall \varphi \in C^\infty(\overline{\Omega}). \tag{4.3}$$

The well-posedness of (4.2) requires the following ellipticity condition:

$(\mathrm{EC})_m$ Let $p > n/2$ and suppose that $a \in L^p(\Omega)$ is a nonnegative function satisfying

$$\int_\Omega (\kappa|\nabla\varphi|^2 + a|\varphi|^2)dx \geq m\|\varphi\|^2_{H_2^1(\Omega)} \quad \forall \varphi \in H_2^1(\Omega) \tag{4.4}$$

for some $m > 0$.

We note that Proposition 2.1 ((i) and (ii)) implies that $H^1_2(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$, if $n \geq 3$, and $H^1_2(\Omega) \hookrightarrow L^s(\Omega)$ for all $1 < s < \infty$, if $n = 2$. Thus, the requirement $p > \frac{n}{2}$ is reasonable to ensure that the second term in the left-hand side of (4.4) is well-defined.

THEOREM 4.2 ([1, Theorem 4]). *Let $p > \frac{n}{2}$ and $a \in L^p(\Omega)$ satisfying* (EC)$_m$ *for some $m > 0$. Then, for every $\mu_{\overline{\Omega}} \in C(\overline{\Omega})^*$, the elliptic problem* (4.2) *admits a unique weak solution $u \in H^1_\tau(\Omega)$ for all $1 \leq \tau < \frac{n}{n-1}$. Moreover, for every $1 \leq \tau < \frac{n}{n-1}$, there exists a constant $C(m, \tau) > 0$, independent of a, $\mu_{\overline{\Omega}}$, and u, such that*

$$\|u\|_{H^1_\tau(\Omega)} \leq C(m, \tau)\|\mu_{\overline{\Omega}}\|_{M(\overline{\Omega})}. \tag{4.5}$$

4.1. Existence and convergence. In all what follows, let $\mu_{\overline{\Omega}} \in C(\overline{\Omega})^*$, $p > \frac{n}{2}$, $\mathfrak{p} \geq p$, $\underline{a} > 0$, $M > 0$ be fixed and

$$D(S) := \{a \in L^{\mathfrak{p}}(\Omega) \mid \|a\|_{L^{\mathfrak{p}}(\Omega)} \leq M \text{ and } \underline{a} \leq a(x) \text{ for a.e. } x \in \Omega\}. \tag{4.6}$$

LEMMA 4.3. *Suppose that $\{a_k\}_{k=1}^\infty \subset D(S)$, and, for every $k \in \mathbb{N}$, let $u_k \in H^1_\tau(\Omega)$ for all $1 \leq \tau < \frac{n}{n-1}$ denote the unique weak solution to* (4.2) *associated with a_k. Then,*

$$a_k \rightharpoonup a \quad \text{weakly in } L^p(\Omega) \quad \Rightarrow \quad u_k \rightharpoonup u \text{ weakly in } H^1_\tau(\Omega) \text{ for all } 1 \leq \tau < \frac{n}{n-1},$$

where $u \in H^1_\tau(\Omega)$ is the unique weak solution to (4.2) *associated with $a \in D(S)$.*

Proof. Suppose that the sequence $\{a_k\}_{k=1}^\infty \subset D(S)$ converges weakly in $L^p(\Omega)$ toward some element $a \in L^p(\Omega)$. Since $D(S)$ is a weakly compact set in $L^{\mathfrak{p}}(\Omega)$ and the embedding $L^{\mathfrak{p}}(\Omega) \hookrightarrow L^p(\Omega)$ is continuous, it follows that the set $D(S)$ is a weakly compact set in $L^p(\Omega)$, which yields that $a \in D(S)$. Furthermore, Theorem 4.2 ensures that for every fixed $1 < \tau < \frac{n}{n-1}$, there exists a subsequence $\{u_{k_m}\}_{m=1}^\infty \subset \{u_k\}_{k=1}^\infty$ weakly converging in $H^1_\tau(\Omega)$ to some $u \in H^1_\tau(\Omega)$.

Let us now fix a $\tau \in (\frac{np}{n(p-1)+p}, \frac{n}{n-1})$, which ensures that $\frac{n\tau}{n-\tau} > \frac{p}{p-1}$. Then, Proposition 2.1(i) implies that the embedding $H^1_\tau(\Omega) \hookrightarrow L^{\frac{p}{p-1}}(\Omega)$ is compact. For this reason, we obtain the strong convergence $u_{k_m} \to u$ in $L^{\frac{p}{p-1}}(\Omega)$, which yields the weak convergence $a_{k_m}u_{k_m} \rightharpoonup au$ in $L^1(\Omega)$. Thus, for any $\varphi \in C^\infty(\overline{\Omega})$, we obtain that

$$\int_\Omega \kappa\nabla u \cdot \nabla\varphi + au\varphi dx = \lim_{m\to\infty} \int_\Omega \kappa\nabla u_{k_m} \cdot \nabla\varphi + a_{k_m}u_{k_m}\varphi dx = \int_{\overline{\Omega}} \varphi d\mu_{\overline{\Omega}}.$$

It follows therefore from Theorem 4.2 that u is the unique weak solution to (4.2), and so a well-known argument implies that the whole sequence $\{u_k\}_{k=1}^\infty$ converges weakly in $H^1_\tau(\Omega)$ toward u. Finally, as the embedding $H^1_\tau(\Omega) \hookrightarrow H^1_{\tilde{\tau}}(\Omega)$ for any $\tilde{\tau} \in [1, \tau]$ is linear and bounded, we conclude that $\{u_k\}_{k=1}^\infty$ converges weakly in $H^1_\tau(\Omega)$ for all $1 \leq \tau < \frac{n}{n-1}$ toward u. ∎

In view of Theorem 4.2, we introduce the solution operator $S : D(S) \subset L^p(\Omega) \to Y$, $a \mapsto u$, where Y denotes a real Banach space satisfying $H^1_\tau(\Omega) \hookrightarrow Y$ for some $1 \leq \tau < \frac{n}{n-1}$. More precisely, the operator S assigns to every coefficient $a \in D(S)$ the unique weak solution of (4.2) $u \in H^1_\tau(\Omega)$ for all $1 \leq \tau < \frac{n}{n-1}$. Applying the solution operator, the mathematical formulation of the elliptic inverse coefficient problem (4.2) reads as follows: Find $a \in D(S)$ such that

$$S(a) = u^\dagger, \tag{4.7}$$

where $u^\dagger \in H^1_\tau(\Omega)$ for all $1 \leq \tau < \frac{n}{n-1}$ denotes the unique weak solution of (4.2) associated with the true coefficient $a^\dagger \in D(S)$. For our convergence analysis, we assume that the (unknown) true solution $a^\dagger$ is the L^p-norm minimizing solution in the sense that $a^\dagger \in D(S)$ solves

$$(4.8)\quad \|a^\dagger - a^*\|_{L^p(\Omega)} = \min_{a\in\Pi(u^\dagger)} \|a - a^*\|_{L^p(\Omega)} \quad \text{with } \Pi(u^\dagger) := \{a \in D(S) \mid S(a) = u^\dagger\}.$$

LEMMA 4.4. *The nonempty set $\Pi(u^\dagger)$ is bounded, convex, and closed in $L^p(\Omega)$. Therefore, the minimization problem (4.8) admits a unique solution.*

Proof. The boundedness follows immediately from the definition of $D(S)$ (see (4.6)) and $L^{\mathfrak{p}}(\Omega) \hookrightarrow L^p(\Omega)$. Moreover, by Definition 4.1, it is straightforward to show that $\Pi(u^\dagger)$ is convex. Let us now prove that $\Pi(u^\dagger) \subset L^p(\Omega)$ is closed. To this aim, let $\{a_k\}_{k=1}^\infty \subset \Pi(u^\dagger)$ such that $a_k \to a$ in $L^p(\Omega)$. This weak limit satisfies $a \in D(S)$ since $D(S) \subset L^p(\Omega)$ is weakly compact (cf. the proof of Lemma 4.3). Furthermore, as the embedding $H^1_\tau(\Omega) \hookrightarrow L^{\frac{p}{p-1}}(\Omega)$ holds true for all $\frac{np}{n(p-1)+p} < \tau < \frac{n}{n-1}$ (cf. the proof of Lemma 4.3) we obtain that $u^\dagger \in L^{\frac{p}{p-1}}(\Omega)$, which implies that

$$a_k u^\dagger \to a u^\dagger \quad \text{in } L^1(\Omega),$$

and consequently

$$\int_\Omega \kappa\nabla u^\dagger \cdot \nabla\varphi + a u^\dagger \varphi dx = \lim_{k\to\infty} \int_\Omega \kappa\nabla u^\dagger \cdot \nabla\varphi + a_k u^\dagger \varphi dx = \int_\Omega \varphi d\mu_{\overline{\Omega}} \quad \forall\, \varphi \in C^\infty(\overline{\Omega}).$$

In conclusion, $a \in \Pi(u^\dagger)$. This completes the proof. ∎

Now, given $\alpha > 0$, the Tikhonov regularization problem associated with (4.7) reads as

$$(4.9)\qquad \min_{a\in D(S)} \left(\frac{1}{\ell} \|S(a) - u^\delta\|_Y^\ell + \frac{\alpha}{\hat{p}} \|a - a^*\|^{\hat{p}}_{L^p(\Omega)} \right)$$

for a fixed constant $\ell > 1$, $\hat{p} = \max\{p, 2\}$, and an arbitrarily fixed a priori estimate $a^* \in L^p(\Omega)$ for $a^\dagger$. Moreover, the noisy data u^δ satisfy

$$\|u^\dagger - u^\delta\|_Y \leq \delta,$$

with the noisy level $\delta > 0$. From the classical theory of Tikhonov regularization (see, e.g., [25, 46]), the sequentially weak-to-weak continuity result (Lemma 4.3) implies the following existence and plain convergence results.

THEOREM 4.5. *The following assertions hold true:*
- (i) *For each $\alpha > 0$ and $u^\delta \in Y$, (4.9) admits a solution $a^\delta_\alpha \in D(S)$.*
- (ii) *Let $\{\delta_k\}_{k=1}^\infty \subset (0, +\infty)$ be a null sequence and $\{u^{\delta_k}\}_{k=1}^\infty \subset Y$ be a sequence satisfying*

$$\|u^{\delta_k} - u^\dagger\|_Y \leq \delta_k \quad \forall\, k \in \mathbb{N}.$$

Moreover, let $\{\alpha_k\}_{k=1}^\infty \subset (0, +\infty)$ fulfill

$$\alpha_k \to 0,\ \frac{\delta_k^\ell}{\alpha_k} \to 0,$$

where $\ell \geq 1$ is as in (4.9). If a_k is a minimizer of (4.9) with u^δ and α replaced by u^{δ_k} and α_k, respectively, then a_k converges strongly to $a^\dagger$ in $L^p(\Omega)$.

4.2. VSC for (4.9). Our goal is to verify VSC for the Tikhonov regularization problem (4.9). We shall apply our abstract result (Theorem 3.3) to the case of $T = S$ and show that the conditional estimate (3.7) is satisfied.

THEOREM 4.6. *Let* $\mathfrak{p} > \frac{n}{2}$,

$$\tau \in \begin{cases} (1, +\infty) & \text{if } \mathfrak{p} \geq n, \\ \left(\frac{\mathfrak{p}n}{n\mathfrak{p}-n+\mathfrak{p}}, \frac{\mathfrak{p}n}{n-\mathfrak{p}}\right) & \text{if } \frac{n}{2} < \mathfrak{p} < n, \end{cases} \tag{4.10}$$

and $1 < r, q < +\infty$, $\gamma > 0$ *such that*

(a) $u^\dagger \in H^1_r(\Omega)$ *and* $|u^\dagger| \geq \gamma$ *a.e. in* Ω*;*

(b) $S(a) - S(a^\dagger) \in H^1_\tau(\Omega)$ *for all* $a \in D(S)$*;*

(c)

$$1 - \frac{1}{\tau} = \frac{1}{q} + \frac{1}{r}. \tag{4.11}$$

Furthermore, $p := \frac{q}{q-1}$, $\hat{p} := \max\{2, p\}$, $\hat{q} := \min\{2, q\}$, *and suppose that* $J_{\hat{p}}(a^\dagger - a^*) := f^\dagger \in H^s_q(\Omega)$ *for some* $s \in (0, 1]$. *Then the following assertions hold true:*

(i) *There exists a constant* $C > 0$ *such that*

$$\|a - a^\dagger\|_{H^1_q(\Omega)^*} \leq C\|S(a) - S(a^\dagger)\|_{H^1_\tau(\Omega)} \quad \forall\, a \in D(S). \tag{4.12}$$

(ii) *If* $\tau < \frac{n}{n-1}$ *and* $Y = H^1_\tau(\Omega)$, *then VSC* (3.3) *holds true for* $T = S$, $\beta = \frac{c_p}{2}$, *and* Ψ *as in* (3.8) *with* $\theta = 1$ *and* $\Psi_0(\delta) = \delta$.

(iii) *If, in addition, there exist* $\overline{\tau} > \tau$ *and* $M_1 > 0$ *such that*

$$\|S(a) - S(a^\dagger)\|_{H^1_{\overline{\tau}}(\Omega)} \leq M_1 \quad \forall\, a \in D(S), \tag{4.13}$$

and $Y = H^1_1(\Omega)$, *then VSC* (3.3) *holds true for* $T = S$, $\beta = \frac{c_p}{2}$, *and* Ψ *as in* (3.8) *with* $\theta = 1$ *and* $\Psi_0(\delta) = \delta^{\frac{\overline{\tau}-\tau}{\tau(\overline{\tau}-1)}}$.

(iv) *If there exists* $M_2 > 0$ *such that*

$$\|S(a) - S(a^\dagger)\|_{H^2_\tau(\Omega)} \leq M_2 \quad \forall\, a \in D(S), \tag{4.14}$$

and $Y = L^\tau(\Omega)$, *then VSC* (3.3) *holds true for* $T = S$, $\beta = \frac{c_p}{2}$, *and* Ψ *as in* (3.8) *with* $\theta = 1$ *and* $\Psi_0(\delta) = \delta^{\frac{1}{2}}$.

Remark 4.7.

(i) The condition (a) implies that $\Pi(u^\dagger) = \{a^\dagger\}$, and so the inverse problem (4.7) has a unique solution. We note that the generalized duality map $J_{\hat{p}} : L^p(\Omega) \to L^q(\Omega)$ satisfies $J_{\hat{p}}(w)(x) = \|w\|_p^{\hat{p}-p}|w(x)|^{p-2}w(x)$ if $w(x) \neq 0$ and $J_{\hat{p}}(w)(x) = 0$ if $w(x) = 0$ (see, e.g., [3, section 1.1]). Therefore, the regularity assumption $f^\dagger := J_{\hat{p}}(a^\dagger - a^*) \in H^s_q(\Omega)$ can be immediately translated to the difference between the true solution and the initial guess as follows:

$$\chi_\omega|a^\dagger - a^*|^{p-2}(a^\dagger - a^*) \in H^s_q(\Omega), \tag{4.15}$$

where χ_ω denotes the characteristic function of $\omega := \{x \in \Omega \,|\, a^\dagger(x) \neq a^*(x)\}$. Introducing

$$\varphi_p(t) := \begin{cases} |t|^{p-2}t, & t \in \mathbb{R}\backslash\{0\}, \\ 0, & t = 0, \end{cases} \qquad \text{and} \quad T_{\varphi_p} f := \varphi_p \circ f,$$

(4.15) can be reformulated as

$$T_{\varphi_p}(a^\dagger - a^*) \in H_q^s(\Omega) \quad \Longleftrightarrow \quad a^\dagger - a^* \in T_{\varphi_p}^{-1}(H_q^s(\Omega)) = T_{\varphi_p^{-1}}(H_q^s(\Omega)).$$

In the literature, there exist numerous contributions to the superposition in Lizorkin–Triebel spaces (see, e.g., [4, 5, 44, 47]). To characterize $T_{\varphi_p^{-1}}(H_q^s(\Omega))$, we first take advantage of [47, Theorem 3]. By an elementary calculation (see the appendix), it holds that

$$\varphi_p(\mathbb{R}) \subset \mathbb{R}, \quad |\varphi_p^{(l)}(t)| \lesssim |t|^{p-1-l},\ l = 0, \ldots, N, \tag{4.16}$$

and

$$\sup_{t_0 \neq t_1} \frac{|\varphi_p^{(N)}(t_1) - \varphi_p^{(N)}(t_0)|}{|t_1 - t_0|^\tau} < \infty, \tag{4.17}$$

where $N \in \mathbb{Z}$ and $0 < \tau := p - 1 - N \leq 1$. Thus, Theorem 3 in [47] is applicable, and [47, Theorem 3] yields that

$$H_q^s(\mathbb{R}^n) \subset T_{\varphi_p}^{-1}(H_q^s(\mathbb{R}^n)) \quad \text{if} \quad p > \max\{2, n\} \text{ and } \quad n/q < s < p - 1.$$

Then, from the definition of $H_q^s(\Omega)$ and the fact that $T_{\varphi_p}^{-1}(U) \mid_\Omega = T_{\varphi_p}^{-1}(u)$ whenever $U \in H_q^s(\mathbb{R}^n)$ and $u = U \mid_\Omega$, we conclude that

$$H_q^s(\Omega) \subset T_{\varphi_p}^{-1}(H_q^s(\Omega)) \quad \text{if} \quad p > \max\{2, n\} \text{ and } \quad n/q < s < p - 1.$$

By [47, Theorem 3] along with the argument above, we obtain that

$$H_q^{s^*}(\Omega) \subset T_{\varphi_p}^{-1}(H_q^{p-1}(\Omega)) \subset T_{\varphi_p}^{-1}(H_q^s(\Omega))$$

if $p > 2$, $0 < s < p-1$, and $n(1-2/p) < s^* < \min\{1+n(1-2/p), n(1-1/p)\}$, where we have used the inclusion $H_q^{p-1}(\Omega) \subset H_q^s(\Omega)$ [56, p. 42]. Let us finally consider the case $1 < p < 2$, $0 < s < p - 1$, and $t > \frac{s}{p-1}$. In this case, introducing $\varphi_{p,+}(t) := |t|^{p-1}$ for $t \in \mathbb{R}$, [44, section 5.4.4, Theorem 1] and the embedding result [44, section 2.2.1, Proposition 1] yield

$$T_{\varphi_{p,+}}(H_p^t(\Omega)) \subset H_q^s(\Omega). \tag{4.18}$$

For the convenience of the reader, we present a proof for (4.18) in the appendix. For the above case, if $a^\dagger - a^* \geq 0$ a.e. on Ω, and $a^\dagger - a^* \in H_p^t(\Omega)$, then $T_{\varphi_p}(a^\dagger - a^*) = T_{\varphi_{p,+}}(a^\dagger - a^*) \in H_q^s(\Omega)$.

(ii) Theorem 4.2 implies that $S(a), S(a^\dagger) \in H_\tau^1(\Omega)$ for all $1 < \tau < \frac{n}{n-1}$. Nevertheless, we shall show in Lemmas 4.13 and 4.14 that $S(a) - S(a^\dagger)$ enjoys a higher regularity property, depending on the regularity of κ, such that the assumptions (b), (4.13), and (4.14) are reasonable.

(iii) The second part of the assumption (a) can be verified under additional assumptions: Suppose that $a^\dagger \in L^\infty(\Omega)$, $\mu_\Omega \in L^{h_1}(\Omega)$ with $h_1 > \frac{n}{2}$, and $\mu_\Gamma \in L^{h_2}(\Gamma)$ with $h_2 > n - 1$. This additional regularity assumption implies that $u^\dagger \in H_2^1(\Omega) \cap C(\overline{\Omega})$ (see [1, Theorem 2]). Suppose further that Ω is convex and piecewise smooth, $\mu_\Omega \geq 0$ a.e. in Ω, and $\mu_\Gamma \geq \hat{c}$ a.e. on $\partial\Omega$ for some constant $\hat{c} > 0$. Let $\hat{u} \in H_2^1(\Omega) \cap C(\overline{\Omega})$ denote the weak solution to the Neumann problem

$$-\nabla \cdot (\kappa \nabla \hat{u}) + a^\dagger \hat{u} = 0 \text{ in } \Omega \quad \text{and} \quad \kappa \frac{\partial \hat{u}}{\partial \nu} = \hat{c} \text{ on } \Gamma.$$

We note again that the continuity of $\hat{u}$ follows from [1, Theorem 2]. Now, according to [39, Theorems 2 and 4], there exists a constant $\gamma > 0$ such that $\hat{u}(x) \geq \gamma$ for all $x \in \overline{\Omega}$. Letting $w := u^\dagger - \hat{u}$, we obtain that

$$\int_\Omega \kappa \nabla w \cdot \nabla \varphi + a^\dagger w \varphi dx - \int_\Gamma \kappa(\mu_\Gamma - \hat{c})\varphi ds = \int_\Omega \mu_\Omega \varphi dx \quad \forall \varphi \in H_2^1(\Omega),$$

from which it follows that

$$\int_\Omega \kappa \nabla w \cdot \nabla \varphi + a^\dagger w \varphi dx = \int_\Omega \mu_\Omega \varphi dx + \int_\Gamma \kappa(\mu_\Gamma - \hat{c})\varphi ds \geq 0 \tag{4.19}$$

for all nonnegative functions $\varphi \in H_2^1(\Omega)$. By [39, Theorem 2], it follows from (4.19) that $w(x) \geq 0$ holds for all $x \in \overline{\Omega}$. As a consequence, we obtain that $u^\dagger(x) \geq \hat{u}(x) \geq \gamma$ for all $x \in \overline{\Omega}$.

Proof. (i) Let τ^* denote the conjugate exponent of τ, i.e., $\tau^* = \frac{\tau}{\tau-1}$. For each $a \in D(S)$, by the definition of the weak solution, we have

$$\int_\Omega \kappa \nabla(S(a) - S(a^\dagger)) \cdot \nabla \varphi + a(S(a) - S(a^\dagger))\varphi dx = \int_\Omega (a^\dagger - a) S(a^\dagger)\varphi dx \quad \forall \varphi \in C^\infty(\overline{\Omega}).$$

Then, in view of (4.1) and Hölder's inequality, it follows that

$$\begin{aligned} \left| \int_\Omega (a^\dagger - a) S(a^\dagger)\varphi dx \right| &\leq \Lambda \|\nabla(S(a) - S(a^\dagger))\|_\tau \|\nabla \varphi\|_{\tau^*} + \left| \int_\Omega a(S(a) - S(a^\dagger))\varphi dx \right| \\ &=: \Lambda \|\nabla(S(a) - S(a^\dagger))\|_\tau \|\nabla \varphi\|_{\tau^*} + \mathbf{J}. \end{aligned} \tag{4.20}$$

By the defiiniton of $D(S) \subset L^{\mathfrak{p}}(\Omega)$ (see (4.6)), we have

$$\mathbf{J} \leq \|a\|_{\mathfrak{p}} \|S(a)\varphi - S(a^\dagger)\varphi\|_{\frac{\mathfrak{p}}{\mathfrak{p}-1}} \leq M \|S(a)\varphi - S(a^\dagger)\varphi\|_{\frac{\mathfrak{p}}{\mathfrak{p}-1}} \ \forall (a, \varphi) \in D(S) \times C^\infty(\overline{\Omega}). \tag{4.21}$$

Let us now prove the following estimate for $\mathbf{J}$:

$$\mathbf{J} \lesssim \|S(a) - S(a^\dagger)\|_{H_\tau^1(\Omega)} \|\varphi\|_{H_{\tau^*}^1(\Omega)} \quad \forall (a, \varphi) \in D(S) \times C^\infty(\overline{\Omega}). \tag{4.22}$$

We first consider the case $1 < \tau, \tau^* < n$, which is only possible for $n \geq 3$. For this case, generalized Hölder's inequality and Proposition 2.1(i) yield that

$$\begin{aligned} \|S(a)\varphi - S(a^\dagger)\varphi\|_{\frac{\mathfrak{p}}{\mathfrak{p}-1}} &\underbrace{\lesssim}_{\mathfrak{p} > \frac{n}{2}} \|(S(a) - S(a^\dagger))\varphi\|_{\frac{n}{n-2}} \leq \|S(a) - S(a^\dagger)\|_{\frac{n\tau}{n-\tau}} \|\varphi\|_{\frac{n\tau^*}{n-\tau^*}} \\ &\lesssim \|S(a) - S(a^\dagger)\|_{H_\tau^1(\Omega)} \|\varphi\|_{H_{\tau^*}^1(\Omega)} \quad \forall (a, \varphi) \in D(S) \times C^\infty(\overline{\Omega}). \end{aligned} \tag{4.23}$$

Applying (4.23) to (4.21) yields (4.22). If $\tau \geq n$ and $\tau^* \geq n$, both $H_\tau^1(\Omega)$ and $H_{\tau^*}^1(\Omega)$ are embedded to $L^s(\Omega)$ for all $1 < s < +\infty$ (Proposition 2.1(ii)), and consequently Hölder's inequality implies

$$\|S(a)\varphi - S(a^\dagger)\varphi\|_{\frac{\mathfrak{p}}{\mathfrak{p}-1}} \lesssim \|S(a) - S(a^\dagger)\|_{H_\tau^1(\Omega)} \|\varphi\|_{H_{\tau^*}^1(\Omega)} \quad \forall (a, \varphi) \in D(S) \times C^\infty(\overline{\Omega}), \tag{4.24}$$

which yields (4.22). Now suppose that $\tau \geq n$ and $\tau^* < n$. From (4.10), we obtain that

$$\frac{1}{\mathfrak{p}} < \frac{1}{\tau} + \frac{1}{n} \quad \Rightarrow \quad 1 - \frac{1}{\mathfrak{p}} > 1 - \frac{1}{\tau} - \frac{1}{n} \quad \Rightarrow \quad \frac{\mathfrak{p}}{\mathfrak{p}-1} < \frac{n\tau^*}{n-\tau^*}. \tag{4.25}$$

Therefore, in view of the generalized Hölder's inequality and Proposition 2.1, we can choose $1 < s < +\infty$ such that

$$\|S(a)\varphi - S(a^\dagger)\varphi\|_{\frac{\mathfrak{p}}{\mathfrak{p}-1}} \leq \|S(a) - S(a^\dagger)\|_s \|\varphi\|_{\frac{n\tau^*}{n-\tau^*}} \lesssim \|S(a) - S(a^\dagger)\|_{H^1_\tau(\Omega)} \|\varphi\|_{H^1_{\tau^*}(\Omega)}$$

for all $(a, \varphi) \in D(S) \times C^\infty(\overline{\Omega})$. Thus, applying the above inequality to (4.21) gives (4.22). Similarly, (4.22) is obtained for the case of $\tau < n$ and $\tau^* \geq n$ as $\frac{\mathfrak{p}}{\mathfrak{p}-1} < \frac{n\tau}{n-\tau}$ is satisfied in this case.

Applying (4.22) to (4.20), we obtain

$$\left| \int_\Omega (a^\dagger - a) S(a^\dagger) \varphi dx \right| \lesssim \|S(a) - S(a^\dagger)\|_{H^1_\tau(\Omega)} \|\varphi\|_{H^1_{\tau^*}(\Omega)} \, \forall \, (a, \varphi) \in D(S) \times H^1_{\tau^*}(\Omega), \tag{4.26}$$

where we have also used the density of $C^\infty(\overline{\Omega})$ in $H^1_{\tau^*}(\Omega)$ (cf. [23]). On the other hand, we observe that

$$\|(a^\dagger - a)\|_{H^1_q(\Omega)^*} = \sup_{\|g\|_{H^1_q(\Omega)}=1} \left| \int_\Omega (a^\dagger - a) g dx \right| = \sup_{\|g\|_{H^1_q(\Omega)}=1} \left| \int_\Omega (a^\dagger - a) S(a^\dagger) \frac{1}{S(a^\dagger)} g dx \right|. \tag{4.27}$$

Now we show that $\frac{1}{S(a^\dagger)}$ is well-defined in $H^1_{\tau^*}(\Omega)$. From the condition (a), it follows that $\frac{1}{S(a^\dagger)} = F(S(a^\dagger))$ holds true for a globally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$ satisfying $F(0) = 0$ and $F(x) = \frac{1}{x}$ for all $|x| \geq \gamma$. For this reason, Proposition 2.2(i) implies that $\frac{1}{S(a^\dagger)} \in H^1_r(\Omega)$. Furthermore, using Proposition 2.2(ii) and the condition (c), we have

$$\left\| \frac{1}{S(a^\dagger)} g \right\|_{H^1_{\tau^*}(\Omega)} \leq C \left\| \frac{1}{S(a^\dagger)} \right\|_{H^1_r(\Omega)} \|g\|_{H^1_q(\Omega)} \quad \forall g \in H^1_q(\Omega). \tag{4.28}$$

As a consequence of (4.26) and (4.27),

$$\|(a^\dagger - a)\|_{H^1_q(\Omega)^*} \lesssim \sup_{\|g\|_{H^1_q(\Omega)}=1} \left(\|S(a) - S(a^\dagger)\|_{H^1_\tau(\Omega)} \left\| \frac{1}{S(a^\dagger)} g \right\|_{H^1_{\tau^*}(\Omega)} \right) \; \forall a \in D(S). \tag{4.29}$$

Then, applying (4.28) to (4.29), we conclude that (4.12) is valid.

(ii) Let $\mathcal{P}_N$ be the PL decomposition as constructed in Example 2.11(ii). In view of (2.13), it holds that

$$F^t_q(\mathcal{P}_N) = H^{2t}_q(\Omega; \mathbb{C}) \quad \forall t \in [0, 1/2] \quad \Rightarrow \quad F^{1/2}_q(\mathcal{P}_N)^* = (H^1_q(\Omega; \mathbb{C}))^* \tag{4.30}$$

with equivalent norms. Then, (4.12) implies that (3.7) holds true for $T = S$, $Y = H^1_\tau(\Omega)$, $\Psi_0(\delta) = \delta$, and $\theta = 1$. In conclusion, the claim (ii) follows from Theorem 3.3.

(iii) Applying the interpolation inequality (2.1) with $s_1 = s_2 = 1$, $\tau_1 = \overline{\tau}$, $\tau_2 = 1$, and $\rho = \frac{\overline{\tau}-\tau}{\tau(\overline{\tau}-1)}$ to the right-hand side of (4.12) along with (4.13), we obtain

$$\|a - a^\dagger\|_{H^1_q(\Omega)^*} \le C M_1^{1-\frac{\overline{\tau}-\tau}{\tau(\overline{\tau}-1)}} \|S(a) - S(a^\dagger)\|_{H^1_1(\Omega)}^{\frac{\overline{\tau}-\tau}{\tau(\overline{\tau}-1)}} \quad \forall\, a \in D(S). \tag{4.31}$$

In view of (4.30) and (4.31), we see that (3.7) holds true for $T = S$, $Y = H^1_1(\Omega)$, $\Psi_0(\delta) = \delta^{\frac{\overline{\tau}-\tau}{\tau(\overline{\tau}-1)}}$, and $\theta = 1$. Thus, by Theorem 3.3, the claim (iii) is valid.

(iv) Similarly, applying the interpolation inequality (2.1) with $s_1 = 2, s_2 = 0, \tau_1 = \tau_2 = \tau$, and $\rho = 1/2$ to the right-hand side of (4.12) together with (4.14), we have

$$\|a - a^\dagger\|_{H^1_q(\Omega)^*} \le C M_2^{\frac{1}{2}} \|S(a) - S(a^\dagger)\|_{L^\tau(\Omega)}^{\frac{1}{2}} \quad \forall\, a \in D(S). \tag{4.32}$$

In view of (4.30) and (4.32), we see that (3.7) holds true for $T = S$, $Y = L^\tau(\Omega)$, $\Psi_0(\delta) = \sqrt{\delta}$, and $\theta = 1$. In conclusion, the claim (iv) follows from Theorem 3.3. $\square$

The conditional stability estimate (4.12) is the main key point to verify VSC (3.3) for $T = S$, as it implies the required condition (3.7) for Theorem 3.3. Concluding from Corollary 3.2, Theorem 4.6, and (3.9), we obtain the following convergence rates.

COROLLARY 4.8. *Under the assumptions of Theorem* 4.6 *and the parameter choice* $\alpha(\delta) := \frac{\delta^\ell}{\Psi(\delta)}$, *the Tikhonov regularization method* (4.9) *yields the convergence rates*

$$\|a^\delta_{\alpha(\delta)} - a^\dagger\|_p^{\hat{p}} = \begin{cases} O\left(\delta^{\frac{\hat{q}s}{1+(\hat{q}-1)s}}\right) & \text{as } \delta \to 0^+ \quad \text{in the case of (ii) with } Y = H^1_\tau(\Omega), \\ O\left(\delta^{\frac{(\overline{\tau}-\tau)\hat{q}s}{\tau(\overline{\tau}-1)(1+(\hat{q}-1)s)}}\right) & \text{as } \delta \to 0^+ \text{ in the case of (iii) with } Y = H^1_1(\Omega), \\ O\left(\delta^{\frac{\hat{q}s}{2+2(\hat{q}-1)s}}\right) & \text{as } \delta \to 0^+ \quad \text{in the case of (iv) with } Y = L^\tau(\Omega). \end{cases}$$

As a conclusion, different choices of Y-norms and estimates (4.12)–(4.14) lead to different convergence rates. The case (iv) with the weakest norm $Y = L^\tau(\Omega)$ is mostly relevant for applications since measurement in higher Sobolev norms is typically difficult to realize in practice.

Remark 4.9. In the case of (iv) of Theorem (4.6) with $p = 2$, one may expect convergence rate $\|a^\delta_{\alpha(\delta)} - a^\dagger\|_2^2 = O(\delta^{\frac{2s}{s+2}})$. Thus, our convergence rate is suboptimal in this case, which may be caused by the treatment of the nonlinearity of the forward operator. It is an open question that we shall investigate as a future goal.

4.3. Discussion of hypotheses in Theorem 4.6. In the following, we discuss the assumptions (b), (4.13), and (4.14) with more details. Although $S(a)$ belongs only to $H^1_\tau(\Omega)$ for all $1 \le \tau < \frac{n}{n-1}$, it turns out that the difference $S(a) - S(b)$ for all $a, b \in D(S)$ enjoys a better regularity property, provided that κ is regular enough. This fact allows us to verify the assumptions (b), (4.13), and (4.14) under the following material assumption:

(A) There exist C^1 domains $\Omega_j \subset \mathbb{R}^n$, $j = 1, \ldots, N$, such that

$$\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset \quad \forall\, i \ne j \text{ and } \overline{\Omega}_j \subset \Omega.$$

Furthermore, it holds that

$$\kappa\,|_{\Omega_c} \in C(\overline{\Omega}_c) \quad \text{and} \quad \kappa\,|_{\Omega_j} \in C(\overline{\Omega}_j) \quad \forall\, j = 0, 1 \ldots, N,$$

where $\Omega_c := \Omega \setminus \bigcup_{j=1}^N$.

Remark 4.10. To model a heterogeneous medium, the assumption of piecewise continuous material functions is reasonable and often used in the mathematical study of elliptic equations (cf. [17]).

LEMMA 4.11 (Theorem 1.1, Remarks 3.17–3.19 in [17]). *Assume that* (A) *holds true and* $1 < r, \tau < +\infty$ *such that*

$$\begin{cases} \tau \in (1, +\infty) & \text{if } r \geq n, \\ \tau \in (1, \frac{nr}{n-r}] & \text{if } 1 < r < n. \end{cases} \tag{4.33}$$

Then, for every $f \in L^r(\Omega)$, *the homogeneous Neumann problem*

$$\begin{cases} -\nabla \cdot \kappa \nabla u + u = f & \text{in } \Omega, \\ \kappa \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma \end{cases} \tag{4.34}$$

admits a unique weak solution $u \in H^1_\tau(\Omega)$ *satisfying*

$$\|u\|_{H^1_\tau(\Omega)} \lesssim \|f\|_{L^r(\Omega)}. \tag{4.35}$$

Remark 4.12. As a special case of [17] and an analogue of [12] for Neumann conditions, the material assumption (A) implies that for every $1 < \tau < \infty$ and $\tau^* = \frac{\tau}{\tau-1}$ the operator $-\nabla \cdot \kappa \nabla + 1 : H^1_\tau(\Omega) \to H^1_{\tau^*}(\Omega)^*$ is a topological isomorphism. We note that (4.33) implies

$$1 - \frac{1}{r} \geq 1 - \frac{1}{\tau} - \frac{1}{n} \quad \Rightarrow \quad \frac{1}{r^*} \geq \frac{1}{\tau^*} - \frac{1}{n}. \tag{4.36}$$

In view of (4.36), Proposition 2.1 yields the continuous embedding $H^1_{\tau^*}(\Omega) \hookrightarrow L^{r^*}(\Omega)$. Therefore, under (A) and (4.33), (4.34) admits for every $f \in L^r(\Omega) \hookrightarrow H^1_{\tau^*}(\Omega)^*$ a unique weak solution $u \in H^1_\tau(\Omega)$ with τ as in (4.33). This unique weak solution satisfies

$$\|u\|_{H^1_\tau(\Omega)} \lesssim \|f\|_{H^1_\tau(\Omega)^*} \lesssim \|f\|_{L^r(\Omega)}.$$

Let us also mention that (A) cannot be relaxed due to the counterexamples in [17].

LEMMA 4.13. *Assume that* (A) *holds and* $\mathfrak{p} > \frac{n}{2}$.

(i) *If* $n = 2$, *then for every*

$$\begin{cases} \tau \in (1, +\infty) & \text{if } \mathfrak{p} > 2, \\ \tau \in \left(1, \frac{2\mathfrak{p}}{2-\mathfrak{p}}\right) & \text{if } 1 < \mathfrak{p} \leq 2 \end{cases}$$

there exists a constant $C > 0$ *such that*

$$\|S(a) - S(b)\|_{H^1_\tau(\Omega)} \leq C \quad \forall a, b \in D(S).$$

(ii) *If* $n = 3$, *then for every* $\tau \in (1, \mathfrak{p})$ *there exists a constant* $C > 0$ *such that*

$$\|S(a) - S(b)\|_{H^1_\tau(\Omega)} \leq C \quad \forall a, b \in D(S).$$

Proof. According to Definition 4.1, we have

$$\begin{aligned} &\int_\Omega \kappa \nabla (S(a) - S(b)) \cdot \nabla \varphi + (S(a) - S(b))\varphi dx \\ &= \int_\Omega (S(a) - S(b) + bS(b) - aS(a))\varphi dx \quad \forall \varphi \in C^\infty(\overline{\Omega}) \quad \forall a, b \in D(S). \end{aligned} \tag{4.37}$$

Let us first consider the case $n = 2$. In view of Theorem 4.2 and Proposition 2.1(i), it holds that

$$\|S(a)\|_s \leq C(s) \quad \forall a \in D(S) \quad \forall 1 \leq s < \infty$$

for some constant $C(s) > 0$, independent of $a \in D(S)$. For this reason, making use of the definition $D(S) \subset L^{\mathfrak{p}}(\Omega)$ (see (4.6)), it follows that

$$\|aS(a)\|_r \leq C(r) \quad \forall a \in D(S) \quad \forall 1 \leq r < \mathfrak{p}$$

for some constant $C(r) > 0$, independent of $a \in D(S)$. Combining the above two inequalities yields that

$$\|S(a) - S(b) + bS(b) - aS(a)\|_r \leq C(r) \quad \forall a, b \in D(S) \quad \forall 1 \leq r < \mathfrak{p}. \tag{4.38}$$

If $\mathfrak{p} > 2$, then we may choose $r = 2 = n$ in (4.38) such that applying Lemma 4.11 to (4.37) yields the claim (i) for $\tau \in (1, +\infty)$ and $\mathfrak{p} > 2$. If $1 < \mathfrak{p} \leq 2 = n$, then for every $\tau \in (1, \frac{2\mathfrak{p}}{2-\mathfrak{p}})$, we can find an $r < \mathfrak{p} \leq n$ such that $\tau < \frac{2r}{2-r} = \frac{nr}{n-r}$. Therefore, in view of (4.38), applying again Lemma 4.11 to (4.37) yields the claim (i) for $\tau \in (1, \frac{2\mathfrak{p}}{2-\mathfrak{p}})$ and $1 < \mathfrak{p} \leq 2$.

Now let us consider the case $n = 3$ and $\mathfrak{p} > \frac{3}{2}$. Theorem 4.2 and Proposition 2.1(i) ensure that

$$\|S(a)\|_s < C(s) \quad \forall a \in D(S) \quad \forall 1 \leq s < 3 \tag{4.39}$$

for some constant $C(s) > 0$, independent of $a \in D(S)$. Then, making use of the definition of $D(S) \subset L^{\mathfrak{p}}(\Omega)$ (see (4.6)), the generalized Hölder inequality implies that

$$\|aS(a)\|_r \leq \|a\|_{\mathfrak{p}} \|S(a)\|_{\frac{r\mathfrak{p}}{\mathfrak{p}-r}} \leq MC(r, \mathfrak{p}) \quad \forall a \in D(S) \quad \forall 1 \leq r < \frac{3\mathfrak{p}}{3+\mathfrak{p}}, \tag{4.40}$$

where we have used (4.39) since $1 \leq \frac{r\mathfrak{p}}{\mathfrak{p}-r} < 3$ holds true for all $1 \leq r < \frac{3\mathfrak{p}}{3+\mathfrak{p}}$. Altogether, since $\frac{3\mathfrak{p}}{3+\mathfrak{p}} < 3$, (4.39) and (4.40) yield

$$\|S(a) - S(b) + bS(b) - aS(a)\|_r \leq C(r, \mathfrak{p}) \quad \forall a, b \in D(S) \quad \forall 1 \leq r < \frac{3\mathfrak{p}}{3+\mathfrak{p}} < 3 = n \tag{4.41}$$

for some constant $C(r, \mathfrak{p}) > 0$, independent of $a, b \in D(S)$. In view of (4.41), applying Lemma 4.11 to (4.37), we come to the conclusion that for every $\tau \in (1, \mathfrak{p})$, there exists a constant $C > 0$ such that

$$\|S(a) - S(b)\|_{H^1_\tau(\Omega)} \leq C \quad \forall a, b \in D(S).$$

This completes the proof. ∎

LEMMA 4.14. *Let $n \in \{2, 3\}$ and $\mathfrak{p} > \frac{n}{2}$. Assume that $\kappa \in C^{0,1}(\overline{\Omega})$. Then, for every $\tau \in (1, \overline{\tau})$ with*

$$\overline{\tau} := \begin{cases} \mathfrak{p}, & n = 2, \\ \frac{3\mathfrak{p}}{3+\mathfrak{p}}, & n = 3, \end{cases} \tag{4.42}$$

there exists a constant $C > 0$ such that

$$\|S(a) - S(b)\|_{H^2_\tau(\Omega)} \leq C \quad \forall a, b \in D(S). \tag{4.43}$$

Proof. Let $a, b \in D(S)$. By the definition of the weak solution (4.1),

$$\int_\Omega \kappa \nabla(S(a) - S(b)) \cdot \nabla\varphi + (S(a) - S(b))\varphi dx$$
$$= \int_\Omega (S(a) - S(b) + bS(b) - aS(a))\varphi dx \quad \forall \varphi \in C^\infty(\overline{\Omega}).$$

Therefore, in view of (4.38) (for $n = 2$) and (4.41) (for $n = 3$), the classical $W^{2,\tau}(\Omega)$-regularity result for elliptic equations [23, Theorem 2.4.1.3] implies (4.43). ∎

In conclusion, we see that assumptions (4.13) and (4.14) can be guaranteed by Lemmas 4.13 and 4.14, respectively.

5. Conclusion. Based on the LP theory and the concept of $\mathcal{R}$-boundedness, we developed sufficient criteria (Theorem 3.3) for VSC (3.3) in $L^p(\Omega, \mu)$-spaces with $1 < p < +\infty$. The proposed sufficient criteria consist of the existence of an LP decomposition for the complex dual space $L^q(\Omega, \mu; \mathbb{C})$ ($q = \frac{p}{p-1}$) together with a conditional stability estimate and a regularity requirement for the true solution in terms of Triebel–Lizorkin-type norms. In section 4, the developed abstract result is applied to the inverse reconstruction problem of unbounded diffusion $L^p(\Omega)$-coefficients in elliptic equations with measure data (4.2). We derived existence and plain convergence results for the associated Tikhonov regularization problem (4.9) with $L^p(\Omega)$-norm penalties (Theorem 4.5). As final results (Theorem 4.6 and Lemmas 4.13 and 4.14)), we prove that all requirements of Theorem 3.3 are satisfied for the inverse problem (4.7), leading to convergence rates for the Tikhonov regularization method (4.9) (Corollary 4.8).

Our future goals are threefold. First, noticing that there has been recent progress on VSC for ℓ^1-regularization (see, e.g., [2, 20, 52]), we aim at extending our study to the Tikhonov regularization method with $L^1(\Omega, \mu)$-penalties. On the other hand, in some applications, the unknown solution could fail to have a finite penalty value if the penalty is oversmoothing. Recently, such oversmoothing regularizations have been studied for inverse problems in Hilbert scales (see, e.g., [13, 26, 28]). As Triebel–Lizorkin-type space allows us to work with scales of Banach space through the use of sectorial operators, it would be attempting to study oversmoothing regularizations under an $L^p(\Omega)$-setting. Third, we would like to extend the developed results to nonlinear and nonsmooth PDEs, in particular for those arising from electromagnetic applications [38, 57, 58, 59, 60, 61].

6. Appendix.

6.1. Properties of $\boldsymbol{\varphi_p}$ from Remark 4.7. It is obvious that $\varphi_p(\mathbb{R}) \subset \mathbb{R}$. Let $N \in \mathbb{Z}$ and $0 < \tau \le 1$ such that $N + \tau = p - 1$. Since $\varphi_p(t) = t^{p-1}$ for $t > 0$, and $\varphi_p(t) = -(-t)^{p-1}$ for $t < 0$, it follows for all $l = 0, 1, 2 \ldots, N$ that

$$\varphi_p^{(l)}(0) = 0 \text{ and } \varphi_p^{(l)}(t) = \begin{cases} c_l t^{p-1-l} & \text{if } t > 0, \\ (-1)^{l+1} c_l (-t)^{p-1-l} & \text{if } t < 0, \end{cases} \tag{6.1}$$

where $c_l := (p-1) \cdot (p-2) \cdots (p-l)$, and hence $|\varphi_p^{(l)}(t)| \le c_l |t|^{p-1-l}$ for all $t \in \mathbb{R}$ with $l = 0, 1, \ldots, N$. Using (6.1) and the inequality $(t+s)^\tau \le t^\tau + s^\tau$ for all $t, s \ge 0$, we have that

$$\frac{|\varphi_p^{(N)}(t_1) - \varphi_p^{(N)}(t_0)|}{|t_1 - t_0|^\tau} = c_l \frac{|t_1^\tau - t_0^\tau|}{|t_1 - t_0|^\tau} = c_l \frac{(t_1 - t_0 + t_0)^\tau - t_0^\tau}{(t_1 - t_0)^\tau} \le c_l,$$

whenever $t_1 > t_0 \geq 0$. The case $t_0 > t_1 \geq 0$ can be handled similarly. Thus,

$$\sup_{t_1,t_0\geq 0, t_0\neq t_1} \frac{|\varphi_p^{(N)}(t_1) - \varphi_p^{(N)}(t_0)|}{|t_1 - t_0|^\tau} \leq c_l. \tag{6.2}$$

In view of (6.2) and the fact that $\varphi_p^{(N)}(t) = (-1)^{l+1}\varphi_p^{(N)}(-t)$ for any $t < 0$ by (6.1), we obtain that

$$\sup_{t_1,t_0\leq 0, t_0\neq t_1} \frac{|\varphi_p^{(N)}(t_1) - \varphi_p^{(N)}(t_0)|}{|t_1 - t_0|^\tau} \leq c_l. \tag{6.3}$$

If $t_1 t_0 < 0$, then from (6.1) it follows that

$$\frac{|\varphi_p^{(N)}(t_1) - \varphi_p^{(N)}(t_0)|}{|t_1 - t_0|^\tau} \leq \frac{|\varphi_p^{(N)}(t_1)|}{|t_1 - t_0|^\tau} + \frac{|\varphi_p^{(N)}(t_0)|}{|t_1 - t_0|^\tau} \leq \frac{|\varphi_p^{(N)}(t_1)|}{|t_1|^\tau} + \frac{|\varphi_p^{(N)}(t_0)|}{|t_0|^\tau} \leq 2c_l. \tag{6.4}$$

Therefore, we conclude from (6.2)–(6.4) that

$$\sup_{t_0\neq t_1} \frac{|\varphi_p^{(N)}(t_1) - \varphi_p^{(N)}(t_0)|}{|t_1 - t_0|^\tau} \leq 2c_l.$$

6.2. Proof of (4.18). For $s \in \mathbb{R}$, $1 \leq q < \infty$, and $1 \leq r \leq \infty$, let $F^s_{q,r}(\mathbb{R}^n;\mathbb{C})$ denote the Triebel–Lizorkin space as defined in [44, p. 8] and $F^s_{q,r}(\mathbb{R}^n)$ the subspace of $F^s_{q,r}(\mathbb{R}^n;\mathbb{C})$ consisting of real-valued functions in $F^s_{q,r}(\mathbb{R}^n;\mathbb{C})$, which satisfies

$$F^s_{q,2}(\mathbb{R}^n) = H^s_q(\mathbb{R}^n)$$

if $s \in \mathbb{R}$ and $1 < q < \infty$ (see [44, section 2.1.2. Proposition 1(vii)]). In addition, the inclusion

$$H^{s'}_q(\mathbb{R}^n) \subset F^{s''}_{q,r''}(\mathbb{R}^n) \tag{6.5}$$

is valid if $1 \leq q < \infty$, $s' > s''$, and $1 \leq r', r'' \leq \infty$ (see [44, section 2.2.1, Proposition 1]). From [44, section 5.4.4, Theorem 1], it follows that the inclusion

$$T_{\varphi_{p,+}}\left(F^{s/(p-1)}_{p,2(p-1)}(\mathbb{R}^n)\right) \subset F^s_{q,2}(\mathbb{R}^n) = H^s_q(\mathbb{R}^n)$$

holds if $1 < p < 2$ and $0 < s < p - 1$, which, together with (6.5), implies the inclusion

$$T_{\varphi_{p,+}}(H^t_p(\mathbb{R}^n)) \subset H^s_q(\mathbb{R}^n) \tag{6.6}$$

if $t > \frac{s}{p-1}$. By the argument used in Remark 4.7(i), we deduce that (6.6) remains true if we replace $H^t_p(\mathbb{R}^n)$ and $H^s_q(\mathbb{R}^n)$ by $H^t_p(\Omega)$ and $H^s_q(\Omega)$, respectively. This completes the proof.

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