An Adaptive Spectral Least-Squares Scheme for the Burgers Equation

by

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Abstract

A least-squares spectral collocation method for the one-dimensional inviscid Burgers equation is proposed. This model problem shows the stability and high accuracy of these schemes for nonlinear hyperbolic scalar equations. Here we make use of a least-squares spectral approach which was already used in an earlier paper for discontinuous and singular perturbation problems [10] . The domain is decomposed in subintervals where continuity is enforced at the interfaces. Equal order polynomials are used on all subdomains. For the spectral collocation scheme Chebyshev polynomials are employed which allow the efficient implementation with Fast Fourier Transforms (FFTs). The collocation conditions and the interface conditions lead to an overdetermined system which can be efficiently solved by least-squares. The solution technique will only involve symmetric positive definite linear systems. The scheme exhibits exponential convergence where the exact solution is smooth. In parts of the domain where the solution contains discontinuities (shocks) the spectral solution displays a Gibbs-like behavior. Here this is overcome by some suitable exponential filtering at each time level. Here we observe that by over-collocation the results remain stable also for increasing filter parameters and also without filtering. Furthermore by an adaptive grid refinement we were able to locate the precise position of the discontinuity. Numerical simulations confirm the high accuracy of our spectral least-squares scheme.

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1 Introduction

Here we consider the nonlinear Burgers equation which plays a fundamental role in computational gasdynamics (see Laney [18]). It models the nonlinear interaction of waves with different wave numbers which may lead to the development of shocks and expansion fans. Since spectral methods (see, e.g., Canuto et al. [1], Gottlieb et al. [8] or Orszag [19]) employ global polynomials they give very accurate approximations for smooth solutions but only poor results for problems with discontinuities. In the part of the domain where the solution contains discontinuities the spectral solution displays a Gibbs-like behavior. Here we approximate the above problem by a spectral least-squares collocation scheme which was already introduced in [10] for discontinuous and singular perturbation problems. Similar methods for spectral elements were already developed by Proot and Gerritsma [7, 21, 22, 23, 24]. For nonlinear hyperbolic equations the method was presented in the paper of De Maerschalck and Gerritsma [2]. They employed an edge detection method to determine the position of the shock. Piecewise reconstruction of the numerical solution retrieves a monotone solution. Here the differential equation is collocated at the usual Chebyshev Gauss-Lobatto nodes including the boundary nodes. Hence on each subdomain we require $N+1$ conditions of collocation. Equal order Chebyshev polynomials of degree N are employed which allow the efficient implementation with Fast Fourier Transforms (FFTs). The collocation conditions together with the interface conditions lead to an overdetermined system which can be efficiently solved by least-squares. The solution technique will only involve symmetric positive definite linear systems. The scheme exhibits exponential convergence where the exact solution is smooth. In parts of the domain where the solution contains discontinuities (shocks) the spectral solution displays a Gibbs-like behavior. Here this is overcome by some suitable exponential filtering at each time level. Furthermore by an adaptive grid refinement we were able to locate the precise position of the discontinuity. Numerical simulations confirm the high accuracy of our spectral least-squares scheme. Summarizing our approach has the following advantages:

- equal order interpolation polynomials can be employed
- iterative solvers for symmetric and positive definite systems (e.g. conjugate gradient methods) can be used
- improved stability properties for hyperbolic problems (see [4, 10, 2]) and the Navier-Stokes equations (see [6, 11, 12, 24] and Jiang et al. [14, 15, 16, 17])

- filtering, over-collocation and adaptivity can easily be combined

The paper is organized as follows. In Section 2, the least-squares scheme for the inviscid Burgers equation is introduced and in section 3 we describe the spectral collocation scheme. This is followed by the methods of filtering, adaptivity and over-collocation in section 4. Finally the numerical results are discussed in section 5.

2 The Burgers Equation

We consider the one-dimensional inviscid Burgers equation

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} = 0, \quad 0 \le x \le L,\ t \ge 0 \tag{1}$$

with $L = 4$ and as initial condition a single cosine hill

$$u_0(x) = \begin{cases} \frac{1}{2}\left(1 - \cos \pi x\right) & \text{for } 0 \le x \le 2, \\ 0 & \text{for } 2 \le x \le L. \end{cases}$$

Since the equation is nonlinear, linearization of the quadratic term is required. Here we prefer the Picard iteration which can be written as

$$\frac{\partial u^l}{\partial t} + 2u^{l-1}\frac{\partial u^l}{\partial x} = 0 \ \text{ for } l = 1, 2, \ldots \tag{2}$$

The above treatment is also called nonconservative formulation. In our numerical computations we observe that for an accuracy of about 10^{-10} only a few steps (3-5) of the above Picard iteration per time level were necessary. The conservative formulation can be obtained by introducing a new function $v = u^2$ which leads to the equations

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial v}{\partial x} &= 0, \\ v - u^2 &= 0. \end{aligned}$$

Because of the bad convergence results (strong oscillations) obtained in [2] we did not follow this approach. Here we only consider the nonconservative Picard iteration (2). For the time discretization we employ standard second order backward differentation formulas (BDF). The numerical results are presented at the fixed time level $T = 2$. More interesting is the discretization in space where we propose a spectral least-squares collocation scheme which

was already introduced in [10] for discontinuous and singular perturbation problems. For this purpose the domain $\Omega = (0, L)$ is decomposed into K subdomains $\Omega_k = (x_{k-1}, x_k),\ k = 1, \ldots, K$ with

$$0 = x_0 < x_1 < x_2 < \ldots < x_{K-1} < x_K = L.$$

Now for each time level l the Picard equation (2) can be decomposed as follows: Find functions $u_k^l, k = 1, \ldots, K$ defined on $\overline{\Omega}_k$ such that

$$\begin{aligned} \frac{\partial u_k^l}{\partial t} + 2u_k^{l-1}\frac{\partial u_k^l}{\partial x} &= 0 \ \text{ in } \Omega_k,\ k = 1, \ldots, K, && (3)\\ u_{k+1}^l(x_k) - u_k^l(x_k) &= 0,\ k = 1, \ldots, K-1, && (4)\\ u_k^1(0) &= 0 && (5) \end{aligned}$$

where we start with $u_k^0 \equiv u_0$ in Ω_k at time level 0. Here we allow no jumps at the interface nodes and hence the solution is required to be continuous.

3 The Spectral Collocation Scheme

For the spectral approximation we introduce the polynomial subspace

$$\mathbb{P}_N = \{Polynomials\ of\ degree\ \leq N\}.$$

Now all unknown functions u_k^l are approximated by polynomials $u_{k,N}^l$ of the same degree N, i.e., $u_{k,N}^l \in \mathbb{P}_N, k = 1, \ldots, K$. Furthermore we have to introduce the standard Chebyshev Gauss-Lobatto collocation nodes. They are explicitly given by

$$\xi_i = -\cos\frac{i\pi}{N},\ i = 0, \ldots, N.$$

In order to define the nodes on Ω_k we have to introduce the mapping from the interval $(-1, 1)$ to Ω_k. The nodes $x_{k,i}, i = 0, \ldots, N, k = 1, \ldots, K$ are explicitly given by

$$x_{k,i} = \frac{1}{2}\left[(x_k - x_{k-1})\xi_i + x_{k-1} + x_k\right].$$

Hence we retrieve the interval left and right bounds by $x_{k,0} = x_{k-1}, x_{k,N} = x_k$. Now the collocation (or pseudo spectral) scheme determines the polynomial functions $u_{k,N}^l \in \mathbb{P}_N$ such that

$$\begin{aligned} \left(\frac{\partial u_{k,N}^l}{\partial t} + 2u_{k,N}^{l-1}\frac{\partial u_{k,N}^l}{\partial x}\right)(x_{k,i}) &= 0,\ i = 0, \ldots, N,\ k = 1, \ldots, K, && (6)\\ u_{k+1,N}^l(x_k) - u_{k,N}^l(x_k) &= 0,\ k = 1, \ldots, K-1, && (7)\\ u_{1,N}^l(0) &= 0. && (8) \end{aligned}$$

These are $K(N+1)$ conditions of collocation for the same number of unknowns. The additional K initial and interface conditions lead to an overdetermined system which can be efficiently solved by least-squares. In the following we write the spectral derivative in matrix notation. First one has to introduce the transformation matrices T from physical space to coefficient space. For the Chebyshev expansion the matrix T is given by

$$T_{i,k} \;=\; \cos(k\frac{(N-i)\pi}{N}),\;\; i,k=0,\ldots,N.$$

Further we need the differentation matrix in the Chebyshev coefficient space which is explicitly given by $\hat{D}=(\hat{d}_{i,j})\in \mathbb{R}^{N+1,N+1}$ with

$$\hat{d}_{i,j} \;=\; \left\{ \begin{array}{l} \frac{2j}{c_i},\; j=i+1,i+3,\ldots,N \\ 0\,, \qquad\qquad \text{else} \end{array} \right.$$

and

$$c_i \;=\; \left\{ \begin{array}{l} 2,\; i=0, \\ 1,\;\; \text{else.} \end{array} \right.$$

Now we are able to write the spectral derivative matrix D for the first derivative. It is explicitly given by

$$D = T\hat{D}T^{-1} \in \mathbb{R}^{N+1,N+1}.$$

The spectral operator can be efficiently evaluated by Fast Fourier Transforms (FFTs) in $O(N\log N)$ arithmetic operations. Because of the above transform the derivative operator on Ω_k is given by

$$D_k = d_k D,\;\; d_k = 2/(x_k - x_{k-1}),\;\; k=1,\ldots,K.$$

The first-order collocation system with $K(N+1)$ equations and the K initial/interface conditions lead to an overdetermined system of $K(N+1)+K$ equations for $K(N+1)$ unknowns. For the corresponding matrix

$$A \in \mathbb{R}^{K(N+2),K(N+1)}$$

the linear system $Az=r$ is solved by least-squares in the discrete L^2-norm which leads to the normal equations

$$A^tAz = A^tr.$$

Hence the linear system resulting from the normal equations becomes symmetric and positive definite which allows the use of efficient iterative solvers like (preconditioned) conjugate gradient methods.

4 Filtering and Adaptive Refinement

In order to determine from the oscillatory numerical solution the location of the discontinuity, the enhanced edge detection method developed by Gelb and Tadmor [5] was used in [2]. The idea is to expand the numerical solution in a filtered conjugate Fourier sum which converges faster than the unfiltered Fourier representation. Once the discontinuity has been detected, it is possible to reconstruct the solution in the piecewise smooth subdomains defined by the position of the edges and the boundary conditions (see [9]). In [2] the piecewise reconstruction method is applied to the least-squares spectral element method and the shock in the space-time domain is now presented by a sharp edge. It is observed that the method improves the accuracy and renders a monotone solution. Here we follow a different approach where we employ techniques of filtering and adaptive refinement. We filter the spectral approximation by some standard exponential filter given by

$$\sigma_q(k) = \exp(-\delta(k/N)^q),\ k = 0, \ldots, N \tag{9}$$

where q denotes the order of the filter. For increasing q the effect of filtering becomes smaller. The parameter δ is chosen such that $\sigma_q(N) = \hat{\epsilon}$ where $\hat{\epsilon}$ denotes the machine precision. This leads to $\delta = -\ln \hat{\epsilon}$. We apply the filter in each time step and the solution $\tilde{u}$ after filtering is given by

$$\tilde{u} = \sum_{k=0}^{N} \sigma_q(k) \hat{u}_k T_k$$

where T_k denotes the k^{th} Chebyshev polynomial and $\hat{u}_k$ the coefficients of the Chebyshev expansion. More information about filters can be obtained from the paper of Vandeven [25]. Furthermore by an adaptive grid refinement we were able to locate the precise position of the discontinuity. On each element we evaluate an a-posteriori error indicator based on the H^1-norm. In particular, we refine where the solution gradients in each time step are large, i.e., we require

$$||\frac{\partial}{\partial x} u_{k,N}|| \leq \epsilon ||u_N||_1 \tag{10}$$

everywhere on the mesh. Here $||\cdot||$ denotes the discrete L^2 norm on element k, $||\cdot||_1$ the discrete H^1 norm on $(0, L)$, and ϵ is a given discretization tolerance. This is a common refinement criteriion in cases where no alternative measure of solution errors is available. For spectral methods there exist other refinement strategies which are based on the decay of the coefficients in the polynomial expansion. A good survey on these methods can be taken from

the review article of Henderson [13]. From the discussion in [13] it is not clear that the other error indicators yield better results. Hence, for simplicity we refer to the above refinement criteria.

In our computations we start with a coarse mesh of 4 elements and polynomial degrees of order 1. In the case that the criterion is not fulfilled the corresponding element is subdivided into two elements with half size. Simultaneously, the degree of the polynomials are increased till to a maximal order of 8. Clearly, the shock is moving in time and in regions where the shock has gone through the solution becomes smooth. Here we return to a coarse mesh with low order polynomial degrees.

For spectral element methods it was observed in [3] that over-integration leads to improved results for nonlinear hyperbolic problems. In the shock region higher-order Gauss-Lobatto integration was used to approximate the integrals involved. Then the discrepancies between the various linearization methods (Picard's method or Newton linearization) were considerably reduced and under-relaxation was no longer necessary. Here we follow a similar approach for the collocation scheme where we collocate in each element at $2N+1$ points by maintaining the polynomial degree N. The nodes are given by

$$\xi_i^o = -\cos\frac{i\pi}{2N},\ i = 0,\ldots,2N$$

with corresponding transformation matrix on $\mathbb{P}_N$

$$T_{i,k}^o \quad = \quad \cos(k\frac{(2N-i)\pi}{2N}),\ i = 0,\ldots,2N, k = 0,\ldots,N.$$

The spectral derivative for the over-collocation scheme is given by

$$D^o = T^o\hat{D}T^{-1} \in \mathbb{R}^{2N+1,N+1}.$$

For the least-squares solver the normal equations still lead to a $K(N+1)\times K(N+1)$ system and hence the computational amount of work is similar to the treatment without over-collocation. But now one really has an overdetermined system which has to be solved by least-squares. Otherwise there is always the danger that the matching conditions dominate the matrix and the continuity conditions can be eliminated, thus leading to a least-squares solution technique, but not to a least-squares problem. In our numerical experiments we compare both approaches.

5 Numerical Results

We calculated the solution of the Burgers equation on the time interval $[0, T]$ with $T = 2$. We perform 1000 time iterations until the time level $T = 2$ is reached. Hence the time step is $\Delta t = 0.002$. The tolerance for the grid refinement in (10) is chosen to be $\epsilon = 0.1$. In table I we present our numerical results for different filter parameters σ_q. We calculated the shock position x_s and checked the accuracy of our spectral solution by computing the integral $I(u_N)$. The latter one is performed by Clenshaw-Curtis quadrature (see, e.g., Peyret [20]) on each element and summing up these values. By integrating the Burgers equation in the variables x, t and noticing that the boundary values of u vanish in $x = 0, L$ we derive

$$\int_0^L u(x, T)dx = \int_0^L u(x, 0)dx \equiv 1.$$

Hence the integral $I(u_N)$ over the spectral solution should approximate the value 1. In the case without over-collocation it can be observed from table I that the shock position is best located for the filter of order $q = 9$. The shock position is located in $x_s = 3.2095$ whereas in [2] it was observed to be in $x_s = 3.225$. Since in [2] only 10 elements were used our shock position is more close to the exact position $x_s = 3.209776$. In fig. 1 we plotted the spectral solutions for filter parameters $q = 9, 12, 14, \infty$. For increasing filter parameters $q = 12, 13$ the results become worse. In particular, in the case without filtering ($q = \infty$) the spectral solution has strong oscillations behind the shock position and hence is far away from the exact shock profile. In the case with over-collocation we observe from table II no significant improvement in accuracy. Further the convergence behaviour of the Picard iteration is slightly worse. The advantage now is that the numerical results remain stable also for increasing filter parameters. Also in the case without filtering ($q = \infty$) we obtain the solution plotted in figure 2. Over-collocation itself leads to a stable solution and hence filtering becomes superfluous.

6 Conclusion

A least-squares spectral collocation scheme for the Burgers equation is presented. The original domain is decomposed into several subdomains. The spectral collocation conditions and the interface conditions lead to an overdetermined system which is efficiently solved by least-squares. The solution technique only involves symmetric positive definite algebraic systems which allow the use of equal order interpolation polynomials in all subdomains. Robust iterative solvers such as the conjugate gradient method can be employed.

The method is stabilized by standard exponential filters and over-collocation. The shock position is located by techniques of adaptive refinement. The presented spectral least-squares scheme shows the ability to produce accurate solutions for nonlinear hyperbolic equations and offers great potential for more complex problems in gasdynamics, such as the Euler equations.

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q	x_s	$I(u_N)$
6	3.2012	0.9956
7	3.2006	0.9951
8	3.2023	0.9963
9	3.2095	1.0024
10	3.2083	1.0002
11	3.2084	1.0001
12	3.1728	0.9770
13	3.1765	0.9794
14	3.1923	0.9897
18	3.2007	0.9955
∞	3.3115	1.0682

Table I. Results without over-collocation.

q	x_s	$I(u_N)$
6	3.2002	0.9949
7	3.2065	0.9985
8	3.2006	0.9955
9	3.2022	0.9957
10	3.2072	0.9994
11	3.2072	0.9995
12	3.2072	0.9995
13	3.2073	0.9995
14	3.2020	0.9960
18	3.2016	0.9957
∞	3.2019	0.9959

Table II. Results with over-collocation.

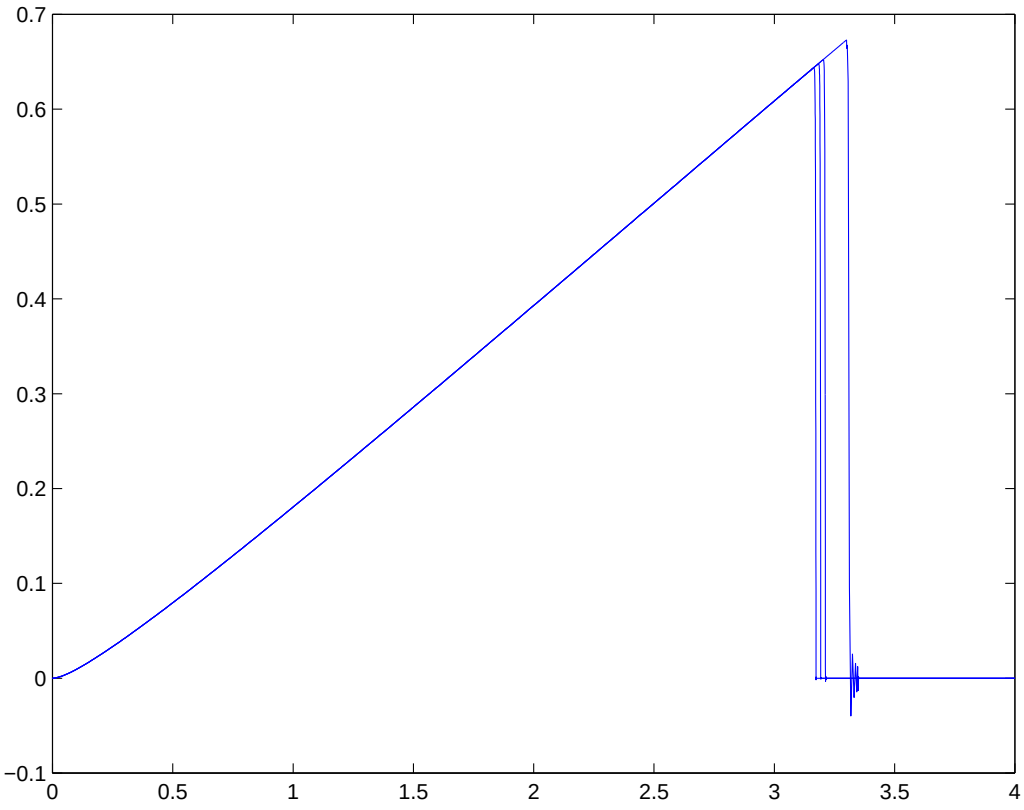

Figure 1: Shock profile for $q = 9, 12, 14, \infty$ without over-collocation.

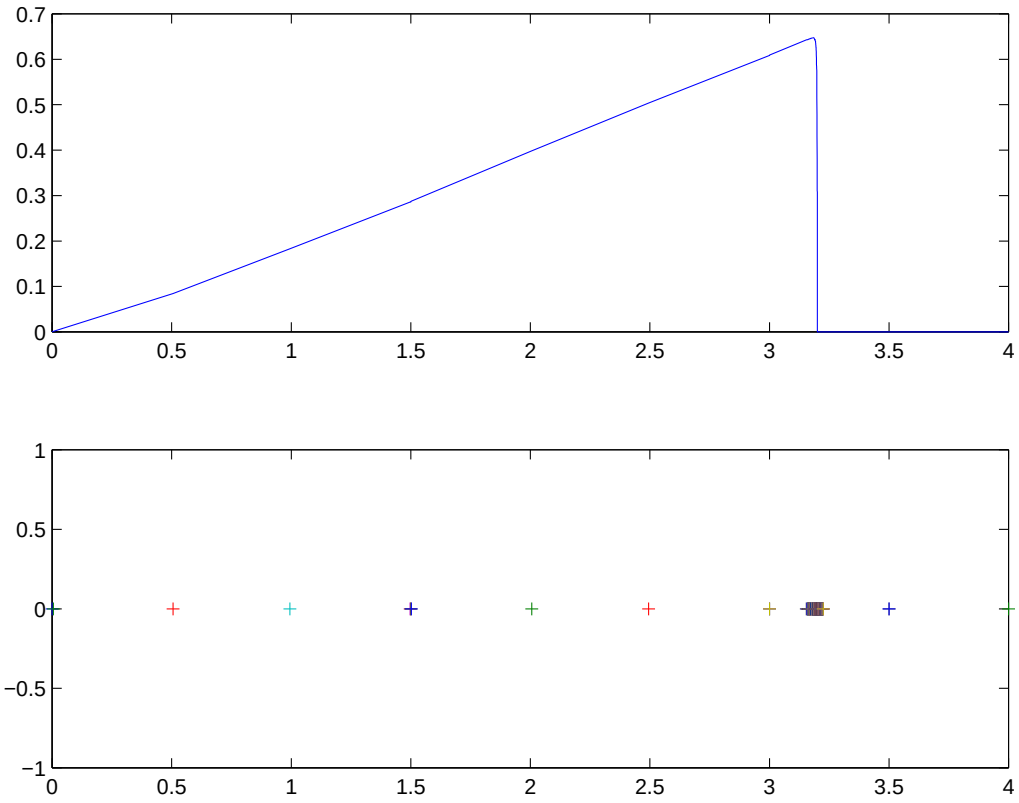

Figure 2: Shock profile and final (adapted) grid.

References

[1] Canuto, C., Hussaini, M.Y., Quarteroni, A., and Zang, T.A.: Spectral Methods in Fluid Dynamics, Springer Series in Computational Physics, Springer-Verlag, Berlin-Heidelberg-New York, 1989

[2] De Maerschalck, B., Gerritsma, M.I.: The use of Chebyshev polynomials in the space-time least-squares spectral element method. Numerical Algorithms **38**, 173-196(2005)

[3] De Maerschalck, B., Gerritsma, M.I.: Higher-order Gauss-Lobatto integration for non-linear hyperbolic equations. Journal of Scientific Computing **27**, Nos. 1-3, 201-214(2006)

[4] Eisen, H., Heinrichs, W.: A new method of stabilization for singular perturbation problems with spectral methods. SIAM J. Numer. Anal. **29**, 107-122(1992)

[5] Gelb, A., Tadmor, E.: Enhanced spectral viscosity approximations for conservation laws. Appl. Numer. Math. **33**, 3-21(2000)

[6] Gerritsma, M.I., Phillips, T.N.: Discontinuous spectral element approximation for the velocity-pressure-stress formulation of the Stokes problem. Internat. J. Numer. Methods Engrg. **43**, 1401-1419(1998)

[7] Gerritsma, M.I., Proot, M.J.: Analysis of a discontinuous least-squares spectral element method. J. Scient. Computing **17**, 297-306(2002)

[8] Gottlieb, D. and Orszag, S.A.: Numerical Analysis of Spectral Methods: Theory and Applications. CBMS-NSF Regional Conference Series in Applied Mathematics No. 26, SIAM, Philadelphia, 1977

[9] Gottlieb, D. and Shu, C.W.: On the Gibbs phenomenon and its resolution. SIAM Rev. **39**, 644-668(1997)

[10] Heinrichs, W.: Least-squares spectral collocation for discontinuous and singular perturbation problems, J. Comput. Appl. Math., **157**, 329-345(2003)

[11] Heinrichs, W.: Least-squares spectral collocation for the Navier-Stokes equations. J. Scient. Comput. **21**, No. 1, 81-90(2004)

[12] Heinrichs, W.: An adaptive least-squares spectral collocation method with triangular elements for the incompressible Navier-Stokes equations. To appear in J. Engrg. Math., 2007.

[13] Henderson, R.D.: Adaptive spectral element methods for turbulence and transition. In T.J. Barth and H. Deconinck,eds., *High Order Methods for Computational Physics*, p. 225-324, Springer, Berlin, 1999

[14] Jiang, B.-N.: A least-squares finite element method for incompressible Navier-Stokes problems. Internat. J. Numer. Methods Fluids **14**, 843-859(1992)

[15] Jiang, B.-N.: On the least-squares method. Comput. Methods Appl. Mech. Engrg. **152**, 239-257(1998)

[16] Jiang, B.-N., Chang, C.L.: Least-squares finite elements for the Stokes problem. Comput. Methods Appl. Mech. Engrg. **78**, 297-311(1990)

[17] Jiang, B.-N., Povinelli, L.: Least-squares finite element method for fluid dynamics. Comput. Methods Appl. Mech. Engrg. **81**, 13-37(1990)

[18] Laney, C.B.: Computational Gasdynamics. Cambridge University Press, Cambridge, 1998

[19] Orszag, S.A.: Spectral methods for problems in complex geometries. J. Comp. Phys. **37**, 70-92(1980)

[20] Peyret, R.: Spectral Methods for Incompressible Viscous Flow. Applied Mathematical Sciences 148, Springer 2001

[21] Proot, M.J.: The least-squares spectral element method. PhD thesis, Delft University of Technology, 2003

[22] Proot, M.J., Gerritsma, M.I..: A least-squares spectral element formulation for the Stokes problem. J. Scient. Computing **17**, 285-296(2002)

[23] Proot, M.J., Gerritsma, M.I..: Least-squares spectral elements applied to the Stokes problem. J. Comput. Phys. **181**, 454-477(2002)

[24] Proot, M.J., Gerritsma, M.I..: Application of the least-squares spectral element method using Chebyshev polynomials to solve the incompressible Navier-Stokes equations. Numerical Algorithms **38**, 155-172(2005)

[25] Vandeven, H.: Family of spectral filters for discontinuous problems. J. Sci. Comput. **6**, 159-192(1991)