Chapter 9

The use of original sources in the mathematics classroom

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Abstract: *The study of original sources is the most ambitious of ways in which history might be integrated into the teaching of mathematics, but also one of the most rewarding for students both at school and at teacher training institutions.*

9.1 Introduction

Among the various possible activities by which historical aspects might be integrated into the teaching of mathematics, the study of an original source is the most demanding and the most time consuming. In many cases a source requires a detailed and deep understanding of the time when it was written and of the general context of ideas; language becomes important in ways which are completely new compared with usual practices of mathematics teaching. Thus, reading a source is an especially ambitious enterprise, but, as we want to show, rewarding and substantially deepening the mathematical understanding. In this chapter we describe some ideas and international experiences concerning the use of original sources in the mathematics classroom, referring to teaching at schools as well as at teacher education institutions.

In principle, the aims and effects which might be pursued by way of an original source will not be different from those attained by other types of historical activities. However, there are three general ideas which might best be suited for describing the special effects of studying a source. These are the notions of *replacement, reorientation* and *cultural understanding*. By these we mean:

John Fauvel, Jan van Maanen (eds.), *History in mathematics education: the ICMI study,* Dordrecht: Kluwer 2000, pp. 291-328

(i) replacement

Integrating history in mathematics replaces the usual with something different: it allows mathematics to be seen as an intellectual activity, rather than as just a corpus of knowledge or a set of techniques.

(ii) reorientation

Integrating history in mathematics challenges one's perceptions through making the familiar unfamiliar. Getting to grips with a historical text can cause a reorientation of our views. History of mathematics has the virtue of 'astonishing with what comes of itself' (Veyne 1971). All too often in teaching, what happens is that concepts appear as if already existing. This is true for the concept of a set, for example, but just as true for the concept of a triangle or a function. And concepts are manipulated with no thought for their construction. History reminds us that these concepts were invented and that this did not happen all by itself.

(iii) cultural understanding

Integrating history of mathematics invites us to place the development of mathematics in the scientific and technological context of a particular time and in the history of ideas and societies, and also to consider the history of teaching mathematics from perspectives that lie outside the established disciplinary subject boundaries.

In this chapter we begin with discussing motivations, aims and uses which are especially connected with the study of original sources (section 9.2). Of course, there is some overlap with the general aims underlying the introduction of historical components, but we concentrate on those dimensions specific for our topic. We discuss especially the hermeneutic process of interpreting a source and the special role of language in it (section 9.3). In a further step we investigate four examples, two taken from the context of teacher education (sections 9.4.1 and 9.4.2) and two from school teaching (sections 9.5.1 and 9.5.2). The special reference to teacher education is motivated by our conviction that the reading of original sources should become an obligatory part of mathematics teacher education at all levels. In section 9.6 we deal with didactic strategies, and in section 9.7 discuss some research questions and issues of concern. Section 9.8 is the bibliography for this chapter, and in the appendix, 9.9, the reader will find hints on useful resources.

9.2 Motivations, aims and uses

9.2.1 The specific value and quality of primary sources

The role of primary sources in the integration of history of mathematics into mathematics education should be considered in the light of different possible purposes. Incorporating primary sources is not good or bad in itself. We need to establish the aims, including the target population, the kind of source that might be suitable and the didactical methodology necessary to support its incorporation.

In the following, we describe some objectives and examples of how primary sources help to pursue them. There are almost certainly further ones we do not mention (cf. Arcavi & Bruckheimer 1998; Fauvel 1990 (see especially the papers by Jozeau, Bühler, Hallez, Horain); Furinghetti 1997; IREM de Montpellier 1995; Jahnke 1995; Laubenbacher & Pengelley 1996, 1998; Lefebvre & Charbonneau 1991; LeGoff 1994; Logarto *et al.* 1996; M:ATH 1991; Métin 1997; Nouet 1992).

In contrast to merely relying on secondary literature the reading of primary sources may help to

a) clarify and extend what is found in secondary material,
b) uncover what is not usually found there,
c) discern general trends in the history of a topic (secondary sources are usually all-topic chronological accounts, and some topics are very briefly treated or omitted altogether), and
d) put in perspective some of the interpretations, value judgements or even misrepresentations found in the literature.

Reading historical texts may produce a cultural shock, by which we may experience the *replacement* and *reorientation* referred to above. This will only happen, however, if the reading is not *teleological,* that is, provided we do not attempt to analyse the text uniquely from the point of view of our current knowledge and understanding. Such a reading could carry with it erroneous interpretations, given that the writer may be using an idea according to a conception quite different from ours. If the value of history lies in reorientation, in understanding rather than judging, then texts need to be *contextualised,* that is located in the context of their time. We need to remind ourselves that the writer was addressing not us, but a contemporary audience.

To have our perspectives of knowledge challenged is beneficial. Thus, it is important to read Descartes' *Geometry* (1637) being aware that the text was not understood by his contemporaries. We would then pay more attention to the changes brought about by Cartesian geometry, for example by the introduction of a unit segment, which appears so 'natural' in coordinate geometry that it passes by almost unnoticed. We can also show that the coordinate geometry system works in a way that can be related with the Section Theorem in the geometry of the triangle (Euclid's *Elements* vi.2, sometimes called Thales' Theorem: that a line parallel to one side of a triangle cuts the other sides in the same ratio), something which appears to be quite absent now from the official curriculum in many countries. This example shows that the *replacement* and *reorientation* aspects of history are directly linked to didactical considerations.

Reading historical texts in class introduces history in an explicit way. Nevertheless, this activity has to be integrated into the mathematics lessons and not provided just as an extra. It also presupposes that the teachers have a sense of history and, of course, that they are able to handle the mathematics involved. Thus, reading sources presupposes adequate preparation (see Chapter 4).

9.2.2 Understanding the evolution of ideas

There is a common belief held by many, teachers and students alike, about the static nature of mathematical concepts: once a concept is defined, it remains unchanged. Even those who do not hold this belief may not have had opportunities to experience the evolving nature of ideas. Take for example the concept of *function*. At some early stage, functions were restricted to those which could be expressed by algebraic relationships. Later, the concept was extended beyond correspondences which can be expressed algebraically, and later still to correspondences not involving sets of numbers at all. Thus we have the more general and formal definition today: a subset of the Cartesian product of two sets with certain properties. In another sense, the concept was restricted to univalent relationships. (For a detailed discussion of the history of the function concept see, for example Youschkevitch 1976 and for a brief survey Kleiner 1989.) We suggest that primary sources can offer the experience of a non-mediated contact with the way in which ideas were defined at a certain time, different from that in use today.

Another example is the notion of a *curve*. Curves seem to be considered the same throughout the school programme. The circle, however, can be variously presented: as a static object in geometry, consisting of points at equal distance from its centre; as a dynamic object produced by the rotation of a line segment about one of its (fixed) extremities; as an object in algebra, namely an equation; or as a functional object. History can make us aware of the significance of these different ways of thinking about a curve through letting us understand the problems that led mathematicians to pass from one notion to the other, and also to see the nature of the changes in conception that came about (Barbin 1996). For example, the dynamic notion of a curve in the 17th century is linked to problems about movement that scientists of the time were considering. In particular, we can see from reading *Dialogues on the two new sciences* how Galileo changes the (static) parabola of his study into the (dynamic) trajectory of a cannon ball. Whereas the parabola of Greek geometry is the intersection of a cone and a plane, the Galilean parabola becomes the trajectory of a moving body, subject to a uniform horizontal and a uniformly accelerated vertical movement.

In order to see how the idea of a curve evolved and became refined, it is interesting to read and compare several historical texts, for example to look at the methods for finding tangents found in the works of Euclid, Apollonius, Roberval, Fermat, Descartes, Leibniz and Newton. Similarly, to see how the idea of function or number has evolved and become refined, it is important to read texts related to stages of their history.

Primary sources provide also lively examples of how different *representational systems* were used in the past. These examples may help students to put into perspective our current representational systems as just one of many possible ways of performing operations and handling and communicating concepts. Moreover, by comparing and contrasting our representations with those in the past as they appear in original sources, students might appreciate the crucial role representations play in the inception and evolution of ideas.

In Arcavi 1987 an activity for elementary school students is described, in which a brief extract from the Rhind Papyrus is presented; with the aid of an accompanying 'dictionary', the challenge consists of deciphering the arithmetical operations performed, explaining how they work, and applying them to further examples. This activity serves as the basis for discussion of the characteristics of the Egyptian numeration system as opposed to ours, including advantages and disadvantages of both. Van Maanen (1997) describes similar experiences with primary sources from a later period. His students report that they find it a difficult but very interesting puzzle, first to find out what the handwritten text says and then what it meant and why it worked. Furthermore such a problem makes students aware that methods and standards are changing. When students compare and contrast the representations they know and use at school with those in original sources, they not only learn about the latter, but most importantly, their attention is re-focused on the former, providing an opportunity to re-discover properties taken for granted and which were "clogged with automatisms" (Freudenthal 1983, 469).

9.2.3 Experiencing the relativity of truth and the human dimension of mathematical activity

The fact that the idea of truth is relative can be seen when we consider how the significance of proof has changed in history (Barbin 1994). While the first reasonings in Greek geometry had to do with explaining real problem situations, like the problem of finding inaccessible distances, the purpose of logical proof in Euclid was to convince, or even defeat, the (supposedly sceptical) reader. This idea of proof was denounced in the 17th century by geometers who preferred to enlighten rather than defeat their readers. As for the idea of proof in Hilbert's geometry, it is conceived of as a way of deciding the validity of a proposition, that is to determine whether or not it is consistent with a set of formal axioms. To obtain a feeling for what proof means, it is interesting to read a variety of proofs of the same theorem, for example the different proofs for the sum of the angles of a triangle given over the two thousand years from Euclid to Hilbert (Barbin 1995).

It is also illuminating to study examples of doubts and errors which arose when mathematicians were working on new problems and concepts. This is different from the usual presentation of mathematical activity, described by Kessel in this way (Kessel 1998, 44):

> This detached style of speaking and writing about mathematics suggests to listeners and readers that mathematics is independent of time and place … ideas that are not tied to specific people, times, and places, but which are abstract and timeless … and which avoids mentioning concrete doers.

Thus, in many classrooms all over the world, mathematical activity is generally perceived as the production of clean and correct answers to problems. Alternative, recent experiences (e.g. de Abreu 1998; Arcavi *et al.* 1998; Farey & Métin 1993; Lampert 1990; Pirie & Schwarzenberger 1988; Voigt 1985; Wood 1998) are beginning to include the sharing of intuitions, conjectures, the development of heuristics, and the encouragement of reflection and communication. All these legitimise the explicit raising of doubts, committing errors, entering blind alleys, and discussing seemingly non-solvable contradictions.

Thus, primary sources can provide lively documented examples of genuine mathematical activity in the making, and reading them may legitimise and humanise it ("if famous mathematicians went through it, why not I?"). Moreover, these doubts become issues for discussion with the potential of enriching students' formal and informal knowledge of a topic, and their ability to 'talk mathematics'.

For example, one could confront teachers with the doubts mathematicians had in the 16th and 17th century regarding the nature of irrational numbers (Arcavi *et al.* 1987). In the discussion of the source, teachers may dare to express their own discomfort and/or uncertainties about the 'infinite' decimal representation of irrational numbers. They can also share in the struggle between the usefulness of the concept of irrationals when rationals fail (e.g. in geometrical measurements) and their uncertain nature as numbers. For those less troubled by such problems, the discussion serves to develop an awareness that the infinite digits in the decimal expansion of an irrational were regarded as problematic to the point that their status as numbers was questioned. By implication, this leads to a recognition that this can be an issue with students as well, and to reflecting on the crucial importance of the role of representations of a concept, their influence on the way the idea is conceptualised, questioned, and ultimately accepted or rejected.

9.2.4 Relations between mathematics and philosophy

The contribution that the history of mathematics makes to our understanding of the cultural context is an excellent opportunity, or a necessary reason, for relating mathematics to other fields of knowledge (see Furinghetti & Somaglia 1998). Frequently, mathematicians were also philosophers and it is quite artificial to separate their disciplines (Barbin & Caveing 1996). In any case, it is often beneficial to read mathematicians with an awareness of the prevailing philosophy of their time. Consideration of the relationship between mathematics and the real world will benefit enormously when mathematics teachers work collaboratively with teachers of the physical sciences. The example we quoted above concerning Galileo illustrates this point.

Reading a source can be the trigger for establishing a dialogue with the ideas expressed. The source then becomes an interlocutor to be interpreted, to be questioned, to be answered and to be argued with. This applies especially to sources which discuss meta-mathematical issues such as the nature of the mathematical objects we handle, and the essence of mathematical activity. For example, one can use extracts taken from *The principles of algebra* (1796) by William Frend (1757-1841) in which it is proposed that negative numbers should be banned. Frend's arguments against negative numbers raised, and continue to raise for students today, serious discussions on issues such as the use of models, analogies, or metaphors in mathematics (such as debts in accounting); the legitimacy of creating new ideas, provided they are well-defined and internally consistent; the ambivalence of symbols when used in allied but yet different meanings; and the need for formal definitions of concepts such as negative numbers.

9.2.5 Simplicity, motivation and didactics

Occasionally, primary sources can be used because they are simpler and friendlier than their later elaboration. One notable example is Dedekind's (1831-1916) definition of real numbers, as it appears in his essay *Stetigkeit und Irrationalzahlen* (1872) (see *Essays on the theory of numbers,* 1924). His style is didactical and clear, first explaining the method to be followed, then using an analogy in order both to engage readers' established knowledge and also to share with readers his sources of insight. Only after that are the formal definitions carefully developed step by step. Simplicity and friendliness can also be found in the sense-making explanations proposed in some primary sources for basic but formal mathematical laws, which teachers and curriculum designers struggle to find. As we progress in history, especially through the 20th century, many texts tend to adopt formal justifications to formal laws, and many students may feel alienated. However, some older texts often resort to everyday language and reasonable explanations which can enrich the didactical repertoire of teachers by appeal to students' sense making. Such is the case with Viète's (1540-1603) presentation of simple algebraic laws, in his *In artem analyticem isagoge* (see Bruckheimer & Arcavi 1997).

9.2.6 Perspectives on mathematics education

Primary sources seem to be a most reasonable way to learn about the central topics taught in schools in the past, curricular trends in general and various approaches to learning and teaching. One activity that Bruckheimer *et al.* (1995) designed for classroom use with 12-13 year old students is based on old arithmetic textbooks, which give the flavour of what and how students studied in the past: methods of calculation were a central topic, and accuracy was a major preoccupation. There are whole sections devoted to calculation checks, such as 'casting out nines'. This checking method, as it appears in primary sources, provides an opportunity to deal with many fundamental topics: why does the method work, which kinds of errors can and which cannot be detected, why 9 is preferred to, say, 2, or 7, and so on.

Besides the flavour of past textbooks and dealing with mathematical issues, the sources provide, by implication, the realisation that the goals for mathematics education have changed rather dramatically over the last 100-150 years. In the past, mathematics instruction for all ('all' in the past was probably more restricted than 'all' is regarded today) may have been mainly devoted to producing good clerks who could calculate accurately. Today, with the emergence of freely available calculators and the demands of a technological society, the emphasis in arithmetic shifts towards estimation, reasonableness of answers, etc., and other signs of mathematical literacy.

9.2.7 Local Mathematics

Primary sources can also be used in mathematics to rediscover and emphasise the heritage of the culture in which students learn. As most cultures have written mathematical documents (and certainly verbal accounts of everyday mathematical practices), it is not hard to find appropriate sources suitable for classroom use or teachers' workshops.

9.3 Sources, hermeneutics and language

Reading an original source is a specific activity of relating the *synchronous* and the *diachronous* mathematical culture to each other (cf. Jahnke 1994, 154 ff.). The term synchronous culture refers to dialogue and work in the classroom as well as the role of mathematics in public life, in economy, technology, science and culture and the image which is attached to it. The diachronous culture means the development of these elements through history and has to be related to the synchronous culture and the life and thinking of the learners. However, it should not simply affirm the synchronous culture, but should rather widen and deepen the understanding of the learner.

In traditional theories of hermeneutics the relation between the historical meaning of a text (the intention of its author) and its meaning for a modern reader is amply reflected and identified as the essential problem of interpretation. In fact, seen under the aspect of method, history of mathematics, like any history, is essentially an hermeneutic effort. If history of mathematics is not to deteriorate into a dead dogma, teachers should have some ideas about the hermeneutic process and the fruitful tension between the meaning of a text in the eyes of its author and the meaning for a modern reader.

The process of interpreting an original mathematical source may be described by a twofold circle. Texts and their authors (or theories and their creators) are interpreted by a modern reader, and the interpreter should always be aware of the hypothetical and intuitive character of his interpretation. The interpretation takes place in a circular process of forming hypotheses and checking them against the text given. In the case of history of science, the objects of this process of interpretation, the scientific subjects (individuals or groups) are themselves involved in a hermeneutic process of creating theories and checking them against phenomena which they want to explain or against intended aims they want to reach. Thus, the whole process of interpreting a source may be described by a twofold circle where in a primary circle a scientist (or a group of scientists) is acting and in a secondary circle the modern reader tries to understand what is going on. Those concerned with history have to engage with a complex network of relations between their own interpretations of a certain concept or theory and the interpretation of the original author.

Teachers should be aware of this twofold circle and able to move in it. Only this will create a climate in the classroom adequate for encouraging students to generate their own hypotheses about a text and so become ready for thinking themselves into other persons who have lived in another time.

This thinking into other persons and into a different world seems to be the core of an educational philosophy underlying the reading of original sources. She who thinks herself into a scientist doing mathematics at a different time has herself to do mathematics; she moves in a mental game in the primary circle reflecting what the person under study might have had in mind. One has to ask for the theoretical conditions this person is explicitly or implicitly supposing, and one will have to mobilise imagination to generate hypotheses about them.

Thinking themselves into other persons motivates students to reflect about their own views of the subject matter. This reflection, in turn, is made objective by the material (the text) they are studying. Certain aspects of the historical persons and their ideas will be easily accessible, others will remain alien. As a crucial point in hermeneutics, the student's self will unavoidably enter the scene, not as a disturbing factor, but as a decisive prerequisite to insight.

Even if an original source is given in the native language of the students its interpretation presupposes a considerable linguistic competence. This requirement should be accepted by teachers and students. Oral and written language are equally important. The students should have the opportunity of extensive discussions, but they should also be asked to produce their own written texts. The idea of a 'mathematical essay' is old and sounds, since it is never realised, a bit antiquated. Historical subjects would provide natural starting points for such activities.

An important aim should be the elaboration of the individual language of the students. In reading a source they are confronted with at least three different languages: the mathematical language of their usual lessons, the language of the original source, and their own way of speaking about mathematics. These three languages have to be related to each other, and the students should be able to move freely from one language into the other. This should be a general educational aim of mathematics teaching beyond the special occasion of history of mathematics. When in their future lives students practise mathematics, they will need above all to communicate and translate ideas and facts into mathematical language and vice versa. History of mathematics contributes considerably to the development of this ability.

9.4 Integrating original sources in pre-service teacher education

As we said above, the reading of original sources should become an obligatory part of mathematics teacher education at all levels. This will not only contribute substantially to their mathematical competence, but is also a necessary condition if they are expected to include historical components into their future mathematics teaching. In the following we describe two experiences with original sources from teacher education institutions, the first from Morocco, the second from Norway.

9.4.1 Example 1: Egyptian measures of angles

In this section, we present an example of using an original text in the pre-service education of mathematics teachers. The objective was to initiate an analysis of trigonometric notions, in particular the concepts of cotangent, tangent and angle. The example was treated at the École Normale Supérieure in Marrakech, Morocco (see El Idrissi 1998). The text used is an extract of the Rhind Papyrus, written in the 17th century B.C. and now in the British Museum, London. It contains problems together with their solutions. The text was originally in Egyptian hieratic script; we refer to a 20th century English translation (Gillings 1972). The example here concerns reckoning a pyramid, problem 56 from the Rhind Papyrus (*RP 56*) and its solution:

A pyramid has a height of 250 cubits and a base of 360 cubits. What is its sekt?

Solution:

1) Find 1/2 of 360: 180.

2) How many times is 250 in 180: 1/2 1/5 1/50 yard

3) Now a yard is 7 palms.

4) Then multiply 7 by 1/2 1/5 1/50: 5 1/25 palms. This is its sekt.

The above extract was presented to teacher students, and they were confronted with questions and proposals for activities. Actually, an analysis of the problem was done even before the questions were posed. The most important elements emerging from this analysis were:

1. The calculation is given by means of unit fractions.
2. The question is asked about an empirical case, a pyramid.
3. The solution is given without any definition or justification. It is an algorithm for calculating.
4. In the first stage of the solution, the student is told to find the half of 360 and not to divide 360 by two. These two seemingly similar operations are conceptually different.
5. In the second stage, the result is given together with a unit, the yard. In principle, there should not be any units as the intention is to divide yards by yards.
6. In the third stage, the students transform a result given in yards into palms.
7. A naive interpretation of the solution could make believe that the sekt is identical with the cotangent (Smith 1958). Taking into account the earlier remarks, however, sekt and cotangent are different.
8. The sekt can be defined as the horizontal shift in palms which corresponds to a vertical shift of one yard (see figure 9.1).

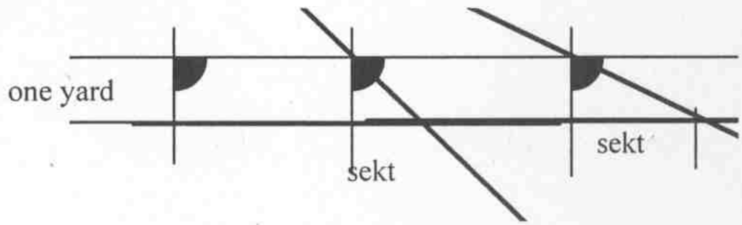

Figure 9.1: The sekt

9. The sekt can be considered as a measure of angles.

On the basis of this analysis, activities were suggested to the students, working in groups of two or three. These activities were to prompt students' reflections on trigonometrical concepts like cotangent, tangent, angle.

The goal of the first activity was to define the sekt. As we have mentioned, the spontaneous answers given by the students tended to identify the sekt with the cotangent. After they had been asked to observe and to note the position of the units in the given solution, several students succeeded in giving more appropriate definitions of the sekt.

With the objective of helping them to consider the sekt as a measure of angle, we asked them to measure the sekt of certain angles while using the metric system, a centimetre corresponding to the cubit. They were also asked to compare an angle of sekt *s* with other angles whose respective sekts were *s*/2 and 2*s*. With the same aim, we asked the students to solve other problems posed in the Rhind Papyrus in which the given and unknown properties are different, while using a reasoning analogous to that of *RP 56*.

We also asked them to guess how the Egyptians, on the basis of the sekt, might have proceeded to construct the pyramids. This question illustrates the fact that a straight line has a constant growth rate. Another and no less fascinating activity consisted in constructing an instrument to measure the sekt of angles, to provide it with a name and to compare the measurements of angles done by means of a sekt and by means of degrees. The activity of constructing real instruments was very dynamic. Indeed, the participants made great efforts to succeed. Some groups achieved classical results, while others showed more originality by providing instruments using glides (see figure 9.2). Two names were proposed for these instruments (in French, as the language of instruction): *seketeur* and *sektomètre*, the second name being maintained.

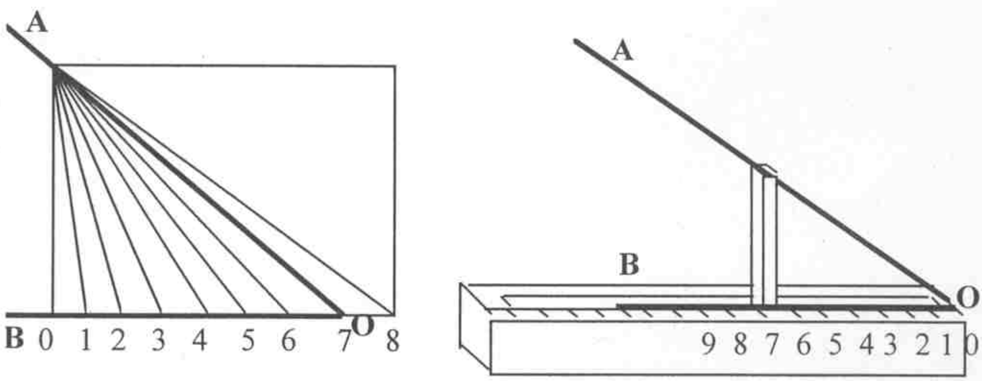

Figure 9.2: Instruments to measure sekts

Comparing degree to sekt raises the problem of the linearity of the concept of cotangent and tangent. Classes discussing the issue are led to understand the advantages of using the degree, and consequently of using to circle arcs to measure angles. Thus, if two angles (OA,OB) and (OB,OC) are given, and S_1 and S_2 are their sekts respectively, the sekt of their sum (OA,OC) is not the sum $(S_1 + S_2)$ of their sekts. Speaking trigonometrically, this signifies that the cotangent function is not linear:

$$\text{ctg}\,(OA,OB) + \text{ctg}\,(OB,OC) > \text{ctg}\,[(OA,OB)+(OB,OC)]$$

The same is true for the tangent function. Measuring in degrees, however, the measure of the sum of angles is equal to the sum of the measures of the angles (see figure 9.3).

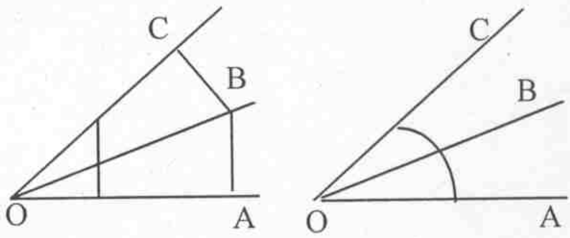

Figure 9.3: Measuring the angle, with the sekt and with degrees

These are the main activities offered to the students. We now describe how history was used and how we were able to profit from it for the education of trainee teachers.

1. The history of mathematics is first involved in introducing the text. The extract is presented and placed into context. Some information is provided about the Egyptian culture and about research into it—the problem *RP 56* also provided an occasion for discussing the notation and concept of the unit fraction, the construction of pyramids, etc.
2. The second part of studying the extract consists in analysing the reasoning of the answer presented in it. This analysis tries to keep as close as possible to the Egyptian way of thinking. While we cannot pretend to have identified the underlying Egyptian reasoning in all its details, an effort was made to draw the students' attention to the contextual components which are involved in analysing this reasoning. In fact, this analysis is in some respects an introduction to the reasoning of future pupils. It can be noted, for instance, that young pupils do not take great pains to justify their own reasoning altogether. They are sometimes quite content with using some ambiguous properties or operations provided these will yield correct results.
3. History is used in the above as a pretext to work on certain practical properties from the concepts of incline, tangent, cotangent and angle. The practical interest of these properties is inspired by the ancient character of the text considered. The historical problem RP56 enabled us to proceed to a comparison of the concepts of sekt, cotangent, and of measuring the angle in degrees.

4. The history of mathematics is in fact used in this example as a crucial motivational element for an epistemological analysis. The latter consists in analysing, from the perspective of teaching, concepts, reasonings and methods used by the ancients, and the difficulties and obstacles which have impeded the evolution of concepts or methods. Thus in this example we have complemented the historical or mathematical analysis proper by activities appropriate for the education of future teachers.

It may be concluded from the above that original texts, even in translation, may be used in a most relevant and fruitful way. To ensure the best contribution to the educational process, however, they must be carefully selected, well analysed, and presented in a dynamic and interactive way.

9.4.2 Example 2: complex numbers in geometry and algebra

A vulgar mechanick can practice what he has been taught, but if he is in error, he knows not how to find out and correct it, and if you put him out of his road, he is at a stand. Whereas he that is able to reason, is never at rest till he gets over every rub. (Newton 1694)

The course MATH 9 at Kristiansand university

This course, first put on in 1978, was intended as preparation for teaching, bearing in mind that there are different ways to integrate history into the mathematics curriculum:

(i) Following genetically the historical development while teaching a theme;
(ii) Using historical problems and examples as a treasure chest to illustrate a subject;
(iii) Opening the student's mind to the fact that mathematics is continually refining its theories, by seeing the historical struggle to develop solutions to problem situations, with new conceptual ideas and theories of understanding.

To read excerpts from original sources should contribute to a critical and more robust understanding of the methods of today. It enables students to work with problems from the origin of a concept, to look at historical mistakes, the etymology of words and the development of notation. The lecture notes (Bekken 1983 and 1994) were put together to help discuss

- issues from our teaching of algebra through historical material,
- the growth of ideas and their forms in algebra,
- in a problem solving style,
- with excerpts from sources, and
- with mathematical problem studies.

Sub-themes were developments of number concepts, like irrationals and imaginaries, symbolisation, and accepted proofs, or demonstrations.

Sources for understanding complex numbers

As Norwegians, we studied the work of a fellow countryman, Caspar Wessel. One of his concerns was how to add and multiply directed lines in the plane. Wessel's solution, first presented in 1796, provides a good introduction to the teaching of complex numbers, because in this source Wessel gave the geometric representation of complex numbers as it is taught today. It is often overlooked that this came out of his attempt to add and multiply directed line segments, vectors as we now call them. In Wessel 1797 (see Nordgaard 1959) we find:

§4. The product of two lines of length 1 in the same plane as the positive unit and with the same starting point, should be in the same plane, with an angle of direction to the unit being the sum of the direction angles of the factors.

§5. Let +1 denote the positive unit, and let a certain perpendicular unit with the same starting point be $+\varepsilon$. The direction angles of +1 $= 0^\circ$, of $-1 = 180^\circ$, of $+\varepsilon = 90^\circ$ and of $-\varepsilon = 270^\circ$. To obtain the rule of §4, we have to multiply according to:

	1	**-1**	**ε**	**-ε**
1	1	-1	ε	-ε
-1	-1	1	-ε	ε
ε	ε	-ε	-1	1
-ε	-ε	ε	1	-1

From this we see that ε *becomes* $= \sqrt{-1}$, and the product follows the usual algebraic rules.

§7. The line having direction angle v to the unit +1 is $\cos v + \varepsilon \sin v$ and when multiplied with the line $\cos u + \varepsilon \sin u$, the product becomes the line with direction angle $v+u$, denoted by $\cos(v+u) + \varepsilon \sin(v+u)$.

§9. The general representation of a line of length r and direction angle v to the positive unit +1 is $r(\cos v + \varepsilon \sin v)$.

Next Wessel demonstrates that he knows very well how this relates to imaginary numbers, and explains the fractional Euler-de Moivre formula. Thus, Wessel had found a new application of imaginary numbers: to the geometry of plane positions.

In this way he also solved another important problem of his time: to give imaginary numbers a geometric representation. This is, in other words, to reconnect the meaning of general numbers to something geometric, but in fact this problem is nowhere mentioned by Wessel.

Glushkov (1977) points to the product of triangles (figure 9.4), introduced by Viète (1591/1983), which we can connect with Wessel's product of directed line segments. Students are asked to explore and explain this.

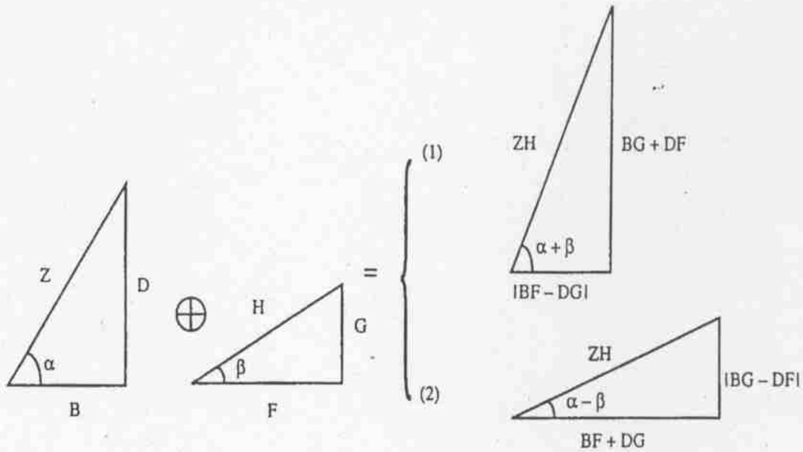

Figure 9.4: Viète's product of triangles

Impossible quantities in algebra

Earlier, imaginaries had come to be useful in algebra, first in the works of Cardano (1545/1968) and Bombelli (1572/1966), later also in Viète (1591/1983), Descartes, and Wallis. The most quoted passages in Cardano's *Ars magna* comes from his chapter 37 'On the rule for postulating a negative' (1545/1968, 219-220):

If it should be said, Divide 10 into two parts the product of which is 40, it is clear that this case is impossible. Nevertheless, we will work thus: We divide 10 into two equal parts, making each 5. These we square, making 25. Subtract 40, if you will, from the 25 thus produced, as I showed you in the chapter on operations in the sixth book, leaving a remainder of −15, the square root of which added to or subtracted from 5 gives parts the product of which is 40. These will be $5+\sqrt{-15}$ and $5-\sqrt{-15}$, ... and you will have that which you seek. ... Putting aside the mental tortures involved, multiply $5+\sqrt{-15}$ by $5-\sqrt{-15}$, making $25-(-15)$. Hence this product is 40. ... This is truly sophisticated since with it one cannot carry out the operations one can in the case of a pure negative. ... So progresses arithmetic subtlety the end of which, as is said, is as refined as it is useless.

which is also worth looking at in Latin:

> ,si quis dicat, diuide 10 in duas partes, ex quarum unius in reliquam ductu, producatur 30, aut 40, manifestum est, quòd casus seu quæstio est impossibilis, sic tamē operabimur, diuidemus 10 per æqualia, & fiet eius medietas 5, duc in se fit 25, auferes ex 25, ipsum producendum, utpote 40, ut docui te, in capitulo operationum, in sexto libro, fiet residuum m: 15, cuius ℞ addita & detracta a 5, ostendit partes, quæ inuicem ductæ producunt 40, erunt igitur hæ, 5 p: ℞ m: 15, & 5 m: ℞ m: 15.
>
> 5 p: ℞ m: 15
> 5 m: ℞ m: 15
> 25 m: m: 15 q̄d. est 40

This is the first known appearance of the square root of negatives, which here reads R m: 15 .

A few paragraphs later we find the following example leading to this case of working with imaginaries, or 'sophistic negatives' as Cardano called them (Cardano 1545/1968, 221):

> If it be said, Divide -6 into two parts the product of which is $+24$, the problem will be one of the sophistic negative and will pertain to the second rule, and the parts will be $-3+\sqrt{-15}$ and $-3-\sqrt{-15}$.

These imaginaries are in *Ars magna* not connected to Cardano's main theme of cubic and quartic equations, but it is interesting to note his point of view on negatives (1545/1968, 154): they may be necessary for intermediate calculations toward a true, i.e. positive, answer.

The same is true for the imaginaries, but Cardano does not comment on this. Instead, we look at an example given by Clairaut in 1746, who wants to solve the cubic equation $x^3 = 63x + 162$. For this equation the Cardano-Tartaglia solution procedure leads to the formula

$$81 \pm 30 \cdot \sqrt{-3} = \left(-3 \pm 2\sqrt{-3}\right)^3$$

where the equality may be verified by direct multiplication. Then the Cardano-Tartaglia solution says that one of the solutions x is found via

$$x = \left(-3 + 2\sqrt{-3}\right) - \left(3 - 2\sqrt{-3}\right) = -6 .$$

Thus the equation has a factor $(x+6)$ and so the other solutions can be found by factoring:

$$\left(x^3 - 63x - 162\right) \div (x+6) = x^2 - 6x - 27 = (x+3)(x-9) .$$

Hence a true positive solution is $x = 9$, but Clairaut reached it only through computations involving both imaginaries and negatives.

Rafael Bombelli (1572/1966) found that in irreducible cases like the one above, there are always three real roots, but most often you are not able to do the actual reduction as simply as in the Clairaut example. Other early examples were given by Bombelli (1572) as well as Leibniz (1676). In this process we have seen Cardano computing with expressions like $\left(a+b\sqrt{-1}\right)\left(a-b\sqrt{-1}\right)$

and Clairaut with

$$\left(a+b\sqrt{-1}\right)\left(c+d\sqrt{-1}\right) = ac - bd + (bc+ad)\sqrt{-1} ,$$

just using what the English mathematician George Peacock (1842) was to call the 'principle of permanence of forms', that such new numbers behave structurally like old ones. But if so, why isn't always $\sqrt{ab} = \sqrt{a}\sqrt{b}$? Because then, we would get, as pointed out by Euler (1770/1984), that

$$-2 = \sqrt{-2}\sqrt{-2} = \sqrt{(-2)(-2)} = \sqrt{4} = +2 .$$

Resolving this apparent paradox will be a helpful discussion item for students exploring the ramifications of symbols they may have come to take for granted.

9.5 Integrating Original Sources in the Classroom

In the following we present two examples of reading original sources in school classrooms, one (§9.5.1) from Germany, the other (§9.5.2) from Italy.

9.5.1 Example 1: Greek surveying: the tunnel of Samos

The story of the tunnel

The ancient Greek historian Herodotus described a tunnel constructed on the island of Samos by the engineer Eupalinos about 530 BC. Knowledge of such a tunnel had become completely lost when it was rediscovered towards the end of the 19th century. First archaeological excavations showed that Herodotus's report was absolutely reliable. Between 1971 and 1978, the tunnel was completely excavated and examined in detail (Kienast 1986/87). The tunnel cuts through a mountain to supply the Samos fortress with water. It is 1040 metres long, 2 metres wide and 2 metres high, consisting of a path for inspections and a canal for the water beside it. It was mined simultaneously from both ends, and the two teams met under the mountain.

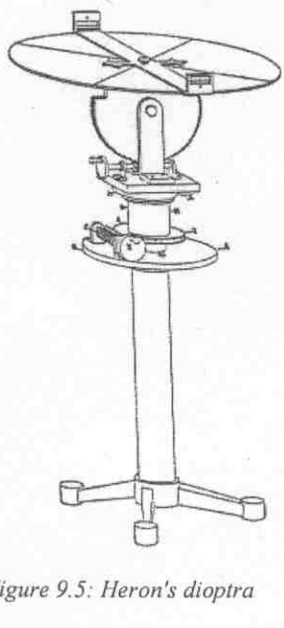

Figure 9.5: Heron's dioptra

The underlying engineering feat is considerable. The standard procedure for tunnels of such length at the time was to dig several shafts to the surface in order to determine the position reached and to correct the direction of the digging. This method was not used here. Since the discovery of the tunnel, a much discussed question has been how Eupalinos surveyed the tunnel's direction with such accuracy.

A possible answer may lie in a source of some 600 years later. In a handbook describing the handling of a surveying instrument called *dioptra* (figure 9.5), Heron of Alexandria (40-120 AD) treats the problem of 'cutting through a mountain in a straight line if the entrances of the tunnel are given' (Schöne 1903, 238 ff). Heron's booklet poses a number of other interesting surveying problems which could be treated in the classroom.

In an introduction, Heron describes the dioptra's uses, naming military applications besides land surveying and astronomy. A specially nice remark says that, frequently storm attacks on fortresses were easily repelled because the besiegers had underestimated the height of the walls, attacking with ladders which were too short. In such cases, Heron said, the dioptra had its uses, for it served to measure the heights in question "out of range" (Schöne 1903, 191).

A teaching unit about Heron's surveying text

For a long time, the experts favoured the hypothesis that Eupalinos had essentially proceeded as described by Heron (cf. Van der Waerden 1956, 168 ff), and it is also the basis of the following lesson. The above mentioned excavations, however, have led archaeologists to prefer another theory. We shall see that the students discovered both these theories on their own.

On the basis of this story about the Samos tunnel, a teaching sequence founded on Heron's text was developed and tested at various schools in the region of Bielefeld, Germany (Jahnke &Habdank-Eichelsbacher 1999).

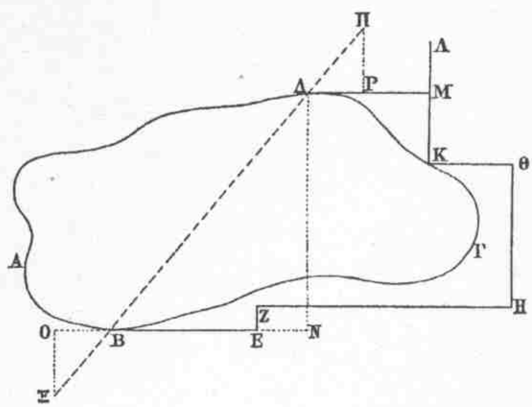

Figure 9.6: Heron's method of surveying a tunnel

While one could expect the story to be attractive to the pupils, the source might raise some difficulties. For fourteen- or fifteen-year-olds it was rather long. As is common in ancient Greek geometrical texts the essential idea is not explicitly mentioned, the argument proceeds step-by-step. Some teachers expected even problems with the Greek letters. Nevertheless, it was decided to present the source unchanged as it was printed in Schöne's Greek-German edition. The students were told that this was a section of an ancient original surveyor's handbook which had not been especially devised for them. While it might not be too easy to read, they would be able to cope with the difficulties. For a number of students, this remark proved to be quite motivating.

Mathematically, Heron's surveying method requires the notion of similarity. This had not been explicitly treated in the 9th grades where the teaching took place. The idea was to rely on the students' intuitive previous knowledge. One could expect that they had a notion of how maps work. In most classes the teaching unit on the tunnel of Samos served as an introduction to the concept of similarity.

All in all, the teaching sequence consisted of $3 + n$ lessons. An introductory lesson about the students' knowledge of history of mathematics ended with the story of the Samos tunnel. In a second lesson the problem of how the direction of the tunnel could have been determined was discussed with the students. In the third lesson the source was analysed, after a first reading had been given as homework. In further lessons other surveying problems were treated.

The classroom experience

All classes had a lot of fun establishing a map of the history of mathematics. To both their teachers' and their own surprise, the students' previous knowledge of history of mathematics was manifold. They knew a lot of facts. Above all, students have historical imagination and find questions such as why mankind started to use and write numbers, or to draw and analyse geometrical figures, quite natural and interesting.

The discussions about how Eupalinos might have determined the direction of the tunnel, under the condition that one end cannot be seen from the other, proved to be very fruitful. All classes developed essentially the same two solutions. And these are exactly those offered by the archaeologists.

The first method is that of the source. It can be understood from Heron's figure (see figure 9.6). Starting from one entrance a sequence of segments around the mountain is measured. From this one can calculate the segments BN and ΔN whose ratio gives the direction of the tunnel. Then, at both entrances beams are constructed showing the right direction. The second possible strategy found by the students results from the question whether it is possible to take bearings from the mountain's summit on both entrances marked by flags. If this is not directly possible, one could put up a sequence of flags connecting the entrances and then adjust the sequence until the flags lie on a straight line from one entrance to the other. Modern archaeologists found signs suggesting that Eupalinos proceeded this way, but it is possible that he used both methods.

After this preliminary and informal discussion with the students which did not end with a clear and definite result, but with a lot of ideas and a feeling for the nature of the problem, they got copies of the source which was to be read as homework. Before the next lesson, there were already discussions among the students about Heron's idea. In the lesson itself the general idea was presented by one or several students, then the source was read step by step. It was a nice experience that in one class the discussion was opened by a student with the statement "Heron has made a mistake!". In fact, if one reads Heron literally the student was right, but others argued that this is a matter of interpretation.

It was interesting to see how the students explained Heron's method without knowing the notion of similarity. In one class, they argued that his idea is the same as that underlying the determination of the slope of a straight line. As this had been treated quite a while ago, this was a compliment to them and their teacher. In the other classes, the argument was a bit vague, but intuitively correct when students argued that Heron constructed a sort of a map.

After discussion of the source, teaching was continued in various ways. It was pointed out to the students that the workers didn't meet exactly, but missed each other by about 10 metres in the middle. It was determined by drawing that the error in measuring the angle of direction had been less than 1 degree. The question how Heron coped with the difference in altitude was raised in all classes.

Written student productions

All students were assigned the task of summarising Heron's method in a small written essay. The results show that more than two thirds of the students had completely understood the text. Many students were able to free themselves from the language of the source and to express the idea in their own words, finding quite convincing descriptions which represented a mixture of everyday language and of the expert language acquired in the classroom. Such written exercises and the skills they develop and demonstrate are an important general objective of integrating historical sources into mathematics teaching.

9.5.2 Example 2: An 18th century treatise on conic sections

The teaching environment

The activity here analysed has been planned and developed in a classroom by a secondary teacher (see Testa 1996). It was carried out in an Italian Scientific Lyceum, a high school in which mathematics is an important (and difficult) subject; 16 students (11 girls, 5 boys) aged 16/17 volunteered to participate. The total time employed was 16 afternoons, after the school time.

The subject taught is conics, which in the official mathematics curriculum is suggested only as optional subject matter. In the first eight afternoons theories about conics of various classical authors such as Pappus and Eutocius were outlined. Also the means for the pointwise construction of the conics (Euclid's *Elements* book II) were discussed. The following eight afternoons were devoted to the study of De la Chapelle's *Traité des sections coniques, et autres courbes anciennes*. This text is a revision with 'didactic eyes' of classic works on optics. There is a systematic application of algebra to geometry, a unifying use of Euclid iii, 35; the links with physics are considered. The text was chosen for its clarity and elegance. The preface shows that the author was aware of the students' difficulties in learning mathematics and looked for ways of overcoming them.

The Italian teacher's choices reveal his view on the use of history in mathematics teaching: to read an ancient text is his favourite way of integrating history in classroom, and doing history of mathematics is nothing other than doing mathematics. The teacher is historically well read and experienced. Thus, to look for original sources and to work with them is not a problem to him.

The experience

In our description, we focus on the teaching of De la Chapelle's text. The main difficulty to face was the unknown language. The teacher rejected the idea of presenting a literal translation, to avoid the temptation for students to participate only passively. Instead, he prepared 34 worksheets containing passages of the French text, with blanks in strategic positions to be filled in by the students. At the beginning the original text was quite fully summarised, in later worksheets the amount of original text was increased, and with the last worksheets the text was almost entirely the original. In the worksheets De la Chapelle's symbols were kept. Since the original figures usually contain elements referring to different propositions, the teacher drew new figures containing only the elements essential to a single proposition, during the first period; later on students were encouraged to use the original figures, and to decode the information contained in them.

At the beginning, mediation by the teacher was important, afterwards the students' work was more and more autonomous. Students worked in groups, and also did homework. They devised their own strategies for handling the difficulties, using coloured pencils to decode figures, and substituting the old notations by new ones. In order to fill the blanks, students had to understand the underlying reasoning. This method of work roused lively discussions among students. After they had worked at the given worksheet, the teacher showed in a transparency the complete original passage and discussed the work performed by the students.

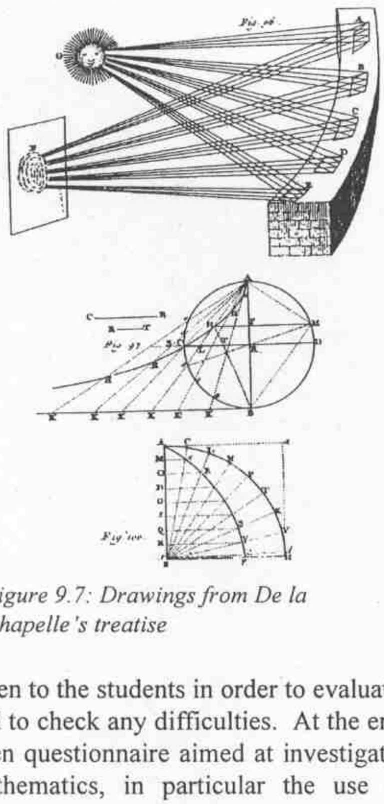

Figure 9.7: Drawings from De la Chapelle's treatise

Evaluation of the experience

After each session a questionnaire was given to the students in order to evaluate the understanding of mathematical contents and to check any difficulties. At the end of the overall experience they answered an open questionnaire aimed at investigating how they perceived the use of history of mathematics, in particular the use of original sources. Students were very collaborative, and gave a great deal of information; their protocols can be considered as written interviews. The most significant points which emerged were:

- doing mathematics became more pleasant
- it was easy to see the evolution of mathematics and to become aware that there are different points of view to face problems
- the method of work led directly to seeing what there is behind a theorem
- the study of the original text was preferred since the participation in the work was more active
- it was more difficult to grasp the language than the spirit of the work

- to work directly with the text required more careful reflection on problems and better understanding of their meaning: thus apparently simple problems revealed unexpected aspects
- it was made possible to go beyond theorems and to arrive at the roots of mathematics
- the experience changed the image of school mathematics.

The words of one student point out the effectiveness of integrating original sources: "The proofs I had to complete helped me to learn working on my own. I liked working with the graded worksheets because they implied a step-by-step reasoning unlike my usual way of thinking." Working with original texts clearly produces changes in the mode of learning.

The experiment had particular features which make it difficult to draw general conclusions. Among these features were that it was an extra-curriculum activity involving only volunteer students; the source concerns a quite unknown author; it deals with a language not mastered by students; the text used was conceived as a textbook; and the teacher here possessed a remarkable competence in history of mathematics and familiarity with the use of original sources.

On the other hand for our study these elements can be seen as positive, since:

- being an optional activity allowed the teacher freedom in planning and developing the didactic procedure
- using a rarely considered author fostered originality and creativity in the experience
- the presence of an unknown language is a quite typical obstacle in using original sources and thus it is interesting to see how the teacher has faced this difficulty
- to use a text written for didactic purposes is an intermediate situation facilitating the approach to an original source
- the teacher's competence has made the experience very rich in cultural values.

The literature on the use of original sources in courses (not specifically for history of mathematics) shows that successful experiments generally refer to university level or, in the case of high school level, to optional courses (see Laubenbacher & Pengelley 1994; 1996). Other interesting examples exist, which concern limited passages in limited activities; this is the case, for example, with using of mediaeval arithmetic word problems. A wider and systematic use of original sources presents difficulties of time, souce availability, and so on. Undoubtedly the main point is the role played by the teacher. He has to really believe in the value of original sources, he has to be competent enough in order to find and to manage materials suitable to the needs of his classroom, he has to plan strategies of mediation very carefully. These strong requirements emphasise the difficulty in the transferability of good experiences from one teacher to another and in making the use of original sources a routine activity.

9.6 Didactical strategies for integrating sources

9.6.1 The triad: text - context - reader

As we explained above (section 9.3), reading a source is a hermeneutic activity and, thus, subject to the rules of hermeneutics. In every teaching where an original source is going to play a role the teacher has to consider the concrete relationship between the text, the context and the readers. Depending on the aims of teaching there should be a certain balance between the proper analysis of the source and the investigation of the context. Usually, students reading a mathematical text are not used to asking for the context. In a way, they are even educated not to consider it: mathematics should be independent of the context and understandable out of itself; the time when a text has been written, the country or the author seem to be irrelevant. Therefore, students have to be guided to asking meaningful questions about the context. Frequently, it will be necessary to do some independent investigations about the context and study the biography of the author before the source can be interpreted adequately. Also, to relate the context information to the meaning of the text under study requires some skills which presuppose some experience and have to be trained. For example, frequently it makes a difference whether a text has been written by a theoretically or a practically minded author, and it is possible to trace indications of this prevalent habit of mind in the text. This is very illuminating, but, of course, requires some experience.

It should be clear that the aim of these activities is not at all an imitation of the professional historian in regard to rigour and sophistication. Rather, the students are led to asking new questions which, in general, they had never asked before.

The use of primary sources in the classroom requires special care, to clarify the proposed objectives and the adequacy of a source to the students' needs. The concrete conditions of the students should be considered, and, of course, it makes a difference whether a source is studied with school students or with future mathematics teachers or with in-service teachers. The chosen contents need to be related to the respective student interests, the availability of texts in the mother language (or, at least, a language known to the students or the teacher) and in accordance with the objectives that the teacher intends to achieve.

9.6.2 Classroom strategies

At the moment there is no elaborated and generally accepted approach available for the reading of sources in the classroom. There are, however, some experiences, and, in the following, we want to give a generalised picture of these experiences. This may be taken as a collection of ideas and guidance (which does not pretend to be exhaustive) from which interested readers may select what is appropriate to their needs.

(i) Introducing a source

To introduce original material in the classroom, two types of strategy are imaginable: direct and indirect. Using a direct strategy, the teacher presents the text without any previous preparation. An indirect strategy is a situation where the source is consulted after some previous activities.

1. A direct strategy to a source might have as an objective to provoke a shock in the students, through perceiving the difference between their modern view of the subject and the view-point expressed in the source. This will provoke questions for study. After reading, the student is required to answer a series of questions previously established by the teacher—or it may be suggested that the student extracts questions from the text. Presenting the text in this way has the objective of challenging the student and raising a polemic around the theme.

2. An indirect strategy might result from solving problems. The teacher presents to the students a non-routine problem, to raise their curiosity and the need for a deeper study of the subject. After this the teacher might present an extract of an original text related to the questions the students had formulated.

3. Another indirect strategy could start with a historical author. The teacher begins by showing how mathematics is connected with the society of a certain time and, together with the students, he points out the mathematicians' names that stood out. The students select one or more authors and try to gather available information about them. Only after interest about the mathematician has been raised does the teacher present a source extract, and the class work culminates with its analysis.

4. Textbooks might be another point of departure. The teacher selects a theme in the textbook used in the classroom. She questions its approach. Then she presents other textbooks, or extracts from an old textbook, for analysis and comparison with the current one. It raises the students' curiosity; they feel the desire to discover who introduced that concept or theory, who formulated or solved that problem. Thus, the original text appears in a natural way and is worked on as a profound study of the text used in the classroom. Further possibilities and problems with respect to the process of interpreting and analysing ancient textbooks are discussed in Glaeser 1983 and Schubring 1987.

5. In the education of adults it might be easiest and most natural to introduce a source through a presentation from the tutor. This is a discourse within which the tutor provides information, formulates a synthesis, or introduces a new question. The tutor sketches the historical background and comments on difficulties, special features, and objectives of the text in question. Switching between different texts or different parts of the same text can also be achieved by short presentations in which the tutor provides a synthesis of the text already treated and introduces the subsequent ones. These presentations should not be over extended; a few minutes will do.

(ii) Analysis of a source and cognitive debates

The analysis of historical texts is a difficult activity in history of mathematics. Sometimes it should be supported and guided through questions from the teacher. Sometimes, it seems to be more adequate to let the students find out the right questions. An important aspect is to find suitable questions so that students become immersed in the historical context of the text under study.

To improve the conditions of analysis, some texts must be modified or translated and adapted to the general context within which they are being introduced. At the same time, they have to be modified so as to remain within the students' or trainees' grasp. Nevertheless, it is imperative that these adaptations remain as closely aligned as possible to the original author's thought (Barbin 1987).

Frequently, the analysis of a text gives rise to cognitive debates. These are discussions within which the students are called to express their own views on a concept's or method's validity and relevance, and above all to give reasons for their own choice. For this, great care is needed in selecting the texts or controversies which are to be the object of debate.

To prompt a successful debate, the educator should suggest to each group of students or trainees that they prepare to argue in favour of one or other point of view. Notwithstanding their directive character, these suggestions tend to motivate students and inspire them to find out for themselves about the advantages of a historical reasoning which at first glance might appear naive or erroneous (Desautels & Larochelle 1989, 1992; Legrand 1988; Lakatos 1976).

(iii) Construction of measuring instruments

Humankind has always been preoccupied with measuring physical or mathematical quantities and this is particularly true for mathematicians and scientists. Historical research reveals different conceptions of measurement. Although these conceptions may easily become apparent in some cases, they may not exist within a structured theory and they may not even have been used to construct instruments. Nevertheless, the ideas encountered through historical study may serve to inspire activities which can help participants to analyse their own reasoning and also encourage them to construct their own measuring instruments. For instance, mathematical machines for drawing curves may be of interest (Dennis 1997; El Idrissi 1998; Ransom 1995; also see section 10.2.2).

(iv) Verbalisation

With regard to acquainting the participants with the reasoning of mathematicians, having them verbalise this reasoning seems to be an excellent strategy. It makes students attentive to original thoughts and helps prevent them from attributing to mathematicians things they never said, and (if trainee or in-service teachers) from passing on such misunderstandings to their pupils in due course. Take care to have them distinguish in these verbalisations between things derived from the texts themselves and interpretations of the latter. This activity is also beneficial in alerting students to the difficulties which may be met when reasoning in mathematics without the support of a formal system.

(v) Translation

As with verbalisation, translations of text extracts are intended to acquaint students and trainees with the thought and conception of mathematicians in regard to mathematical reasonings and concepts. At least two types of translation can be distinguished here: translations into modern mathematical language, and translation from one language into another. While the former serves in particular to reconstruct a mathematical argument, the latter has promising educational advantages insofar as it initiates students and trainees into mastering a language and to conceptual analysis (Arcavi *et al.* 1982, 1987; Testa 1996).

(vi) Validation of reasonings

During their first studies of historical works, students and trainees sometimes disparage the mathematical value of reasonings found in historical texts, especially if they have been accustomed to continuous praise of recent mathematical progress. This attitude may prevent students from realising the educational and mathematical potential contained in ancient reasoning.

To challenge this attitude, one may ask students to validate the reasoning of the mathematicians of old. Such validations are intended to demonstrate how well-founded are the methods used in history, in the light of more elaborate mathematical knowledge. This prompts the students both to give historical reasonings the same status as present-day thought with regard to mathematical foundations and also to challenge their own conceptions of present-day methods, in particular as regards the learning of mathematics. These ancient methods often have the advantage of being within the pupils' grasp and of providing interesting hints for teaching (Arcavi *et al.* 1982, 1987).

(vii) Comparison

To compare different texts or text extracts is also a fascinating approach, in particular in history of mathematics. The comparison may include texts of the same period or of different periods, having the same or different objects. These comparisons must be accompanied by activities and questions of understanding aimed at making analysis more purposeful and more attractive.

Comparing historical texts permits students to realise how the notation and symbols of mathematics have evolved. It helps them to focus on the essential in historical mathematical writings. In addition, the comparison of mathematical textbooks is a promising approach to the history of teaching.

(viii) Synthesis

Activities of synthesis should be done by students outside of the course; for this external type of work, all the strategies mentioned above can be used. This homework should be designed either as a preparation for future courses or as a work of synthesis. It may be planned for the end of course sections.

9.7 Evaluation, research questions and issues of concern

Though the idea of integrating history of mathematics into mathematics teaching originated more than a hundred years ago, practical efforts on a larger scale beyond isolated activities of individuals have been made only in the last twenty years. Since reading sources belongs to the most demanding of possible activities, it is not surprising that up till now there is no systematic empirical research, investigating opportunities, difficulties and outcomes of sources as part of mathematics teaching.

At present, there are essentially two types of contributions to the field. On the one hand we have a number of reports reflecting personal experiences with reading original sources in various contexts, be it school teaching, or the education of future teachers (cf. Arcavi 1987; Furinghetti 1997; Jahnke 1995; Laubenbacher & Pengelley 1998; Silene & Testa 1998). On the other hand, there are quite a few papers with proposals on what could be done. However, to reach a new conceptual and practical level we do need more research. In this section we sketch some directions of work which could be followed.

First of all, we should know better whether reading a source does in fact make a difference compared to other possible activities. Given the large amount of time required for using original sources, we should be sure that the effort is really worthwhile. From theoretical reflections we are quite sure that a source will open up new dimensions of understanding. We have mentioned above experiments where the integration of historical sources has been successful. The problem is to ensure adequate conditions. It is clear that the role of the teacher/tutor is crucial for creating the right atmosphere and providing the necessary intellectual tools for students.

This leads to further questions. Reading a source demands in a specific way a feeling for the intellectual, social and cultural context in which it has been written and the ability to ask questions concerning these dimensions. This in turn presupposes that the learner has already a certain historical background and an ability which we would like to call historical imagination. Under conditions where history of science is a curricular subject neither in regular school teaching nor at universities, a historical background in science and mathematics can only result from personal reading or from the media (television, films etc.). Thus we should investigate what previous knowledge about these things our students have and how much we can rely on this as a historical foundation to build on (see Demattè 1994; Demattè & Furinghetti 1999).

Because of this context dependence, reading a source is quite different from reading a normal text of mathematics. Thus, one has to change one's reading habits, and, again, we should know more about this, theoretically and empirically.

It is very important to investigate the reading strategies and the strategies of interpretation as well as the difficulties students encounter with sources. How do students react to a text, how do they work with terms whose meaning they do not know? Are they able to identify essential elements of a text? How do they translate the meaning of a text into their own language? Only with a better understanding about this shall we be able to devise more effective teaching strategies.

One of the essential ideas connected with the reading of a source is that this will influence the students on their meta-cognitive level and contribute to their ability to reflect about mathematics. Again, we need to know more whether this is really the case, and if so, to what degree.

It would be worthwhile also to know more about processes of mathematical understanding which might not be intended, but nevertheless happen. Students, or teachers, may see in a historical document a source of insight, which may add to their understanding, regardless of the historical context, and far from the intention of the original writer. Thus the question to explore is: can original documents be the trigger for re-thinking the mathematics, even by way of erroneously attributing intentions to the text, which are not there, or misrepresenting its ideas? In other words, the source can be the motivation and inspiration for thinking differently about a mathematical topic, in a way which has nothing to do with what any historian would have seen in the source.

It is obvious that all these questions might be answered differently for young pupils or adults. Thus, the age factor is important.

There is a practical problem which will continue to remain a task of great importance for future work: the identification and editing of adequate source material.

This overview shows that we are only at the beginning of a process in which history of mathematics might become an organic part of mathematics teaching. To achieve this goal we have to solve a lot of problems. Fortunately, these problems turn out to be interesting and demanding.

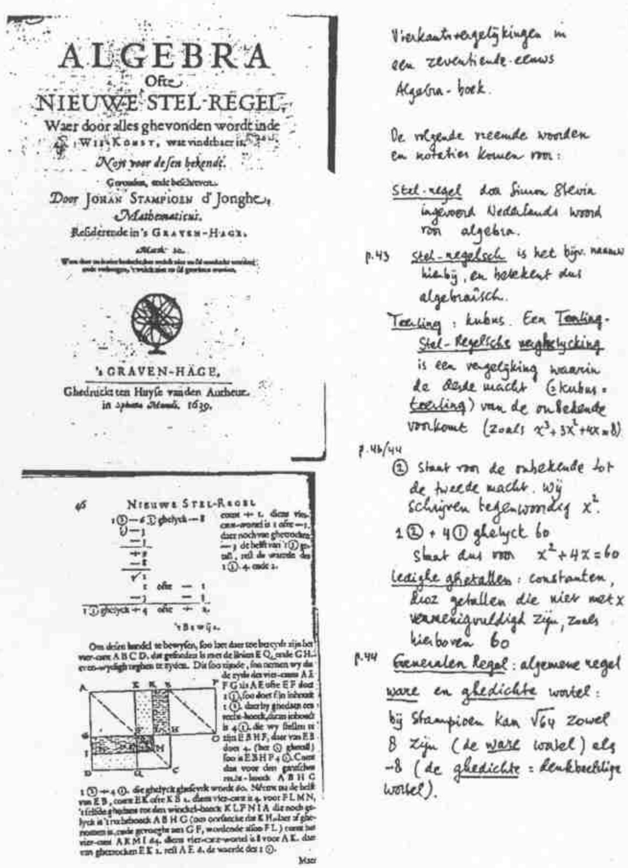

Figure 9.8: Grade 9 pupils (13 year olds) in a Dutch secondary school explored this 17th century Dutch algebra text in learning about quadratic equations. The title page, and a page with the geometrical proof of an equation-solving rule, were supplied together with the teacher's hand-written glossary to help pupils to study the text at home before the classroom discussion (from van Maanen 1997)

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APPENDIX: RESOURCES

Sources of original mathematical material

The following references are to sources where original works can be found. The selection of material has to be somewhat restricted and we have chosen material that is currently, or recently, in print or material that is widely available in libraries. There are a number of excellent histories of mathematics which, while being histories, also contain a great deal of illustrative original material. These are not listed here but a selection of them will be found directly after this appendix. Nor have we included here references to Complete Works of mathematicians, assuming that the interested reader would know how to access such material.

Archimedes: Dijksterhuis, E. J., *Archimedes*, Princeton, New Jersey: Princeton University Press, 1987; Heath, T. L. *The Works of Archimedes*, New York: Dover Publications.
The Heath edition, with the 1912 supplement, presents a translation of the extant works of Archimedes, using modern notation to make the mathematics easier to follow for the modern reader, but this has the disadvantage of re-interpreting the original line of thought. Dijksterhuis uses a notation that allows the reader to come closer to the original Greek thinking. On the other hand, Dijksterhuis does not give a translation of all the propositions, preferring to guide the reader through the essential material.

Argand, R.: *Essai sur une manière de représenter les quantités imaginaires dans les constructions géométriques*. new print. Paris: Albert Blanchard, 1971.
English text books continue to use the name Argand diagram for the representation

of complex numbers on the plane and this facsimile of the 1874 second edition of Argand's 1806 essay is a clear and simple presentation of his argument. An English translation of the earlier publication by Caspar Wessel on this subject will be found in D. E. Smith.

Barrow-Green, J. 1998, 'History of mathematics: resources on the World Wide Web', *Mathematics in school* **27** (4), 16-22.
This paper annotates web addresses useful for historians of mathematics. Cf. §10.3.2.

Berggren, L., Borwein, J, Borwein, P. 1997. *Pi: a source book*, New York: Springer
Not so much a history of π – is that possible? – but a collection of articles about the number. Thus we find essays on the series formula, algorithms, computer calculations and the Gauss Arithmetic-Geometric mean. The book deserves a mention here because of the wealth of original material. Many of the original papers appear in the original Latin, French or German and without translations. The photocopies of original printed works are of variable quality and no editorial corrections of typos, etc. has been undertaken. Nonetheless, having original works in their original presentations brings its own excitement to the interested reader.

Bernoulli, Johann. *Lectiones de calculo differentialium. Mscrpt.* German edition: P. Schafheitlin (ed. & transl.), *Die Differentialrechnung von Johann Bernoulli aus dem Jahre 1691/92*. Ostwalds Klassiker der exakten Wissenschaften 211, Leipzig: Akademische Verlagsgesellschaft, 1924
This is a German translation of the first textbook on calculus ever written, though not published at its time. It can be read after some introduction into calculus.

Bibby, J. 1986. *Notes towards a history of teaching statistics*, Edinburgh: John Bibby Books
Much of interest here for projects – it includes many original sources and pictures – as well as giving the statistics teacher useful information on how the teaching of the subject has changed over that past century or two.

Cantor, G. 1915. *Contributions to the Founding of the Theory of Transfinite Numbers*, New York: Open Court Publications
An English translation of Cantor's 'Beiträge zur Begründung der transfiniten Mengenlehre'.

Cardano, G. 1545/1968. *The great art or The rules of algebra*, (T. Richard Witmer tr. 1968), Cambridge: MIT Press
First published as *Ars magna* in 1545, this a cornerstone book in the history of mathematics reveals the author's solution to cubic and biquadratic equations. Long unavailable, except in rare Latin editions, now available through a Dover reprint.

Chabert, J.-L. *et al.* (ed.), *Histoire d'algorithmes*, Paris: Belin, 1994; English tr. *A history of algorithms*, Berlin: Springer, 1999
A rich source of historical material, including many non-European works. Each chapter shows the development of a topic with extensive extracts from original writing. This would allow the teacher to introduce a topic directly from the original publications of mathematicians. Topics covered include: methods of false position, Euclid's algorithm, interpolation, approximate solutions and convergence.

Cullen, C. 1998. *Astronomy and mathematics in ancient China: the Zhou Bi Suan Jing*, Cambridge: University Press
This complete translation of an important 1st century Chinese text provides rich material for the mathematics classroom. It is also a very beautifully produced book with an easily

accessible introduction to the developing mathematical and astronomical practices of ancient Chinese astronomers and shows how the generation and validation of knowledge was closely related to statecraft and politics.

Descartes: D. E. Smith & M. L. Latham (ed., tr.), *The Geometry of René Descartes*, Open Court, 1925; New York: Dover Publications, 1954
La Géométrie, which appeared originally as an appendix to *Discours de la Méthode* (1637), presents Descartes' algebraic treatment of geometry. The English translation is in a simple and direct style, while the parallel facsimile of the first edition provides the possibility of comparison with the original French, as well the opportunity of comparing modern algebraic usage with the original French typography. The whole is enriched with numerous explanatory footnotes.

Dhombres, J. *et al.*, *Mathématiques au fil des âges*, Paris: Gauthier-Villars 1987
For readers of French this is a valuable collection of over 100 extracts, some quite extensive, grouped together to reflect ideas in the use of mathematics, arithmetic, algebra, analysis, probability and geometry. The chapters on analysis and geometry are subdivided to deal with themes, such as the origin of the infinitesimal calculus and the representation of space. The selection of material naturally reflects French interests and contributions. Here you will find Fermat's use of geometric progression to determine areas under the hyperbola and Condorcet on combining probabilities. The whole is most attractively produced, with fine illustrations, and concludes with brief biographies of more than 150 mathematicians.

Dürer, A.: *Unterwisung der Messung: Um einiges gekürzt und neuerem Sprachgebrauch angepaßt herausgegeben sowie mit einem Nachwort versehen.* Reproduction of the edition München 1908: Wiesbaden, Sändig 1970. Original edition: Nürnberg 1525, reproduced in facsimile: Nördlingen: Verlag Dr. Alfons Uhl, 1983
Contains a lot of geometrical constructions.

Eagle, R. E., *Exploring mathematics through history*, Cambridge: Univ. Press, 1995
A collection of sources, from the earliest number recordings up to Fermat and Pascal's discussion of probability, prepared to be used in the secondary mathematics classroom. Each topic contains a description of the context and a simple explanation of the mathematics for use by the teacher or by a student. The material for use in class contains brief extracts of original material. The whole is delightfully illustrated and is presented so that it can be used by a teacher who has little or no historical background knowledge.

Fauvel, J. (ed.), *History in the Mathematics Classroom: the IREM Papers*, Leicester: Mathematical Association, 1990
Nine articles by French mathematics teachers showing how they have used original material in their classrooms. Each article contains the original material in English translation, providing the teacher with lesson material. A wealth of ideas and experiences.

Fauvel, John and Gray, Jeremy, *The history of mathematics: a reader*, Basingstoke and London: Macmillan Press, 1987
This selection of over 400 extracts was originally prepared for the Open University course *Topics in the History of Mathematics* and covers mathematical writings from the earliest ideas of numbers and counting up to the mechanisation of calculation. The collection includes many comments on the nature of mathematical activity by mathematicians and other philosophers to sit alongside the original mathematical material. The contribution of Islamic mathematics is given its rightful place and of particular note is the chapter on the

mathematical sciences in Tudor and Stuart England which contains material unlikely to be encountered in other collections. The extracts have been carefully chosen to be easily accessible and include, for example, the proof by Gauss that the regular 17-*gon* is constructible.

Hay, Cynthia (ed.), *Mathematics from Manuscript to Print 1300 – 1600*, Oxford: Clarendon Press, 1988

Papers on aspects of mediaeval mathematics, containing extensive extracts from the works of Maurolico, Nicolas Chuquet and Agrippa not easily found elsewhere.

Heath, Thomas L. *Aristarchus of Samos, the ancient Copernicus: a history of Greek astronomy to Aristarchus* together with Aristarchus's *treatise On the sizes and distances of the sun and moon*; a new Greek text with translation and notes, Oxford: Clarendon Press 1966

This book contains Aristarchus' famous paper on the relative distances of the sun and the moon from the earth. The hypotheses and theorems can be discussed in a course on trigonometry.

Heronis Alexandrini opera quae supersunt omnia. Greek-German edition. Stuttgart: Teubner 1976.

Contains a lot of valuable sources on measurement, optics, geometry.

Hilbert, D. *Foundations of geometry*, Chicago: Open Court 1902.

A translation of the 1899 *Grundlagen der Geometrie*, in which Hilbert showed that it is possible to construct a geometry based on a complete system of axioms. In the first chapter, Hilbert began by stating 21 axioms involving six primitive or indefined terms. He presents five groups of axioms: incidence, order, congruence, parallels and continuity.

l'Hospital, G. M. *L'analyse des infiniment petits, pour l'intelligence des lignes courbes*. Paris: Imprimérie Royale, 1696. Reproduction Paris: ACL-éditions, 1988

The first calculus textbook ever published. See Bernoulli.

IREM: Images, Imaginaires, Imaginations, Une perspective historique pour l'introduction des numbres complexes. Paris: Ellipses 1998

Historical sources on complex numbers, and experiences with these texts in the classrooom.

IREM de Basse Normandie (ed.): Une histoire des équations par les textes. 1994

A collection of sources on the solution of equations from the Babylonians to Lagrange.

IREM de Basse Normandie (ed.): La question des parallèles: une histoire de l'émergence des géométries non-euclidiennes. 1995

Texts by Euclid, Al Khayyam, Wallis, Saccheri, Gauss, Lobatchevsky.

IREM de Basse Normandie (ed.): La création du calcul des probabilités et la loi des grands nombres de Pascal à Poisson. 1995

Texts by Pascal, Huygens, Bernoulli, de Moivre, Laplace, Poisson.

Klein, F. *et al.*, *Famous Problems and other monologues*, New York: Chelsea Publishing Company, 1955

Of the four monographs brought together in this single volume, the most useful from our point of view is the translation of Klein's *Famous Problems of Elementary Geometry.* Not only do we have the presentation of the three classical problems – the duplication of the cube, the trisection of an angle and the quadrature of the circle – as well as a detailed explanation for the construction of a 17–*gon*, but also, in part II, a discussion of the transcendence of π and a very nice presentation of the countability of algebraic numbers. This last is at a level that could be used as a rich source, accessible to school mathematicians.

Lietzmann, W. Bd. 1: *Aus der Mathematik der Alten: Quellen zur Arithmetik, . . .* 1928, Bd. 2: *Aus der neueren Mathematik: Quellen zum Zahlbegriff und zur Gleichungslehre, zum Funktionsbegriff und zur Analysis* 1929, Leipzig: Teubner
A useful collection, unfortunately no longer in print.

Midonick, H. (ed.), *A treasury of mathematics*, New York: Philosophical Library, Inc., 1965.
An attractively produced volume of fifty four original sources selected to illustrate contributions which changed or altered the course of the development of mathematics. More extracts from non-European sources than in other comparable collections. Each selections is preceded by a short introductory essay.

Newman, J. R., *The world of mathematics*, London: George Allen & Unwin, 4 vols. 1960
Described as a 'small library of mathematics', this four volume collection of articles contains many examples of original mathematical writing. Here will be found, for example, Newton's letters of 1676 in which he explains the extension of the binomial theorem to fractional and negative exponents (as well as his use of $a^{1/2}$ and a^{-1}), *The Sand Reckoner* by Archimedes, Euler's original article on the seven bridges of Königsberg and Alan Turing's article 'Can a Machine Think?'

Newton, I., *The mathematical papers of Isaac Newton*, ed. D. T. Whiteside: Vol. V: *Lectures on Algebra*. Cambridge: University Press, 1972
Newton's lectures on algebra, from 1683 to 1684. A bilingual edition, Latin and English, containing a full commentary by Whiteside and facsimiles of Newton. It starts with 'First book of universal arithmetic', where it is possible to detect the author's conception of algebra. Particularly interesting is Newton's didactic approach to show the use of algebra in a mathematical problem, translating a word problem from natural everyday discourse to mathematical symbolism. Clearly expressed, the text can easily be read by mathematical beginners.

Open University, *Topics in the history of mathematics,* (General ed. John Fauvel), Milton Keynes: Open University Press, 1987
The Open University course material for the degree level unit of this title contains 17 booklets, each of which can be obtained separately. While being a teaching course, each booklet contains extracts of original material. Video materials are also available.

Pappas, T., *Mathematics appreciation*, John Bibby Books, 1988.
A source book containing ten lessons, each with photocopiable assignment pages. Historical material at the level of elementary mathematics.

Rhind mathematical papyrus: Chace, A. B. *et al*: Oberlin, Ohio: Mathematical Association of America, 1927, 1929; reprt. National Council of Teachers of Mathematics, 1978; Robins, G & Shute, C., London: British Museum Publications, 1987, 1998.
The Chace edition includes almost all of the problems from the Rhind Papyrus, with attractively presented text in hieroglyphic and hieratic writing alongside the English

translation. The British Museum publication only has some sample problems but contains attractive full colour plates of the papyrus.

Riese, Adam, *Rechenbuch*, facsimile of 1574 edition, Hanover: Th. Schäfer, 1992
This book is perhaps the most famous of the early printed arithmetics and the name Adam Riese has come into the German language to signify accurate calculation. The fact that it is in German, and in Gothic script as well, makes it difficult for the non German reader to use, but the beautiful woodcut illustrations alone recommend the book as an important stage in the change from abacus calculation to written methods.

Schneider, I. *Die Entwicklung der Wahrscheinlichkeitstheorie von den Anfängen bis 1933 : Einführungen und Texte*. Darmstadt: Wissenschaftliche Buchgesellschaft 1988
A comprehensive collection of sources from the history of probability theory, translated into German.

Smith, David E., *A source book in mathematics*, New York: Dover Publications, 1929, 1959.
A collection of 125 extracts, mostly not available in English elsewhere. The book is divided into five sections (number, algebra, geometry, probability and calculus/functions). The extracts have been chosen to illustrate significant incidents or 'discoveries'. Some of the extracts, such as Cardan on imaginary roots or the correspondence between Pascal and Fermat on the notion of probability, are capable of being used in upper secondary school mathematics.

Struik, Dirk J., *A source book in mathematics, 1200–1800*, Princeton, New Jersey: Princeton University Press, 1969, 1986
A selection of mathematical writings of authors from the Latin world who lived between the thirteenth and the end of eighteenth century. By Latin, Struik means that there are no Arabic or Oriental sources, except where much used Latin translations are available, for example in the case of Al-Khwarizmi. Struik intersperses helpful explanatory commentary on the selected texts but substantial blocks of original writing remain intact. There is a great deal of rich material here, ranging from Stevin's description of decimal notation to Euler's theory of zeros of different values.

Swetz, F. (ed.) *Learn from the masters!*, Washington, DC: Mathematical Association of America, 1995
A collection of twenty-three articles by contributors who are actively engaged in using history in the teaching of mathematics. The intention is to show how one can use history in mathematics teaching and many of the articles contain direct extracts from original material which could be used by the teacher. An excellent starting point for the interested mathematics teacher.

Swetz, F., *Capitalism & arithmetic: the new math of the fifteenth century*. La Salle, Illinois: Open Court Publishing Co., 1987
The Treviso Arithmetic of 1478 is the earliest known dated printed arithmetic book and this English translation from the Venetian dialect comes with a useful commentary. Many of the problems, for example on the rule of three or problems of inheritance, could be used directly in the mathematics classroom. Students will also benefit from seeing so many ways of setting out 'long' multiplication.

Thomas [=Bulmer-Thomas], Ivor, *Greek mathematical works*, Cambridge, Mass & London: Harvard University Press, vol. 1, 1939, 1980, vol. 2, 1941, 1993.
This valuable collection of writings is arranged roughly chronologically with the first volume dealing with the mathematics up to Euclid (*fl.* 300 BC) and the second volume taking the story on as far as Pappus of Alexandria (*fl.* 300 AD). Thomas arranges his material around themes so some later writings appear in the first volume, when giving examples of Greek writing on arithmetic, for example. The whole work is set with the Greek original alongside the English translation and helpful footnotes are used to explain the text. Among the gems for use in the mathematics classroom are: Nicomachus on figurate numbers and his description of the sieve of Eratosthenes, selections from Archimedes and early ideas of trigonometry, including Ptolemy's table of chords and Diophantus on types of equations.

Wieleitner, H. Mathematische Quellenbücher. Bd. 1: Rechnen und Algebra, 1927, Bd. 2: Geometrie und Trigonometrie, 1927, Bd. 3: Analytische und synthetische Geometrie, 1928, Bd. 4: Analysis, 1928. Berlin: Salle
A useful collection, unfortunately no longer in print.

Viète, François, *Introduction to the Analytical Art*, in: J. Klein, *Greek Mathematical Thought and the Origin of Algebra*, Cambridge: MIT Press, 1968, 313-353
Klein's study of the revival of Greek mathematics, via Arabic science, in the 13th to 16th centuries, contains an English translation (by J. Winfree Smith) of Viète's important work which marks the beginning of the use of symbolism in mathematics.