FULLY DISCRETE SCHEME FOR BEAN'S CRITICAL-STATE MODEL WITH TEMPERATURE EFFECTS IN SUPERCONDUCTIVITY *

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Abstract. This paper considers a hyperbolic Maxwell variational inequality with temperature effects arising from Bean's critical-state model in type-II (high-temperature) superconductivity. Here, temperature dependence is included in the critical current density due to its main importance for the realization of superconducting effects, as confirmed through physical measurements. We propose a fully discrete scheme based on the implicit Euler in time and a mixed FEM in space consisting of Nédélec's edge elements for the electric field and piecewise constant elements for the magnetic induction. Furthermore, the initial approximation is specified by a compatibility system given by an elliptic curl-curl variational inequality. This specific setting enables us to derive the well-posedness of the discrete solution with a certain magnetic induction regularity. Our main result is the uniform convergence of the proposed fully discrete method. To prove this result, first of all, we establish stability estimates for the zero-order and first-order terms of the fully discrete solution. These stability estimates along with the underlying nonlinear structure allow us to derive a weak-star convergence result, which in particular yields the well-posedness of the governing Maxwell variational inequality with temperature effects. Finally, through the use of the solution operator for a discrete mixed variational problem in combination with the involved magnetic induction regularity and the weak-star convergence result, we are able to complete the proof of the uniform convergence. The last part of the paper is devoted to the a priori error analysis under a low Sobolev regularity assumption on the electric field. We close this paper by presenting some 3D numerical results, which especially confirm the physical Meissner-Ochsenfeld effect in superconductivity.

Key words. Maxwell variational inequality, Bean's critical-state model with temperature effects, superconductivity, fully discrete scheme, convergence analysis, error estimates.

AMS subject classifications. 35Q61, 35L87, 78M10

1. Introduction.

The physical phenomenon of superconductivity is characterized by zero electrical resistance and repulsion of magnetic fields (Meissner-Ochsenfeld effect) under the condition that the temperature is below some critical level. It was first discovered in 1911 by H. Kamerlingh-Onnes and has gained tremendous theoretical and practical attentions ever since. Nowadays, modern magnetic levitation trains, distributed superconducting magnetic energy storage (D-SMES), magnetic resonance imaging (MRI) and magnetic confinement fusion cannot be realized without the use of superconductors, just to mention a few key technologies. A critical-state model describing the magnetization process of penetration and exit of magnetic flux in type-II (high-temperature) superconductors was proposed by Bean [5,6]. More precisely, his model describes a nonlinear and non-smooth constitutive relation between the (total) current density and the electric field as follows:

(B1) The current density strength $|\mathbf{J}|$ cannot exceed the critical current j_c;
(B2) if $|\mathbf{J}|$ is strictly less than j_c, then the electric field $\mathbf{E}$ vanishes;
(B3) the electric field $\mathbf{E}$ is parallel to $\mathbf{J}$.

We underline that the (unknown) superconductive region is determined by points $(x,t) \in \Omega \times (0,T)$, for which the strict inequality $|\mathbf{J}(x,t)| < j_c$ is satisfied. Thus, in

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this region, there is no electrical resistance so that the electric field must vanish.

We refer to Bossavit [7–9] for early contributions towards extended Bean's law and the corresponding finite element method. The Bean critical-state model (B1)-(B3) governed by the eddy current equations with magnetic field dependence $j_c = j_c(\mathbf{H})$ leads to a parabolic quasi-variational inequality (QVI) of obstacle type. Prigozhin [31, 32] was the first, who introduced and analyzed this formulation. Barrett and Prigozhin [3] analyzed it in a scalar two-dimensional (2D) setting and its dual formulation. The finite element analysis for the associated parabolic variational inequality in a 2D setting was investigated in [16] (see also [17] for a similar 2D model using an **E-J**-formulation). Furthermore, the numerical analysis for the three-dimensional (3D) setting was investigated in [15]. Recent results on the numerical analysis for the parabolic QVI in a 2D setting were obtained in [4]. All the previously mentioned contributions were devoted to the numerical analysis for the eddy current case. In the full 3D Maxwell case (cf. [22]), the Bean's critical state model (B1)-(B3) with $j_c = j_c(x)$ leads to a hyperbolic Maxwell variational inequality of the second kind [35] (see [33] for the mathematical analysis in a more general setting). The numerical analysis for this variational inequality is still in its earlier state. We are only aware of the recent work [35] for the analysis of the semi-discrete spatial Galerkin approximations.

This paper is devoted to the fully discrete analysis of the Bean critical-state model (B1)-(B3) governed by the full 3D Maxwell equations with temperature effects. Let us underline that in all previously mentioned contributions, temperature dependence was neglected. However, by the nature of superconductivity, temperature effects play a major role, since superconducting effects strongly depend on the temperature itself and can only be reached, if the temperature is underneath some critical level. We refer to [2,14] concerning experimental measurements showing the strong temperature dependence in the critical current $j_c = j_c(x, \theta(x,t))$ and its physical properties. Let us now formulate the variational inequality we focused on in this paper:

$$
\text{(VI)} \qquad \begin{cases} \displaystyle\int_\Omega \epsilon \partial_t \mathbf{E}(t) \cdot (\mathbf{v} - \mathbf{E}(t)) + \mu^{-1} \partial_t \mathbf{B}(t) \cdot (\mathbf{w} - \mathbf{B}(t))\, dx \\ \displaystyle\quad + \int_\Omega \mu^{-1}\, \mathbf{curl}\, \mathbf{E}(t) \cdot \mathbf{w} - \mu^{-1} \mathbf{B}(t) \cdot \mathbf{curl}\, \mathbf{v}\, dx \\ \displaystyle\quad + \varphi(\theta(t), \mathbf{v}) - \varphi(\theta(t), \mathbf{E}(t)) \geq \int_\Omega \mathbf{f}(t) \cdot (\mathbf{v} - \mathbf{E}(t))\, dx \\ \quad \text{for a.e. } t \in (0,T) \text{ and every } (\mathbf{v}, \mathbf{w}) \in \mathbf{H}_0(\mathbf{curl}) \times \mathbf{L}^2(\Omega), \\ \quad (\mathbf{E}(0), \mathbf{B}(0)) = (\mathbf{E}_0, \mathbf{B}_0), \end{cases}
$$

with a nonsmooth L^1-type functional

$$
\varphi \colon L^\infty(\Omega) \times \mathbf{L}^1(\Omega) \to \mathbb{R}, \quad (y, \mathbf{v}) \mapsto \int_\Omega j_c(x, y(x)) |\mathbf{v}(x)|\, dx.
$$

In this setting, $\Omega \subset \mathbb{R}^3$ is a bounded polyhedral domain with a connected Lipschitz-boundary $\partial\Omega$. The assumption of the connected boundary guarantees that $\{\mathbf{v} \in \mathbf{H}_0(\mathbf{curl}) \cap \mathbf{H}(\mathrm{div}) \mid \mathbf{curl}\, \mathbf{v} = 0,\ \mathrm{div}\ \mathbf{v} = 0\} = \{0\}$ (cf. [1, Proposition 3.18.]), which is required for our analysis in connection with the application of the (discrete) Poincaré-Friedrichs-type inequality [20, Theorem 4.7]. Furthermore, $\mathbf{E} : \Omega \times (0,T) \to \mathbb{R}^3$ denotes the electric field, $\mathbf{B} : \Omega \times (0,T) \to \mathbb{R}^3$ the magnetic induction, $\mathbf{f} : \Omega \times (0,T) \to \mathbb{R}^3$ the applied current source and $\theta : \Omega \times (0,T) \to \mathbb{R}^3$ the temperature distribution. Note that in (VI) and all what follows, we use the abbreviation $\mathbf{E}(t) = \mathbf{E}(\cdot, t)$ (the same notation is also used for other quantities). The precise assumptions for the data involved in (VI) will be given in Section 2.

In ($\mathrm{VI}_{N,h}$), we propose a fully discrete scheme for (VI) based on the implicit Euler in time and a mixed FEM in space consisting of Nédélec's edge elements [30] for $\mathbf{E}$ and piecewise constant elements for $\mathbf{B}$. Furthermore, we consider finite element approximations for the initial data $(\mathbf{E}_0, \mathbf{B}_0)$ by solving an elliptic **curl-curl** variational inequality (3.2). This specific setting enables us to prove the well-posedness of ($\mathrm{VI}_{N,h}$) with a magnetic induction regularity in $\mathbf{curl}\,\mathbf{V}_h$ (see Theorem 3.4), where $\mathbf{V}_h$ denotes the Nédélec edge element space.

Our main goal is the uniform convergence of ($\mathrm{VI}_{N,h}$) towards (VI) (Theorem 3.10), which in particular yields the global well-posedness for (VI). The proof follows the following consecutive steps: First of all, by the compatibility system (3.2) and exploiting the regularity properties of the critical current density and the given data, we derive stability estimates for the zero-order and first-order terms of the fully discrete solution (Lemmas 3.6 and 3.7). These a priori estimates together with the mathematical properties of φ allow us to extract weakly-* converging subsequences whose limits turn out to solve the original variational inequality (Theorem 3.8). In particular, this implies the well-posedness of (VI). Hereafter, we consider the solution operator $\mathbf{\Phi}_h\colon \mathbf{H}_0(\mathbf{curl}) \to \mathbf{V}_h$ associated with a discrete mixed variational problem (Definition 3.2) and use its properties in combination with the magnetic induction regularity in $\mathbf{curl}\,\mathbf{V}_h$ and the weak-star convergence result to complete the proof of the uniform convergence. The final part of the paper is devoted to the a priori error analysis for the proposed fully discrete scheme ($\mathrm{VI}_{N,h}$). Under a low Sobolev regularity assumption on the electric field $\mathbf{E}$ of (VI), we derive a priori estimates for the error between the fully discrete solution and the continuous one (Theorem 4.4). The proof is based on the use of the operator $\mathbf{\Phi}_h\colon \mathbf{H}_0(\mathbf{curl}) \to \mathbf{V}_h$ and the recent sharp quasi-interpolation results [13, 18, 19]. Last but not least, we refer the reader to some existing works [11, 12, 24, 25, 28] concerning fully discrete approximations for time-dependent Maxwell's equations.

2. Preliminaries.

For a given Banach space X, we denote its norm by $\|\cdot\|_X$ and the duality pairing with the corresponding dual space X^* by $\langle\cdot,\cdot\rangle$. If X is a Hilbert space, then $(\cdot,\cdot)_X$ stands for its scalar product and $\|\cdot\|_X$ for the induced norm. In the case of $X = \mathbb{R}^n$, we renounce the subscript in the (Euclidean) norm and write $|\cdot|$. The Euclidean scalar product is denoted by a dot. Unless otherwise stated, we identify the dual space X^* with the Hilbert space X itself. The embedding between two Banach spaces X, Y is denoted by $X \hookrightarrow Y$. Now, we introduce some important Hilbert spaces throughout this paper:

$$\mathbf{H}(\mathbf{curl}) \coloneqq \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \mathbf{curl}\,\mathbf{v} \in \mathbf{L}^2(\Omega)\} \ \text{ and } \ \mathbf{H}(\mathrm{div}) \coloneqq \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \mathrm{div}\,\mathbf{v} \in L^2(\Omega)\},$$

where **curl** and div are understood in the distributional sense. Also, note that we use bold letters for vector-valued functions and the respective spaces. As usual, $\mathbf{C}_0^\infty(\Omega)$ denotes the space of all infinitely differentiable functions with compact support in Ω. The spaces $\mathbf{H}_0(\mathbf{curl})$ and $\mathbf{H}_0(\mathrm{div})$ stand for the closure of $\mathbf{C}_0^\infty(\Omega)$ with respect to the $\mathbf{H}(\mathbf{curl})$-norm and the $\mathbf{H}(\mathrm{div})$-norm, respectively. Furthermore, the spaces of divergence-free vector functions are

$$\mathbf{H}(\mathrm{div}{=}0) \coloneqq \{\mathbf{v} \in \mathbf{L}^2(\Omega) \ : \ (\mathbf{v}, \nabla\phi)_{\mathbf{L}^2(\Omega)} = 0 \quad \forall \phi \in H_0^1(\Omega)\},$$
$$\mathbf{H}_0(\mathrm{div}{=}0) \coloneqq \{\mathbf{v} \in \mathbf{L}^2(\Omega) \ : \ (\mathbf{v}, \nabla\phi)_{\mathbf{L}^2(\Omega)} = 0 \quad \forall \phi \in H^1(\Omega)\},$$

which are endowed with the $\mathbf{L}^2(\Omega)$-norm. Material parameters will occur on the problem statement, and thus, for a given positive function $\alpha \in L^\infty(\Omega)$, we denote by $\mathbf{L}^2_\alpha(\Omega)$ the weighted $\mathbf{L}^2(\Omega)$-space with the weighted scalar product $(\alpha\cdot,\cdot)_{\mathbf{L}^2(\Omega)}$. Moreover, we denote by $C > 0$ a generic constant, that can change during an estimation. Let us close this section by presenting all the mathematical assumptions for (VI).

Assumption 2.1 (Regularity assumptions on the material parameters).

(A1) The material parameters $\epsilon, \mu \in L^\infty(\Omega)$ are strictly positive, i.e., there exist positive constants $\underline{\epsilon}, \overline{\epsilon}, \underline{\mu}, \overline{\mu} \in \mathbb{R}_{>0}$ such that

$$\underline{\epsilon} \leq \epsilon(x) \leq \overline{\epsilon} \quad \text{and} \quad \underline{\mu} \leq \mu(x) \leq \overline{\mu} \text{ for a.e. } x \in \Omega.$$

(A2) For every $y \in \mathbb{R}$, $j_c(\cdot, y)\colon \Omega \to \mathbb{R}$ is Lebesgue-measurable and nonnegative.

(A3) For every $M > 0$, there exists a constant $C(M) > 0$ such that

$$0 \leq j_c(x, y) \leq C(M)$$

for a.e. $x \in \Omega$ and every $y \in [-M, M]$.

(A4) For every $M > 0$, there exists a constant $L(M) > 0$ such that

$$|j_c(x, y) - j_c(x, z)| \leq L(M)|y - z|$$

for a.e. $x \in \Omega$ and every $y, z \in [-M, M]$.

Let us remark that the local Lipschitz property (A4) and the local boundedness property (A3) for the temperature dependence in the critical current are justified by experimental measurements reported in [2, 14]. Note that assumptions (A2)–(A4) seem to be sharp for our mathematical analysis. In contrast to (A2)–(A4), from the mathematical point of view, (A1) is not sharp, as our results can be extended to matrix-valued material parameters ϵ and μ. However, this case leads to a physical model of an anisotropic material, for which Bean's law (B1)–(B3) is not suitable. Indeed, (B3) is only reasonable for a scalar-valued resistivity, i.e., not for anisotropic materials (see [3, 4, 8, 15, 31, 32]). Therefore, due to this physical reason, we only consider scalar-valued material parameters.

Assumption 2.2 (Regularity assumptions on the given data).

(A5) Suppose that

$$\mathbf{f} \in \mathcal{C}^{0,1}([0,T], \mathbf{L}^2(\Omega)) \quad \text{and} \quad \theta \in \mathcal{C}^{0,1}([0,T], L^2(\Omega)) \cap \mathcal{C}([0,T], L^\infty(\Omega)).$$

(A6) The initial data $(\mathbf{E}_0, \mathbf{B}_0) \in \mathbf{H}_0(\mathbf{curl}) \times \mathbf{H}_0(\text{div=0})$ satisfies the compatibility system

$$\begin{cases} \displaystyle\int_\Omega \epsilon \mathbf{E}_0 \cdot (\mathbf{v} - \mathbf{E}_0) + \mu^{-1}\mathbf{B}_0 \cdot (\mathbf{w} - \mathbf{B}_0)\, dx \\ \quad \displaystyle + \int_\Omega \mu^{-1}\, \mathbf{curl}\, \mathbf{E}_0 \cdot \mathbf{w} - \mu^{-1}\mathbf{B}_0 \cdot \mathbf{curl}\, \mathbf{v}\, dx \\ \quad \displaystyle + \varphi(\theta(0), \mathbf{v}) - \varphi(\theta(0), \mathbf{E}_0) \geq \int_\Omega \mathbf{f}(0) \cdot (\mathbf{v} - \mathbf{E}_0)\, dx \\ \text{for all } (\mathbf{v}, \mathbf{w}) \in \mathbf{H}_0(\mathbf{curl}) \times \mathbf{L}^2(\Omega). \end{cases} \tag{2.1}$$

3. Fully discrete scheme. As pointed out in the introduction, we focus on the implicit Euler scheme for the time discretization in (VI). To this aim, let us fix $N \in \mathbb{N}$

and define an equidistant partition of $[0,T]$ in the following way:

$$\tau := \frac{T}{N}, \quad 0 = t_0 < t_1 < \cdots < t_N = T \quad \text{with} \quad t_n := n\tau$$

for all $n \in \{0,\ldots,N\}$. Furthermore, we define

$$\mathbf{f}^n := \mathbf{f}(t_n) \in \mathbf{L}^2(\Omega), \quad \varphi^n(\mathbf{v}) := \int_\Omega j_c(x,\theta(x,t_n))|\mathbf{v}(x)|\,dx \quad \forall n \in \{0,\ldots,N\}.$$

We choose a family of quasi-uniform triangulations $\{\mathcal{T}_h\}_{h>0}$, i.e.,

$$\overline{\Omega} = \bigcup_{T\in\mathcal{T}_h} T \quad \forall h > 0,$$

and, for h_T denoting the diameter of T and ρ_T denoting the diameter of the largest ball contained in T, there exist constants $\rho > 0$ and $\nu > 0$ such that

$$\frac{h_T}{\rho_T} \le \rho \quad \text{and} \quad \frac{h}{h_T} \le \nu \quad \forall T \in \mathcal{T}_h, \quad \forall h > 0.$$

The subscript h denotes the maximum of h_T for $T \in \mathcal{T}_h$. The finite element space of Nédélec's first family of edge elements is defined by

$$\mathbf{V}_h := \{\mathbf{v}_h \in \mathbf{H}_0(\mathbf{curl}) \ : \ \mathbf{v}_h|_T = \mathbf{a}_T + \mathbf{b}_T \times x \text{ with } \mathbf{a}_T, \mathbf{b}_T \in \mathbb{R}^3,\ \forall T \in \mathcal{T}_h\},$$

and the finite element space of piecewise constant functions is denoted by

$$\mathbf{W}_h := \{\mathbf{w}_h \in \mathbf{L}^2(\Omega) \ : \ \mathbf{w}_h|_T = \mathbf{a}_T \text{ with } \mathbf{a}_T \in \mathbb{R}^3,\ \forall T \in \mathcal{T}_h\},$$

which satisfy $\mathbf{curl}\,\mathbf{V}_h \subset \mathbf{W}_h$. In addition to these spaces, we introduce the space of continuous piecewise linear elements with vanishing traces by

$$\Theta_h := \{\phi_h \in H_0^1(\Omega) \ : \ \phi_h|_T = \mathbf{a}_T \cdot x + b_T \text{ with } \mathbf{a}_T \in \mathbb{R}^3, b_T \in \mathbb{R} \quad \forall T \in \mathcal{T}_h\}.$$

Moreover, the family $\{\mathcal{T}_h\}_{h>0}$ is chosen such that there exists $\bar{h} > 0$ with

$$\text{(3.1)} \qquad \mathbf{V}_{\tilde{h}} \subset \mathbf{V}_h \quad \text{and} \quad \mathbf{W}_{\tilde{h}} \subset \mathbf{W}_h \quad \forall\, 0 < h \le \tilde{h} \le \bar{h}.$$

Having introduced all the required finite element spaces, we now propose the following fully discrete scheme to (VI):

$$(\mathrm{VI}_{N,h}) \qquad \begin{cases} \displaystyle\int_\Omega \epsilon\delta\mathbf{E}_h^n \cdot (\mathbf{v}_h - \mathbf{E}_h^n) + \mu^{-1}\delta\mathbf{B}_h^n \cdot (\mathbf{w}_h - \mathbf{B}_h^n)\,dx \\ \quad \displaystyle + \int_\Omega \mu^{-1}\,\mathbf{curl}\,\mathbf{E}_h^n \cdot \mathbf{w}_h - \mu^{-1}\mathbf{B}_h^n \cdot \mathbf{curl}\,\mathbf{v}_h\,dx \\ \quad \displaystyle + \varphi^n(\mathbf{v}_h) - \varphi^n(\mathbf{E}_h^n) \ge \int_\Omega \mathbf{f}^n \cdot (\mathbf{v}_h - \mathbf{E}_h^n)\,dx \\ \quad \text{for every } (\mathbf{v}_h, \mathbf{w}_h) \in \mathbf{V}_h \times \mathbf{W}_h \text{ and } n \in \{1,\ldots,N\} \\ \quad (\mathbf{E}_h^0, \mathbf{B}_h^0) = (\mathbf{E}_{0h}, \mathbf{B}_{0h}), \end{cases}$$

where

$$\delta\mathbf{E}_h^n := \frac{\mathbf{E}_h^n - \mathbf{E}_h^{n-1}}{\tau} \quad \text{and} \quad \delta\mathbf{B}_h^n := \frac{\mathbf{B}_h^n - \mathbf{B}_h^{n-1}}{\tau} \quad \forall n \in \{1,\ldots,N\},$$

Moreover, $(\mathbf{E}_{0h}, \mathbf{B}_{0h}) \in \mathbf{V}_h \times \mathbf{W}_h$ denotes the finite element approximation of the initial data $(\mathbf{E}_0, \mathbf{B}_0)$, which is defined as the solution to the discrete mixed problem

$$(3.2) \quad \begin{cases} \int_\Omega \epsilon \mathbf{E}_{0h} \cdot (\mathbf{v}_h - \mathbf{E}_{0h}) + \mu^{-1} \mathbf{B}_{0h} \cdot (\mathbf{w}_h - \mathbf{B}_{0h})\, dx \\ \quad + \int_\Omega \mu^{-1} \, \mathbf{curl}\, \mathbf{E}_{0h} \cdot \mathbf{w}_h - \mu^{-1} \mathbf{B}_{0h} \cdot \mathbf{curl}\, \mathbf{v}_h \, dx \\ \quad + \varphi(\theta(0), \mathbf{v}_h) - \varphi(\theta(0), \mathbf{E}_{0h}) \geq \int_\Omega \mathbf{f}(0) \cdot (\mathbf{v}_h - \mathbf{E}_{0h})\, dx \\ \quad \text{for all } (\mathbf{v}_h, \mathbf{w}_h) \in \mathbf{V}_h \times \mathbf{W}_h. \end{cases}$$

The well-posedness of (3.2) follows from the classical theory of variational inequalities [26, Theorem 2.2], as (3.2) is equivalent to an elliptic **curl-curl** variational inequality (cf. the proof of Theorem 3.4). In view of (3.2), it makes sense to set $(\delta \mathbf{E}_h^0, \delta \mathbf{B}_h^0) := (\mathbf{E}_{0h}, \mathbf{B}_{0h})$. Indeed, if we replace $(\delta \mathbf{E}_h^n, \delta \mathbf{B}_h^n)$ by $(\mathbf{E}_{0h}, \mathbf{B}_{0h})$ in $(\mathrm{VI}_{N,h})$ and set $n = 0$, then we arrive exactly at (3.2). Note that $(\delta \mathbf{E}_h^0, \delta \mathbf{B}_h^0) = (\mathbf{E}_{0h}, \mathbf{B}_{0h})$ is important for our stability analysis (see (3.22) in the proof of Lemma 3.7).

Remark 3.1. All mathematical findings in this paper remain true, if we replace $\mathbf{W}_h$ by $\mathbf{W}_h \cap \mathbf{H}_0(\mathrm{div})$. Both $\mathbf{W}_h$ and $\mathbf{W}_h \cap \mathbf{H}_0(\mathrm{div})$ are dense in $\mathbf{L}^2(\Omega)$ and contain $\mathbf{curl}\, \mathbf{V}_h$. The condition of $\mathbf{curl}\, \mathbf{V}_h$ being a subspace is necessary to prove the regularity properties $\mathbf{B}_{0h}, \mathbf{B}_h^n \in \mathbf{curl}\, \mathbf{V}_h$ for the solutions to (3.2) and $(\mathrm{VI}_{N,h})$; see Lemma 3.3 and Theorem 3.4. Furthermore, the density property is required for the derivation of the weak-* convergence result (Theorem 3.8). We note that the choice $\mathbf{W}_h = \mathbf{curl}\, \mathbf{V}_h$ is not suitable for our analysis, as $\mathbf{curl}\, \mathbf{V}_h$ is not dense in $\mathbf{L}^2(\Omega)$.

DEFINITION 3.2. *For every $h > 0$ and $\mathbf{y} \in \mathbf{H}_0(\mathbf{curl})$, we denote the solution operator of the discrete variational mixed problem*

$$(3.3) \quad \begin{cases} (\mathbf{curl}\, \mathbf{y}_h, \mathbf{curl}\, \mathbf{v}_h)_{\mathbf{L}^2_{1/\mu}(\Omega)} = (\mathbf{curl}\, \mathbf{y}, \mathbf{curl}\, \mathbf{v}_h)_{\mathbf{L}^2_{1/\mu}(\Omega)} & \forall \mathbf{v}_h \in \mathbf{V}_h, \\ (\mathbf{y}_h, \nabla \psi_h)_{\mathbf{L}^2(\Omega)} = (\mathbf{y}, \nabla \psi_h)_{\mathbf{L}^2(\Omega)} & \forall \psi_h \in \Theta_h \end{cases}$$

by $\mathbf{\Phi}_h \colon \mathbf{H}_0(\mathbf{curl}) \to \mathbf{V}_h$ with $\mathbf{\Phi}_h \mathbf{y} := \mathbf{y}_h$.

The theory of mixed problems (cf. [29, Theorem 2.45]) in combination with the discrete Poincaré-Friedrichs-type inequality [20, Theorem 4.7] and the discrete LBB condition (cf. [37, pp. 2802-2803]) implies that for every $h > 0$ and $\mathbf{y} \in \mathbf{H}_0(\mathbf{curl})$, (3.3) admits a unique solution $\mathbf{y}_h = \mathbf{\Phi}_h \mathbf{y} \in \mathbf{V}_h$ satisfying

$$(3.4) \qquad \|\mathbf{\Phi}_h \mathbf{y} - \mathbf{y}\|_{\mathbf{H}(\mathbf{curl})} \leq C \left(\inf_{\boldsymbol{\chi}_h \in \mathbf{V}_h} \|\mathbf{y} - \boldsymbol{\chi}_h\|_{\mathbf{H}(\mathbf{curl})} \right) \quad \forall \mathbf{y} \in \mathbf{H}_0(\mathbf{curl}).$$

with a constant $C > 0$, independent of h and $\mathbf{y}$. In particular, (3.4) yields

$$(3.5) \qquad \|\mathbf{\Phi}_h \mathbf{y}\|_{\mathbf{H}(\mathbf{curl})} \leq (C+1) \|\mathbf{y}\|_{\mathbf{H}(\mathbf{curl})} \quad \forall h > 0, \quad \forall \mathbf{y} \in \mathbf{H}_0(\mathbf{curl}).$$

Moreover, (3.4) along with the density property of $\mathbf{V}_h$:

$$\forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl})\ \forall \delta > 0\ \exists \tilde{h} > 0\ \forall h \in (0, \tilde{h})\ \exists \mathbf{v}_h \in \mathbf{V}_h : \ \|\mathbf{v} - \mathbf{v}_h\|_{\mathbf{H}(\mathbf{curl})} \leq \delta$$

implies that

$$(3.6) \qquad \lim_{h \to 0} \|\mathbf{\Phi}_h \mathbf{y} - \mathbf{y}\|_{\mathbf{H}(\mathbf{curl})} = 0 \quad \forall \mathbf{y} \in \mathbf{H}_0(\mathbf{curl}).$$

This operator enables us to show the strong convergence of the discrete initial values towards $(\mathbf{E}_0, \mathbf{B}_0)$.

LEMMA 3.3. *Under Assumptions 2.1 and 2.2, the discrete approximation of the initial value* $(\mathbf{E}_{0h}, \mathbf{B}_{0h}) \in \mathbf{V}_h \times \mathbf{W}_h$ *satisfies* $\mathbf{B}_{0h} \in \mathbf{curl}\,\mathbf{V}_h$ *for all* $h > 0$ *and*

$$\lim_{h\to 0} \|\mathbf{E}_{0h} - \mathbf{E}_0\|_{\mathbf{L}^2_\epsilon(\Omega)} = \lim_{h\to 0} \|\mathbf{B}_{0h} - \mathbf{B}_0\|_{\mathbf{L}^2_{1/\mu}(\Omega)} = 0.$$

Proof. Let $h > 0$. Inserting $\mathbf{v}_h = \mathbf{E}_{0h}$ in (3.2), we obtain that

$$\int_\Omega \mu^{-1}(\mathbf{B}_{0h} + \mathbf{curl}\,\mathbf{E}_{0h}) \cdot (\mathbf{w}_h - \mathbf{B}_{0h}) dx = 0 \quad \forall \mathbf{w}_h \in \mathbf{W}_h. \tag{3.7}$$

Since $\mathbf{curl}\,\mathbf{V}_h \subset \mathbf{W}_h$, we may set $\mathbf{w}_h := 2\mathbf{B}_{0h} + \mathbf{curl}\,\mathbf{E}_{0h}$ in (3.7), which implies

$$\mathbf{B}_{0h} = -\,\mathbf{curl}\,\mathbf{E}_{0h}. \tag{3.8}$$

Thus, $\mathbf{B}_{0h} \in \mathbf{curl}\,\mathbf{V}_h \subset \mathbf{H}_0(\text{div=0})$ follows. Moreover, testing (3.2) with $(\mathbf{v}_h, \mathbf{w}_h) = (2\mathbf{E}_{0h}, 2\mathbf{B}_{0h})$ as well as $(\mathbf{v}_h, \mathbf{w}_h) = (0, 0)$ yields

$$\begin{aligned} \|\mathbf{E}_{0h}\|^2_{\mathbf{L}^2_\epsilon(\Omega)} + \|\mathbf{B}_{0h}\|^2_{\mathbf{L}^2_{1/\mu}(\Omega)} + \int_\Omega j_c(x, \theta(x, 0))|\mathbf{E}_{0h}(x)|\, dx &= \int_\Omega \mathbf{f}(0) \cdot \mathbf{E}_{0h}\, dx \\ \Rightarrow \|\mathbf{E}_{0h}\|_{\mathbf{L}^2_\epsilon(\Omega)} &\le \|\epsilon^{-1/2}\mathbf{f}(0)\|_{\mathbf{L}^2(\Omega)}. \end{aligned} \tag{3.9}$$

Next, we insert $(\mathbf{v}, \mathbf{w}) = (\mathbf{E}_{0h}, \mathbf{B}_{0h})$ into (2.1) and $(\mathbf{v}_h, \mathbf{w}_h) = (\mathbf{\Phi}_h\mathbf{E}_0, 0)$ into (3.2) and obtain after adding the resulting inequalities together that

$$\begin{aligned} &\|\mathbf{E}_{0h} - \mathbf{E}_0\|^2_{\mathbf{L}^2_\epsilon(\Omega)} + \|\mathbf{B}_{0h} - \mathbf{B}_0\|^2_{\mathbf{L}^2_{1/\mu}(\Omega)} \\ &\le \int_\Omega \mathbf{f}(0) \cdot (\mathbf{E}_0 - \mathbf{\Phi}_h\mathbf{E}_0)\, dx + \int_\Omega \epsilon\mathbf{E}_{0h} \cdot (\mathbf{\Phi}_h\mathbf{E}_0 - \mathbf{E}_0)\, dx \\ &\quad - \int_\Omega \mu^{-1}(\mathbf{B}_{0h} + \mathbf{curl}\,\mathbf{E}_{0h}) \cdot \mathbf{B}_0\, dx + \int_\Omega \mu^{-1}\mathbf{B}_{0h} \cdot \mathbf{curl}(\mathbf{E}_0 - \mathbf{\Phi}_h\mathbf{E}_0)\, dx \\ &\quad + \int_\Omega j_c(x, \theta(x, 0))\big(|\mathbf{\Phi}_h\mathbf{E}_0| - |\mathbf{E}_0|\big)\, dx. \end{aligned} \tag{3.10}$$

Due to (3.8), the third term on the right-hand side of (3.10) vanishes. Moreover,

$$\int_\Omega \mu^{-1}\mathbf{B}_{0h} \cdot \mathbf{curl}(\mathbf{E}_0 - \mathbf{\Phi}_h\mathbf{E}_0)\, dx \underbrace{=}_{(3.8)} \int_\Omega \mu^{-1}\,\mathbf{curl}\,\mathbf{E}_{0h} \cdot \mathbf{curl}(\mathbf{E}_0 - \mathbf{\Phi}_h\mathbf{E}_0)\, dx \underbrace{=}_{(3.3)} 0.$$

Thus, applying Hölder's inequality together with (A3), (A5) and (3.9) to (3.10) yields

$$\|\mathbf{E}_{0h} - \mathbf{E}_0\|^2_{\mathbf{L}^2_\epsilon(\Omega)} + \|\mathbf{B}_{0h} - \mathbf{B}_0\|^2_{\mathbf{L}^2_{1/\mu}(\Omega)} \le C\|\mathbf{\Phi}_h\mathbf{E}_0 - \mathbf{E}_0\|_{\mathbf{L}^2_\epsilon(\Omega)} \tag{3.11}$$

with a constant $C > 0$ only depending on $\epsilon, \mathbf{f}$, j_c and θ. Finally, passing to the limit $h \to 0$ in (3.11), (3.6) yields the assertion. ∎

The following theorem proves the well-posedness of ($\mathrm{VI}_{N,h}$) and gives an important regularity property for the discrete magnetic induction.

THEOREM 3.4. *Let Assumptions 2.1 and 2.2 hold. Then, for every* $h > 0$ *and* $N \in \mathbb{N}$*, the system of discrete variational inequalities* ($\mathrm{VI}_{N,h}$) *admits a unique solution* $\{(\mathbf{E}^n_h, \mathbf{B}^n_h)\}_{n=1}^N \subset \mathbf{V}_h \times \mathbf{curl}\,\mathbf{V}_h$.

Proof. Let $N \in \mathbb{N}$, $h > 0$ and $n \in \{1, \dots, N\}$. Furthermore, assume that $(\mathbf{E}_h^{n-1}, \mathbf{B}_h^{n-1})$ is already known. Using the same arguments as in (3.8), we may decouple the variational inequality into two parts by testing (VI$_{N,h}$) with $\mathbf{v}_h = \mathbf{E}_h^n$ to obtain that

$$\delta \mathbf{B}_h^n = -\operatorname{\mathbf{curl}} \mathbf{E}_h^n. \tag{3.12}$$

By definition, (3.12) yields the following explicit formula for $\mathbf{B}_h^n$:

$$\mathbf{B}_h^n = \mathbf{B}_h^{n-1} - \tau \operatorname{\mathbf{curl}} \mathbf{E}_h^n. \tag{3.13}$$

Next, we insert $\mathbf{w}_h = \mathbf{B}_h^n$ in (VI$_{N,h}$) and employ (3.13) to obtain the variational inequality

$$\begin{aligned} &\int_\Omega \epsilon \delta \mathbf{E}_h^n \cdot (\mathbf{v}_h - \mathbf{E}_h^n)\, dx + \int_\Omega \tau \mu^{-1} \operatorname{\mathbf{curl}} \mathbf{E}_h^n \cdot \operatorname{\mathbf{curl}}(\mathbf{v}_h - \mathbf{E}_h^n)\, dx + \varphi^n(\mathbf{v}_h) \\ &- \varphi^n(\mathbf{E}_h^n) \geq \int_\Omega \mathbf{f}^n \cdot (\mathbf{v}_h - \mathbf{E}_h^n) + \mu^{-1} \mathbf{B}_h^{n-1} \cdot \operatorname{\mathbf{curl}}(\mathbf{v}_h - \mathbf{E}_h^n)\, dx \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \end{aligned} \tag{3.14}$$

The well-posedness of (3.14) is covered by a classical result in [26, Theorem 2.2], because it is equivalent to an elliptic **curl-curl** variational inequality of the form

$$a(\mathbf{E}_h^n, \mathbf{v}_h - \mathbf{E}_h^n) + \varphi^n(\mathbf{v}_h) - \varphi^n(\mathbf{E}_h^n) \geq \langle \tilde{\mathbf{f}}^n, \mathbf{v}_h - \mathbf{E}_h^n \rangle \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{3.15}$$

with the continuous and coercive bilinear form $a \colon \mathbf{V}_h \times \mathbf{V}_h \to \mathbb{R}$ defined by

$$a(\mathbf{u}_h, \mathbf{v}_h) := \int_\Omega \tau^{-1} \epsilon \mathbf{u}_h \cdot \mathbf{v}_h\, dx + \int_\Omega \tau \mu^{-1} \operatorname{\mathbf{curl}} \mathbf{u}_h \cdot \operatorname{\mathbf{curl}} \mathbf{v}_h\, dx \quad \forall \mathbf{u}_h, \mathbf{v}_h \in \mathbf{V}_h$$

and the right-hand side $\tilde{\mathbf{f}}^n \in \mathbf{H}_0(\mathbf{curl})^*$ by

$$\langle \tilde{\mathbf{f}}^n, \mathbf{v} \rangle := \int_\Omega (\mathbf{f}^n + \tau^{-1} \epsilon \mathbf{E}_h^{n-1}) \cdot \mathbf{v}\, dx + \int_\Omega \mu^{-1} \mathbf{B}_h^{n-1} \cdot \operatorname{\mathbf{curl}} \mathbf{v}\, dx \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}).$$

Inserting the solution $\mathbf{E}_h^n$ of (3.14) into (3.13), we finally obtain a unique solution

$$\{(\mathbf{E}_h^n, \mathbf{B}_h^n)\}_{n=1}^N \subset \mathbf{V}_h \times \mathbf{W}_h$$

of (VI$_{N,h}$). Finally, (3.13) and Lemma 3.3 give $\mathbf{B}_h^n \in \operatorname{\mathbf{curl}} \mathbf{V}_h$ by inductive reasoning. ∎

Remark 3.5. The formulas (3.13) and (3.14) will be the foundation for the computation of the numerical solution.

The following Lemmas prove the zero-order and first-order stability estimates for the fully discrete solution to (VI$_{N,h}$):

LEMMA 3.6. *Let Assumptions 2.1 and 2.2 be satisfied. Then, there exists a constant $C > 0$, depending only on $\mathbf{f}, \mathbf{E}_0, \mathbf{B}_0$ and T, ϵ, μ, such that for every $N \in \mathbb{N}$ and $h > 0$, the solution $\{(\mathbf{E}_h^n, \mathbf{B}_h^n)\}_{n=1}^N$ of (VI$_{N,h}$) fulfills the estimate*

$$\begin{aligned} &\max_{n \in \{1,\dots,N\}} \|\mathbf{E}_h^n\|^2_{\mathbf{L}^2_\epsilon(\Omega)} + \max_{n \in \{1,\dots,N\}} \|\mathbf{B}_h^n\|^2_{\mathbf{L}^2_{1/\mu}(\Omega)} \\ &+ \sum_{n=1}^N \|\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\|^2_{\mathbf{L}^2_\epsilon(\Omega)} + \sum_{n=1}^N \|\mathbf{B}_h^n - \mathbf{B}_h^{n-1}\|^2_{\mathbf{L}^2_{1/\mu}(\Omega)} + 2\tau \sum_{n=1}^N \varphi^n(\mathbf{E}_h^n) \leq C. \end{aligned} \tag{3.16}$$

Proof. Let $N \in \mathbb{N}$ and $h > 0$. We start by testing ($\mathrm{VI}_{N,h}$) with $(\mathbf{v}_h, \mathbf{w}_h) = (2\mathbf{E}_h^n, 2\mathbf{B}_h^n)$ as well as with $(\mathbf{v}_h, \mathbf{w}_h) = (0, 0)$ to obtain

(3.17)
$$\int_\Omega \epsilon \delta \mathbf{E}_h^n \cdot \mathbf{E}_h^n \, dx + \int_\Omega \mu^{-1} \delta \mathbf{B}_h^n \cdot \mathbf{B}_h^n \, dx + \varphi^n(\mathbf{E}_h^n) = \int_\Omega \mathbf{f}^n \cdot \mathbf{E}_h^n \, dx \quad \forall n \in \{1, \ldots, N\}.$$

Now, fix $i_0 \in \{1, \ldots, N\}$ and sum (3.17) up over $\{1, \ldots, i_0\}$. Then, applying the binomial formulas along with the Hölder and Young inequalities, we deduce that

$$\begin{aligned} (3.18) \qquad & \|\mathbf{E}_h^{i_0}\|^2_{\mathbf{L}^2_\epsilon(\Omega)} + \|\mathbf{B}_h^{i_0}\|_{\mathbf{L}^2_{1/\mu}(\Omega)} + \sum_{n=1}^{i_0} \|\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\|^2_{\mathbf{L}^2_\epsilon(\Omega)} \\ & + \sum_{n=1}^{i_0} \|\mathbf{B}_h^n - \mathbf{B}_h^{n-1}\|^2_{\mathbf{L}^2_{1/\mu}(\Omega)} + 2\tau \sum_{n=1}^{i_0} \varphi^n(\mathbf{E}_h^n) \\ & \leq \frac{2T\tau}{\underline{\epsilon}} \sum_{n=1}^{i_0} \|\mathbf{f}^n\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{E}_{0h}\|^2_{\mathbf{L}^2_\epsilon(\Omega)} + \|\mathbf{B}_{0h}\|^2_{\mathbf{L}^2_{1/\mu}(\Omega)} + \frac{\tau}{2T} \sum_{n=1}^{i_0} \|\mathbf{E}_h^n\|^2_{\mathbf{L}^2_\epsilon(\Omega)}. \end{aligned}$$

This, combined with (A5) and Lemma 3.3 and the fact that $\tau/T = 1/N \leq 1$ gives us an estimate of the form

$$\|\mathbf{E}_h^{i_0}\|^2_{\mathbf{L}^2_\epsilon(\Omega)} \leq C + \sum_{n=1}^{i_0-1} \frac{1}{N} \|\mathbf{E}_h^n\|^2_{\mathbf{L}^2_\epsilon(\Omega)} \quad \Rightarrow \quad \|\mathbf{E}_h^{i_0}\|^2_{\mathbf{L}^2_\epsilon(\Omega)} \leq C \exp\left(\sum_{n=1}^{i_0-1} \frac{1}{N} \right) \leq C,$$

where we have used the discrete Gronwall inequality. Since i_0 was arbitrary, we see from (3.18) that the proof is finished. ∎

LEMMA 3.7. *Let Assumptions 2.1 and 2.2 hold. Then, there exists a constant $C > 0$, depending only on $\mathbf{f}, \mathbf{E}_0, \mathbf{B}_0$ and $T, \epsilon, \mu, \theta, j_c$, such that for every $N \in \mathbb{N}$ and $h > 0$ the solution $\{(\mathbf{E}_h^n, \mathbf{B}_h^n)\}_{n=1}^N$ of ($\mathrm{VI}_{N,h}$) satisfies*

$$\begin{aligned} (3.19) \qquad & \max_{n \in \{1,\ldots,N\}} \|\delta \mathbf{E}_h^n\|^2_{\mathbf{L}^2_\epsilon(\Omega)} + \max_{n \in \{1,\ldots,N\}} \|\delta \mathbf{B}_h^n\|^2_{\mathbf{L}^2_{1/\mu}(\Omega)} \\ & + \sum_{n=1}^N \|\delta \mathbf{E}_h^n - \delta \mathbf{E}_h^{n-1}\|^2_{\mathbf{L}^2_\epsilon(\Omega)} + \sum_{n=1}^N \|\delta \mathbf{B}_h^n - \delta \mathbf{B}_h^{n-1}\|^2_{\mathbf{L}^2_{1/\mu}(\Omega)} \leq C \end{aligned}$$

and

$$(3.20) \qquad \max_{n \in \{1,\ldots,N\}} \|\mathbf{curl}\, \mathbf{E}_h^n\|^2_{\mathbf{L}^2_{1/\mu}(\Omega)} \leq C.$$

Proof. Let $N \in \mathbb{N}$, $h > 0$ and $n \in \{1, \ldots, N\}$. Inserting $(\mathbf{v}_h, \mathbf{w}_h) = (\mathbf{E}_h^{n-1}, \mathbf{B}_h^{n-1})$ in the n-th inequality of ($\mathrm{VI}_{N,h}$) and then adding it with the $(n-1)$-th inequality of ($\mathrm{VI}_{N,h}$) tested with $(\mathbf{v}_h, \mathbf{w}_h) = (\mathbf{E}_h^n, \mathbf{B}_h^n)$ lead to

$$\begin{aligned} (3.21) \qquad & \int_\Omega \epsilon (\delta \mathbf{E}_h^n - \delta \mathbf{E}_h^{n-1}) \cdot (\mathbf{E}_h^n - \mathbf{E}_h^{n-1}) + \mu^{-1} (\delta \mathbf{B}_h^n - \delta \mathbf{B}_h^{n-1}) \cdot (\mathbf{B}_h^n - \mathbf{B}_h^{n-1}) \, dx \\ & \leq \int_\Omega (\mathbf{f}^n - \mathbf{f}^{n-1}) \cdot (\mathbf{E}_h^n - \mathbf{E}_h^{n-1}) \, dx + \int_\Omega j_c(x, \theta(x, t_n))(|\mathbf{E}_h^{n-1}| - |\mathbf{E}_h^n|) \, dx \\ & \quad + \int_\Omega j_c(x, \theta(x, t_{n-1}))(|\mathbf{E}_h^n| - |\mathbf{E}_h^{n-1}|) \, dx. \end{aligned}$$

We sum (3.21) up over $\{1,\dots,i_0\}$ for a fixed $i_0 \in \{1,\dots,N\}$ and divide the resulting inequality by τ to get

$$\sum_{n=1}^{i_0}\left[\int_\Omega \epsilon(\delta\mathbf{E}_h^n - \delta\mathbf{E}_h^{n-1})\cdot\delta\mathbf{E}_h^n\,dx + \int_\Omega \mu^{-1}(\delta\mathbf{B}_h^n - \delta\mathbf{B}_h^{n-1})\cdot\delta\mathbf{B}_h^n\,dx\right]$$
$$\le \sum_{n=1}^{i_0}\left[\int_\Omega(\mathbf{f}^n - \mathbf{f}^{n-1})\cdot\delta\mathbf{E}_h^n\,dx + \int_\Omega \big(j_c(x,\theta(x,t_n)) - j_c(x,\theta(x,t_{n-1}))\big)\left(\frac{|\mathbf{E}_h^{n-1}| - |\mathbf{E}_h^n|}{\tau}\right)dx\right].$$

Then, as in the proof of Lemma 3.6, the binomial formulas along with the Hölder and Young inequalities yield

$$\|\delta\mathbf{E}_h^{i_0}\|^2_{\mathbf{L}^2_\epsilon(\Omega)} + \|\delta\mathbf{B}_h^{i_0}\|^2_{\mathbf{L}^2_{1/\mu}(\Omega)} + \sum_{n=1}^{i_0}\|\delta\mathbf{E}_h^n - \delta\mathbf{E}_h^{n-1}\|^2_{\mathbf{L}^2_\epsilon(\Omega)} + \|\delta\mathbf{B}_h^n - \delta\mathbf{B}_h^{n-1}\|^2_{\mathbf{L}^2_{1/\mu}(\Omega)}$$
$$\le \frac{4T\tau}{\underline{\epsilon}}\sum_{n=1}^{i_0}\left\|\frac{\mathbf{f}^n - \mathbf{f}^{n-1}}{\tau}\right\|^2_{\mathbf{L}^2(\Omega)} + \frac{4T\tau}{\underline{\epsilon}}\sum_{n=1}^{i_0}\left\|\frac{j_c(x,\theta(x,t_n)) - j_c(x,\theta(x,t_{n-1}))}{\tau}\right\|^2_{\mathbf{L}^2(\Omega)}$$
$$+ \|\mathbf{E}_{0h}\|^2_{\mathbf{L}^2_\epsilon(\Omega)} + \|\mathbf{B}_{0h}\|^2_{\mathbf{L}^2_{1/\mu}(\Omega)} + \underbrace{\frac{\tau}{2T}}_{=\frac{1}{2N}}\sum_{n=1}^{i_0}\|\delta\mathbf{E}_h^n\|^2_{\mathbf{L}^2_\epsilon(\Omega)}, \tag{3.22}$$

where we have also used $\delta\mathbf{E}_{0h} = \mathbf{E}_{0h}$ and $\delta\mathbf{B}_{0h} = \mathbf{B}_{0h}$. Therefore, (A4), (A5) and Lemma 3.3 applied to (3.22) imply

$$\|\delta\mathbf{E}_h^{i_0}\|^2_{\mathbf{L}^2_\epsilon(\Omega)} \le C + \sum_{n=1}^{i_0-1}\frac{1}{N}\|\delta\mathbf{E}_h^n\|^2_{\mathbf{L}^2_\epsilon(\Omega)} \quad\Rightarrow\quad \|\delta\mathbf{E}_h^{i_0}\|^2_{\mathbf{L}^2_\epsilon(\Omega)} \le C\exp\left(\sum_{n=1}^{i_0-1}\frac{1}{N}\right) \le C,$$

by the discrete Gronwall inequality. Since i_0 was chosen arbitrarily, applying the above estimate to (3.22) yields (3.19). Finally, (3.20) follows immediately from (3.12) and (3.19). ∎

With these stability estimates at hand, we will establish a weak-* convergence result for ($\mathrm{VI}_{N,h}$), which particularly implies the well-posedness of (VI). First, we denote

$$\begin{cases} \mathbf{E}_{N,h}(0) := \mathbf{E}_{0h} \\ \mathbf{E}_{N,h}(t) := \mathbf{E}_h^{n-1} + (t - t_{n-1})\delta\mathbf{E}_h^n \end{cases} \quad\text{and}\quad \begin{cases} \overline{\mathbf{E}}_{N,h}(0) := \mathbf{E}_{0h} \\ \overline{\mathbf{E}}_{N,h}(t) := \mathbf{E}_h^n \end{cases} \tag{3.23}$$

for $t \in (t_{n-1}, t_n]$ and $n \in \{1,\dots,N\}$. In the same way, we define $\mathbf{B}_{N,h}, \overline{\mathbf{B}}_{N,h}$ and $\overline{\mathbf{f}}_N$. Furthermore, we introduce the function $\varphi_N\colon [0,T]\times\mathbf{L}^2(\Omega)\to\mathbb{R}$ by

$$\begin{cases} \varphi_N(0,\mathbf{v}) := \varphi(\theta(0),\mathbf{v}) = \displaystyle\int_\Omega j_c(x,\theta(x,0))|\mathbf{v}(x)|\,dx \\ \varphi_N(t,\mathbf{v}) := \varphi^n(\mathbf{v}) = \displaystyle\int_\Omega j_c(x,\theta(x,t_n))|\mathbf{v}(x)|\,dx \quad \forall t \in (t_{n-1}, t_n], \end{cases} \tag{3.24}$$

for $\mathbf{v} \in \mathbf{L}^2(\Omega)$ and $n \in \{1, \ldots, N\}$. Now, we can rewrite (VI$_{N,h}$) in the following manner:

$$
\begin{cases}
\int_\Omega \epsilon \partial_t \mathbf{E}_{N,h}(t) \cdot (\mathbf{v}_h - \overline{\mathbf{E}}_{N,h}(t)) + \mu^{-1} \partial_t \mathbf{B}_{N,h}(t) \cdot (\mathbf{w}_h - \overline{\mathbf{B}}_{N,h}(t))\, dx \\
\quad + \int_\Omega \mu^{-1}\, \mathbf{curl}\, \overline{\mathbf{E}}_{N,h}(t) \cdot \mathbf{w}_h - \mu^{-1} \overline{\mathbf{B}}_{N,h}(t) \cdot \mathbf{curl}\, \mathbf{v}_h \, dx \\
\quad + \varphi_N(t, \mathbf{v}_h) - \varphi_N(t, \overline{\mathbf{E}}_{N,h}(t)) \geq \int_\Omega \overline{\mathbf{f}}_N(t) \cdot (\mathbf{v}_h - \overline{\mathbf{E}}_{N,h}(t))\, dx \\
\quad \text{for every } (\mathbf{v}_h, \mathbf{w}_h) \in \mathbf{V}_h \times \mathbf{W}_h \text{ and a.e. } t \in (0,T) \\
\quad (\mathbf{E}_{N,h}(0), \mathbf{B}_{N,h}(0)) = (\mathbf{E}_{0h}, \mathbf{B}_{0h}).
\end{cases} \tag{3.25}
$$

THEOREM 3.8. *Let Assumptions 2.1 and 2.2 hold. Then, there exists a pair*

$$(\mathbf{E}, \mathbf{B}) \in W^{1,\infty}((0,T), \mathbf{L}^2_\epsilon(\Omega) \times \mathbf{H}_0(\text{div=0})) \cap L^\infty((0,T), \mathbf{H}_0(\mathbf{curl}) \times \mathbf{H}_0(\text{div=0}))$$

such that for $N = N(h)$ with $N(h) \to \infty$ as $h \to 0$ it holds that

$$
\begin{aligned}
&\mathbf{E}_{N,h} \rightharpoonup^* \mathbf{E} \text{ and } \overline{\mathbf{E}}_{N,h} \rightharpoonup^* \mathbf{E} && \text{weakly-* in } L^\infty((0,T), \mathbf{H}_0(\mathbf{curl})),\\
&\mathbf{B}_{N,h} \rightharpoonup^* \mathbf{B} \text{ and } \overline{\mathbf{B}}_{N,h} \rightharpoonup^* \mathbf{B} && \text{weakly-* in } L^\infty((0,T), \mathbf{H}_0(\text{div=0})),\\
&\partial_t \mathbf{E}_{N,h} \rightharpoonup^* \partial_t \mathbf{E} && \text{weakly-* in } L^\infty((0,T), \mathbf{L}^2_\epsilon(\Omega)),\\
&\partial_t \mathbf{B}_{N,h} \rightharpoonup^* \partial_t \mathbf{B} && \text{weakly-* in } L^\infty((0,T), \mathbf{H}_0(\text{div=0})),
\end{aligned}
$$

and $(\mathbf{E}, \mathbf{B})$ is the unique solution to (VI).

Proof. First of all, we emphasize that $N = N(h)$ denotes a family of natural numbers with $N(h) \to \infty$ for $h \to 0$. As shown in Lemma 3.6 and Lemma 3.7, $\{\mathbf{E}_{N,h}\}_{h>0}, \{\mathbf{B}_{N,h}\}_{h>0}$, $\{\overline{\mathbf{E}}_{N,h}\}_{h>0}, \{\overline{\mathbf{B}}_{N,h}\}_{h>0}$, and $\{\partial_t \mathbf{E}_{N,h}\}_{h>0}, \{\partial_t \mathbf{B}_{N,h}\}_{h>0}$ are bounded in their respective spaces. Therefore, we may extract weakly-* converging subsequences, which will not be denoted in a special way:

$$
\begin{cases}
\overline{\mathbf{E}}_{N,h} \rightharpoonup^* \overline{\mathbf{E}} & \text{weakly-* in } L^\infty((0,T), \mathbf{H}_0(\mathbf{curl})),\\
\overline{\mathbf{B}}_{N,h} \rightharpoonup^* \overline{\mathbf{B}} & \text{weakly-* in } L^\infty((0,T), \mathbf{H}_0(\text{div=0})),\\
\mathbf{E}_{N,h} \rightharpoonup^* \mathbf{E} & \text{weakly-* in } L^\infty((0,T), \mathbf{H}_0(\mathbf{curl})),\\
\mathbf{B}_{N,h} \rightharpoonup^* \mathbf{B} & \text{weakly-* in } L^\infty((0,T), \mathbf{H}_0(\text{div=0})),\\
\partial_t \mathbf{E}_{N,h} \rightharpoonup^* \xi & \text{weakly-* in } L^\infty((0,T), \mathbf{L}^2_\epsilon(\Omega)),\\
\partial_t \mathbf{B}_{N,h} \rightharpoonup^* \chi & \text{weakly-* in } L^\infty((0,T), \mathbf{H}_0(\text{div=0})),
\end{cases} \tag{3.26}
$$

for some $\mathbf{E}, \mathbf{B}, \overline{\mathbf{E}}, \overline{\mathbf{B}}, \xi, \chi$ as $h \to 0$. First of all, we verify that $\mathbf{E} = \overline{\mathbf{E}}$ and $\mathbf{B} = \overline{\mathbf{B}}$. However, this is readily seen by the definition (3.23) and Lemma 3.7 since

$$
\begin{aligned}
&\|\overline{\mathbf{E}}_{N,h} - \mathbf{E}_{N,h}\|_{L^\infty((0,T), \mathbf{L}^2_\epsilon(\Omega))} \leq \tau \max_{n \in \{1,\ldots,N\}} \|\delta \mathbf{E}^n_h\|_{\mathbf{L}^2_\epsilon(\Omega)} \leq C\tau,\\
&\|\overline{\mathbf{B}}_{N,h} - \mathbf{B}_{N,h}\|_{L^\infty((0,T), \mathbf{L}^2_{1/\mu}(\Omega))} \leq \tau \max_{n \in \{1,\ldots,N\}} \|\delta \mathbf{B}^n_h\|_{\mathbf{L}^2_{1/\mu}(\Omega)} \leq C\tau.
\end{aligned} \tag{3.27}
$$

Next, derivation in the sense of distributions gives

$$\int_0^T (\xi(t), \mathbf{v})_{\mathbf{L}^2_\epsilon(\Omega)} \phi(t) dt \underbrace{\leftarrow}_{(3.26)} \int_0^T (\partial_t \mathbf{E}_{N,h}(t), \mathbf{v})_{\mathbf{L}^2_\epsilon(\Omega)} \phi(t) dt$$
$$= - \int_0^T (\mathbf{E}_{N,h}(t), \mathbf{v})_{\mathbf{L}^2_\epsilon(\Omega)} \phi'(t) dt \underbrace{\rightarrow}_{(3.26)} - \int_0^T (\mathbf{E}(t), \mathbf{v})_{\mathbf{L}^2_\epsilon(\Omega)} \phi'(t) dt$$

for every $\mathbf{v} \in \mathbf{L}^2(\Omega)$ and $\phi \in \mathcal{C}_0^\infty(0,T)$, which yields $\xi = \partial_t \mathbf{E}$ and so

$$\mathbf{E} \in W^{1,\infty}((0,T), \mathbf{L}^2_\epsilon(\Omega)) \cap L^\infty((0,T), \mathbf{H}_0(\mathbf{curl})).$$

Obviously, the same conclusion can be drawn for $\chi = \partial_t \mathbf{B}$, which implies that $\mathbf{B} \in W^{1,\infty}((0,T), \mathbf{H}_0(\text{div=}0))$. Note that

$$(\mathbf{E}, \mathbf{B}) \in W^{1,\infty}((0,T), \mathbf{L}^2_\epsilon(\Omega) \times \mathbf{H}_0(\text{div=}0)) \hookrightarrow \mathcal{C}([0,T], \mathbf{L}^2_\epsilon(\Omega) \times \mathbf{H}_0(\text{div=}0))$$

implies possibly after a modification on a subset of $[0,T]$ with measure zero that $(\mathbf{E}, \mathbf{B}) \in \mathcal{C}([0,T], \mathbf{L}^2_\epsilon(\Omega) \times \mathbf{H}_0(\text{div=}0))$. Next, we prove the pointwise weak convergence

(3.28)
$$\mathbf{E}_{N,h}(t) \rightharpoonup \mathbf{E}(t) \text{ weakly in } \mathbf{L}^2_\epsilon(\Omega) \quad \text{and} \quad \mathbf{B}_{N,h}(t) \rightharpoonup \mathbf{B}(t) \text{ weakly in } \mathbf{H}_0(\text{div=}0)$$

for every $t \in [0,T]$. For that purpose, we fix $t \in (0,T]$, $\mathbf{w} \in \mathbf{L}^2(\Omega)$ and $\phi \in \mathcal{C}^1([0,t])$. Then, integration by parts yields

$$\int_0^t (\partial_t \mathbf{E}(s), \mathbf{w})_{\mathbf{L}^2_\epsilon(\Omega)} \phi(s) ds \leftarrow \int_0^t (\partial_t \mathbf{E}_{N,h}(s), \mathbf{w})_{\mathbf{L}^2_\epsilon(\Omega)} \phi(s) ds \tag{3.29}$$
$$= - \int_0^t (\mathbf{E}_{N,h}(s), \mathbf{w})_{\mathbf{L}^2_\epsilon(\Omega)} \phi'(s) ds + (\mathbf{E}_{N,h}(t), \mathbf{w})_{\mathbf{L}^2_\epsilon(\Omega)} \phi(t) - (\mathbf{E}_{N,h}(0), \mathbf{w})_{\mathbf{L}^2_\epsilon(\Omega)} \phi(0).$$

Choosing $\phi(0) = 0$ as well as $\phi(t) \neq 0$ and applying integration by parts again gives

$$\lim_{h \to 0} (\mathbf{E}_{N,h}(t), \mathbf{w})_{\mathbf{L}^2_\epsilon(\Omega)} = (\mathbf{E}(t), \mathbf{w})_{\mathbf{L}^2_\epsilon(\Omega)} \quad \forall \mathbf{w} \in \mathbf{L}^2(\Omega).$$

Applying the above convergence to (3.29) and choosing $\phi(0) \neq 0$ leads to

$$\lim_{h \to 0} (\mathbf{E}_{N,h}(0), \mathbf{w})_{\mathbf{L}^2_\epsilon(\Omega)} = (\mathbf{E}(0), \mathbf{w})_{\mathbf{L}^2_\epsilon(\Omega)} \quad \forall \mathbf{w} \in \mathbf{L}^2(\Omega).$$

The same results hold also for $\mathbf{B}_{N,h}$, and so we conclude that (3.28) is valid. From Lemma 3.3, (3.23) and (3.28) with $t = 0$, it follows that

$$\mathbf{E}(0) = \mathbf{E}_0 \text{ and } \mathbf{B}(0) = \mathbf{B}_0. \tag{3.30}$$

We continue and recall the classical identity:

$$\int_0^t (\partial_t \mathbf{E}_{N,h}(s), \mathbf{E}_{N,h}(s))_{\mathbf{L}^2_\epsilon(\Omega)} ds = \frac{1}{2} \|\mathbf{E}_{N,h}(t)\|^2_{\mathbf{L}^2_\epsilon(\Omega)} - \frac{1}{2} \|\mathbf{E}_{0h}\|^2_{\mathbf{L}^2_\epsilon(\Omega)}. \tag{3.31}$$

Combining (3.31) with (3.28) and Lemma 3.3 yields

$$
\begin{aligned}
(3.32)\qquad &\liminf_{h\to 0}\int_0^t (\partial_t \mathbf{E}_{N,h}(s), \overline{\mathbf{E}}_{N,h}(s))_{\mathbf{L}^2_\epsilon(\Omega)} ds\\
&\underbrace{=}_{(3.27)} \liminf_{h\to 0}\int_0^t (\partial_t \mathbf{E}_{N,h}(s), \mathbf{E}_{N,h}(s))_{\mathbf{L}^2_\epsilon(\Omega)} ds\\
&\underbrace{=}_{(3.31)} \liminf_{h\to 0} \frac{1}{2}\|\mathbf{E}_{N,h}(t)\|^2_{\mathbf{L}^2_\epsilon(\Omega)} - \frac{1}{2}\|\mathbf{E}_{0h}\|^2_{\mathbf{L}^2_\epsilon(\Omega)}\\
&\geq \frac{1}{2}\|\mathbf{E}(t)\|^2_{\mathbf{L}^2_\epsilon(\Omega)} - \frac{1}{2}\|\mathbf{E}(0)\|^2_{\mathbf{L}^2_\epsilon} = \int_0^t (\partial_t \mathbf{E}(s), \mathbf{E}(s))_{\mathbf{L}^2_\epsilon(\Omega)} ds,
\end{aligned}
$$

where the above inequality holds due to the fact that the squared norm is weakly lower semicontinuous. Analogously, we obtain

$$
(3.33)\qquad \liminf_{h\to 0}\int_0^t (\partial_t \mathbf{B}_{N,h}(s), \overline{\mathbf{B}}_{N,h}(s))_{\mathbf{L}^2_{1/\mu}(\Omega)} ds \geq \int_0^t (\partial_t \mathbf{B}(s), \mathbf{B}(s))_{\mathbf{L}^2_{1/\mu}(\Omega)} ds.
$$

Next, we prove

$$
(3.34)\qquad \liminf_{h\to 0} \varphi_N(t, \overline{\mathbf{E}}_{N,h}(t)) ds \geq \varphi(\theta(t), \mathbf{E}(t)) \quad \forall t \in [0,T].
$$

For $t=0$, Lemma 3.3 and (3.23), (3.24), and (3.30) grant even the strong convergence

$$
\lim_{h\to 0} \varphi_N(0, \overline{\mathbf{E}}_{N,h}(0)) = \lim_{h\to 0}\int_\Omega j_c(x,\theta(x,0))|\mathbf{E}_{0h}|\, dx = \varphi(\theta(0), \mathbf{E}(0)).
$$

Let now $t \in (0,T]$. Then, for every $N \in \mathbb{N}$, there exists a unique $n \in \{1,\dots,N\}$ such that $t \in (t_{n-1}, t_n]$. Hence, the sequence $\tilde{t}_{N,h} := t_n$ fulfills $\tilde{t}_{N,h} \to t$ as $h \to 0$. Making use of this sequence, we obtain that

$$
\begin{aligned}
(3.35)\quad &\liminf_{h\to 0} \varphi_N(t, \overline{\mathbf{E}}_{N,h}(t))\\
&= \liminf_{h\to 0}\left(\varphi(\theta(t), \overline{\mathbf{E}}_{N,h}(t)) + \int_\Omega (j_c(x,\theta(x,\tilde{t}_{N,h})) - j_c(x,\theta(x,t)))|\overline{\mathbf{E}}_{N,h}(t)|\, dx\right)\\
&\geq \varphi(\theta(t), \mathbf{E}(t)) + \liminf_{h\to\infty}\int_\Omega (j_c(x,\theta(x,\tilde{t}_{N,h})) - j_c(x,\theta(x,t)))|\overline{\mathbf{E}}_{N,h}(t)|\, dx,
\end{aligned}
$$

where we have employed (3.28) and the fact that $\varphi(\theta(t),\cdot)\colon \mathbf{L}^2_\epsilon(\Omega) \to \mathbb{R}$, for every fixed $t \in [0,T]$, is sequentially weakly lower semicontinuous. In order to pass to the limit in the second term in (3.35), we make use of (A4) and (A5) to deduce after selecting a subsequence that

$$
(3.36)\qquad \lim_{h\to 0} j_c(x,\theta(x,\tilde{t}_{N,h})) - j_c(x,\theta(x,t)) = 0 \quad \text{for a.e. } x \in \Omega,
$$

and so, thanks to (A3)–(A5) and Lemma 3.6, Lebesgue's dominated convergence theorem yields

$$
(3.37)\qquad \lim_{h\to 0}\int_\Omega |j_c(x,\theta(\tilde{t}_{N,h},x)) - j_c(x,\theta(t,x))||\overline{\mathbf{E}}_{N,h}(t)| dx = 0.
$$

In conclusion, (3.34) is valid.

Now, we show that $(\mathbf{E},\mathbf{B})$ is a solution to (VI): Fix $t \in (0,T]$, $\tilde{h} \in (0,\bar{h}]$, integrate (3.25) for $h < \tilde{h}$ over $[0,t]$ and test it with $(\mathbf{v}_{\tilde{h}}, \mathbf{w}_{\tilde{h}}) \in \mathbf{V}_{\tilde{h}} \times \mathbf{W}_{\tilde{h}} \subset \mathbf{V}_h \times \mathbf{W}_h$ (cf. (3.1)). Afterwards, we apply the limit superior to the resulting inequality to deduce

$$
\begin{aligned}
(3.38) \qquad & \int_0^t (\mathbf{f}(s), \mathbf{v}_h - \mathbf{E}(s))_{\mathbf{L}^2(\Omega)} ds = \lim_{h\to 0} \int_0^t (\overline{\mathbf{f}}_N(s), \mathbf{v}_h - \overline{\mathbf{E}}_{N,h}(s))_{\mathbf{L}^2(\Omega)} ds \\
\underbrace{\leq}_{(3.25)} \quad & \limsup_{h\to 0} \Big[\int_0^t (\partial_t \mathbf{E}_{N,h}(s), \mathbf{v}_h - \overline{\mathbf{E}}_{N,h}(s))_{\mathbf{L}^2_\epsilon(\Omega)} ds \\
& + \int_0^t (\partial_t \mathbf{B}_{N,h}(s), \mathbf{w}_h - \overline{\mathbf{B}}_{N,h}(s))_{\mathbf{L}^2_{1/\mu}(\Omega)} + (\mathbf{curl}\, \overline{\mathbf{E}}_{N,h}(s), \mathbf{w}_h)_{\mathbf{L}^2_{1/\mu}(\Omega)} ds \\
& - \int_0^t (\overline{\mathbf{B}}_{N,h}(s), \mathbf{curl}\, \mathbf{v}_h)_{\mathbf{L}^2_{1/\mu}(\Omega)} ds + \int_0^t \varphi_N(s, \mathbf{v}_h) - \varphi_N(s, \overline{\mathbf{E}}_{N,h}(s)) ds \Big] \\
\underbrace{\leq}_{(3.26),(3.32),(3.33)} \quad & \int_0^t (\partial_t \mathbf{E}(s), \mathbf{v} - \mathbf{E}(s))_{\mathbf{L}^2_\epsilon(\Omega)} + (\partial_t \mathbf{B}(s), \mathbf{w} - \mathbf{B}(s))_{\mathbf{L}^2_{1/\mu}(\Omega)} ds \\
& + \int_0^t (\mathbf{curl}\, \mathbf{E}(s), \mathbf{w})_{\mathbf{L}^2_{1/\mu}(\Omega)} - (\mathbf{B}(s), \mathbf{curl}\, \mathbf{v})_{\mathbf{L}^2_{1/\mu}(\Omega)} ds \\
& + \int_0^t \varphi(\theta(s), \mathbf{v}) - \varphi(\theta(s), \mathbf{E}(s)) ds,
\end{aligned}
$$

where we have also used (3.34) and Fatou's lemma to obtain convergence of the last time integral. Since there is no restriction to $\tilde{h} > 0$, the density of $\mathbf{V}_h \subset \mathbf{H}_0(\mathbf{curl})$ and $\mathbf{W}_h \subset \mathbf{L}^2(\Omega)$ yields, if we differentiate (3.38) with respect to t, that $(\mathbf{E},\mathbf{B}) \in W^{1,\infty}((0,T), \mathbf{L}^2_\epsilon(\Omega) \times \mathbf{H}_0(\text{div}{=}0)) \cap L^\infty((0,T), \mathbf{H}_0(\mathbf{curl}) \times \mathbf{H}_0(\text{div}{=}0))$ satisfies the evolutionary variational inequality (VI).

The uniqueness of the solution to (VI) follows by an energy argument: Let $(\tilde{\mathbf{E}}, \tilde{\mathbf{B}}) \in W^{1,\infty}((0,T), \mathbf{L}^2_\epsilon(\Omega) \times \mathbf{L}^2_{1/\mu}(\Omega)) \cap L^\infty((0,T), \mathbf{H}_0(\mathbf{curl}) \times \mathbf{L}^2_{1/\mu}(\Omega))$ be another solution to (VI). Then, inserting $(\mathbf{v},\mathbf{w}) = (\tilde{\mathbf{E}}(t), \tilde{\mathbf{B}}(t))$ in (VI) associated with $(\mathbf{E},\mathbf{B})$ and $(\mathbf{v},\mathbf{w}) = (\mathbf{E}(t), \mathbf{B}(t))$ in (VI) associated with $(\tilde{\mathbf{E}}, \tilde{\mathbf{B}})$, and then adding the resulting inequalities together, we obtain

$$
\int_\Omega \epsilon(\partial_t \mathbf{E}(t) - \partial_t \tilde{\mathbf{E}}(t)) \cdot (\mathbf{E}(t) - \tilde{\mathbf{E}}(t)) + \mu^{-1}(\partial_t \mathbf{B}(t) - \partial_t \tilde{\mathbf{B}}(t)) \cdot (\mathbf{B}(t) - \tilde{\mathbf{B}}(t))\, dx \leq 0,
$$

which implies that the difference $(\mathbf{e}(t), \mathbf{b}(t)) = (\mathbf{E}(t) - \tilde{\mathbf{E}}(t), \mathbf{B}(t) - \tilde{\mathbf{B}}(t))$ fulfills

$$
\frac{1}{2}\frac{d}{dt}\|\mathbf{e}(t)\|^2_{\mathbf{L}^2_\epsilon(\Omega)} + \frac{1}{2}\frac{d}{dt}\|\mathbf{b}(t)\|^2_{\mathbf{L}^2_{1/\mu}(\Omega)} \leq 0.
$$

Since $\mathbf{e}(0) = \mathbf{b}(0) = 0$, the above inequality yields that $\mathbf{e}(t) = \mathbf{b}(t) = 0$ for all $t \in [0,T]$. Hence, $(\mathbf{E},\mathbf{B})$ is the unique solution to (VI). ∎

Remark 3.9. A main consequence of Theorem 3.8 is the global well-posedness for (VI). We point out that, based on a direct approach, i.e., without discretization techniques, [33] proved existence and uniqueness results for hyperbolic Maxwell variational inequalities with a general nonlinearity. However, due to the temperature-dependent critical current density j_c, [33] cannot be applied to deduce the well-posedness of (VI). Here, the direct approach requires a substantial extension of [33] to the case of time-dependent nonlinearities.

We now prove our main result on the uniform convergence of (3.25) towards (VI).

THEOREM 3.10. *Let $N = N(h)$ be a family of natural numbers with $N(h) \to \infty$ for $h \to 0$. Then, under Assumptions 2.1 and 2.2, the solution $(\mathbf{E}_{N,h}, \mathbf{B}_{N,h})$ to (3.25) converges uniformly to the solution $(\mathbf{E}, \mathbf{B})$ of (VI), i.e.,*

$$\lim_{h\to 0} \|\mathbf{E}_{N,h} - \mathbf{E}\|_{\mathcal{C}([0,T],\mathbf{L}^2_\epsilon(\Omega))} = \lim_{h\to 0} \|\mathbf{B}_{N,h} - \mathbf{B}\|_{\mathcal{C}([0,T],\mathbf{L}^2_{1/\mu}(\Omega))} = 0,$$
$$\lim_{h\to 0} \|\overline{\mathbf{E}}_{N,h} - \mathbf{E}\|_{L^\infty((0,T),\mathbf{L}^2_\epsilon(\Omega))} = \lim_{h\to 0} \|\overline{\mathbf{B}}_{N,h} - \mathbf{B}\|_{L^\infty((0,T),\mathbf{L}^2_{1/\mu}(\Omega))} = 0.$$

Proof. First of all, we test (VI) with $(\mathbf{v}, \mathbf{w}) = (\overline{\mathbf{E}}_{N,h}(t), \overline{\mathbf{B}}_{N,h}(t))$ to obtain

$$\begin{aligned}(3.39)\qquad &\int_\Omega \epsilon\partial_t\mathbf{E}(t)\cdot(\overline{\mathbf{E}}_{N,h}(t) - \mathbf{E}(t)) + \mu^{-1}\partial_t\mathbf{B}(t)\cdot(\overline{\mathbf{B}}_{N,h}(t) - \mathbf{B}(t))\,dx \\ &\quad + \int_\Omega \mu^{-1}\,\mathbf{curl}\,\mathbf{E}(t)\cdot\overline{\mathbf{B}}_{N,h}(t) - \mu^{-1}\mathbf{B}(t)\cdot\mathbf{curl}\,\overline{\mathbf{E}}_{N,h}(t)\,dx \\ &\quad + \varphi(\theta(t), \overline{\mathbf{E}}_{N,h}(t)) - \varphi(\theta(t), \mathbf{E}(t)) \geq \int_\Omega \mathbf{f}(t)\cdot(\overline{\mathbf{E}}_{N,h}(t) - \mathbf{E}(t))\,dx\end{aligned}$$

for a.e. $t \in (0,T)$. Next, inserting $(\mathbf{v}_h, \mathbf{w}_h) = (\mathbf{\Phi}_h\mathbf{E}(t), 0) \in \mathbf{V}_h \times \mathbf{W}_h$ in (3.25) leads to

$$\begin{aligned}(3.40)\qquad &\int_\Omega \epsilon\partial_t\mathbf{E}_{N,h}(t)\cdot(\mathbf{E}(t) - \overline{\mathbf{E}}_{N,h}(t)) + \mu^{-1}\partial_t\mathbf{B}_{N,h}(t)\cdot(\mathbf{B}(t) - \overline{\mathbf{B}}_{N,h}(t))\,dx \\ &\quad + \int_\Omega \epsilon\partial_t\mathbf{E}_{N,h}(t)\cdot(\mathbf{\Phi}_h\mathbf{E}(t) - \mathbf{E}(t))\,dx - \int_\Omega \mu^{-1}\partial_t\mathbf{B}_{N,h}(t)\cdot\mathbf{B}(t)\,dx \\ &\quad - \int_\Omega \mu^{-1}\overline{\mathbf{B}}_{N,h}(t)\cdot\mathbf{curl}\,\mathbf{\Phi}_h\mathbf{E}(t)\,dx + \varphi_N(t, \mathbf{\Phi}_h\mathbf{E}(t)) - \varphi_N(t, \overline{\mathbf{E}}_{N,h}(t)) \\ &\geq \int_\Omega \overline{\mathbf{f}}_N(t)\cdot(\mathbf{\Phi}_h\mathbf{E}(t) - \overline{\mathbf{E}}_{N,h}(t))\,dx\end{aligned}$$

for a.e. $t \in (0,T)$. Now, by using the fact that $\partial_t\mathbf{B}_{N,h}(t) = -\,\mathbf{curl}\,\overline{\mathbf{E}}_{N,h}(t)$ holds for a.e. $t \in (0,T)$ (see (3.13) and (3.23)), we obtain

$$(3.41)\qquad \int_\Omega \mu^{-1}(\partial_t\mathbf{B}_{N,h}(t) + \mathbf{curl}\,\overline{\mathbf{E}}_{N,h}(t))\cdot\mathbf{B}(t)\,dx = 0 \quad \text{for a.e. } t \in (0,T).$$

Moreover, we know from Theorem 3.4 that $\overline{\mathbf{B}}_{N,h}(t) \in \mathbf{curl}\,\mathbf{V}_h$, which implies by (3.3) that

$$(3.42)\qquad \int_\Omega \mu^{-1}\overline{\mathbf{B}}_{N,h}(t)\cdot\mathbf{curl}(\mathbf{E}(t) - \mathbf{\Phi}_h\mathbf{E}(t))\,dx = 0 \quad \text{for a.e. } t \in (0,T).$$

In view of (3.41)–(3.42), adding (3.39) and (3.40) together and then integrating the resulting inequality over the time interval $[0,\sigma]$ with $\sigma \in (0,T]$ yield that

$$
\begin{aligned}
(3.43)\quad & \int_0^\sigma \int_\Omega \epsilon(\partial_t \mathbf{E}_{N,h}(t) - \partial_t \mathbf{E}(t)) \cdot (\overline{\mathbf{E}}_{N,h}(t) - \mathbf{E}(t))\, dxdt \\
&+ \int_0^\sigma \int_\Omega \mu^{-1}(\partial_t \mathbf{B}_{N,h}(t) - \partial_t \mathbf{B}(t)) \cdot (\overline{\mathbf{B}}_{N,h}(t) - \mathbf{B}(t))\, dxdt \\
&\le \int_0^\sigma \Big[\int_\Omega \overline{\mathbf{f}}_N(t) \cdot (\mathbf{E}(t) - \mathbf{\Phi}_h \mathbf{E}(t)) + (\mathbf{f}(t) - \overline{\mathbf{f}}_N(t)) \cdot (\mathbf{E}(t) - \overline{\mathbf{E}}_{N,h}(t))\, dx \\
&+ \int_\Omega \epsilon \partial_t \mathbf{E}_{N,h}(t) \cdot (\mathbf{\Phi}_h \mathbf{E}(t) - \mathbf{E}(t))\, dx + \big(\varphi_N(t, \mathbf{\Phi}_h \mathbf{E}(t)) - \varphi(\theta(t), \mathbf{E}(t))\big) \\
&+ \big(\varphi(\theta(t), \overline{\mathbf{E}}_{N,h}(t)) - \varphi_N(t, \overline{\mathbf{E}}_{N,h}(t))\big)\Big] dt =: \sum_{i=1}^5 C_i.
\end{aligned}
$$

We proceed by showing the convergence of C_i, $i \in \{1,\dots,5\}$, towards 0 as $h \to 0$. This obviously exploits the convergence property of $\mathbf{\Phi}_h$. Therefore, we use (3.5) and (3.6) to deduce by Lebesgue's dominated convergence theorem that

$$
(3.44)\qquad \lim_{h\to 0} \int_0^\sigma \|\mathbf{\Phi}_h \mathbf{E}(t) - \mathbf{E}(t)\|_{\mathbf{L}^2_\epsilon(\Omega)}\, dt = 0 \quad \forall \sigma \in [0,T].
$$

Now, (A5), Lemma 3.7 and (3.44) imply for $i \in \{1,3\}$ that $|C_i| \to 0$ as $h \to 0$. Also, the Lipschitz continuity of $\mathbf{f}$ (A5) together with Theorem 3.8 implies that $|C_2| \to 0$ as $h \to 0$. Next, the convergence for C_4 is shown: We begin with

$$
\begin{aligned}
(3.45)\quad & \left| \int_0^\sigma \varphi_N(t, \mathbf{\Phi}_h \mathbf{E}(t)) - \varphi(\theta(t), \mathbf{E}(t)) dt \right| \\
&\le \int_0^T |\varphi_N(t, \mathbf{\Phi}_h \mathbf{E}(t)) - \varphi_N(t, \mathbf{E}(t))| dt + \int_0^T |\varphi_N(t, \mathbf{E}(t)) - \varphi(\theta(t), \mathbf{E}(t))| dt.
\end{aligned}
$$

Because of (A3) and (A5), the first term on the right-hand side of (3.45) satisfies

$$
(3.46)\qquad \int_0^T |\varphi_N(t, \mathbf{\Phi}_h \mathbf{E}(t)) - \varphi_N(t, \mathbf{E}(t))| dt \le C \int_0^T \|\mathbf{\Phi}_h \mathbf{E}(t) - \mathbf{E}(t)\|_{\mathbf{L}^2_\epsilon(\Omega)} dt,
$$

with a constant $C > 0$, independent of h. On the other hand, the second term in (3.45) is estimated by using (A4) and (A5):

$$
\begin{aligned}
(3.47)\quad & \int_0^T |\varphi_N(t, \mathbf{E}(t)) - \varphi(\theta(t), \mathbf{E}(t))| dt \\
& \underbrace{=}_{(3.24)} \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \int_\Omega |j_c(x, \theta(t_n, x)) - j_c(x, \theta(t, x))|\, |\mathbf{E}(t)|\, dxdt \\
& \le\ C \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \tau \|\mathbf{E}(t)\|_{\mathbf{L}^2_\epsilon(\Omega)} dt = C\tau \|\mathbf{E}\|_{L^1((0,T),\mathbf{L}^2_\epsilon(\Omega))}.
\end{aligned}
$$

Thus, combining (3.45)–(3.47) gives

(3.48)

$$
|C_4| \le C \left(\int_0^T \|\mathbf{\Phi}_h \mathbf{E}(t) - \mathbf{E}(t)\|_{\mathbf{L}^2_\epsilon(\Omega)} dt + \tau \|\mathbf{E}\|_{L^1((0,T),\mathbf{L}^2_\epsilon(\Omega))} \right) \overbrace{\to}^{(3.44)} 0 \quad \text{as } h \to 0.
$$

We reuse the arguments from (3.47) in combination with Lemma 3.6 to obtain the convergence $|C_5| \to 0$ as $h \to 0$. Finally, we extract the desired norms on the left hand side of (3.43) as follows:

(3.49)
$$\int_0^\sigma \int_\Omega \epsilon(\partial_t \mathbf{E}_{N,h}(t) - \partial_t \mathbf{E}(t)) \cdot (\overline{\mathbf{E}}_{N,h}(t) - \mathbf{E}(t))\, dxdt = \frac{1}{2}\|\mathbf{E}_{N,h}(\sigma) - \mathbf{E}(\sigma)\|^2_{\mathbf{L}^2_\epsilon(\Omega)}$$
$$-\frac{1}{2}\|\mathbf{E}_{0h} - \mathbf{E}_0\|^2_{\mathbf{L}^2_\epsilon(\Omega)} + \int_0^\sigma \int_\Omega \epsilon(\partial_t \mathbf{E}_{N,h}(t) - \partial_t \mathbf{E}(t)) \cdot (\overline{\mathbf{E}}_{N,h}(t) - \mathbf{E}_{N,h}(t))\, dxdt$$

and

$$\begin{aligned} &\int_0^\sigma \int_\Omega \mu^{-1}(\partial_t \mathbf{B}_{N,h}(t) - \partial_t \mathbf{B}(t)) \cdot (\overline{\mathbf{B}}_{N,h}(t) - \mathbf{B}(t))\, dxdt \\ &= \frac{1}{2}\|\mathbf{B}_{N,h}(\sigma) - \mathbf{B}(\sigma)\|^2_{\mathbf{L}^2_{1/\mu}(\Omega)} - \frac{1}{2}\|\mathbf{B}_{0h} - \mathbf{B}_0\|^2_{\mathbf{L}^2_{1/\mu}(\Omega)} \\ &\quad + \int_0^\sigma \int_\Omega \mu^{-1}(\partial_t \mathbf{B}_{N,h}(t) - \partial_t \mathbf{B}(t)) \cdot (\overline{\mathbf{B}}_{N,h}(t) - \mathbf{B}_{N,h}(t))\, dxdt. \end{aligned} \tag{3.50}$$

In view of (3.27) and Lemma 3.7, we have

$$\begin{aligned} &\left| \int_0^\sigma \int_\Omega \epsilon(\partial_t \mathbf{E}_{N,h}(t) - \partial_t \mathbf{E}(t)) \cdot (\overline{\mathbf{E}}_{N,h}(t) - \mathbf{E}_{N,h}(t))\, dxdt \right| \\ &\le \|\partial_t \mathbf{E}_{N,h} - \partial_t \mathbf{E}\|_{L^1((0,T),\mathbf{L}^2_\epsilon(\Omega))} \|\overline{\mathbf{E}}_{N,h} - \mathbf{E}_{N,h}\|_{L^\infty((0,T),\mathbf{L}^2_\epsilon(\Omega))} \\ &\le C\tau \|\partial_t \mathbf{E}_{N,h} - \partial_t \mathbf{E}\|_{L^1((0,T),\mathbf{L}^2_\epsilon(\Omega))} \le C\tau(\|\partial_t \mathbf{E}\|_{L^1((0,T),\mathbf{L}^2_\epsilon(\Omega))} + 1), \end{aligned} \tag{3.51}$$

and analoguously

$$\begin{aligned} &\left| \int_0^\sigma \int_\Omega \mu^{-1}(\partial_t \mathbf{B}_{N,h}(t) - \partial_t \mathbf{B}(t)) \cdot (\overline{\mathbf{B}}_{N,h}(t) - \mathbf{B}_{N,h}(t))\, dxdt \right| \\ &\le C\tau(\|\partial_t \mathbf{B}\|_{L^1((0,T),\mathbf{L}^2_{1/\mu}(\Omega))} + 1). \end{aligned} \tag{3.52}$$

From (3.43) and (3.49)–(3.52) combined with the previously proved convergence for C_i for all $i \in \{1, \dots, 5\}$ and Lemma 3.3, we obtain

$$\lim_{h\to 0} \|\mathbf{E}_{N,h}(t) - \mathbf{E}(t)\|_{\mathbf{L}^2_\epsilon(\Omega)} = \lim_{h \to 0} \|\mathbf{B}_{N,h}(t) - \mathbf{B}(t)\|_{\mathbf{L}^2_{1/\mu}(\Omega)} = 0 \quad \forall t \in [0,T]. \tag{3.53}$$

On the other hand, Lemmas 3.6 and 3.7 imply the existence of a positive constant $C > 0$, independent of N and h, such that

$$\|(\mathbf{E}_{N,h}, \mathbf{B}_{N,h})\|_{W^{1,\infty}((0,T),\mathbf{L}^2_\epsilon(\Omega)\times \mathbf{L}^2_{1/\mu}(\Omega))} \le C \quad \forall h > 0, \tag{3.54}$$

which yields the uniform boundedness and the equicontinuity of $\{(\mathbf{E}_{N,h}, \mathbf{B}_{N,h})\}_{h>0} \subset \mathcal{C}([0,T], \mathbf{L}^2_\epsilon(\Omega) \times \mathbf{L}^2_{1/\mu}(\Omega))$. Therefore, by (3.53) and (3.54), the Arzelá-Ascoli theorem for Banach space-valued functions (cf. [23, Theorem 3.1]) implies the existence of a subsequence of $\{(\mathbf{E}_{N,h}, \mathbf{B}_{N,h})\}_{h>0}$ converging uniformly towards $(\mathbf{E}, \mathbf{B})$. As $(\mathbf{E}, \mathbf{B})$ is the unique solution of (VI), independent of the choice of the converging subsequence, a standard argument implies that the whole sequence converges uniformly, i.e.,

$$\lim_{h\to 0} \|\mathbf{E}_{N,h} - \mathbf{E}\|_{\mathcal{C}([0,T],\mathbf{L}^2_\epsilon(\Omega))} = 0 \underbrace{\Rightarrow}_{(3.27)} \lim_{h\to 0} \|\overline{\mathbf{E}}_{N,h} - \mathbf{E}\|_{L^\infty((0,T),\mathbf{L}^2_\epsilon(\Omega))} = 0.$$
$$\lim_{h\to 0} \|\mathbf{B}_{N,h} - \mathbf{B}\|_{\mathcal{C}([0,T],\mathbf{L}^2_{1/\mu}(\Omega))} = 0 \underbrace{\Rightarrow}_{(3.27)} \lim_{h\to 0} \|\overline{\mathbf{B}}_{N,h} - \mathbf{B}\|_{L^\infty((0,T),\mathbf{L}^2_{1/\mu}(\Omega))} = 0.$$

This completes the proof. □

4. A priori error analysis. We start by providing an error estimate result with low regularity fields for $\boldsymbol{\Phi}_h\colon \mathbf{H}_0(\mathbf{curl}) \to \mathbf{V}_h$ introduced in Definition 3.2.

LEMMA 4.1. *Let $s \in (0,1]$. There exists a constant $C > 0$, independent of h and $\mathbf{y}$, such that*

$$\|\mathbf{y} - \boldsymbol{\Phi}_h \mathbf{y}\|_{\mathbf{H}(\mathbf{curl})} \le C h^s \|\mathbf{y}\|_{\mathbf{H}^s_0(\mathbf{curl})} \quad \forall \mathbf{y} \in \mathbf{H}^s_0(\mathbf{curl})$$

for all $h > 0$. Here, $\mathbf{H}^s_0(\mathbf{curl}) \coloneqq \{\mathbf{y} \in \mathbf{H}^s(\Omega) \cap \mathbf{H}_0(\mathbf{curl}) \ : \ \mathbf{curl}\,\mathbf{y} \in \mathbf{H}^s(\Omega)\}$.

The proof is completely analogous to the one of [19, Theorem 3.3], which follows from (3.4) in combination with the stable commuting quasi-interpolation operator [19, Theorem 2.2] (cf. [10]) and the sharp approximation result [18, Corollary 6.5] (cf. [13]).

Assumption 4.2 (Additional assumptions on the initial data and the solution).

(A7) There exists $s \in (0,1]$ such that $\mathbf{E}_0 \in \mathbf{H}^s_0(\mathbf{curl})$ and the solution of (VI) satisfies $\mathbf{E} \in L^1((0,T), \mathbf{H}^s_0(\mathbf{curl}))$.

Assumption 4.2 yields the following error estimate for the initial value, which follows readily from (3.11) by using (A7) and Lemma 4.1.

LEMMA 4.3. *Let Assumptions 2.1, 2.2, and 4.2 hold. Then there exists a constant $C > 0$, independent of $h > 0$, such that*

$$\|\mathbf{E}_{0h} - \mathbf{E}_0\|^2_{\mathbf{L}^2_\epsilon(\Omega)} + \|\mathbf{B}_{0h} - \mathbf{B}_0\|^2_{\mathbf{L}^2_{1/\mu}(\Omega)} \le C h^s \quad \forall h > 0. \tag{4.1}$$

THEOREM 4.4. *Let Assumptions 2.1, 2.2, and 4.2 hold. Then, there exists a constant $C > 0$, independent of N and h, such that*

$$\|\mathbf{E}_{N,h} - \mathbf{E}\|^2_{\mathcal{C}([0,T],\mathbf{L}^2_\epsilon(\Omega))} + \|\mathbf{B}_{N,h} - \mathbf{B}\|^2_{\mathcal{C}([0,T],\mathbf{L}^2_{1/\mu}(\Omega))}$$
$$\le C(h^s + \tau)(\|\mathbf{E}\|_{L^1((0,T),\mathbf{H}^s_0(\mathbf{curl}))} + \|\partial_t \mathbf{E}\|_{L^1((0,T),\mathbf{L}^2_\epsilon(\Omega))} + \|\partial_t \mathbf{B}\|_{L^1((0,T),\mathbf{L}^2_{1/\mu}(\Omega))} + 1)$$

holds for every $h > 0$ and every $N \in \mathbb{N}$.

Proof. The lines of the proof are similar to the proof of Theorem 3.10, but, due to the regularity assumption on $\mathbf{E}$ (Assumption 4.2), we may use Lemma 4.1 in place of (3.6). Thus, we consider again (3.43) and give estimates for C_i, $i \in \{1, \dots, 5\}$, instead of simply proving their convergence towards 0. The stability results in Lemma 3.6 and Lemma 3.7 combined with the regularity of $\mathbf{E}$ (see (A7)) as well as the error estimates for $\boldsymbol{\Phi}_h$ in Lemma 4.1 lead to

$$|C_i| \le C h^s \|\mathbf{E}\|_{L^1((0,T),\mathbf{H}^s_0(\mathbf{curl}))} \quad \forall i \in \{1, 3\}, \tag{4.2}$$

with a constant C, independent of the time variable, N, and h. To estimate C_2, we use the Lipschitz continuity of $\mathbf{f}$ (see (A5)) and Theorem 3.10:

$$|C_2| \le C\tau \int_0^\sigma \|\mathbf{E}(t) - \overline{\mathbf{E}}_{N,h}(t)\|_{\mathbf{L}^2_\epsilon(\Omega)} dt \le C\tau. \tag{4.3}$$

Next, C_4 is estimated by applying Lemma 4.1 to (3.48)

$$|C_4| \le C(h^s + \tau)\|\mathbf{E}\|_{L^1((0,T),\mathbf{H}^s_0(\mathbf{curl}))}. \tag{4.4}$$

Last but not least, the arguments from (3.47) in combination with Lemma 3.6 imply $|C_5| \le C\tau$. The combination of (3.43) and (3.49)-(3.50) with (3.51)-(3.52) as well as the previously proved estimation for C_i, $i \in \{1, \dots, 5\}$ and Lemma 4.3 finally yields the desired error estimate. ∎

Remark 4.5. The results by Ern and Guermond [18, 19] are also valid for higher-order finite elements. Therefore, [18,19] together with the higher-order FEM for linear Maxwell's equations [28] would serve as an important basis for the extension of our approach to the higher-order case.

5. Numerical Results. We close this paper by presenting numerical results for some particular examples for (VI). When it comes to computing the solution $(\mathbf{E}, \mathbf{B})$ to (VI), Euler's implicit method provides an iterative algorithm, which also enables us to split the mixed problem into two associated problems as we did in Theorem 3.4. We recall (3.13), which gives an explicit formula for $\mathbf{B}_h^n$:

$$\mathbf{B}_h^n = \mathbf{B}_h^{n-1} - \tau \, \mathbf{curl} \, \mathbf{E}_h^n \tag{5.1}$$

provided that $\mathbf{E}_h^n$ is already computed. In view of (3.15), $\mathbf{E}_h^n$ solves an elliptic **curl-curl** variational inequality of the form

$$a(\mathbf{E}_h^n, \mathbf{v}_h - \mathbf{E}_h^n) + \varphi^n(\mathbf{v}_h) - \varphi^n(\mathbf{E}_h^n) \geq \langle \tilde{\mathbf{f}}^n, \mathbf{v}_h - \mathbf{E}_h^n \rangle \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \tag{5.2}$$

We solve this variational inequality using the semi-smooth Newton method (cf. [21]).

Our computational domain is the cube $\Omega = (-1, 1)^3$ and we apply a circular current $\mathbf{f} \colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\mathbf{f}(x, y, z) = \begin{cases} 1/R \left(0, \; -z/(y^2 + z^2)^{1/2}, \; y/(y^2 + z^2)^{1/2}\right) & \text{for } (x, y, z) \in \Omega_p \\ 0 & \text{for } (x, y, z) \notin \Omega_p \end{cases}$$

to a cylindrical pipe coil $\Omega_p \coloneqq \{(x, y, z) \in \mathbb{R}^3 \; : \; |x| \leq 0.5, \; \sqrt{y^2 + z^2} \in [0.3, 0.5]\}$. The constant $R > 0$ denotes the electrical resistance of the pipe (here: $R = 1$). All implementations were done with the open-source finite-element computational platform FEniCS [27] and as a visualization tool Paraview was used. For this study, the uniform tetrahedral mesh was refined around the coil. If we do not include a superconductor in this setup, the applied current induces an orthogonal magnetic field, which admits its greatest field strength in the center of the coil.

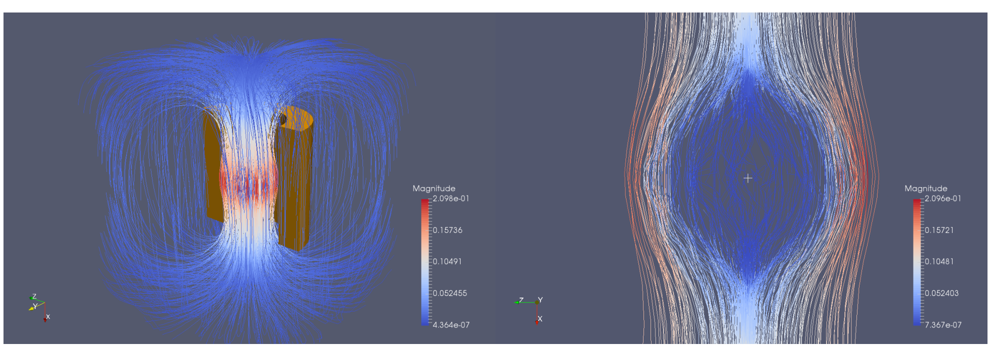

Fig. 1. *First numerical example. Left: Magnetic field lines and the clipped pipe coil. Right: 2D-slice of the magnetic field along the x-z axis.*

In the first example, we place a type-II superconducting ball Ω_{sc} with radius 0.2 in the center of the pipe, set $j_c = 80\chi_{\Omega_{sc}}$, $\epsilon = \mu = 1$ and solve the compatibility system (3.2) for the discrete initial value $(\mathbf{E}_{0h}, \mathbf{B}_{0h})$. In our computation, the mesh

n	0	1	2	3	4	5	6
$j_c(\cdot, \theta(t_n))$	$80\chi_{\Omega_{sc}}$	$50\chi_{\Omega_{sc}}$	$35\chi_{\Omega_{sc}}$	$20\chi_{\Omega_{sc}}$	$10\chi_{\Omega_{sc}}$	$5\chi_{\Omega_{sc}}$	$0,5\chi_{\Omega_{sc}}$
$\theta(t_n)$	$60,0K$	$65,0K$	$67,5K$	$70,0K$	$72,5K$	$75,0K$	$80,0K$
t_n	0	1/6	1/3	1/2	2/3	5/6	1

TABLE 1

Critical current j_c and temperature θ of the superconductor at each the time step.

was refined around the superconductor such that we end up with roughly 240.000 cells and 1.020.000 degrees of freedom (DOFs) for the mixed finite element space. The resulting solution $(\mathbf{E}_{0h}, \mathbf{B}_{0h})$ of (3.2) exhibits the physical phenomenon of the Meissner-Ochsenfeld effect. In Figure 1, we see how the magnetic field lines get repelled by the superconductor and since they are squashed between the superconductor and the coil, one observes the highest magnetic field strength in this area (see Figure 1).

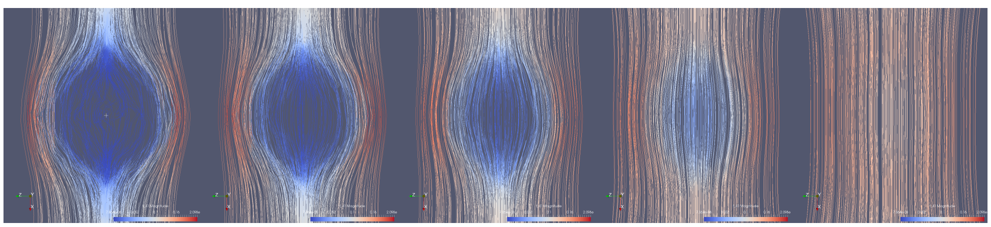

FIG. 2. *Evolution of the magnetic field around the superconductor in the time-steps t_n for $n \in \{0, 1, 2, 3, 4\}$.*

Keeping the observations of the first example in mind, we continue and compute a time-dependent problem, where the solution of the first example serves as the discrete initial electromagnetic field, since it satisfies the discrete compatibility system (3.2). We consider the temperature dependence in the critical current density j_c for a superconductor with the nominal composition $Y_{1.2}Ba_{0.8}Cu_2O_x$ as it was suggested in [2]. Moreover, we set $T = 1$ as well as $\tau = 1/6$ and use the same amount of DOFs and cells as in the first example. We place the cooled down superconductor inside the coil in the same way it was done in the first example, but now the temperature θ increases over time (see Table 1), whereas the applied current source $\mathbf{f}$ stays constant. The evolution of the magnetic field over time is shown in Figure 2. One observes that the magnetic field lines in the squashed area start penetrating the superconductor as the temperature becomes larger and larger. As soon as the temperature θ exceeds the threshold $75K$, the magnetical field completely penetrates the superconductor and we can no longer observe the Meissner-Ochsenfeld effect.

Further Research. As pointed out in the introduction, the Bean critical-state model is a free boundary problem, as it involves unknown superconductive and normal regions, which may change their locations in course of time, depending on the temperature distribution θ and the applied current source $\mathbf{f}$. Thus, an adaptive mesh refinement strategy based on rigorous a posteriori error estimators will be useful, not only for increasing numerical accuracy, but also for capturing the unknown interfaces between the superconductive and normal regions. We also point out that Theorem 3.8 opens a way to study the temperature control in the magnetization process of type-II superconductivity. This leads to a state-constrained optimal control problem governed by a fully coupled system consisting of (VI) and non-smooth heat equations. This problem requires a substantial extension of the recently developed optimal control techniques for electromagnetic problems [34, 36, 37].

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