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On Closed and Exact Grad grad- and div Div-Complexes,
Corresponding Compact Embeddings for Symmetric Rotations,
and a Related Decomposition Result for Biharmonic Problems in 3D

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ABSTRACT. It is shown that the first biharmonic boundary value problem on a topologically simple domain in 3D is equivalent to three (consecutively to solve) second-order problems. This decomposition result is based on a Helmholtz-like decomposition of an involved non-standard Sobolev space of tensor fields and a proper characterization of the operator $\operatorname{div} \operatorname{Div}$ acting on this space. Similar results for biharmonic problems in 2D and their impact on the construction and analysis of finite element methods have been recently published in [14]. The discussion of the kernel of $\operatorname{div} \operatorname{Div}$ leads to (de Rham-like) closed and exact Hilbert complexes, the $\operatorname{div} \operatorname{Div}$ -complex and its adjoint the Grad grad-complex, involving spaces of trace-free and symmetric tensor fields. For these tensor fields we show Helmholtz type decompositions and, most importantly, new compact embedding results. Almost all our results hold and are formulated for general bounded strong Lipschitz domains of arbitrary topology. There is no reasonable doubt that our results extend to strong Lipschitz domains in \mathbb{R}^N .

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1. INTRODUCTION

In [14] it was shown that the fourth-order biharmonic boundary value problem

$$(1.1) \quad \Delta^2 u = f \quad \text{in } \Omega, \quad u = \partial_n u = 0 \quad \text{on } \Gamma,$$

where Ω is a bounded and simply connected domain in \mathbb{R}^2 with a (strong) Lipschitz boundaryⁱ Γ , can be decomposed into three second-order problems. The first problem is a Poisson problem for an auxiliary

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ⁱLocally Γ is a graph of a Lipschitz function.

scalar field p

$$\Delta p = f \quad \text{in } \Omega, \quad p = 0 \quad \text{on } \Gamma,$$

the second problem is a linear elasticity problem for an auxiliary vector field E

$$-\text{Div } \varepsilon(E) = -\text{Div}(\text{sym Grad } E) = \text{grad } p \quad \text{in } \Omega, \quad (\text{sym Grad } E) n = -p n = 0 \quad \text{on } \Gamma,$$

i.e.,

$$\text{Div}(\text{sym Grad } E + p \mathbf{I}) = 0 \quad \text{in } \Omega, \quad (\text{sym Grad } E + p \mathbf{I}) n = 0 \quad \text{on } \Gamma,$$

and, finally, the third problem is a Poisson problem for the original scalar field u

$$\Delta u = 2p + \text{div } E \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Gamma.$$

Here f is a given right-hand side, Δ , n , and ∂_n denote the Laplace operator, the outward normal vector to the boundary, and the derivative in this direction, respectively. The differential operators grad , div , and (for later use) rot denote the gradient of a scalar field and the divergence and rotation of a vector field, the corresponding capitalized differential operators Grad , Div , and Rot denote the row-wise application of grad to a vector field, div and rot to a tensor field. The prefix sym is used for the symmetric part of a matrix, for the skew-symmetric part we use the prefix skw . This decomposition is of triangular structure, i.e., the first problem is a well-posed second-order problem in p , the second problem is a well-posed second-order problem in E for given p , and the third problem is a well-posed second-order problem in u for given p and E . This allows to solve them consecutely analytically or numerically by means of techniques for second-order problems.

This is - in the first place - a new analytic result for fourth-order problems. But it also has interesting implications for discretization methods applied to (1.1). It allows to re-interpret known finite element methods as well as to construct new discretization methods for (1.1) by exploiting the decomposable structure of the problem. In particular, it was shown in [14] that the Hellan-Herrmann-Johnson mixed method (see [8, 9, 13]) for (1.1) allows a similar decomposition as the continuous problem, which leads to a new and simpler assembling procedure for the discretization matrix and to more efficient solution techniques for the discretized problem. Moreover, a novel conforming variant of the Hellan-Herrmann-Johnson mixed method was found based on the decomposition.

The aim of this paper is to derive a similar decomposition result for biharmonic problems on bounded and topologically simple three-dimensional domains Ω with a (strong) Lipschitz boundary Γ . For this we proceed as in [14] and reformulate (1.1) using $\Delta^2 = \text{div Div Grad grad}$ as a mixed problem by introducing the (negative) Hessian of the original scalar field u as an auxiliary tensor field

$$(1.2) \quad \mathbf{M} = -\text{Grad grad } u.$$

Then the biharmonic differential equation reads

$$(1.3) \quad -\text{div Div } \mathbf{M} = f \quad \text{in } \Omega.$$

For an appropriate non-standard Sobolev space for \mathbf{M} it can be shown that the mixed problem in \mathbf{M} and u is well-posed. Then the decomposition of the biharmonic problem follows from a Helmholtz-like decomposition of this non-standard Sobolev space. This part of the analysis carries over completely from the two-dimensional case to the three-dimensional case and is shortly recalled in Section 6. To efficiently utilize this Helmholtz-like decomposition for the decomposition of the biharmonic problem an appropriate characterization of the kernel of the operator div Div is required, which is well understood for the two-dimensional case, see, e.g., [3, 11, 14]. Its extension to the three-dimensional case is the central topic of this paper. We expect - as in the two-dimensional case - similar interesting implications for the study of appropriate discretization methods for four-order problems in the three-dimensional case.

The paper is organized as follows. After some preliminaries in Section 2 and introducing our general functional analytical setting, we will discuss the relevant unbounded linear operators, show closed and exact Hilbert complex properties, and present a suitable representation of the kernel of div Div for the three-dimensional case in Section 3. We also prove Helmholtz type decompositions and two new and crucial compact embeddings. Based on the representation of the kernel of div Div a decomposition of the three-dimensional biharmonic problem into three (consecutely to solve) second-order problems will be derived in Section 6. The proofs of some useful identities are presented in an appendix.

2. PRELIMINARIES

We start by recalling some basic concepts and abstract results from functional analysis concerning Helmholtz decompositions, closed ranges, Friedrichs/Poincaré type estimates, and bounded or even compact inverse operators. Since we will need both the Banach space setting for bounded linear operators as well as the Hilbert space setting for (possibly unbounded) closed and densely defined linear operators, we will shortly recall these two variants.

2.1. Functional Analysis Toolbox. Let X and Y be real Banach spaces. With $BL(X, Y)$ we introduce the space of bounded linear operators mapping X to Y . The dual spaces of X and Y are denoted by $X' := BL(X, \mathbb{R})$ and $Y' := BL(Y, \mathbb{R})$. For a given $A \in BL(X, Y)$ we write $A' \in BL(Y', X')$ for its Banach space dual or adjoint operator defined by $A' y'(x) := y'(Ax)$ for all $y' \in Y'$ and all $x \in X$. Norms and duality in X resp. X' are denoted by $|\cdot|_X$, $|\cdot|_{X'}$, and $\langle \cdot, \cdot \rangle_{X'}$.

Suppose H_1 and H_2 are Hilbert spaces. For a (possibly unbounded) densely defined linear operator $A : D(A) \subset H_1 \rightarrow H_2$ we recall that its Hilbert space dual or adjoint $A^* : D(A^*) \subset H_2 \rightarrow H_1$ can be defined via its Banach space adjoint A' and the Riesz isomorphisms of H_1 and H_2 or directly as follows: $y \in D(A^*)$ if and only if $y \in H_2$ and

$$\exists f \in H_1 \quad \forall x \in D(A) \quad \langle Ax, y \rangle_{H_2} = \langle x, f \rangle_{H_1}.$$

In this case we define $A^* y := f$. We note that A^* has maximal domain of definition and that A^* is characterized by

$$\forall x \in D(A) \quad \forall y \in D(A^*) \quad \langle Ax, y \rangle_{H_2} = \langle x, A^* y \rangle_{H_1}.$$

Here $\langle \cdot, \cdot \rangle_H$ denotes the scalar product in a Hilbert space H and D is used for the domain of definition of a linear operator. Additionally, we introduce the notation N for the kernel or null space and R for the range of a linear operator.

Let $A : D(A) \subset H_1 \rightarrow H_2$ be a (possibly unbounded) closed and densely defined linear operator on two Hilbert spaces H_1 and H_2 with adjoint $A^* : D(A^*) \subset H_2 \rightarrow H_1$. Note $(A^*)^* = \overline{A} = A$, i.e., (A, A^*) is a dual pair. By the projection theorem the Helmholtz type decompositions

$$(2.1) \quad H_1 = N(A) \oplus_{H_1} \overline{R(A^*)}, \quad H_2 = N(A^*) \oplus_{H_2} \overline{R(A)}$$

hold and we can define the reduced operators

$$\begin{aligned} \mathcal{A} &:= A|_{\overline{R(A^*)}} : D(\mathcal{A}) \subset \overline{R(A^*)} \rightarrow \overline{R(A)}, & D(\mathcal{A}) &:= D(A) \cap \overline{R(A^*)} = D(A) \cap N(A)^{\perp_{H_1}}, \\ \mathcal{A}^* &:= A^*|_{\overline{R(A)}} : D(\mathcal{A}^*) \subset \overline{R(A)} \rightarrow \overline{R(A^*)}, & D(\mathcal{A}^*) &:= D(A^*) \cap \overline{R(A)} = D(A^*) \cap N(A^*)^{\perp_{H_2}}, \end{aligned}$$

which are also closed and densely defined linear operators. We note that \mathcal{A} and \mathcal{A}^* are indeed adjoint to each other, i.e., $(\mathcal{A}, \mathcal{A}^*)$ is a dual pair as well. Now the inverse operators

$$\mathcal{A}^{-1} : R(\mathcal{A}) \rightarrow D(\mathcal{A}), \quad (\mathcal{A}^*)^{-1} : R(\mathcal{A}^*) \rightarrow D(\mathcal{A}^*)$$

exist and they are bijective, since \mathcal{A} and \mathcal{A}^* are injective by definition. Furthermore, by (2.1) we have the refined Helmholtz type decompositions

$$(2.2) \quad D(A) = N(A) \oplus_{H_1} D(\mathcal{A}), \quad D(A^*) = N(A^*) \oplus_{H_2} D(\mathcal{A}^*)$$

and thus we obtain for the ranges

$$(2.3) \quad R(A) = R(\mathcal{A}), \quad R(A^*) = R(\mathcal{A}^*).$$

By the closed range theorem or the bounded inverse theorem we get immediately the following.

Lemma 2.1. *The following assertions are equivalent:*

- (i) $\exists c_A \in (0, \infty) \quad \forall x \in D(\mathcal{A}) \quad |x|_{H_1} \leq c_A |Ax|_{H_2}$
- (i*) $\exists c_{A^*} \in (0, \infty) \quad \forall y \in D(\mathcal{A}^*) \quad |y|_{H_2} \leq c_{A^*} |A^*y|_{H_1}$
- (ii) $R(A) = R(\mathcal{A})$ is closed in H_2 .
- (ii*) $R(A^*) = R(\mathcal{A}^*)$ is closed in H_1 .
- (iii) $\mathcal{A}^{-1} : R(\mathcal{A}) \rightarrow D(\mathcal{A})$ is continuous and bijective with norm bounded by $(1 + c_A^2)^{1/2}$.
- (iii*) $(\mathcal{A}^*)^{-1} : R(\mathcal{A}^*) \rightarrow D(\mathcal{A}^*)$ is continuous and bijective with norm bounded by $(1 + c_{A^*}^2)^{1/2}$.

In case that one of the assertions of Lemma 2.1 is true, e.g., $R(A)$ is closed, we have

$$\begin{aligned} \mathbf{H}_1 &= N(A) \oplus_{\mathbf{H}_1} R(A^*), & \mathbf{H}_2 &= N(A^*) \oplus_{\mathbf{H}_2} R(A), \\ D(A) &= N(A) \oplus_{\mathbf{H}_1} D(\mathcal{A}), & D(A^*) &= N(A^*) \oplus_{\mathbf{H}_2} D(\mathcal{A}^*), \\ D(\mathcal{A}) &= D(A) \cap R(A^*), & D(\mathcal{A}^*) &= D(A^*) \cap R(A). \end{aligned}$$

For the “best” constants c_A, c_{A^*} we have the following lemma.

Lemma 2.2. *The Rayleigh quotients*

$$\frac{1}{c_A} := \inf_{0 \neq x \in D(\mathcal{A})} \frac{|Ax|_{\mathbf{H}_2}}{|x|_{\mathbf{H}_1}} = \inf_{0 \neq y \in D(\mathcal{A}^*)} \frac{|A^*y|_{\mathbf{H}_1}}{|y|_{\mathbf{H}_2}} =: \frac{1}{c_{A^*}}$$

coincide, i.e., $c_A = c_{A^*}$, if either c_A or c_{A^*} exists in $(0, \infty)$. Otherwise they also coincide, i.e., it holds $c_A = c_{A^*} = \infty$.

From now on and throughout this paper, we always pick the best possible constants in the various Friedrichs/Poincaré type estimates.

Proof. Let, e.g., c_{A^*} exist in $(0, \infty)$. By Lemma 2.1 also c_A exists in $(0, \infty)$, $R(A^*)$ is closed in \mathbf{H}_1 , and for any $0 \neq x \in D(\mathcal{A}) = D(A) \cap R(A^*)$ there exists $y \in D(\mathcal{A}^*)$ with $x = A^*y$. More precisely, $y := (\mathcal{A}^*)^{-1}x \in D(\mathcal{A}^*)$ is uniquely determined and we have $|y|_{\mathbf{H}_2} \leq c_{A^*}|A^*y|_{\mathbf{H}_1}$. But then

$$|x|_{\mathbf{H}_1}^2 = \langle x, A^*y \rangle_{\mathbf{H}_1} = \langle Ax, y \rangle_{\mathbf{H}_2} \leq |Ax|_{\mathbf{H}_2}|y|_{\mathbf{H}_2} \leq c_{A^*}|Ax|_{\mathbf{H}_2}|A^*y|_{\mathbf{H}_1},$$

yielding $|x|_{\mathbf{H}_1} \leq c_{A^*}|Ax|_{\mathbf{H}_2}$. Therefore, $0 < c_A \leq c_{A^*}$ and by symmetry we obtain $c_A = c_{A^*}$. \square

A standard indirect argument shows the following.

Lemma 2.3. *Let $D(\mathcal{A}) = D(A) \cap \overline{R(A^*)} \hookrightarrow \mathbf{H}_1$ be compact. Then the assertions of Lemma 2.1 hold. Moreover, the inverse operators*

$$\mathcal{A}^{-1} : R(A) \rightarrow R(A^*), \quad (\mathcal{A}^*)^{-1} : R(A^*) \rightarrow R(A)$$

are compact with norms $|\mathcal{A}^{-1}|_{R(A), R(A^*)} = |(\mathcal{A}^*)^{-1}|_{R(A^*), R(A)} = c_A$.

Proof. If, e.g., Lemma 2.1 (i) was wrong, there exists a sequence $(x_n) \subset D(\mathcal{A})$ with $|x_n|_{\mathbf{H}_1} = 1$ and $Ax_n \rightarrow 0$. As (x_n) is bounded in $D(\mathcal{A})$ we can extract a subsequence, again denoted by (x_n) , with $x_n \rightarrow x \in \mathbf{H}_1$ in \mathbf{H}_1 . Since A is closed, we have $x \in D(A)$ and $Ax = 0$. Hence $x \in N(A)$. On the other hand, $(x_n) \subset D(\mathcal{A}) \subset \overline{R(A^*)} = N(A)^\perp$ implies $x \in N(A)^\perp$. Thus $x = 0$, in contradiction to $1 = |x_n|_{\mathbf{H}_1} \rightarrow |x|_{\mathbf{H}_1} = 0$. \square

Lemma 2.4. *$D(\mathcal{A}) \hookrightarrow \mathbf{H}_1$ is compact, if and only if $D(\mathcal{A}^*) \hookrightarrow \mathbf{H}_2$ is compact.*

Proof. By symmetry it is enough to show one direction. Let $D(\mathcal{A}) \hookrightarrow \mathbf{H}_1$ be compact and let $(y_n) \subset D(\mathcal{A}^*)$ be a bounded sequence. By Lemma 2.1 and Lemma 2.3 we get $y_n = Ax_n$ with $(x_n) \subset D(\mathcal{A})$, which is bounded in $D(\mathcal{A})$ by Lemma 2.1 (i). Hence we may extract a subsequence, again denoted by (x_n) converging in \mathbf{H}_1 . Thus with $x_{n,m} := x_n - x_m$ and $y_{n,m} := y_n - y_m$ we see

$$|y_{n,m}|_{\mathbf{H}_2}^2 = \langle y_{n,m}, Ax_{n,m} \rangle_{\mathbf{H}_2} = \langle A^*(y_{n,m}), x_{n,m} \rangle_{\mathbf{H}_1} \leq c|x_{n,m}|_{\mathbf{H}_1},$$

and hence (y_n) is a Cauchy sequence in \mathbf{H}_2 . \square

Now, let $A_0 : D(A_0) \subset \mathbf{H}_0 \rightarrow \mathbf{H}_1$ and $A_1 : D(A_1) \subset \mathbf{H}_1 \rightarrow \mathbf{H}_2$ be (possibly unbounded) closed and densely defined linear operators on three Hilbert spaces $\mathbf{H}_0, \mathbf{H}_1$ and \mathbf{H}_2 with adjoints $A_0^* : D(A_0^*) \subset \mathbf{H}_1 \rightarrow \mathbf{H}_0$ and $A_1^* : D(A_1^*) \subset \mathbf{H}_2 \rightarrow \mathbf{H}_1$ as well as reduced operators $\mathcal{A}_0, \mathcal{A}_0^*$, and $\mathcal{A}_1, \mathcal{A}_1^*$. Furthermore, we assume the sequence property of A_0 and A_1 , that is, $A_1 A_0 = 0$, i.e.,

$$(2.4) \quad R(A_0) \subset N(A_1).$$

Then also $A_0^* A_1^* = 0$, i.e., $R(A_1^*) \subset N(A_0^*)$. The Helmholtz type decompositions of (2.1) for $A = A_1$ and $A = A_0$ read

$$(2.5) \quad \mathbf{H}_1 = N(A_1) \oplus_{\mathbf{H}_1} \overline{R(A_1^*)}, \quad \mathbf{H}_1 = N(A_0^*) \oplus_{\mathbf{H}_1} \overline{R(A_0)}$$

and by (2.4) we see

$$(2.6) \quad N(A_0^*) = N_{0,1} \oplus_{H_1} \overline{R(A_1^*)}, \quad N(A_1) = N_{0,1} \oplus_{H_1} \overline{R(A_0)}, \quad N_{0,1} := N(A_1) \cap N(A_0^*)$$

yielding the refined Helmholtz type decomposition

$$(2.7) \quad H_1 = \overline{R(A_0)} \oplus_{H_1} N_{0,1} \oplus_{H_1} \overline{R(A_1^*)}.$$

The previous results of this section imply immediately the following.

Lemma 2.5. *Let A_0, A_1 be as introduced before with $A_1 A_0 = 0$, i.e., (2.4). Moreover, let $R(A_0)$ and $R(A_1)$ be closed. Then, the assertions of Lemma 2.1 and Lemma 2.2 hold for A_0 and A_1 . Moreover, the refined Helmholtz type decompositions*

$$\begin{aligned} H_1 &= R(A_0) \oplus_{H_1} N_{0,1} \oplus_{H_1} R(A_1^*), & N_{0,1} &= N(A_1) \cap N(A_0^*), \\ N(A_1) &= R(A_0) \oplus_{H_1} N_{0,1}, & N(A_0^*) &= N_{0,1} \oplus_{H_1} R(A_1^*), \\ R(A_0) &= R(A_0) = N(A_1) \ominus_{H_1} N_{0,1}, & R(A_1^*) &= R(A_1^*) = N(A_0^*) \ominus_{H_1} N_{0,1}, \\ D(A_1) &= R(A_0) \oplus_{H_1} N_{0,1} \oplus_{H_1} D(A_1), & D(A_0^*) &= D(A_0^*) \oplus_{H_1} N_{0,1} \oplus_{H_1} R(A_1^*), \\ D(A_1) \cap D(A_0^*) &= D(A_0^*) \oplus_{H_1} N_{0,1} \oplus_{H_1} D(A_1) \end{aligned}$$

hold. Especially, $R(A_0)$, $R(A_0^*)$, $R(A_1)$, and $R(A_1^*)$ are closed, the respective inverse operators are continuous and there exist positive constants c_{A_0} , c_{A_1} , such that the Friedrichs/Poincaré type estimates

$$\begin{aligned} \forall x \in D(A_0) & \quad |x|_{H_0} \leq c_{A_0} |A_0 x|_{H_1}, & \forall y \in D(A_1) & \quad |y|_{H_1} \leq c_{A_1} |A_1 y|_{H_2}, \\ \forall y \in D(A_0^*) & \quad |y|_{H_1} \leq c_{A_0} |A_0^* y|_{H_0}, & \forall z \in D(A_1^*) & \quad |z|_{H_2} \leq c_{A_1} |A_1^* z|_{H_1} \end{aligned}$$

hold.

Remark 2.6. *Note that $R(A_0)$ resp. $R(A_1)$ is closed, if e.g. $D(A_0) \hookrightarrow H_0$ resp. $D(A_1) \hookrightarrow H_1$ is compact.*

Remark 2.7. *Observe $D(A_1) = D(A_1) \cap \overline{R(A_1^*)} \subset D(A_1) \cap N(A_0^*) \subset D(A_1) \cap D(A_0^*)$. Utilizing the Helmholtz type decompositions of Lemma 2.5 we can immediately show: The embeddings $D(A_0) \hookrightarrow H_0$, $D(A_1) \hookrightarrow H_1$, and $N_{0,1} \hookrightarrow H_1$ are compact, if and only if the embedding $D(A_1) \cap D(A_0^*) \hookrightarrow H_1$ is compact. In this case $N_{0,1}$ has finite dimension.*

Remark 2.8. *The assumptions in Lemma 2.5 on A_0 and A_1 are equivalent to the assumption that*

$$D(A_0) \subset H_0 \xrightarrow{A_0} D(A_1) \subset H_1 \xrightarrow{A_1} H_2$$

is a closed Hilbert complex, meaning that the ranges are closed. As a result of the previous lemmas, the adjoint complex

$$H_0 \xleftarrow{A_0^*} D(A_0^*) \subset H_1 \xleftarrow{A_1^*} D(A_1^*) \subset H_2.$$

is a closed Hilbert complex as well.

A special situation is the following.

Lemma 2.9. *Let A_0, A_1 be as introduced before with $R(A_0) = N(A_1)$ and $R(A_1)$ closed in H_2 . Then $R(A_0^*)$ is closed in H_0 and the simplified Helmholtz type decompositions*

$$\begin{aligned} H_1 &= R(A_0) \oplus_{H_1} R(A_1^*), & N_{0,1} &= \{0\}, \\ N(A_1) &= R(A_0) = R(A_0), & N(A_0^*) &= R(A_1^*) = R(A_1^*), \\ D(A_1) &= R(A_0) \oplus_{H_1} D(A_1), & D(A_0^*) &= D(A_0^*) \oplus_{H_1} R(A_1^*), \\ D(A_1) \cap D(A_0^*) &= D(A_0^*) \oplus_{H_1} D(A_1) \end{aligned}$$

are valid. Moreover, the respective inverse operators are continuous and the corresponding Friedrichs/Poincaré type estimates hold.

Remark 2.10. Note that $R(A_1^*) = N(A_0^*)$ and $R(A_0^*)$ closed are equivalent assumptions for Lemma 2.9 to hold.

Lemma 2.11. Let A_0, A_1 be as introduced before with the sequence property (2.4), i.e., $R(A_0) \subset N(A_1)$. If the embedding $D(A_1) \cap D(A_0^*) \hookrightarrow H_1$ is compact and $N_{0,1} = \{0\}$, then the assumptions of Lemma 2.9 are satisfied.

Remark 2.12. The assumptions in Lemma 2.9 on A_0 and A_1 are equivalent to the assumption that

$$D(A_0) \subset H_0 \xrightarrow{A_0} D(A_1) \subset H_1 \xrightarrow{A_1} H_2$$

is a closed and exact Hilbert complex. By Lemma 2.9 the adjoint complex

$$H_0 \xleftarrow{A_0^*} D(A_0^*) \subset H_1 \xleftarrow{A_1^*} D(A_1^*) \subset H_2.$$

is a closed and exact Hilbert complex as well.

Parts of Lemma 2.9 hold also in the Banach space setting. As a direct consequence of the closed range theorem and the bounded inverse theorem the following abstract result holds.

Lemma 2.13. Let X_0, X_1, X_2 be Banach spaces and suppose $A_0 \in BL(X_0, X_1)$, $A_1 \in BL(X_1, X_2)$ with $R(A_0) = N(A_1)$ and that $R(A_1)$ is closed in X_2 . Then $R(A_0')$ is closed in X_0' and $R(A_1') = N(A_0')$. Moreover, $(A_1')^{-1} \in BL(R(A_1'), R(A_1)')$.

Note that in the latter context we consider the operators

$$A_1 : X_1 \longrightarrow R(A_1), \quad A_1' : R(A_1)' \longrightarrow R(A_1') \quad (A_1')^{-1} : R(A_1') \longrightarrow R(A_1)',$$

with $N(A_1') = R(A_1)^\circ = \{0\}$.

Remark 2.14. The conditions on A_0 and A_1 in Lemma 2.13 are identical to the assumption that

$$X_0 \xrightarrow{A_0} X_1 \xrightarrow{A_1} X_2$$

is a closed and exact complex of Banach spaces. The consequences of Lemma 2.13 can be rephrased as follows. The adjoint complex of Banach spaces

$$X_0' \xleftarrow{A_0'} X_1' \xleftarrow{A_1'} X_2'$$

is closed and exact as well.

We refer to [1] for a presentation of some results of this section from a numerical analysis perspective.

Remark 2.15. $(A_1')^{-1} \in BL(R(A_1'), R(A_1)')$ is equivalent to

$$(2.8) \quad \exists c_{A_1'} > 0 \quad \forall x \in R(A_1') \quad |(A_1')^{-1}x|_{R(A_1)'} \leq c_{A_1'} |x|_{X_1'}.$$

We have $A_1'(A_1')^{-1} = \text{id}$ on $R(A_1')$. Moreover, $(A_1')^{-1}A_1' = \text{id}$ holds on $R(A_1)'$ since for $y \in R(A_1)'$ and $x := A_1'y$ we get $y - (A_1')^{-1}x \in N(A_1') = \{0\}$ and hence $y = (A_1')^{-1}x$ as A_1' is injective. Therefore (2.8) is equivalent to

$$(2.9) \quad \exists c_{A_1'} > 0 \quad \forall y \in R(A_1) \quad |y|_{R(A_1)'} \leq c_{A_1'} |A_1'y|_{X_1'}$$

with the same (best) constant $c_{A_1'}$. Moreover, for $y \in X_2'$

$$|A_1'y|_{X_1'} = \sup_{0 \neq x \in X_1} \frac{\langle A_1'y, x \rangle_{X_1'}}{|x|_{X_1}} = \sup_{0 \neq x \in X_1} \frac{\langle y, A_1x \rangle_{X_2'}}{|x|_{X_1}},$$

and hence for $y \in R(A_1)'$

$$|A_1'y|_{X_1'} = \sup_{0 \neq x \in X_1} \frac{\langle y, A_1x \rangle_{R(A_1)'}}{|x|_{X_1}}.$$

Choosing the best constant $c_{A_1'}$, (2.9) is equivalent to the general inf-sup-condition

$$(2.10) \quad 0 < \frac{1}{c_{A_1'}} = \inf_{0 \neq y \in R(A_1)'} \frac{|A_1'y|_{X_1'}}{|y|_{R(A_1)'}} = \inf_{0 \neq y \in R(A_1)'} \sup_{0 \neq x \in X_1} \frac{\langle y, A_1x \rangle_{R(A_1)'}}{|y|_{R(A_1)'} |x|_{X_1}}.$$

In the special case that $X_2 = H_2$ is a Hilbert space and that H_2' is identified with H_2 and hence $R(A_1)'$ with $R(A_1)$ as a closed subspace of H_2 , we obtain the following inf-sup-condition

$$(2.11) \quad 0 < \frac{1}{c_{A_1'}} = \inf_{0 \neq y \in R(A_1)} \frac{|A_1' y|_{X_1'}}{|y|_{H_2}} = \inf_{0 \neq y \in R(A_1)} \sup_{0 \neq x \in X_1} \frac{\langle y, A_1 x \rangle_{H_2}}{|y|_{H_2} |x|_{X_1}}.$$

2.2. Sobolev Spaces. Next we introduce our notations for several classes of Sobolev spaces on a bounded domain $\Omega \subset \mathbb{R}^3$. Let $m \in \mathbb{N}_0$. We denote by $L^2(\Omega)$ and $H^m(\Omega)$ the standard Lebesgue and Sobolev spaces and write $H^0(\Omega) = L^2(\Omega)$. Our notation of spaces will not indicate whether the elements are scalar functions or vector fields. For the rotation and divergence we define the Sobolev spaces

$$R(\Omega) := \{E \in L^2(\Omega) : \text{rot } E \in L^2(\Omega)\}, \quad D(\Omega) := \{E \in L^2(\Omega) : \text{div } E \in L^2(\Omega)\}$$

with the respective graph norms, where rot and div have to be understood in the distributional or weak sense. We introduce spaces with boundary conditions in the weak sense in the natural way by

$$\overset{\circ}{H}^m(\Omega) := \overline{C^\infty(\Omega)}^{H^m(\Omega)}, \quad \overset{\circ}{R}(\Omega) := \overline{C^\infty(\Omega)}^{R(\Omega)}, \quad \overset{\circ}{D}(\Omega) := \overline{C^\infty(\Omega)}^{D(\Omega)},$$

i.e., as closures of test functions or fields under the respective graph norms, which generalizes homogeneous scalar, tangential and normal boundary conditions, respectively. We also introduce the well known dual spaces

$$H^{-m}(\Omega) := (\overset{\circ}{H}^m(\Omega))'$$

with the standard dual or operator norm defined by

$$|u|_{H^{-m}(\Omega)} := \sup_{0 \neq \varphi \in \overset{\circ}{H}^m(\Omega)} \frac{\langle u, \varphi \rangle_{H^{-m}(\Omega)}}{|\varphi|_{\overset{\circ}{H}^m(\Omega)}} \quad \text{for } u \in H^{-m}(\Omega),$$

where we recall the duality pairing $\langle \cdot, \cdot \rangle_{H^{-m}(\Omega)}$ in $(H^{-m}(\Omega), \overset{\circ}{H}^m(\Omega))$. Moreover, we define with respective graph norms

$$R^{-m}(\Omega) := \{E \in H^{-m}(\Omega) : \text{rot } E \in H^{-m}(\Omega)\}, \\ D^{-m}(\Omega) := \{E \in H^{-m}(\Omega) : \text{div } E \in H^{-m}(\Omega)\}.$$

A vanishing differential operator will be indicated by a zero at the lower right corner of the spaces, e.g.,

$$R_0(\Omega) = \{E \in R(\Omega) : \text{rot } E = 0\}, \quad \overset{\circ}{D}_0(\Omega) = \{E \in \overset{\circ}{D}(\Omega) : \text{div } E = 0\}, \\ R_0^{-m}(\Omega) = \{E \in R^{-m}(\Omega) : \text{rot } E = 0\}, \quad D_0^{-1}(\Omega) = \{E \in D^{-1}(\Omega) : \text{div } E = 0\}.$$

Let us also introduce

$$L_0^2(\Omega) := \{u \in L^2(\Omega) : u \perp_{L^2(\Omega)} \mathbb{R}\} = \{u \in L^2(\Omega) : \int_{\Omega} u = 0\},$$

where $\perp_{L^2(\Omega)}$ denotes orthogonality in $L^2(\Omega)$.

2.3. General Assumptions. We will impose the following regularity and topology assumptions on our domain Ω .

Definition 2.16. Let Ω be an open subset of \mathbb{R}^3 with boundary $\Gamma := \partial\Omega$. We will call Ω

- (i) *strong Lipschitz*, if Γ is locally a graph of a Lipschitz function $\psi : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}$,
- (ii) *topologically trivial*, if Ω is simply connected with connected boundary Γ .

General Assumption 2.17. From now on and throughout this paper it is assumed that $\Omega \subset \mathbb{R}^3$ is a bounded strong Lipschitz domain.

If the domain Ω has to be topologically trivial, we will always indicate this in the respective result. Note that several results will hold for arbitrary open subsets Ω of \mathbb{R}^3 . All results are valid for bounded and topologically trivial strong Lipschitz domains $\Omega \subset \mathbb{R}^3$. Nevertheless, most of the results will remain true for bounded strong Lipschitz domains $\Omega \subset \mathbb{R}^3$.

2.4. Vector Analysis. In this last part of the preliminary section we summarize and prove several results related to scalar and vector potentials of various smoothness, corresponding Friedrichs/Poincaré type estimates, and related Helmholtz decompositions of $L^2(\Omega)$ and other Hilbert and Sobolev spaces. This is a first application of the functional analysis toolbox Section 2.1 for the operators $\overset{\circ}{\text{grad}}$, $\overset{\circ}{\text{rot}}$, $\overset{\circ}{\text{div}}$, and their adjoints $-\text{div}$, rot , $-\text{grad}$. Although these are well known facts, we recall and collect them here, as we will use later similar techniques to obtain related results for the more complicated operators $\overset{\circ}{\text{Gradgrad}}$, $\overset{\circ}{\text{Rot}}_{\mathbb{S}}$, $\overset{\circ}{\text{Div}}_{\mathbb{T}}$, and their adjoints $\text{div Div}_{\mathbb{S}}$, $\text{sym Rot}_{\mathbb{T}}$, $-\text{dev Grad}$. Let

$$A_0 := \overset{\circ}{\text{grad}} : \overset{\circ}{\mathbf{H}}^1(\Omega) \subset L^2(\Omega) \longrightarrow L^2(\Omega),$$

$$A_1 := \overset{\circ}{\text{rot}} : \overset{\circ}{\mathbf{R}}(\Omega) \subset L^2(\Omega) \longrightarrow L^2(\Omega),$$

$$A_2 := \overset{\circ}{\text{div}} : \overset{\circ}{\mathbf{D}}(\Omega) \subset L^2(\Omega) \longrightarrow L^2(\Omega).$$

Then A_0 , A_1 , and A_2 are unbounded, densely defined, and closed linear operators with adjoints

$$A_0^* = \overset{\circ}{\text{grad}}^* = -\text{div} : \mathbf{D}(\Omega) \subset L^2(\Omega) \longrightarrow L^2(\Omega),$$

$$A_1^* := \overset{\circ}{\text{rot}}^* = \text{rot} : \mathbf{R}(\Omega) \subset L^2(\Omega) \longrightarrow L^2(\Omega),$$

$$A_2^* := \overset{\circ}{\text{div}}^* = -\text{grad} : \mathbf{H}^1(\Omega) \subset L^2(\Omega) \longrightarrow L^2(\Omega)$$

and the sequence properties

$$R(A_0) = \overset{\circ}{\text{grad}} \overset{\circ}{\mathbf{H}}^1(\Omega) \subset \overset{\circ}{\mathbf{R}}_0(\Omega) = N(A_1), \quad R(A_1^*) = \text{rot } \mathbf{R}(\Omega) \subset \mathbf{D}_0(\Omega) = N(A_0^*),$$

$$R(A_1) = \overset{\circ}{\text{rot}} \overset{\circ}{\mathbf{R}}(\Omega) \subset \overset{\circ}{\mathbf{D}}_0(\Omega) = N(A_2), \quad R(A_2^*) = \text{grad } \mathbf{H}^1(\Omega) \subset \mathbf{R}_0(\Omega) = N(A_1^*)$$

hold. Note $N(A_0) = \{0\}$ and $N(A_2^*) = \mathbb{R}$. Moreover, the embeddings

$$D(A_1) \cap D(A_0^*) = \overset{\circ}{\mathbf{R}}(\Omega) \cap \mathbf{D}(\Omega) \hookrightarrow L^2(\Omega), \quad D(A_2) \cap D(A_1^*) = \overset{\circ}{\mathbf{D}}(\Omega) \cap \mathbf{R}(\Omega) \hookrightarrow L^2(\Omega)$$

are compact. The latter compact embeddings are called Maxwell compactness properties or Weck's selection theorems. The first proof for strong Lipschitz domains (uniform cone like domains) avoiding smoothness of Γ was given by Weck in [27]. Generally, Weck's selection theorems hold e.g. for weak Lipschitz domains, see [22], or even for more general domains with p -cusps or antennas, see [28, 23]. See also [26] for a different proof in the case of a strong Lipschitz domain. Weck's selection theorem for mixed boundary conditions has been proved in [12] for strong Lipschitz domains and recently in [2] for weak Lipschitz domains. Similar to Rellich's selection theorem, i.e., the compact embedding of $\overset{\circ}{\mathbf{H}}^1(\Omega)$ resp. $\mathbf{H}^1(\Omega)$ into $L^2(\Omega)$, it is crucial that the domain Ω is bounded. Finally, the kernels

$$N(A_1) \cap N(A_0^*) = \overset{\circ}{\mathbf{R}}_0(\Omega) \cap \mathbf{D}_0(\Omega) =: \mathcal{H}_{\mathbf{D}}(\Omega) \quad \text{resp.} \quad N(A_2) \cap N(A_1^*) = \overset{\circ}{\mathbf{D}}_0(\Omega) \cap \mathbf{R}_0(\Omega) =: \mathcal{H}_{\mathbf{N}}(\Omega),$$

are finite dimensional, as the unit balls are compact, i.e., the spaces of Dirichlet or Neumann fields are finite dimensional. More precisely, the dimension of the Dirichlet resp. Neumann fields depends on the topology or cohomology of Ω , i.e., second resp. first Betti number, see e.g. [20, 21]. Especially we have

$$\mathcal{H}_{\mathbf{D}}(\Omega) = \{0\}, \text{ if } \Gamma \text{ is connected,} \quad \mathcal{H}_{\mathbf{N}}(\Omega) = \{0\}, \text{ if } \Omega \text{ is simply connected.}$$

Remark 2.18. *Our general assumption on Ω to be bounded and strong Lipschitz ensures that Weck's selection theorems (and thus also Rellich's) hold. The additional assumption that Ω is also topologically trivial excludes the existence of non-trivial Dirichlet or Neumann fields, as Ω is simply connected with a connected boundary Γ .*

By the results of the functional analysis toolbox Section 2.1 we see that all ranges are closed with

$$\begin{aligned} R(A_0) &= R(\mathcal{A}_0), & R(A_1) &= R(\mathcal{A}_1), & R(A_2) &= R(\mathcal{A}_2), \\ R(A_0^*) &= R(\mathcal{A}_0^*), & R(A_1^*) &= R(\mathcal{A}_1^*), & R(A_2^*) &= R(\mathcal{A}_2^*), \end{aligned}$$

i.e., the ranges

$$(2.12) \quad \begin{aligned} \mathring{\text{grad}} \mathring{H}^1(\Omega), & & \text{grad } H^1(\Omega) &= \text{grad } (H^1(\Omega) \cap L_0^2(\Omega)), \\ \mathring{\text{rot}} \mathring{R}(\Omega) &= \mathring{\text{rot}} (\mathring{R}(\Omega) \cap \text{rot } R(\Omega)), & \text{rot } R(\Omega) &= \text{rot} (R(\Omega) \cap \mathring{\text{rot}} \mathring{R}(\Omega)), \\ \mathring{\text{div}} \mathring{D}(\Omega) &= \mathring{\text{div}} (\mathring{D}(\Omega) \cap \text{grad } H^1(\Omega)), & \text{div } D(\Omega) &= \text{div} (D(\Omega) \cap \mathring{\text{grad}} \mathring{H}^1(\Omega)) \end{aligned}$$

are closed, and the reduced operators are

$$\begin{aligned} \mathcal{A}_0 &:= \mathring{\text{grad}} : \mathring{H}^1(\Omega) \subset L^2(\Omega) \longrightarrow \mathring{\text{grad}} \mathring{H}^1(\Omega), \\ \mathcal{A}_1 &:= \mathring{\text{rot}} : \mathring{R}(\Omega) \cap \text{rot } R(\Omega) \subset \text{rot } R(\Omega) \longrightarrow \mathring{\text{rot}} \mathring{R}(\Omega), \\ \mathcal{A}_2 &:= \mathring{\text{div}} : \mathring{D}(\Omega) \cap \text{grad } H^1(\Omega) \subset \text{grad } H^1(\Omega) \longrightarrow L_0^2(\Omega), \\ \mathcal{A}_0^* &:= -\mathring{\text{div}} : D(\Omega) \cap \mathring{\text{grad}} \mathring{H}^1(\Omega) \subset \mathring{\text{grad}} \mathring{H}^1(\Omega) \longrightarrow L^2(\Omega), \\ \mathcal{A}_1^* &:= \mathring{\text{rot}} : R(\Omega) \cap \mathring{\text{rot}} \mathring{R}(\Omega) \subset \mathring{\text{rot}} \mathring{R}(\Omega) \longrightarrow \text{rot } R(\Omega), \\ \mathcal{A}_2^* &:= -\text{grad} : H^1(\Omega) \cap L_0^2(\Omega) \subset L_0^2(\Omega) \longrightarrow \text{grad } H^1(\Omega). \end{aligned}$$

Moreover, we have the following well known Helmholtz decompositions of L^2 -vector fields into irrotational and solenoidal vector fields, corresponding Friedrichs/Poincaré type estimates and continuous or compact inverse operators.

Lemma 2.19. *The Helmholtz decompositions*

$$\begin{aligned} L^2(\Omega) &= \mathring{\text{div}} \mathring{D}(\Omega) \oplus_{L^2(\Omega)} \mathbb{R}, & \mathring{\text{div}} \mathring{D}(\Omega) &= L_0^2(\Omega), \\ L^2(\Omega) &= \text{div } D(\Omega), \\ L^2(\Omega) &= \mathring{\text{grad}} \mathring{H}^1(\Omega) \oplus_{L^2(\Omega)} D_0(\Omega) \\ &= \mathring{R}_0(\Omega) \oplus_{L^2(\Omega)} \text{rot } R(\Omega) \\ &= \mathring{\text{grad}} \mathring{H}^1(\Omega) \oplus_{L^2(\Omega)} \mathcal{H}_D(\Omega) \oplus_{L^2(\Omega)} \text{rot } R(\Omega), \\ L^2(\Omega) &= \text{grad } H^1(\Omega) \oplus_{L^2(\Omega)} \mathring{D}_0(\Omega) \\ &= R_0(\Omega) \oplus_{L^2(\Omega)} \mathring{\text{rot}} \mathring{R}(\Omega) \\ &= \text{grad } H^1(\Omega) \oplus_{L^2(\Omega)} \mathcal{H}_N(\Omega) \oplus_{L^2(\Omega)} \mathring{\text{rot}} \mathring{R}(\Omega) \end{aligned}$$

holds with

$$\begin{aligned} \mathring{\text{grad}} \mathring{H}^1(\Omega) &= \mathring{R}_0(\Omega) \ominus_{L^2(\Omega)} \mathcal{H}_D(\Omega), & \text{grad } H^1(\Omega) &= R_0(\Omega) \ominus_{L^2(\Omega)} \mathcal{H}_N(\Omega) \\ \mathring{\text{rot}} \mathring{R}(\Omega) &= \mathring{D}_0(\Omega) \ominus_{L^2(\Omega)} \mathcal{H}_N(\Omega), & \text{rot } R(\Omega) &= D_0(\Omega) \ominus_{L^2(\Omega)} \mathcal{H}_D(\Omega). \end{aligned}$$

Moreover, (2.12) is true for the respective ranges and the “better” potentials in (2.12) are uniquely determined and depend continuously in the right hand sides. If Γ is connected, it holds $\mathcal{H}_D(\Omega) = \{0\}$ and, e.g.,

$$L^2(\Omega) = \mathring{R}_0(\Omega) \oplus D_0(\Omega) \quad \text{and} \quad \mathring{R}_0(\Omega) = \mathring{\text{grad}} \mathring{H}^1(\Omega), \quad D_0(\Omega) = \text{rot } R(\Omega) = \text{rot} (R(\Omega) \cap \mathring{D}_0(\Omega)).$$

If Ω is simply connected, it holds $\mathcal{H}_N(\Omega) = \{0\}$ and, e.g.,

$$L^2(\Omega) = R_0(\Omega) \oplus \mathring{D}_0(\Omega) \quad \text{and} \quad R_0(\Omega) = \text{grad } H^1(\Omega), \quad \mathring{D}_0(\Omega) = \mathring{\text{rot}} \mathring{R}(\Omega) = \mathring{\text{rot}} (\mathring{R}(\Omega) \cap D_0(\Omega)).$$

Lemma 2.20. *The following Friedrichs/Poincaré type estimates hold. There exist positive constants c_g , c_r , c_d , such that*

$$\begin{aligned}
\forall u \in \mathring{H}^1(\Omega) & & |u|_{L^2(\Omega)} & \leq c_g |\text{grad } u|_{L^2(\Omega)}, \\
\forall E \in D(\Omega) \cap \mathring{\text{grad}} \mathring{H}^1(\Omega) & & |E|_{L^2(\Omega)} & \leq c_g |\text{div } E|_{L^2(\Omega)}, \\
\forall E \in \mathring{R}(\Omega) \cap \text{rot } R(\Omega) & & |E|_{L^2(\Omega)} & \leq c_r |\text{rot } E|_{L^2(\Omega)}, \\
\forall E \in R(\Omega) \cap \mathring{\text{rot}} \mathring{R}(\Omega) & & |E|_{L^2(\Omega)} & \leq c_r |\text{rot } E|_{L^2(\Omega)}, \\
\forall E \in \mathring{D}(\Omega) \cap \text{grad } H^1(\Omega) & & |E|_{L^2(\Omega)} & \leq c_d |\text{div } E|_{L^2(\Omega)}, \\
\forall u \in H^1(\Omega) \cap L_0^2(\Omega) & & |u|_{L^2(\Omega)} & \leq c_d |\text{grad } u|_{L^2(\Omega)}.
\end{aligned}$$

Moreover, the reduced versions of the operators

$$\mathring{\text{grad}}, \quad \mathring{\text{rot}}, \quad \mathring{\text{div}}, \quad \text{grad}, \quad \text{rot}, \quad \text{div}$$

have continuous resp. compact inverse operators

$$\begin{aligned}
\mathring{\text{grad}}^{-1} : \mathring{\text{grad}} \mathring{H}^1(\Omega) & \longrightarrow \mathring{H}^1(\Omega), & \mathring{\text{grad}}^{-1} : \mathring{\text{grad}} \mathring{H}^1(\Omega) & \longrightarrow L^2(\Omega), \\
\text{div}^{-1} : L^2(\Omega) & \longrightarrow D(\Omega) \cap \mathring{\text{grad}} \mathring{H}^1(\Omega), & \text{div}^{-1} : L^2(\Omega) & \longrightarrow \mathring{\text{grad}} \mathring{H}^1(\Omega) \subset L^2(\Omega), \\
\mathring{\text{rot}}^{-1} : \mathring{\text{rot}} \mathring{R}(\Omega) & \longrightarrow \mathring{R}(\Omega) \cap \text{rot } R(\Omega), & \mathring{\text{rot}}^{-1} : \mathring{\text{rot}} \mathring{R}(\Omega) & \longrightarrow \text{rot } R(\Omega) \subset L^2(\Omega), \\
\text{rot}^{-1} : \text{rot } R(\Omega) & \longrightarrow R(\Omega) \cap \mathring{\text{rot}} \mathring{R}(\Omega), & \text{rot}^{-1} : \text{rot } R(\Omega) & \longrightarrow \mathring{\text{rot}} \mathring{R}(\Omega) \subset L^2(\Omega), \\
\mathring{\text{div}}^{-1} : L_0^2(\Omega) & \longrightarrow \mathring{D}(\Omega) \cap \text{grad } H^1(\Omega), & \mathring{\text{div}}^{-1} : L_0^2(\Omega) & \longrightarrow \text{grad } H^1(\Omega) \subset L^2(\Omega), \\
\text{grad}^{-1} : \text{grad } H^1(\Omega) & \longrightarrow H^1(\Omega) \cap L_0^2(\Omega), & \text{grad}^{-1} : \text{grad } H^1(\Omega) & \longrightarrow L_0^2(\Omega),
\end{aligned}$$

with norms $(1 + c_g^2)^{1/2}$, $(1 + c_r^2)^{1/2}$, $(1 + c_d^2)^{1/2}$ resp. c_g , c_r , c_d . In other words, the operators

$$\begin{aligned}
\mathring{\text{grad}} : \mathring{H}^1(\Omega) & \longrightarrow \mathring{\text{grad}} \mathring{H}^1(\Omega), & \text{div} : D(\Omega) \cap \mathring{\text{grad}} \mathring{H}^1(\Omega) & \longrightarrow L^2(\Omega), \\
u & \longmapsto \text{grad } u & E & \longmapsto \text{div } E \\
\mathring{\text{rot}} : \mathring{R}(\Omega) \cap \text{rot } R(\Omega) & \longrightarrow \mathring{\text{rot}} \mathring{R}(\Omega), & \text{rot} : R(\Omega) \cap \mathring{\text{rot}} \mathring{R}(\Omega) & \longrightarrow \text{rot } R(\Omega), \\
E & \longmapsto \text{rot } E & E & \longmapsto \text{rot } E \\
\mathring{\text{div}} : \mathring{D}(\Omega) \cap \text{grad } H^1(\Omega) & \longrightarrow L_0^2(\Omega), & \text{grad} : H^1(\Omega) \cap L_0^2(\Omega) & \longrightarrow \text{grad } H^1(\Omega), \\
E & \longmapsto \text{div } E & u & \longmapsto \text{grad } u
\end{aligned}$$

are topological isomorphisms. If Ω is topologically trivial, then

$$\begin{aligned}
\mathring{\text{grad}} : \mathring{H}^1(\Omega) & \longrightarrow \mathring{R}_0(\Omega), & \text{div} : D(\Omega) \cap \mathring{R}_0(\Omega) & \longrightarrow L^2(\Omega), \\
u & \longmapsto \text{grad } u & E & \longmapsto \text{div } E \\
(2.13) \quad \mathring{\text{rot}} : \mathring{R}(\Omega) \cap D_0(\Omega) & \longrightarrow \mathring{D}_0(\Omega), & \text{rot} : R(\Omega) \cap \mathring{D}_0(\Omega) & \longrightarrow D_0(\Omega), \\
E & \longmapsto \text{rot } E & E & \longmapsto \text{rot } E \\
\mathring{\text{div}} : \mathring{D}(\Omega) \cap R_0(\Omega) & \longrightarrow L_0^2(\Omega), & \text{grad} : H^1(\Omega) \cap L_0^2(\Omega) & \longrightarrow R_0(\Omega), \\
E & \longmapsto \text{div } E & u & \longmapsto \text{grad } u
\end{aligned}$$

are topological isomorphisms.

Remark 2.21. Recently it has been shown in [17, 18, 19], that for bounded and convex $\Omega \subset \mathbb{R}^3$ it holds

$$c_r \leq c_d \leq \frac{\text{diam } \Omega}{\pi},$$

i.e., the Maxwell constant c_r can be estimated from above by the Poincaré constant.

Remark 2.22. Some of the previous results can be formulated equivalently in terms of complexes: The sequence

$$\{0\} \xrightarrow{0} \mathring{H}^1(\Omega) \xrightarrow{\text{grad}} \mathring{R}(\Omega) \xrightarrow{\text{rot}} \mathring{D}(\Omega) \xrightarrow{\text{div}} \mathring{L}^2(\Omega) \xrightarrow{\pi_{\mathbb{R}}} \mathbb{R}$$

and thus also its dual or adjoint sequence

$$\{0\} \xleftarrow{0} \mathring{L}^2(\Omega) \xleftarrow{-\text{div}} \mathring{D}(\Omega) \xleftarrow{\text{rot}} \mathring{R}(\Omega) \xleftarrow{-\text{grad}} \mathring{H}^1(\Omega) \xleftarrow{\iota_{\mathbb{R}}} \mathbb{R}$$

are closed Hilbert complexes. Here $\pi_{\mathbb{R}} : \mathring{L}^2(\Omega) \rightarrow \mathbb{R}$ denotes the orthogonal projector onto \mathbb{R} with adjoint $\pi_{\mathbb{R}}^* = \iota_{\mathbb{R}} : \mathbb{R} \rightarrow \mathring{L}^2(\Omega)$, the canonical embedding. If Ω is additionally topologically trivial, then the complexes are also exact.

Let Ω be additionally topologically trivial. For irrotational vector fields in $\mathring{H}^m(\Omega)$ resp. $H^m(\Omega)$ we have smooth potentials, which follows immediately by $\mathring{R}_0(\Omega) = \text{grad } \mathring{H}^1(\Omega)$ resp. $R_0(\Omega) = \text{grad } H^1(\Omega)$ from the previous lemma.

Lemma 2.23. Let Ω be additionally topologically trivial and $m \in \mathbb{N}_0$. Then

$$\mathring{H}^m(\Omega) \cap \mathring{R}_0(\Omega) = \text{grad } \mathring{H}^{m+1}(\Omega), \quad H^m(\Omega) \cap R_0(\Omega) = \text{grad } H^{m+1}(\Omega)$$

hold with linear and continuous potential operators P_{grad}° , P_{grad} .

Note that the potential in $H^{m+1}(\Omega)$ is uniquely determined only up to a constant. For solenoidal vector fields in $\mathring{H}^m(\Omega)$ resp. $H^m(\Omega)$ we have smooth potentials, too.

Lemma 2.24. Let Ω be additionally topologically trivial and $m \in \mathbb{N}_0$. Then

$$\mathring{H}^m(\Omega) \cap \mathring{D}_0(\Omega) = \text{rot } \mathring{H}^{m+1}(\Omega), \quad H^m(\Omega) \cap D_0(\Omega) = \text{rot } H^{m+1}(\Omega)$$

hold with linear and continuous potential operators P_{rot}° , P_{rot} .

For a proof see e.g. [6, Corollary 4.7] or with slight modifications the generalized lifting lemma [10, Corollary 5.4] for the case $d = 3$, $k = m$, $l = 2$. Moreover, the potential in $\mathring{H}^{m+1}(\Omega)$ resp. $H^{m+1}(\Omega)$ is no longer uniquely determined. For the divergence operator we have the following result.

Lemma 2.25. Let $m \in \mathbb{N}_0$. Then

$$\mathring{H}^m(\Omega) \cap \mathring{L}_0^2(\Omega) = \text{div } \mathring{H}^{m+1}(\Omega), \quad H^m(\Omega) = \text{div } H^{m+1}(\Omega)$$

hold with linear and continuous potential operators P_{div}° , P_{div} .

Again, the potential in $\mathring{H}^{m+1}(\Omega)$ resp. $H^{m+1}(\Omega)$ is no longer uniquely determined. Also Lemma 2.23 resp. Lemma 2.25 has been proved in [6, Corollary 4.7(b)] and in [10, Corollary 5.4] for the case $d = 3$, $k = m$, $l = 1$ resp. $d = 3$, $k = m$, $l = 3$.

Remark 2.26. Lemma 2.25, which shows a classical result on the solvability and on the properties of the solution operator of the divergence equation, is an important tool in fluid dynamics, i.e., in the theory of Stokes or Navier-Stokes equations. The potential operator is often called Bogovskiĭ operator, see [4, 5] for the original works and also [7, p. 179, Theorem III.3.3], [25, Lemma 2.1.1]. Moreover, there are also versions of Lemma 2.23 and Lemma 2.24, if Ω is not topologically trivial, which we will not need in the paper at hand.

Using the latter three results and Lemma 2.13, irrotational and solenoidal vector fields in $H^{-m}(\Omega)$ can be characterized.

Corollary 2.27. *Let Ω be additionally topologically trivial and $m \in \mathbb{N}$. Then*

$$R_0^{-m}(\Omega) = \text{grad } H^{-m+1}(\Omega) = \text{grad } (\mathring{H}^{m-1}(\Omega) \cap L_0^2(\Omega))'$$

is closed in $H^{-m}(\Omega)$ with continuous inverse, i.e., $\text{grad}^{-1} \in BL(R_0^{-m}(\Omega), (\mathring{H}^{m-1}(\Omega) \cap L_0^2(\Omega))')$. Especially for $m = 1$,

$$R_0^{-1}(\Omega) = \text{grad } L^2(\Omega) = \text{grad } L_0^2(\Omega)$$

is closed in $H^{-1}(\Omega)$ with continuous inverse $\text{grad}^{-1} \in BL(R_0^{-1}(\Omega), L_0^2(\Omega))$ and uniquely determined potential in $L_0^2(\Omega)$. Moreover,

$$\exists c_{g,-1} > 0 \quad \forall u \in L_0^2(\Omega) \quad |u|_{L^2(\Omega)} \leq c_{g,-1} |\text{grad } u|_{H^{-1}(\Omega)} \leq \sqrt{3} c_{g,-1} |u|_{L^2(\Omega)}$$

and the inf-sup-condition

$$0 < \frac{1}{c_{g,-1}} = \inf_{0 \neq u \in L_0^2(\Omega)} \frac{|\text{grad } u|_{H^{-1}(\Omega)}}{|u|_{L^2(\Omega)}} = \inf_{0 \neq u \in L_0^2(\Omega)} \sup_{0 \neq E \in \mathring{H}^1(\Omega)} \frac{\langle u, \text{div } E \rangle_{L^2(\Omega)}}{|u|_{L^2(\Omega)} |\text{Grad } E|_{L^2(\Omega)}}.$$

holds.

Proof. Let $X_0 := \mathring{H}^{m+1}(\Omega)$, $X_1 := \mathring{H}^m(\Omega)$, $X_2 := \mathring{H}^{m-1}(\Omega)$ and

$$A_0 := \mathring{\text{rot}} : \mathring{H}^{m+1}(\Omega) \rightarrow \mathring{H}^m(\Omega), \quad A_1 := -\mathring{\text{div}} : \mathring{H}^m(\Omega) \rightarrow \mathring{H}^{m-1}(\Omega).$$

These linear operators are bounded, $R(A_0) = \mathring{\text{rot}} \mathring{H}^{m+1}(\Omega) = \mathring{H}^m(\Omega) \cap \mathring{D}_0(\Omega) = N(A_1)$ by Lemma 2.24, and $R(A_1) = \mathring{\text{div}} \mathring{H}^m(\Omega) = \mathring{H}^{m-1}(\Omega) \cap L_0^2(\Omega)$ by Lemma 2.25. Therefore, $R(A_1)$ is closed. For the adjoint operators we get

$$A'_0 = \text{rot} = \mathring{\text{rot}}' : H^{-m}(\Omega) \rightarrow H^{-m-1}(\Omega), \quad A'_1 = \text{grad} = -\mathring{\text{div}}' : H^{-m+1}(\Omega) \rightarrow H^{-m}(\Omega)$$

and obtain from Lemma 2.13 that

$$R_0^{-m}(\Omega) = N(A'_0) = R(A'_1) = \text{grad } H^{-m+1}(\Omega)$$

is closed and

$$\text{grad}^{-1} = (A'_1)^{-1} \in BL(R(A'_1), R(A_1)') = BL(R_0^{-m}(\Omega), (\mathring{H}^{m-1}(\Omega) \cap L_0^2(\Omega))').$$

which completes the proof for general m . If $m = 1$, we identify $(\mathring{H}^0(\Omega) \cap L_0^2(\Omega))' = L_0^2(\Omega)' = L_0^2(\Omega)$ and get the assertions about the Friedrichs/Poincaré/Nečas inequality and inf-sup-condition by Remark 2.15, i.e., (2.9) and (2.11). \square

Corollary 2.28. *Let Ω be additionally topologically trivial and $m \in \mathbb{N}$. Then*

$$D_0^{-m}(\Omega) = \text{rot } H^{-m+1}(\Omega) = \text{rot } (\mathring{H}^{m-1}(\Omega) \cap \mathring{D}_0(\Omega))'$$

is closed in $H^{-m}(\Omega)$ with continuous inverse, i.e., $\text{rot}^{-1} \in BL(D_0^{-m}(\Omega), (\mathring{H}^{m-1}(\Omega) \cap \mathring{D}_0(\Omega))')$. Especially for $m = 1$,

$$D_0^{-1}(\Omega) = \text{rot } L^2(\Omega) = \text{rot } \mathring{D}_0(\Omega)$$

is closed in $H^{-1}(\Omega)$ with continuous inverse $\text{rot}^{-1} \in BL(D_0^{-1}(\Omega), \mathring{D}_0(\Omega))$ and uniquely determined potential in $\mathring{D}_0(\Omega)$. Moreover,

$$\exists c_{r,-1} > 0 \quad \forall E \in \mathring{D}_0(\Omega) \quad |E|_{L^2(\Omega)} \leq c_{r,-1} |\text{rot } E|_{H^{-1}(\Omega)} \leq \sqrt{2} c_{r,-1} |E|_{L^2(\Omega)}$$

and the inf-sup-condition

$$0 < \frac{1}{c_{r,-1}} = \inf_{0 \neq E \in \mathring{D}_0(\Omega)} \frac{|\operatorname{rot} E|_{\mathring{H}^{-1}(\Omega)}}{|E|_{L^2(\Omega)}} = \inf_{0 \neq E \in \mathring{D}_0(\Omega)} \sup_{0 \neq H \in \mathring{H}^1(\Omega)} \frac{\langle E, \operatorname{rot} H \rangle_{L^2(\Omega)}}{|E|_{L^2(\Omega)} |\operatorname{Grad} H|_{L^2(\Omega)}}.$$

holds.

Proof. Let $X_0 := \mathring{H}^{m+1}(\Omega)$, $X_1 := \mathring{H}^m(\Omega)$, $X_2 := \mathring{H}^{m-1}(\Omega)$ and

$$A_0 := \operatorname{grad} : \mathring{H}^{m+1}(\Omega) \rightarrow \mathring{H}^m(\Omega), \quad A_1 := \operatorname{rot} : \mathring{H}^m(\Omega) \rightarrow \mathring{H}^{m-1}(\Omega).$$

These linear operators are bounded, $R(A_0) = \operatorname{grad} \mathring{H}^{m+1}(\Omega) = \mathring{H}^m(\Omega) \cap \mathring{R}_0(\Omega) = N(A_1)$ by Lemma 2.23, and $R(A_1) = \operatorname{rot} \mathring{H}^m(\Omega) = \mathring{H}^{m-1}(\Omega) \cap \mathring{D}_0(\Omega)$ by Lemma 2.24. Therefore, $R(A_1)$ is closed. For the adjoint operators we get

$$A'_0 = -\operatorname{div} = \operatorname{grad}' : \mathring{H}^{-m}(\Omega) \rightarrow \mathring{H}^{-m-1}(\Omega), \quad A'_1 = \operatorname{rot} = \operatorname{rot}' : \mathring{H}^{-m+1}(\Omega) \rightarrow \mathring{H}^{-m}(\Omega)$$

and obtain from Lemma 2.13 that

$$\mathring{D}_0^{-m}(\Omega) = N(A'_0) = R(A'_1) = \operatorname{rot} \mathring{H}^{-m+1}(\Omega)$$

is closed and

$$\operatorname{rot}^{-1} = (A'_1)^{-1} \in BL(R(A'_1), R(A_1)') = BL(\mathring{D}_0^{-m}(\Omega), (\mathring{H}^{m-1}(\Omega) \cap \mathring{D}_0(\Omega))'),$$

which completes the proof for general m . If $m = 1$, we identify $(\mathring{H}^0(\Omega) \cap \mathring{D}_0(\Omega))' = \mathring{D}_0(\Omega)' = \mathring{D}_0(\Omega)$ and get the assertions about the Friedrichs/Poincaré/Nečas inequality and inf-sup-condition by Remark 2.15, i.e., (2.9) and (2.11). \square

For completeness let us present the corresponding result for the divergence as well.

Corollary 2.29. *Let Ω be additionally topologically trivial and $m \in \mathbb{N}$. Then*

$$\mathring{H}^{-m}(\Omega) = \operatorname{div} \mathring{H}^{-m+1}(\Omega) = \operatorname{div} (\mathring{H}^{m-1}(\Omega) \cap \mathring{R}_0(\Omega))'$$

(is closed in $\mathring{H}^{-m}(\Omega)$) with continuous inverse, i.e., $\operatorname{div}^{-1} \in BL(\mathring{H}^{-m}(\Omega), (\mathring{H}^{m-1}(\Omega) \cap \mathring{R}_0(\Omega))')$. Especially for $m = 1$,

$$\mathring{H}^{-1}(\Omega) = \operatorname{div} L^2(\Omega) = \operatorname{div} \mathring{R}_0(\Omega)$$

(is closed in $\mathring{H}^{-1}(\Omega)$) with continuous inverse $\operatorname{div}^{-1} \in BL(\mathring{H}^{-1}(\Omega), \mathring{R}_0(\Omega))$ and uniquely determined potential in $\mathring{R}_0(\Omega)$. Moreover,

$$\exists c_{d,-1} > 0 \quad \forall E \in \mathring{R}_0(\Omega) \quad |E|_{L^2(\Omega)} \leq c_{d,-1} |\operatorname{div} E|_{\mathring{H}^{-1}(\Omega)} \leq c_{d,-1} |E|_{L^2(\Omega)}$$

and the inf-sup-condition

$$0 < \frac{1}{c_{d,-1}} = \inf_{0 \neq E \in \mathring{R}_0(\Omega)} \frac{|\operatorname{div} E|_{\mathring{H}^{-1}(\Omega)}}{|E|_{L^2(\Omega)}} = \inf_{0 \neq E \in \mathring{D}_0(\Omega)} \sup_{0 \neq u \in \mathring{H}^1(\Omega)} \frac{\langle E, \operatorname{grad} u \rangle_{L^2(\Omega)}}{|E|_{L^2(\Omega)} |\operatorname{grad} u|_{L^2(\Omega)}}.$$

holds.

Proof. Let $X_1 := \mathring{H}^m(\Omega)$, $X_2 := \mathring{H}^{m-1}(\Omega)$ and $A_1 := -\operatorname{grad} : \mathring{H}^m(\Omega) \rightarrow \mathring{H}^{m-1}(\Omega)$. A_1 is linear and bounded with $R(A_1) = \operatorname{grad} \mathring{H}^m(\Omega) = \mathring{H}^{m-1}(\Omega) \cap \mathring{R}_0(\Omega)$ by Lemma 2.23. Therefore, $R(A_1)$ is closed. The adjoint is $A'_1 = \operatorname{div} = -\operatorname{grad}' : \mathring{H}^{-m+1}(\Omega) \rightarrow \mathring{H}^{-m}(\Omega)$ with closed range $R(A'_1) = \operatorname{div} \mathring{H}^{-m+1}(\Omega)$ by the closed range theorem. Moreover, $N(A_1) = \{0\}$. Hence A'_1 is surjective as A_1 is injective, i.e.,

$$\mathring{H}^{-m}(\Omega) = N(A_1)^\circ = R(A'_1) = \operatorname{div} \mathring{H}^{-m+1}(\Omega).$$

As A_1 is also surjective onto its range, $A'_1 = \text{div} : \mathbf{H}^{-m+1}(\Omega) \rightarrow R(A'_1)$ is bijective. By the bounded inverse theorem we get

$$\text{div}^{-1} = (A'_1)^{-1} \in BL(R(A'_1), R(A_1)') = BL(\mathbf{H}^{-m}(\Omega), (\mathring{\mathbf{H}}^{m-1}(\Omega) \cap \mathring{\mathbf{R}}_0(\Omega))'),$$

which completes the proofⁱⁱ for general m . If $m = 1$, we identify $(\mathring{\mathbf{H}}^0(\Omega) \cap \mathring{\mathbf{R}}_0(\Omega))' = \mathring{\mathbf{R}}_0(\Omega)' = \mathring{\mathbf{R}}_0(\Omega)$ and get the assertions about the Friedrichs/Poincaré/Nečas inequality and inf-sup-condition by Remark 2.15, i.e., (2.9) and (2.11). \square

Remark 2.30. *The results of the latter three lemmas and corollaries can be formulated equivalently in terms of complexes: Let Ω be additionally topologically trivial and $m \in \mathbb{N}$. Then the sequence*

$$\{0\} \xrightarrow{0} \mathring{\mathbf{H}}^{m+1}(\Omega) \xrightarrow{\mathring{\text{grad}}} \mathring{\mathbf{H}}^m(\Omega) \xrightarrow{\mathring{\text{rot}}} \mathring{\mathbf{H}}^{m-1}(\Omega) \xrightarrow{\mathring{\text{div}}} \mathring{\mathbf{H}}^{m-2}(\Omega) \xrightarrow{\pi_{\mathbb{R}}} \mathbb{R}$$

and thus also its dual or adjoint sequence

$$\{0\} \xleftarrow{0} \mathbf{H}^{-m-1}(\Omega) \xleftarrow{-\text{div}} \mathbf{H}^{-m}(\Omega) \xleftarrow{\text{rot}} \mathbf{H}^{-m+1}(\Omega) \xleftarrow{-\text{grad}} \mathbf{H}^{-m+2}(\Omega) \xleftarrow{\iota_{\mathbb{R}}} \mathbb{R}$$

are closed and exact Banach complexes.

3. THE Grad grad- AND div Div-COMPLEXES

We will use the following standard notations from linear algebra. For vectors $a, b \in \mathbb{R}^3$ and matrices $A, B \in \mathbb{R}^{3 \times 3}$ the expressions $a \cdot b$ and $A : B$ denote the inner product of vectors and the Frobenius inner product of matrices, respectively. For a vector $a \in \mathbb{R}^3$ with components a_i for $i = 1, 2, 3$ the matrix $\text{spn } a \in \mathbb{R}^{3 \times 3}$ is defined by

$$\text{spn } a = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.$$

Observe that $(\text{spn } a) b = a \times b$ for $a, b \in \mathbb{R}^3$, where $a \times b$ denotes the exterior product of vectors. The exterior product $a \times B$ of a vector $a \in \mathbb{R}^3$ and a matrix $B \in \mathbb{R}^{3 \times 3}$ is defined as the matrix which is obtained by applying the exterior product row-wise. In addition to $\text{sym } A$ and $\text{skw } A$ for the symmetric part and the skew-symmetric part of a matrix A , we use $\text{dev } A$ and $\text{tr } A$ for denoting the deviatoric part and the trace of a matrix A . Finally, the set of symmetric matrices in $\mathbb{R}^{3 \times 3}$ is denoted by \mathbb{S} , the set of matrices in $\mathbb{R}^{3 \times 3}$ with vanishing trace is denoted by \mathbb{T} .

In this section we need several spaces of tensor fields. The spaces

$$\mathring{\mathbf{C}}^\infty(\Omega), \quad \mathbf{L}^2(\Omega), \quad \mathbf{H}^1(\Omega), \quad \mathring{\mathbf{H}}^1(\Omega), \quad \mathbf{D}(\Omega), \quad \mathring{\mathbf{D}}(\Omega), \quad \mathring{\mathbf{R}}_0(\Omega), \quad \dots$$

are introduced as those spaces of tensor fields, whose rows are in the corresponding spaces of vector fields $\mathring{\mathbf{C}}^\infty(\Omega)$, $\mathbf{L}^2(\Omega)$, $\mathbf{H}^1(\Omega)$, $\mathring{\mathbf{H}}^1(\Omega)$, $\mathbf{D}(\Omega)$, $\mathring{\mathbf{D}}(\Omega)$, $\mathring{\mathbf{R}}_0(\Omega)$, \dots , respectively. Additionally, we will need spaces allowing for a deviatoric gradient, a symmetric rotation, and a double divergence, i.e.,

$$\mathbf{G}_{\text{dev}}(\Omega) := \{E \in \mathbf{L}^2(\Omega) : \text{dev Grad } E \in \mathbf{L}^2(\Omega)\}, \quad \mathbf{G}_{\text{dev},0}(\Omega) := \{E \in \mathbf{L}^2(\Omega) : \text{dev Grad } E = 0\},$$

ⁱⁱAn alternative proof using Lemma 2.13: Let $\mathbf{X}_0 := \mathring{\mathbf{H}}^{m+1}(\Omega)$, $\mathbf{X}_1 := \mathring{\mathbf{H}}^m(\Omega)$, $\mathbf{X}_2 := \mathring{\mathbf{H}}^{m-1}(\Omega)$ and

$$A_0 := 0 : \mathring{\mathbf{H}}^{m+1}(\Omega) \rightarrow \mathring{\mathbf{H}}^m(\Omega), \quad A_1 := -\mathring{\text{grad}} : \mathring{\mathbf{H}}^m(\Omega) \rightarrow \mathring{\mathbf{H}}^{m-1}(\Omega).$$

These linear operators are bounded, $R(A_0) = \{0\} = N(A_1)$, and $R(A_1) = \text{grad } \mathring{\mathbf{H}}^m(\Omega) = \mathring{\mathbf{H}}^{m-1}(\Omega) \cap \mathring{\mathbf{R}}_0(\Omega)$ by Lemma 2.23. Therefore, $R(A_1)$ is closed. For the adjoint operators we get

$$A'_0 = 0 = 0' : \mathbf{H}^{-m}(\Omega) \rightarrow \mathbf{H}^{-m-1}(\Omega), \quad A'_1 = \text{div} = -\mathring{\text{grad}}' : \mathbf{H}^{-m+1}(\Omega) \rightarrow \mathbf{H}^{-m}(\Omega)$$

and obtain from Lemma 2.13 that

$$\mathbf{H}^{-m}(\Omega) = N(A'_0) = R(A'_1) = \text{div } \mathbf{H}^{-m+1}(\Omega)$$

is closed and

$$\text{div}^{-1} = (A'_1)^{-1} \in BL(R(A'_1), R(A_1)') = BL(\mathbf{H}^{-m}(\Omega), (\mathring{\mathbf{H}}^{m-1}(\Omega) \cap \mathring{\mathbf{R}}_0(\Omega))').$$

$$\begin{aligned} \mathbf{R}_{\text{sym}}(\Omega) &:= \{\mathbf{N} \in \mathbf{L}^2(\Omega) : \text{sym Rot } \mathbf{N} \in \mathbf{L}^2(\Omega)\}, & \mathbf{R}_{\text{sym},0}(\Omega) &:= \{\mathbf{N} \in \mathbf{L}^2(\Omega) : \text{sym Rot } \mathbf{N} = 0\}, \\ \mathbf{DD}(\Omega) &:= \{\mathbf{M} \in \mathbf{L}^2(\Omega) : \text{div Div } \mathbf{M} \in \mathbf{L}^2(\Omega)\}, & \mathbf{DD}_0(\Omega) &:= \{\mathbf{M} \in \mathbf{L}^2(\Omega) : \text{div Div } \mathbf{M} = 0\}. \end{aligned}$$

Moreover, we introduce various spaces of symmetricⁱⁱⁱ tensor fields without prescribed boundary conditions, i.e.,

$$\mathbf{L}^2(\Omega, \mathbb{S}) := \{\mathbf{M} \in \mathbf{L}^2(\Omega) : \mathbf{M}^\top = \mathbf{M}\}, \quad \mathbf{DD}(\Omega, \mathbb{S}) := \mathbf{DD}(\Omega) \cap \mathbf{L}^2(\Omega, \mathbb{S}), \quad \dots,$$

and with homogeneous boundary conditions as closures of symmetric test tensor fields, i.e.,

$$\overset{\circ}{\mathbf{R}}(\Omega, \mathbb{S}) := \overline{\overset{\circ}{\mathbf{C}}^\infty(\Omega) \cap \mathbf{L}^2(\Omega, \mathbb{S})}^{\mathbf{R}(\Omega)}, \quad \dots,$$

as well as spaces of tensor fields with vanishing trace^{iv} and without prescribed boundary conditions, i.e.,

$$\mathbf{L}^2(\Omega, \mathbb{T}) := \{\mathbf{N} \in \mathbf{L}^2(\Omega) : \text{tr } \mathbf{N} = 0\}, \quad \mathbf{H}^1(\Omega, \mathbb{T}) := \mathbf{H}^1(\Omega) \cap \mathbf{L}^2(\Omega, \mathbb{T}), \quad \dots,$$

and with homogeneous boundary conditions as closures of trace-free test tensor fields, i.e.,

$$\overset{\circ}{\mathbf{D}}(\Omega, \mathbb{T}) := \overline{\overset{\circ}{\mathbf{C}}^\infty(\Omega) \cap \mathbf{L}^2(\Omega, \mathbb{T})}^{\mathbf{D}(\Omega)}, \quad \dots$$

We note

$$\overset{\circ}{\mathbf{R}}(\Omega, \mathbb{S}) \subset \overset{\circ}{\mathbf{R}}(\Omega) \cap \mathbf{L}^2(\Omega, \mathbb{S}), \quad \overset{\circ}{\mathbf{D}}(\Omega, \mathbb{T}) \subset \overset{\circ}{\mathbf{D}}(\Omega) \cap \mathbf{L}^2(\Omega, \mathbb{T}), \quad \dots$$

Let us also mention that

$$\text{dev Grad } \mathbf{G}_{\text{dev}}(\Omega) \subset \mathbf{L}^2(\Omega, \mathbb{T}), \quad \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega) \subset \mathbf{L}^2(\Omega, \mathbb{S})$$

hold. This can be seen as follows. Pick $\Phi \in \mathbf{G}_{\text{dev}}(\Omega)$ with $\mathbf{N} := \text{dev Grad } \Phi$ and $\Phi \in \mathbf{R}_{\text{sym}}(\Omega)$ with $\mathbf{M} := \text{sym Rot } \Phi$. Then for all $\psi \in \overset{\circ}{\mathbf{C}}^\infty(\Omega)$ and $\Psi \in \overset{\circ}{\mathbf{C}}^\infty(\Omega)$

$$\begin{aligned} \langle \text{tr } \mathbf{N}, \psi \rangle_{\mathbf{L}^2(\Omega)} &= \langle \mathbf{N}, \psi \mathbf{I} \rangle_{\mathbf{L}^2(\Omega)} = -\langle \Phi, \text{Div dev } \psi \mathbf{I} \rangle_{\mathbf{L}^2(\Omega)} = 0, \\ \langle \text{skw } \mathbf{M}, \Psi \rangle_{\mathbf{L}^2(\Omega)} &= \langle \mathbf{M}, \text{skw } \Psi \rangle_{\mathbf{L}^2(\Omega)} = \langle \Phi, \text{Rot sym skw } \Psi \rangle_{\mathbf{L}^2(\Omega)} = 0. \end{aligned}$$

Before we proceed we need a few technical lemmas.

Lemma 3.1. *For any distributional vector field E it holds for $i, j, k = 1, \dots, 3$*

$$\partial_k(\text{Grad } E)_{ij} = \begin{cases} \partial_k(\text{dev Grad } E)_{ij} & , \text{ if } i \neq j, \\ \partial_j(\text{dev Grad } E)_{ik} & , \text{ if } i \neq k, \\ \frac{3}{2} \partial_i(\text{dev Grad } E)_{ii} + \frac{1}{2} \sum_{l \neq i} \partial_l(\text{dev Grad } E)_{li} & , \text{ if } i = j = k. \end{cases}$$

Proof. Let $\Phi \in \overset{\circ}{\mathbf{C}}^\infty(\mathbb{R}^3)$ be a vector field. We want to express the second derivatives of Φ by the derivatives of the deviatoric part of the Jacobian, i.e., of $\text{dev Grad } \Phi$. Recall that we have $\text{dev } \mathbf{M} = \mathbf{M} - \frac{1}{3}(\text{tr } \mathbf{M}) \mathbf{I}$ for a tensor \mathbf{M} . Hence $\text{dev Grad } \Phi$ coincides with $\text{Grad } \Phi$ outside the diagonal entries, i.e., we have $(\text{Grad } \Phi)_{ij} = (\text{dev Grad } \Phi)_{ij}$ for $i \neq j$. Hence, looking at second derivatives, we see immediately

$$\begin{aligned} \partial_k \partial_j \Phi_i &= \partial_k(\text{Grad } \Phi)_{ij} = \partial_k(\text{dev Grad } \Phi)_{ij} & \text{for } i \neq j, \\ \partial_k \partial_j \Phi_i &= \partial_j \partial_k \Phi_i = \partial_j(\text{Grad } \Phi)_{ik} = \partial_j(\text{dev Grad } \Phi)_{ik} & \text{for } i \neq k. \end{aligned}$$

Thus it remains to represent $\partial_i^2 \Phi_i$ by the derivatives of $\text{dev Grad } \Phi$. By

$$\partial_i^2 \Phi_i = \partial_i(\text{Grad } \Phi)_{ii} = \partial_i(\text{dev Grad } \Phi)_{ii} + \frac{1}{3} \partial_i \text{div } \Phi$$

ⁱⁱⁱBy $\text{sym } \mathbf{L}^2(\Omega) \subset \mathbf{L}^2(\Omega, \mathbb{S}) = \text{sym } \mathbf{L}^2(\Omega, \mathbb{S}) \subset \text{sym } \mathbf{L}^2(\Omega)$ we see $\mathbf{L}^2(\Omega, \mathbb{S}) = \text{sym } \mathbf{L}^2(\Omega)$.

^{iv}By $\text{dev } \mathbf{L}^2(\Omega) \subset \mathbf{L}^2(\Omega, \mathbb{T}) = \text{dev } \mathbf{L}^2(\Omega, \mathbb{T}) \subset \text{dev } \mathbf{L}^2(\Omega)$ we see $\mathbf{L}^2(\Omega, \mathbb{T}) = \text{dev } \mathbf{L}^2(\Omega)$.

we get

$$\frac{2}{3} \partial_i^2 \Phi_i = \partial_i (\operatorname{dev Grad} \Phi)_{ii} + \frac{1}{3} \sum_{l \neq i} \partial_i \partial_l \Phi_l = \partial_i (\operatorname{dev Grad} \Phi)_{ii} + \frac{1}{3} \sum_{l \neq i} \partial_l (\operatorname{dev Grad} \Phi)_{li},$$

yielding the stated result for test vector fields. Testing extends the formulas to distributions, which finishes the proof. \square

We note that the latter trick is similar to the well known fact that second derivatives of a vector field can always be written as derivatives of the symmetric gradient of the vector field, leading by Nečas estimate to Korn's second and first inequalities. We will now do the same for the operator $\operatorname{dev Grad}$.

Lemma 3.2. *It holds:*

(i) *There exists $c > 0$, such that for all vector fields $E \in \mathbf{H}^1(\Omega)$*

$$|\operatorname{Grad} E|_{\mathbf{L}^2(\Omega)} \leq c (|E|_{\mathbf{L}^2(\Omega)} + |\operatorname{dev Grad} E|_{\mathbf{L}^2(\Omega)}).$$

(ii) $\mathbf{G}_{\operatorname{dev}}(\Omega) = \mathbf{H}^1(\Omega)$.

(iii) *For $\operatorname{dev Grad} : \mathbf{G}_{\operatorname{dev}}(\Omega) \subset \mathbf{L}^2(\Omega) \longrightarrow \mathbf{L}^2(\Omega, \mathbb{T})$ it holds $D(\operatorname{dev Grad}) = \mathbf{G}_{\operatorname{dev}}(\Omega) = \mathbf{H}^1(\Omega)$, and the kernel of $\operatorname{dev Grad}$ equals the space of (global) shape functions of the lowest order Raviart-Thomas elements, i.e.,*

$$N(\operatorname{dev Grad}) = \mathbf{G}_{\operatorname{dev},0}(\Omega) = \operatorname{RT}_0 := \{P : P(x) = ax + b, a \in \mathbb{R}, b \in \mathbb{R}^3\},$$

which dimension is $\dim \operatorname{RT}_0 = 4$.

(iv) *There exists $c > 0$, such that for all vector fields $E \in \mathbf{H}^1(\Omega) \cap \operatorname{RT}_0^{\perp \mathbf{L}^2(\Omega)}$*

$$|E|_{\mathbf{H}^1(\Omega)} \leq c |\operatorname{dev Grad} E|_{\mathbf{L}^2(\Omega)}.$$

Proof. Let $E \in \mathbf{H}^1(\Omega)$. By the latter lemma and Nečas estimate, i.e.,

$$\exists c > 0 \quad \forall u \in \mathbf{L}^2(\Omega) \quad c |u|_{\mathbf{L}^2(\Omega)} \leq |\operatorname{grad} u|_{\mathbf{H}^{-1}(\Omega)} + |u|_{\mathbf{H}^{-1}(\Omega)} \leq (\sqrt{3} + 1) |u|_{\mathbf{L}^2(\Omega)},$$

we get

$$\begin{aligned} |\operatorname{Grad} E|_{\mathbf{L}^2(\Omega)} &\leq c \left(\sum_{k=1}^3 |\partial_k \operatorname{Grad} E|_{\mathbf{H}^{-1}(\Omega)} + |\operatorname{Grad} E|_{\mathbf{H}^{-1}(\Omega)} \right) \\ &\leq c \left(\sum_{k=1}^3 |\partial_k \operatorname{dev Grad} E|_{\mathbf{H}^{-1}(\Omega)} + |\operatorname{Grad} E|_{\mathbf{H}^{-1}(\Omega)} \right) \\ &\leq c (|\operatorname{dev Grad} E|_{\mathbf{L}^2(\Omega)} + |E|_{\mathbf{L}^2(\Omega)}), \end{aligned}$$

which shows (i). As Ω has the segment property and by standard mollification we obtain that restrictions of $\mathring{C}^\infty(\mathbb{R}^3)$ -vector fields are dense in $\mathbf{G}_{\operatorname{dev}}(\Omega)$. Especially $\mathbf{H}^1(\Omega)$ is dense in $\mathbf{G}_{\operatorname{dev}}(\Omega)$. Let $E \in \mathbf{G}_{\operatorname{dev}}(\Omega)$ and $(E_n) \subset \mathbf{H}^1(\Omega)$ with $E_n \rightarrow E$ in $\mathbf{G}_{\operatorname{dev}}(\Omega)$. By (i) (E_n) is a Cauchy sequence in $\mathbf{H}^1(\Omega)$ converging to E in $\mathbf{H}^1(\Omega)$, which proves $E \in \mathbf{H}^1(\Omega)$ and hence (ii). For $P \in \operatorname{RT}_0$ it holds $\operatorname{dev Grad} P = a \operatorname{dev} \mathbf{I} = 0$. Let $\operatorname{dev Grad} E = 0$ for some vector field $E \in \mathbf{G}_{\operatorname{dev}}(\Omega) = \mathbf{H}^1(\Omega)$. By Lemma 3.1 we get $\partial_k \operatorname{Grad} E = 0$ for all $k = 1, \dots, 3$, and therefore $E(x) = Ax + b$ for some matrix $A \in \mathbb{R}^{3 \times 3}$ and vector $b \in \mathbb{R}^3$. Then $0 = \operatorname{dev Grad} E = \operatorname{dev} A$, if and only if $A = \frac{1}{3}(\operatorname{tr} A) \mathbf{I}$, which shows (iii). If (iv) was wrong, there exists a sequence $(E_n) \subset \mathbf{H}^1(\Omega) \cap \operatorname{RT}_0^{\perp \mathbf{L}^2(\Omega)}$ with $|E_n|_{\mathbf{H}^1(\Omega)} = 1$ and $\operatorname{dev Grad} E_n \rightarrow 0$. As (E_n) is bounded in $\mathbf{H}^1(\Omega)$, by Rellich's selection theorem there exists a subsequence, again denoted by (E_n) , and some $E \in \mathbf{L}^2(\Omega)$ with $E_n \rightarrow E$ in $\mathbf{L}^2(\Omega)$. By (i), (E_n) is a Cauchy sequence in $\mathbf{H}^1(\Omega)$. Hence $E_n \rightarrow E$ in $\mathbf{H}^1(\Omega)$ and $E \in \mathbf{H}^1(\Omega) \cap \operatorname{RT}_0^{\perp \mathbf{L}^2(\Omega)}$. As $0 \leftarrow \operatorname{dev Grad} E_n \rightarrow \operatorname{dev Grad} E$, we have by (iii) $E \in \operatorname{RT}_0 \cap \operatorname{RT}_0^{\perp \mathbf{L}^2(\Omega)} = \{0\}$, a contradiction to $1 = |E_n|_{\mathbf{H}^1(\Omega)} \rightarrow 0$. The proof is complete. \square

Lemma 3.3. Let $\mathring{\mathbf{G}}\mathring{\mathbf{G}}(\Omega) := \{u \in \mathbf{L}^2(\Omega) : \text{Grad grad } u \in \mathbf{L}^2(\Omega)\}$ and $\mathring{\mathbf{G}}\mathring{\mathbf{G}}(\Omega) := \overline{\mathring{\mathbf{C}}^\infty(\Omega)}^{\mathring{\mathbf{G}}\mathring{\mathbf{G}}(\Omega)}$. Then

$$\mathring{\mathbf{G}}\mathring{\mathbf{G}}(\Omega) = \mathring{\mathbf{H}}^2(\Omega)$$

and there exists $c > 0$ such that for all $u \in \mathring{\mathbf{H}}^2(\Omega)$

$$|u|_{\mathring{\mathbf{H}}^2(\Omega)} \leq c |\Delta u|_{\mathbf{L}^2(\Omega)}.$$

Proof. Let $\varphi \in \mathring{\mathbf{C}}^\infty(\Omega)$. Then by the Friedrichs/Poincaré inequality, i.e., $|\varphi|_{\mathbf{L}^2(\Omega)} \leq c_{\mathbf{g}} |\text{grad } \varphi|_{\mathbf{L}^2(\Omega)}$, and

$$|\text{grad } \varphi|_{\mathbf{L}^2(\Omega)}^2 = -\langle \varphi, \Delta \varphi \rangle_{\mathbf{L}^2(\Omega)} \leq c_{\mathbf{g}} |\text{grad } \varphi|_{\mathbf{L}^2(\Omega)} |\Delta \varphi|_{\mathbf{L}^2(\Omega)},$$

we obtain $|\text{grad } \varphi|_{\mathbf{L}^2(\Omega)} \leq c_{\mathbf{g}} |\Delta \varphi|_{\mathbf{L}^2(\Omega)}$. Moreover, by

$$\sum_{i,j=1}^3 |\partial_i \partial_j \varphi|_{\mathbf{L}^2(\Omega)}^2 = \sum_{i,j=1}^3 \langle \partial_i \partial_i \varphi, \partial_i \partial_i \varphi \rangle_{\mathbf{L}^2(\Omega)} = |\Delta \varphi|_{\mathbf{L}^2(\Omega)}^2$$

we see

$$(3.1) \quad |\varphi|_{\mathring{\mathbf{H}}^2(\Omega)}^2 \leq (1 + c_{\mathbf{g}}^2(1 + c_{\mathbf{g}}^2)) |\Delta \varphi|_{\mathbf{L}^2(\Omega)}^2.$$

Let $u \in \mathring{\mathbf{G}}\mathring{\mathbf{G}}(\Omega)$ and $(u_n) \subset \mathring{\mathbf{C}}^\infty(\Omega)$ with $u_n \rightarrow u$ in $\mathring{\mathbf{G}}\mathring{\mathbf{G}}(\Omega)$. By (3.1), (u_n) is a Cauchy sequence in $\mathring{\mathbf{H}}^2(\Omega)$ converging to u in $\mathring{\mathbf{H}}^2(\Omega)$, which proves $u \in \mathring{\mathbf{H}}^2(\Omega)$. By continuity, (3.1) holds for $u \in \mathring{\mathbf{H}}^2(\Omega)$, which finishes the proof. \square

By straight forward calculations and standard arguments for distributions, see the Appendix, we get the following.

Lemma 3.4. *It holds:*

(i) $\text{skw Grad grad } \mathring{\mathbf{H}}^2(\Omega) = 0$, i.e., *Hessians are symmetric.*

(ii) $\text{tr Rot } \mathbf{R}(\Omega, \mathbb{S}) = 0$, i.e., *rotations of symmetric tensors are trace free.*

These formulas extend to distributions as well.

With Lemma 3.3 and Lemma 3.4 let us now consider the linear operators

$$(3.2) \quad \text{Grad grad} : \mathring{\mathbf{G}}\mathring{\mathbf{G}}(\Omega) = \mathring{\mathbf{H}}^2(\Omega) \subset \mathbf{L}^2(\Omega) \longrightarrow \mathbf{L}^2(\Omega, \mathbb{S}), \quad u \mapsto \text{Grad grad } u,$$

$$(3.3) \quad \text{Rot}_{\mathbb{S}} : \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \subset \mathbf{L}^2(\Omega, \mathbb{S}) \longrightarrow \mathbf{L}^2(\Omega, \mathbb{T}), \quad \mathbf{M} \mapsto \text{Rot } \mathbf{M},$$

$$(3.4) \quad \text{Div}_{\mathbb{T}} : \mathring{\mathbf{D}}(\Omega, \mathbb{T}) \subset \mathbf{L}^2(\Omega, \mathbb{T}) \longrightarrow \mathbf{L}^2(\Omega), \quad \mathbf{N} \mapsto \text{Div } \mathbf{N}.$$

These are well and densely defined and closed. Closedness is clear. For densely definedness we look e.g. at $\text{Rot}_{\mathbb{S}}$. For $\mathbf{M} \in \mathbf{L}^2(\Omega, \mathbb{S})$ pick $(\Phi_n) \subset \mathring{\mathbf{C}}^\infty(\Omega)$ with $\Phi_n \rightarrow \mathbf{M}$ in $\mathbf{L}^2(\Omega)$. Then

$$|\mathbf{M} - \text{sym } \Phi_n|_{\mathbf{L}^2(\Omega)}^2 + |\text{skw } \Phi_n|_{\mathbf{L}^2(\Omega)}^2 = |\mathbf{M} - \Phi_n|_{\mathbf{L}^2(\Omega)}^2 \rightarrow 0,$$

showing $(\text{sym } \Phi_n) \subset \mathring{\mathbf{C}}^\infty(\Omega) \cap \mathbf{L}^2(\Omega, \mathbb{S}) \subset \mathring{\mathbf{R}}(\Omega, \mathbb{S})$ and $\text{sym } \Phi_n \rightarrow \mathbf{M}$ in $\mathbf{L}^2(\Omega, \mathbb{S})$. By Lemma 3.3 the kernels are

$$N(\text{Grad grad}) = \{0\}, \quad N(\text{Rot}_{\mathbb{S}}) = \mathring{\mathbf{R}}_0(\Omega, \mathbb{S}), \quad N(\text{Div}_{\mathbb{T}}) = \mathring{\mathbf{D}}_0(\Omega, \mathbb{T}).$$

Lemma 3.5. *The adjoints of (3.2), (3.3), (3.4) are*

$$(\text{Grad grad})^* = \text{div Div}_{\mathbb{S}} : \mathring{\mathbf{D}}\mathring{\mathbf{D}}(\Omega, \mathbb{S}) \subset \mathbf{L}^2(\Omega, \mathbb{S}) \longrightarrow \mathbf{L}^2(\Omega), \quad \mathbf{M} \mapsto \text{div Div } \mathbf{M},$$

$$(\text{Rot}_{\mathbb{S}})^* = \text{sym Rot}_{\mathbb{T}} : \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \subset \mathbf{L}^2(\Omega, \mathbb{T}) \longrightarrow \mathbf{L}^2(\Omega, \mathbb{S}), \quad \mathbf{N} \mapsto \text{sym Rot } \mathbf{N},$$

$$(\text{Div}_{\mathbb{T}})^* = -\text{dev Grad} : \mathbf{G}_{\text{dev}}(\Omega) = \mathbf{H}^1(\Omega) \subset \mathbf{L}^2(\Omega) \longrightarrow \mathbf{L}^2(\Omega, \mathbb{T}), \quad E \mapsto -\text{dev Grad } E.$$

with kernels

$$N(\operatorname{div} \operatorname{Div}_{\mathbb{S}}) = \mathbf{DD}_0(\Omega, \mathbb{S}), \quad N(\operatorname{sym} \operatorname{Rot}_{\mathbb{T}}) = \mathbf{R}_{\operatorname{sym},0}(\Omega, \mathbb{T}), \quad N(\operatorname{dev} \operatorname{Grad}) = \operatorname{RT}_0.$$

Proof. We have $\mathbf{M} \in D((\operatorname{Grad} \operatorname{grad})^*) \subset \mathbf{L}^2(\Omega, \mathbb{S})$ and $(\operatorname{Grad} \operatorname{grad})^* \mathbf{M} = u \in \mathbf{L}^2(\Omega)$, if and only if $\mathbf{M} \in \mathbf{L}^2(\Omega, \mathbb{S})$ and there exists $u \in \mathbf{L}^2(\Omega)$, such that

$$\begin{aligned} \forall \varphi \in D(\operatorname{Grad} \operatorname{grad}) &= \mathring{\mathbf{H}}^2(\Omega) & \langle \operatorname{Grad} \operatorname{grad} \varphi, \mathbf{M} \rangle_{\mathbf{L}^2(\Omega, \mathbb{S})} &= \langle \varphi, u \rangle_{\mathbf{L}^2(\Omega)} \\ \Leftrightarrow \forall \varphi \in \mathring{\mathbf{C}}^\infty(\Omega) & & \langle \operatorname{Grad} \operatorname{grad} \varphi, \mathbf{M} \rangle_{\mathbf{L}^2(\Omega)} &= \langle \varphi, u \rangle_{\mathbf{L}^2(\Omega)}, \end{aligned}$$

if and only if $\mathbf{M} \in \mathbf{DD}(\Omega) \cap \mathbf{L}^2(\Omega, \mathbb{S}) = \mathbf{DD}(\Omega, \mathbb{S})$ and $\operatorname{div} \operatorname{Div} \mathbf{M} = u$. Moreover, we observe that $\mathbf{N} \in D((\operatorname{Rot}_{\mathbb{S}})^*) \subset \mathbf{L}^2(\Omega, \mathbb{T})$ and $(\operatorname{Rot}_{\mathbb{S}})^* \mathbf{N} = \mathbf{M} \in \mathbf{L}^2(\Omega, \mathbb{S})$, if and only if $\mathbf{N} \in \mathbf{L}^2(\Omega, \mathbb{T})$ and there exists $\mathbf{M} \in \mathbf{L}^2(\Omega, \mathbb{S})$, such that (note $\operatorname{sym}^2 = \operatorname{sym}$)

$$\begin{aligned} \forall \Phi \in D(\operatorname{Rot}_{\mathbb{S}}) &= \mathring{\mathbf{R}}(\Omega, \mathbb{S}) & \langle \operatorname{Rot} \Phi, \mathbf{N} \rangle_{\mathbf{L}^2(\Omega, \mathbb{T})} &= \langle \Phi, \mathbf{M} \rangle_{\mathbf{L}^2(\Omega, \mathbb{S})} \\ \Leftrightarrow \forall \Phi \in \mathring{\mathbf{C}}^\infty(\Omega) \cap \mathbf{L}^2(\Omega, \mathbb{S}) & & \langle \operatorname{Rot} \operatorname{sym} \Phi, \mathbf{N} \rangle_{\mathbf{L}^2(\Omega)} &= \langle \operatorname{sym} \Phi, \mathbf{M} \rangle_{\mathbf{L}^2(\Omega)} \\ \Leftrightarrow \forall \Phi \in \mathring{\mathbf{C}}^\infty(\Omega) & & \langle \operatorname{Rot} \operatorname{sym} \Phi, \mathbf{N} \rangle_{\mathbf{L}^2(\Omega)} &= \langle \operatorname{sym} \Phi, \mathbf{M} \rangle_{\mathbf{L}^2(\Omega)} \\ \Leftrightarrow \forall \Phi \in \mathring{\mathbf{C}}^\infty(\Omega) & & \langle \operatorname{Rot} \operatorname{sym} \Phi, \mathbf{N} \rangle_{\mathbf{L}^2(\Omega)} &= \langle \Phi, \mathbf{M} \rangle_{\mathbf{L}^2(\Omega)}, \end{aligned}$$

if and only if $\mathbf{N} \in \mathbf{R}_{\operatorname{sym}}(\Omega) \cap \mathbf{L}^2(\Omega, \mathbb{T}) = \mathbf{R}_{\operatorname{sym}}(\Omega, \mathbb{T})$ and $\operatorname{sym} \operatorname{Rot} \mathbf{N} = \mathbf{M}$. Similarly, we see that $E \in D((\operatorname{Div}_{\mathbb{T}})^*) \subset \mathbf{L}^2(\Omega)$ and $(\operatorname{Div}_{\mathbb{T}})^* E = \mathbf{N} \in \mathbf{L}^2(\Omega, \mathbb{T})$, if and only if $E \in \mathbf{L}^2(\Omega)$ and there exists $\mathbf{N} \in \mathbf{L}^2(\Omega, \mathbb{T})$, such that (note $\operatorname{dev}^2 = \operatorname{dev}$)

$$\begin{aligned} \forall \Phi \in D(\operatorname{Div}_{\mathbb{S}}) &= \mathring{\mathbf{D}}(\Omega, \mathbb{T}) & \langle \operatorname{Div} \Phi, E \rangle_{\mathbf{L}^2(\Omega)} &= \langle \Phi, \mathbf{N} \rangle_{\mathbf{L}^2(\Omega, \mathbb{T})} \\ \Leftrightarrow \forall \Phi \in \mathring{\mathbf{C}}^\infty(\Omega) \cap \mathbf{L}^2(\Omega, \mathbb{T}) & & \langle \operatorname{Div} \operatorname{dev} \Phi, E \rangle_{\mathbf{L}^2(\Omega)} &= \langle \operatorname{dev} \Phi, \mathbf{N} \rangle_{\mathbf{L}^2(\Omega)} \\ \Leftrightarrow \forall \Phi \in \mathring{\mathbf{C}}^\infty(\Omega) & & \langle \operatorname{Div} \operatorname{dev} \Phi, E \rangle_{\mathbf{L}^2(\Omega)} &= \langle \operatorname{dev} \Phi, \mathbf{N} \rangle_{\mathbf{L}^2(\Omega)} \\ \Leftrightarrow \forall \Phi \in \mathring{\mathbf{C}}^\infty(\Omega) & & \langle \operatorname{Div} \operatorname{dev} \Phi, E \rangle_{\mathbf{L}^2(\Omega)} &= \langle \Phi, \mathbf{N} \rangle_{\mathbf{L}^2(\Omega)}, \end{aligned}$$

if and only if $E \in \mathbf{G}_{\operatorname{dev}}(\Omega) = \mathbf{H}^1(\Omega)$ and $-\operatorname{dev} \operatorname{Grad} E = \mathbf{N}$ using Lemma 3.2. Lemma 3.2 also shows $N(\operatorname{dev} \operatorname{Grad}) = \mathbf{G}_{\operatorname{dev},0}(\Omega) = \operatorname{RT}_0$, completing the proof. \square

Remark 3.6. Note that, e.g., the second order operator $\operatorname{Grad} \operatorname{grad}$ is “one” operator and not a composition of the two first order operators Grad and grad . Similarly the operator $\operatorname{div} \operatorname{Div}_{\mathbb{S}}$, $\operatorname{sym} \operatorname{Rot}_{\mathbb{T}}$, resp. $\operatorname{dev} \operatorname{Grad}$ has to be understood as “one” operator.

Lemma 3.7. The ranges

$$\begin{aligned} R(\operatorname{Grad} \operatorname{grad}) &= \operatorname{Grad} \operatorname{grad} \mathring{\mathbf{H}}^2(\Omega), & R(\operatorname{div} \operatorname{Div}_{\mathbb{S}}) &= \operatorname{div} \operatorname{Div} \mathbf{DD}(\Omega, \mathbb{S}), \\ R(\operatorname{dev} \operatorname{Grad}) &= \operatorname{dev} \operatorname{Grad} \mathbf{H}^1(\Omega) = \operatorname{dev} \operatorname{Grad} (\mathbf{H}^1(\Omega) \cap \operatorname{RT}_0^{\perp \mathbf{L}^2(\Omega)}), & R(\operatorname{Div}_{\mathbb{T}}) &= \operatorname{Div} \mathring{\mathbf{D}}(\Omega, \mathbb{T}) \end{aligned}$$

are closed in $\mathbf{L}^2(\Omega)$ resp. $\mathbf{L}^2(\Omega)$.

Proof. For $\mathbf{M} \in \operatorname{Grad} \operatorname{grad} \mathring{\mathbf{H}}^2(\Omega)$ there is a sequence $(u_n) \subset \mathring{\mathbf{H}}^2(\Omega)$ with $\operatorname{Grad} \operatorname{grad} u_n \rightarrow \mathbf{M}$. By Lemma 3.3, (u_n) is a Cauchy sequence in $\mathbf{H}^2(\Omega)$, converging to some $u \in \mathring{\mathbf{H}}^2(\Omega)$ in $\mathbf{H}^2(\Omega)$. Therefore, $\mathbf{M} \leftarrow \operatorname{Grad} \operatorname{grad} u_n \rightarrow \operatorname{Grad} \operatorname{grad} u \in \operatorname{Grad} \operatorname{grad} \mathring{\mathbf{H}}^2(\Omega)$. Similarly for $\mathbf{N} \in \operatorname{dev} \operatorname{Grad} \mathbf{H}^1(\Omega)$ there is a

sequence $(E_n) \subset \mathbf{H}^1(\Omega)$ with $\text{dev Grad } E_n \rightarrow \mathbf{N}$. Let $\pi_{\text{RT}_0} : \mathbf{L}^2(\Omega) \rightarrow \text{RT}_0$ be the orthogonal projector onto RT_0 subject to the orthogonal decomposition

$$\mathbf{L}^2(\Omega) = \text{RT}_0 \oplus_{\mathbf{L}^2(\Omega)} \text{RT}_0^{\perp \mathbf{L}^2(\Omega)}.$$

Then $H_n := (1 - \pi_{\text{RT}_0})E_n \in \mathbf{H}^1(\Omega) \cap \text{RT}_0^{\perp \mathbf{L}^2(\Omega)}$ with $\text{dev Grad } H_n = \text{dev Grad } E_n \rightarrow \mathbf{N}$. By Lemma 3.2 (iv), (H_n) is a Cauchy sequence in $\mathbf{H}^1(\Omega)$, converging to some $H \in \mathbf{H}^1(\Omega) \cap \text{RT}_0^{\perp \mathbf{L}^2(\Omega)}$ in $\mathbf{H}^1(\Omega)$. Therefore, $\mathbf{N} \leftarrow \text{dev Grad } H_n \rightarrow \text{dev Grad } H \in \text{dev Grad}(\mathbf{H}^1(\Omega) \cap \text{RT}_0^{\perp \mathbf{L}^2(\Omega)})$. As $R(\text{Grad grad})$ and $R(\text{dev Grad})$ are closed, so are the ranges of the adjoints by the closed range theorem, finishing the proof. \square

Using the results of the functional analysis toolbox Section 2.1, e.g., (2.1), (2.5), (2.6), (2.7), for the densely defined, closed and unbounded linear operators Grad grad , $\text{Rot}_{\mathbb{S}}$, $\text{Div}_{\mathbb{T}}$, and their adjoints $\text{div Div}_{\mathbb{S}}$, $\text{sym Rot}_{\mathbb{T}}$, dev Grad as well as the corresponding sequence properties, i.e.,

$$\overset{\circ}{\text{Rot}}_{\mathbb{S}} \overset{\circ}{\text{Grad}} \overset{\circ}{\text{grad}} = 0, \quad \overset{\circ}{\text{Div}}_{\mathbb{T}} \overset{\circ}{\text{Rot}}_{\mathbb{S}} = 0$$

and hence also for the adjoints

$$\text{div Div}_{\mathbb{S}} \text{sym Rot}_{\mathbb{T}} = 0, \quad \text{sym Rot}_{\mathbb{T}} \text{dev Grad} = 0,$$

we have the following results for the ranges and corresponding Helmholtz type decompositions.

Lemma 3.8. *For the ranges it holds*

$$\begin{aligned} \mathbf{DD}_0(\Omega, \mathbb{S})^{\perp \mathbf{L}^2(\Omega)} &= N(\text{div Div}_{\mathbb{S}})^{\perp \mathbf{L}^2(\Omega)} = R(\text{Grad grad}) \subset N(\overset{\circ}{\text{Rot}}_{\mathbb{S}}) = \overset{\circ}{\mathbf{R}}_0(\Omega, \mathbb{S}), \\ \mathbf{L}^2(\Omega) = \{0\}^{\perp \mathbf{L}^2(\Omega)} &= N(\overset{\circ}{\text{Grad}} \overset{\circ}{\text{grad}})^{\perp \mathbf{L}^2(\Omega)} = R(\text{div Div}_{\mathbb{S}}), \\ \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T})^{\perp \mathbf{L}^2(\Omega)} &= N(\text{sym Rot}_{\mathbb{T}})^{\perp \mathbf{L}^2(\Omega)} = \overline{R(\overset{\circ}{\text{Rot}}_{\mathbb{S}})}^{\mathbf{L}^2(\Omega)} \subset N(\overset{\circ}{\text{Div}}_{\mathbb{T}}) = \overset{\circ}{\mathbf{D}}_0(\Omega, \mathbb{T}), \\ \overset{\circ}{\mathbf{R}}_0(\Omega, \mathbb{S})^{\perp \mathbf{L}^2(\Omega)} &= N(\overset{\circ}{\text{Rot}}_{\mathbb{S}})^{\perp \mathbf{L}^2(\Omega)} = \overline{R(\text{sym Rot}_{\mathbb{T}})}^{\mathbf{L}^2(\Omega)} \subset N(\text{div Div}_{\mathbb{S}}) = \mathbf{DD}_0(\Omega, \mathbb{S}), \\ \text{RT}_0^{\perp \mathbf{L}^2(\Omega)} &= N(\text{dev Grad})^{\perp \mathbf{L}^2(\Omega)} = R(\overset{\circ}{\text{Div}}_{\mathbb{T}}), \\ \overset{\circ}{\mathbf{D}}_0(\Omega, \mathbb{T})^{\perp \mathbf{L}^2(\Omega)} &= N(\overset{\circ}{\text{Div}}_{\mathbb{T}})^{\perp \mathbf{L}^2(\Omega)} = R(\text{dev Grad}) \subset N(\text{sym Rot}_{\mathbb{T}}) = \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}), \end{aligned}$$

and we have

$$\begin{aligned} \mathbf{L}^2(\Omega) &= R(\text{div Div}_{\mathbb{S}}) = \text{div Div } \mathbf{DD}(\Omega, \mathbb{S}) = \text{div Div}(\mathbf{DD}(\Omega, \mathbb{S}) \cap R(\overset{\circ}{\text{Grad}} \overset{\circ}{\text{grad}})) \\ &= \text{div Div}(\mathbf{DD}(\Omega, \mathbb{S}) \cap \text{Grad grad } \overset{\circ}{\mathbf{H}}^2(\Omega)), \\ R(\overset{\circ}{\text{Rot}}_{\mathbb{S}}) &= \text{Rot } \overset{\circ}{\mathbf{R}}(\Omega, \mathbb{S}) = \text{Rot}(\overset{\circ}{\mathbf{R}}(\Omega, \mathbb{S}) \cap \overline{R(\text{sym Rot}_{\mathbb{T}})}^{\mathbf{L}^2(\Omega)}) \\ &= \text{Rot}(\overset{\circ}{\mathbf{R}}(\Omega, \mathbb{S}) \cap \overline{\text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T})}^{\mathbf{L}^2(\Omega)}), \\ R(\text{sym Rot}_{\mathbb{T}}) &= \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) = \text{sym Rot}(\mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \overline{R(\overset{\circ}{\text{Rot}}_{\mathbb{S}})}^{\mathbf{L}^2(\Omega)}) \\ &= \text{sym Rot}(\mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \overline{\text{Rot } \overset{\circ}{\mathbf{R}}(\Omega, \mathbb{S})}^{\mathbf{L}^2(\Omega)}), \\ \text{RT}_0^{\perp \mathbf{L}^2(\Omega)} &= R(\overset{\circ}{\text{Div}}_{\mathbb{T}}) = \text{Div } \overset{\circ}{\mathbf{D}}(\Omega, \mathbb{T}) = \text{Div}(\overset{\circ}{\mathbf{D}}(\Omega, \mathbb{T}) \cap R(\text{dev Grad})) \\ &= \text{Div}(\overset{\circ}{\mathbf{D}}(\Omega, \mathbb{T}) \cap \text{dev Grad } \overset{\circ}{\mathbf{H}}^1(\Omega)), \\ R(\text{dev Grad}) &= \text{dev Grad } \overset{\circ}{\mathbf{H}}^1(\Omega) = \text{dev Grad}(\overset{\circ}{\mathbf{H}}^1(\Omega) \cap R(\overset{\circ}{\text{Div}}_{\mathbb{T}})) \end{aligned}$$

$$\begin{aligned}
&= \operatorname{dev} \operatorname{Grad} (\mathbf{H}^1(\Omega) \cap \operatorname{Div} \mathring{\mathbf{D}}(\Omega, \mathbb{T})) \\
&= \operatorname{dev} \operatorname{Grad} (\mathbf{H}^1(\Omega) \cap \operatorname{RT}_0^{\perp \mathbf{L}^2(\Omega)}).
\end{aligned}$$

These more regular potentials on the right hand sides are uniquely determined.

Remark 3.9. Lemma 3.8 can be formulated equivalently in terms of complexes: The sequence

$$\{0\} \xrightarrow{0} \mathring{\mathbf{H}}^2(\Omega) \xrightarrow{\operatorname{Grad} \operatorname{grad}} \mathring{\mathbf{R}}(\Omega; \mathbb{S}) \xrightarrow{\mathring{\operatorname{Rot}}_{\mathbb{S}}} \mathring{\mathbf{D}}(\Omega, \mathbb{T}) \xrightarrow{\mathring{\operatorname{Div}}_{\mathbb{T}}} \mathbf{L}^2(\Omega) \xrightarrow{\pi_{\operatorname{RT}_0}} \operatorname{RT}_0$$

and thus also its dual or adjoint sequence

$$\{0\} \xleftarrow{0} \mathbf{L}^2(\Omega) \xleftarrow{\operatorname{div} \operatorname{Div}_{\mathbb{S}}} \mathbf{DD}(\Omega, \mathbb{S}) \xleftarrow{\operatorname{sym} \operatorname{Rot}_{\mathbb{T}}} \mathbf{R}_{\operatorname{sym}}(\Omega, \mathbb{T}) \xleftarrow{-\operatorname{dev} \operatorname{Grad}} \mathbf{H}^1(\Omega) \xleftarrow{\iota_{\operatorname{RT}_0}} \operatorname{RT}_0$$

are Hilbert complexes. As before, $\pi_{\operatorname{RT}_0} : \mathbf{L}^2(\Omega) \rightarrow \operatorname{RT}_0$ denotes the orthogonal projector onto RT_0 with adjoint $\pi_{\operatorname{RT}_0}^* = \iota_{\operatorname{RT}_0} : \operatorname{RT}_0 \rightarrow \mathbf{L}^2(\Omega)$, the canonical embedding. Except of the central operators

$$\mathring{\operatorname{Rot}}_{\mathbb{S}} \quad \text{and} \quad (\mathring{\operatorname{Rot}}_{\mathbb{S}})^* = \operatorname{sym} \operatorname{Rot}_{\mathbb{T}}$$

the sequences is already closed.

Lemma 3.10. The following Helmholtz type decompositions hold.

$$\begin{aligned}
\mathbf{L}^2(\Omega) &= \operatorname{div} \operatorname{Div} \mathbf{DD}(\Omega, \mathbb{S}), \\
\mathbf{L}^2(\Omega) &= \operatorname{Div} \mathring{\mathbf{D}}(\Omega, \mathbb{T}) \oplus_{\mathbf{L}^2(\Omega)} \operatorname{RT}_0, \quad \operatorname{Div} \mathring{\mathbf{D}}(\Omega, \mathbb{T}) = \operatorname{RT}_0^{\perp \mathbf{L}^2(\Omega)} = N(\pi_{\operatorname{RT}_0}), \\
\mathbf{L}^2(\Omega, \mathbb{S}) &= \operatorname{Grad} \operatorname{grad} \mathring{\mathbf{H}}^2(\Omega) \oplus_{\mathbf{L}^2(\Omega, \mathbb{S})} \mathbf{DD}_0(\Omega, \mathbb{S}) \\
&= \mathring{\mathbf{R}}_0(\Omega, \mathbb{S}) \oplus_{\mathbf{L}^2(\Omega, \mathbb{S})} \overline{\operatorname{sym} \operatorname{Rot} \mathbf{R}_{\operatorname{sym}}(\Omega, \mathbb{T})}^{\mathbf{L}^2(\Omega)} \\
&= \operatorname{Grad} \operatorname{grad} \mathring{\mathbf{H}}^2(\Omega) \oplus_{\mathbf{L}^2(\Omega, \mathbb{S})} (\mathring{\mathbf{R}}_0(\Omega, \mathbb{S}) \cap \mathbf{DD}_0(\Omega, \mathbb{S})) \oplus_{\mathbf{L}^2(\Omega, \mathbb{S})} \overline{\operatorname{sym} \operatorname{Rot} \mathbf{R}_{\operatorname{sym}}(\Omega, \mathbb{T})}^{\mathbf{L}^2(\Omega)}, \\
\mathbf{L}^2(\Omega, \mathbb{T}) &= \operatorname{Rot} \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \oplus_{\mathbf{L}^2(\Omega, \mathbb{T})} \mathbf{R}_{\operatorname{sym}, 0}(\Omega, \mathbb{T}) \\
&= \mathring{\mathbf{D}}_0(\Omega, \mathbb{T}) \oplus_{\mathbf{L}^2(\Omega, \mathbb{T})} \operatorname{dev} \operatorname{Grad} \mathbf{H}^1(\Omega) \\
&= \operatorname{Rot} \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \oplus_{\mathbf{L}^2(\Omega, \mathbb{T})} (\mathring{\mathbf{D}}_0(\Omega, \mathbb{T}) \cap \mathbf{R}_{\operatorname{sym}, 0}(\Omega, \mathbb{T})) \oplus_{\mathbf{L}^2(\Omega, \mathbb{T})} \operatorname{dev} \operatorname{Grad} \mathbf{H}^1(\Omega).
\end{aligned}$$

For the operators with already closed ranges we can apply Lemma 2.1 and Lemma 2.2 of the functional analysis toolbox Section 2.1.

Lemma 3.11. The following Friedrichs/Poincaré type estimates hold: There exist positive constants c_{Gg} , c_{D} , such that

$$\begin{aligned}
\forall u \in \mathring{\mathbf{H}}^2(\Omega) & \quad |u|_{\mathbf{L}^2(\Omega)} \leq c_{\operatorname{Gg}} |\operatorname{Grad} \operatorname{grad} u|_{\mathbf{L}^2(\Omega)}, \\
\forall \mathbf{M} \in \mathbf{DD}(\Omega, \mathbb{S}) \cap \operatorname{Grad} \operatorname{grad} \mathring{\mathbf{H}}^2(\Omega) & \quad |\mathbf{M}|_{\mathbf{L}^2(\Omega)} \leq c_{\operatorname{Gg}} |\operatorname{div} \operatorname{Div} \mathbf{M}|_{\mathbf{L}^2(\Omega)}, \\
\forall E \in \mathbf{H}^1(\Omega) \cap \operatorname{RT}_0^{\perp \mathbf{L}^2(\Omega)} & \quad |E|_{\mathbf{L}^2(\Omega)} \leq c_{\operatorname{D}} |\operatorname{dev} \operatorname{Grad} E|_{\mathbf{L}^2(\Omega)}, \\
\forall \mathbf{N} \in \mathring{\mathbf{D}}(\Omega, \mathbb{T}) \cap \operatorname{dev} \operatorname{Grad} \mathbf{H}^1(\Omega) & \quad |\mathbf{N}|_{\mathbf{L}^2(\Omega)} \leq c_{\operatorname{D}} |\operatorname{Div} \mathbf{N}|_{\mathbf{L}^2(\Omega)}.
\end{aligned}$$

Moreover, the reduced versions of the operators $\operatorname{Grad} \operatorname{grad}$, $\operatorname{div} \operatorname{Div}_{\mathbb{S}}$, $\operatorname{dev} \operatorname{Grad}$, $\mathring{\operatorname{Div}}_{\mathbb{T}}$ have continuous inverse operators

$$(\operatorname{Grad} \operatorname{grad})^{-1} : \operatorname{Grad} \operatorname{grad} \mathring{\mathbf{H}}^2(\Omega) \longrightarrow \mathring{\mathbf{H}}^2(\Omega),$$

$$\begin{aligned}
 (\operatorname{div} \operatorname{Div}_{\mathbb{S}})^{-1} : \mathbf{L}^2(\Omega) &\longrightarrow \mathbf{DD}(\Omega, \mathbb{S}) \cap \operatorname{Grad} \operatorname{grad} \overset{\circ}{\mathbf{H}}^2(\Omega), \\
 (\operatorname{dev} \operatorname{Grad})^{-1} : \operatorname{dev} \operatorname{Grad} \mathbf{H}^1(\Omega) &\longrightarrow \mathbf{H}^1(\Omega) \cap \operatorname{RT}_0^{\perp \mathbf{L}^2(\Omega)}, \\
 (\operatorname{Div}_{\mathbb{T}})^{-1} : \operatorname{RT}_0^{\perp \mathbf{L}^2(\Omega)} &\longrightarrow \overset{\circ}{\mathbf{D}}(\Omega, \mathbb{T}) \cap \operatorname{dev} \operatorname{Grad} \mathbf{H}^1(\Omega)
 \end{aligned}$$

with norms $(1 + c_{\mathbb{G}}^2)^{1/2}$ resp. $(1 + c_{\mathbb{D}}^2)^{1/2}$.

We note that stronger versions of the Friedrichs/Poincaré estimates for $\operatorname{Grad} \operatorname{grad}$ and $\operatorname{dev} \operatorname{Grad}$ have already been proved in Lemma 3.2 (iv) and Lemma 3.3. It remains to show that the ranges

$$(3.5) \quad R(\operatorname{Rot}_{\mathbb{S}}) = \operatorname{Rot} \overset{\circ}{\mathbf{R}}(\Omega, \mathbb{S}), \quad R(\operatorname{sym} \operatorname{Rot}_{\mathbb{T}}) = \operatorname{sym} \operatorname{Rot} \mathbf{R}_{\operatorname{sym}}(\Omega, \mathbb{T})$$

are closed as well, where, again by the closed range theorem, it is again sufficient to show that one of them is closed as $(A, A^*) = (\operatorname{Rot}_{\mathbb{S}}, \operatorname{sym} \operatorname{Rot}_{\mathbb{T}})$ is a dual pair. By the considerations in the functional analysis toolbox Section 2.1, one possibility of proving this is to show the compactness of one of the embeddings (then the other one follows automatically, see the Lemma 2.4)

$$(3.6) \quad \begin{aligned} \overset{\circ}{\mathbf{R}}(\Omega, \mathbb{S}) \cap \operatorname{sym} \operatorname{Rot} \mathbf{R}_{\operatorname{sym}}(\Omega, \mathbb{T}) &\xrightarrow{\mathbf{L}^2(\Omega)} \mathbf{L}^2(\Omega, \mathbb{S}), \\ \mathbf{R}_{\operatorname{sym}}(\Omega, \mathbb{T}) \cap \operatorname{Rot} \overset{\circ}{\mathbf{R}}(\Omega, \mathbb{S}) &\xrightarrow{\mathbf{L}^2(\Omega)} \mathbf{L}^2(\Omega, \mathbb{T}) \end{aligned}$$

or, even better, of one of the embeddings

$$(3.7) \quad \begin{aligned} \overset{\circ}{\mathbf{R}}(\Omega, \mathbb{S}) \cap \mathbf{DD}_0(\Omega, \mathbb{S}) &\subset \overset{\circ}{\mathbf{R}}(\Omega, \mathbb{S}) \cap \mathbf{DD}(\Omega, \mathbb{S}) \hookrightarrow \mathbf{L}^2(\Omega, \mathbb{S}), \\ \mathbf{R}_{\operatorname{sym}}(\Omega, \mathbb{T}) \cap \overset{\circ}{\mathbf{D}}_0(\Omega, \mathbb{T}) &\subset \mathbf{R}_{\operatorname{sym}}(\Omega, \mathbb{T}) \cap \overset{\circ}{\mathbf{D}}(\Omega, \mathbb{T}) \hookrightarrow \mathbf{L}^2(\Omega, \mathbb{T}). \end{aligned}$$

Indeed we will show in the next section that (3.7) are compact. As a first step, we will prove directly the closedness of the ranges (3.5) by showing that they are kernels, if the topology of the underlying domain is trivial. In the next lemma, which will be shown in the Appendix, a few identities are collected, which will be used in the proof of the subsequent theorem.

Lemma 3.12. *Let u , E , and \mathbf{M} be distributional scalar, vector, and tensor fields. Then*

- (i) $\operatorname{Div}(u \mathbf{I}) = \operatorname{grad} u$ and $\operatorname{Rot}(u \mathbf{I}) = -\operatorname{spn} \operatorname{grad} u$,
- (ii) $\operatorname{Div}(\operatorname{spn} E) = -\operatorname{rot} E$,
- (iii) $\operatorname{skw} \operatorname{Rot} \mathbf{N} = \operatorname{spn} H$ and $\operatorname{Div}(\operatorname{sym} \operatorname{Rot} \mathbf{N}) = \operatorname{rot} H$ with $H = \frac{1}{2}(\operatorname{Div} \mathbf{N}^{\top} - \operatorname{grad}(\operatorname{tr} \mathbf{N}))$.

We can characterize the kernels of $\operatorname{div} \operatorname{Div}_{\mathbb{S}}$ and $\operatorname{sym} \operatorname{Rot}_{\mathbb{T}}$.

Lemma 3.13. *Let Ω be additionally topologically trivial. Then*

- (i) $N(\operatorname{div} \operatorname{Div}_{\mathbb{S}}) = \mathbf{DD}_0(\Omega, \mathbb{S}) = \operatorname{sym} \operatorname{Rot} \mathbf{R}_{\operatorname{sym}}(\Omega, \mathbb{T}) = \operatorname{sym} \operatorname{Rot} \mathbf{H}^1(\Omega, \mathbb{T}) = \operatorname{sym} \operatorname{Rot} (\overset{\circ}{\mathbf{R}}(\Omega) \cap \overset{\circ}{\mathbf{D}}_0(\Omega))$, especially the range $R(\operatorname{sym} \operatorname{Rot}_{\mathbb{T}}) = \operatorname{sym} \operatorname{Rot} \mathbf{R}_{\operatorname{sym}}(\Omega, \mathbb{T})$ is closed.
- (ii) $N(\operatorname{sym} \operatorname{Rot}_{\mathbb{T}}) = \mathbf{R}_{\operatorname{sym},0}(\Omega, \mathbb{T}) = \operatorname{dev} \operatorname{Grad} \mathbf{H}^1(\Omega) = \operatorname{dev} \operatorname{Grad} (\mathbf{H}^1(\Omega) \cap \operatorname{RT}_0^{\perp \mathbf{L}^2(\Omega)})$ and the potential in $\mathbf{H}^1(\Omega) \cap \operatorname{RT}_0^{\perp \mathbf{L}^2(\Omega)}$ is uniquely determined.

The corresponding potential operators, e.g.,

$$\mathbf{P}_{\operatorname{sym} \operatorname{Rot}} : \mathbf{DD}_0(\Omega, \mathbb{S}) \longrightarrow \mathbf{H}^1(\Omega, \mathbb{T}), \quad \mathbf{P}_{\operatorname{dev} \operatorname{Grad}} : \mathbf{R}_{\operatorname{sym},0}(\Omega, \mathbb{T}) \longrightarrow \mathbf{H}^1(\Omega),$$

are linear and continuous.

Note that^v

$$\mathbf{DD}_0(\Omega, \mathbb{S}) = \operatorname{sym} \operatorname{Rot} \operatorname{dev} \mathbf{H}^1(\Omega) \quad \text{as} \quad \mathbf{H}^1(\Omega, \mathbb{T}) = \operatorname{dev} \mathbf{H}^1(\Omega).$$

^vBy $\operatorname{dev} \mathbf{H}^1(\Omega) \subset \mathbf{H}^1(\Omega, \mathbb{T}) = \operatorname{dev} \mathbf{H}^1(\Omega, \mathbb{T}) \subset \operatorname{dev} \mathbf{H}^1(\Omega)$ we see $\mathbf{H}^1(\Omega, \mathbb{T}) = \operatorname{dev} \mathbf{H}^1(\Omega)$.

Proof. By Lemma 3.8 we already know

$$\text{sym Rot } \mathbf{H}^1(\Omega, \mathbb{T}) \subset \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) = R(\text{sym Rot}_{\mathbb{T}}) \subset N(\text{div Div}_{\mathbb{S}}) = \mathbf{DD}_0(\Omega, \mathbb{S}).$$

Moreover, for $\Phi \in \mathbf{R}_{\text{sym}}(\Omega)$ we have

$$\langle \text{div Div sym Rot } \Phi, \varphi \rangle_{\mathbf{H}^{-2}(\Omega)} = \langle \text{sym Rot } \Phi, \text{Grad grad } \varphi \rangle_{\mathbf{L}^2(\Omega)} = \langle \Phi, \text{Rot sym Grad grad } \varphi \rangle_0 = 0$$

for all $\varphi \in \mathring{C}^\infty(\Omega)$ and hence by continuity for all $\varphi \in \mathring{\mathbf{H}}^2(\Omega)$, which shows $\text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega) \subset \mathbf{DD}_0(\Omega, \mathbb{S})$. Hence

$$\text{sym Rot } (\mathbf{R}(\Omega) \cap \mathring{\mathbf{D}}_0(\Omega)) \subset \text{sym Rot } \mathbf{R}(\Omega) \subset \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega) \subset \mathbf{DD}_0(\Omega, \mathbb{S}).$$

Let $\mathbf{M} \in \mathbf{DD}_0(\Omega, \mathbb{S})$. So $\text{Div } \mathbf{M} \in \mathbf{D}_0^{-1}(\Omega)$ and there is a unique vector field $E := \text{rot}^{-1} \text{Div } \mathbf{M} \in \mathring{\mathbf{D}}_0(\Omega)$, depending linearly and continuously on \mathbf{M} , i.e.,

$$|E|_{\mathbf{L}^2(\Omega)} \leq c |\text{Div } \mathbf{M}|_{\mathbf{H}^{-1}(\Omega)} \leq c |\mathbf{M}|_{\mathbf{L}^2(\Omega)},$$

such that

$$\text{Div } \mathbf{M} = \text{rot } E = -\text{Div}(\text{spn } E),$$

see Corollary 2.28 for $m = 1$, and Lemma 3.12 (ii). Hence $\text{Div}(\mathbf{M} + \text{spn } E) = 0$, i.e., $\mathbf{M} + \text{spn } E \in \mathbf{D}_0(\Omega)$, and by Lemma 2.24 there is a tensor field $\Phi := \text{P}_{\text{Rot}}(\mathbf{M} + \text{spn } E) \in \mathbf{H}^1(\Omega)$, depending linearly and continuously on \mathbf{M} , i.e.,

$$|\Phi|_{\mathbf{H}^1(\Omega)} \leq c (|\mathbf{M}|_{\mathbf{L}^2(\Omega)} + |E|_{\mathbf{L}^2(\Omega)}) \leq c |\mathbf{M}|_{\mathbf{L}^2(\Omega)},$$

such that

$$\mathbf{M} + \text{spn } E = \text{Rot } \Phi.$$

Note that we can also choose $\Phi := \text{Rot}^{-1}(\mathbf{M} + \text{spn } E) \in \mathbf{R}(\Omega) \cap \mathring{\mathbf{D}}_0(\Omega)$ by (2.13) and that in this case Φ is uniquely determined and depends also linearly and continuously on \mathbf{M} , i.e.,

$$|\Phi|_{\mathbf{R}(\Omega)} \leq c (|\mathbf{M}|_{\mathbf{L}^2(\Omega)} + |E|_{\mathbf{L}^2(\Omega)}) \leq c |\mathbf{M}|_{\mathbf{L}^2(\Omega)}.$$

Observe that \mathbf{M} is symmetric and $\text{spn } E$ is skew-symmetric. Thus

$$\mathbf{M} = \text{sym Rot } \Phi \quad \text{and} \quad \text{spn } E = \text{skw Rot } \Phi,$$

which completes the proof of (i) for the potential $\Phi \in \mathbf{R}(\Omega) \cap \mathring{\mathbf{D}}_0(\Omega)$ as we have shown

$$\mathbf{DD}_0(\Omega, \mathbb{S}) \subset \text{sym Rot } (\mathbf{R}(\Omega) \cap \mathring{\mathbf{D}}_0(\Omega))$$

with a linear and continuous potential operator. If we choose the potential $\Phi \in \mathbf{H}^1(\Omega)$, we see

$$\mathbf{M} = \text{sym Rot } \Phi = \text{sym Rot } \tilde{\Phi} \quad \text{with} \quad \tilde{\Phi} := \text{dev } \Phi \in \mathbf{H}^1(\Omega, \mathbb{T}),$$

as $\text{dev } \Phi = \Phi - \frac{1}{3}(\text{tr } \Phi) \mathbf{I}$ and $\text{Rot}((\text{tr } \Phi) \mathbf{I}) = -\text{spn grad}(\text{tr } \Phi)$ is skew-symmetric by the second identity in Lemma 3.12 (i), which completes the proof of part (i) since we have proved

$$\mathbf{DD}_0(\Omega, \mathbb{S}) \subset \text{sym Rot } \mathbf{H}^1(\Omega, \mathbb{T}) = \text{sym Rot dev } \mathbf{H}^1(\Omega)$$

with a linear and continuous potential operator. Let us note

$$\Phi = \text{Rot}^{-1}(\mathbf{M} + \text{spn rot}^{-1} \text{Div } \mathbf{M}) \in \mathbf{R}(\Omega) \cap \mathring{\mathbf{D}}_0(\Omega),$$

$$\tilde{\Phi} = \text{dev P}_{\text{Rot}}(\mathbf{M} + \text{spn rot}^{-1} \text{Div } \mathbf{M}) \in \mathbf{H}^1(\Omega, \mathbb{T}).$$

Moreover, the Helmholtz projection $\tilde{\tilde{\Phi}} \in \mathbf{R}(\Omega) \cap \mathring{\mathbf{D}}_0(\Omega)$ of $\tilde{\Phi}$ onto $\mathring{\mathbf{D}}_0(\Omega)$, see Lemma 2.19, gives another potential $\tilde{\tilde{\Phi}}$ with the same properties as $\Phi \in \mathbf{R}(\Omega) \cap \mathring{\mathbf{D}}_0(\Omega)$ itself.

Let us prove (ii). We know already by Lemma 3.7 and Lemma 3.8 that

$$\text{dev Grad } (\mathbf{H}^1(\Omega) \cap \text{RT}_0^{\perp \mathbf{L}^2(\Omega)}) = \text{dev Grad } \mathbf{H}^1(\Omega) = R(\text{dev Grad}) \subset N(\text{sym Rot}_{\mathbb{T}}) = \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}),$$

with unique potentials in $H^1(\Omega) \cap \mathbf{RT}_0^{\perp L^2(\Omega)}$, and that the ranges are closed. Let $\mathbf{N} \in \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T})$. Then (trivially) $\text{Div sym Rot } \mathbf{N} = 0$ and it follows from Lemma 3.12 (iii) that

$$\text{rot } H = 0 \quad \text{with} \quad H := \frac{1}{2} (\text{Div } \mathbf{N}^\top - \text{grad}(\text{tr } \mathbf{N})) = \frac{1}{2} \text{Div } \mathbf{N}^\top$$

and

$$(3.8) \quad \text{skw Rot } \mathbf{N} = \text{spn } H.$$

So $H \in \mathbf{R}_0^{-1}(\Omega)$. Therefore, there is a unique scalar field $u := \text{grad}^{-1} H \in \mathbf{L}_0^2(\Omega)$, depending linearly and continuously on H resp. \mathbf{N} , i.e.,

$$|u|_{\mathbf{L}^2(\Omega)} \leq c |H|_{\mathbf{H}^{-1}(\Omega)} \leq c |\mathbf{N}|_{\mathbf{L}^2(\Omega)},$$

such that

$$H = \text{grad } u,$$

see Corollary 2.27 for $m = 1$. By the second identity in Lemma 3.12 (i) we see $\text{Rot}(u \mathbf{I}) = -\text{spn grad } u$ and thus $\text{sym Rot}(u \mathbf{I}) = 0$, implying

$$\hat{\mathbf{N}} := \mathbf{N} + u \mathbf{I} \in \mathbf{R}_{\text{sym},0}(\Omega).$$

Moreover, by (3.8)

$$\text{skw Rot } \hat{\mathbf{N}} = \text{skw Rot } \mathbf{N} + \text{skw Rot}(u \mathbf{I}) = \text{spn } H - \text{spn grad } u = 0.$$

Hence $\hat{\mathbf{N}} \in \mathbf{R}_0(\Omega)$. Therefore, there is a unique vector field $\Phi := \text{Grad}^{-1} \hat{\mathbf{N}} \in H^1(\Omega)$, depending linearly and continuously on \mathbf{N} , i.e.,

$$|\Phi|_{\mathbf{H}^1(\Omega)} \leq c |\hat{\mathbf{N}}|_{\mathbf{L}^2(\Omega)} \leq c |\mathbf{N}|_{\mathbf{L}^2(\Omega)},$$

such that $\hat{\mathbf{N}} = \text{Grad } \Phi$, see (2.13) or Lemma 2.23. So we have

$$\mathbf{N} = \text{Grad } \Phi - u \mathbf{I}.$$

From the additional condition $\text{tr } \mathbf{N} = 0$ it follows that $3u = \text{tr Grad } \Phi = \text{div } \Phi$ leading to

$$\mathbf{N} = \text{dev Grad } \Phi, \quad \Phi \in H^1(\Omega),$$

which completes the proof as we have shown

$$\mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}) \subset \text{dev Grad } H^1(\Omega) = \text{dev Grad } (H^1(\Omega) \cap \mathbf{RT}_0^{\perp L^2(\Omega)})$$

and by Lemma 3.2 (iv) the potential in $H^1(\Omega) \cap \mathbf{RT}_0^{\perp L^2(\Omega)}$ depends linearly and continuously on the right hand side. Note that $\Phi = \text{Grad}^{-1} (\mathbf{N} + \frac{1}{2} (\text{grad}^{-1} \text{Div } \mathbf{N}^\top) \mathbf{I}) \in H^1(\Omega)$. \square

Now, if Ω has trivial topology, we can improve Lemma 3.13 and the potentials, the Helmholtz decompositions, as well as the Friedrichs/Poincaré estimates of Lemma 3.8, Lemma 3.10, and Lemma 3.11. Let Ω be additionally topologically trivial. By Lemma 3.13 and the closed range theorem, see (3.5), the ranges

$$R(\text{sym Rot}_\mathbb{T}) = \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}), \quad R(\text{Rot}_\mathbb{S}) = \text{Rot } \overset{\circ}{\mathbf{R}}(\Omega, \mathbb{S})$$

are closed in $\mathbf{L}^2(\Omega)$. Lemma 3.8 shows

$$R(\text{sym Rot}_\mathbb{T}) = \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) = \text{sym Rot } (\mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \text{Rot } \overset{\circ}{\mathbf{R}}(\Omega, \mathbb{S})),$$

$$R(\text{Rot}_\mathbb{S}) = \text{Rot } \overset{\circ}{\mathbf{R}}(\Omega, \mathbb{S}) = \text{Rot } (\overset{\circ}{\mathbf{R}}(\Omega, \mathbb{S}) \cap \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T})),$$

where the potentials on the right hand sides are uniquely determined. As in Lemma 3.11 two important Friedrichs/Poincaré type estimates follow, i.e., There exists $c_{\mathbb{R}} > 0$, such that

$$\begin{aligned} \forall \mathbf{N} \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \text{Rot } \overset{\circ}{\mathbf{R}}(\Omega, \mathbb{S}) & \quad |\mathbf{N}|_{\mathbf{L}^2(\Omega)} \leq c_{\mathbb{R}} |\text{sym Rot } \mathbf{N}|_{\mathbf{L}^2(\Omega)}, \\ \forall \mathbf{M} \in \overset{\circ}{\mathbf{R}}(\Omega, \mathbb{S}) \cap \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) & \quad |\mathbf{M}|_{\mathbf{L}^2(\Omega)} \leq c_{\mathbb{R}} |\text{Rot } \mathbf{M}|_{\mathbf{L}^2(\Omega)}. \end{aligned}$$

Moreover, the reduced versions of $\text{sym Rot}_{\mathbb{T}}$ and $\mathring{\text{Rot}}_{\mathbb{S}}$ have continuous inverse operators

$$\begin{aligned} (\text{sym Rot}_{\mathbb{T}})^{-1} : \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) &\longrightarrow \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \text{Rot } \mathring{\mathbf{R}}(\Omega, \mathbb{S}), \\ (\mathring{\text{Rot}}_{\mathbb{S}})^{-1} : \text{Rot } \mathring{\mathbf{R}}(\Omega, \mathbb{S}) &\longrightarrow \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \cap \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \end{aligned}$$

with norms $(1 + c_{\mathbb{R}}^2)^{1/2}$. By Lemma 3.10 and Lemma 3.13 we obtain

$$\begin{aligned} \mathring{\mathbf{R}}_0(\Omega, \mathbb{S}) \cap \text{DD}_0(\Omega, \mathbb{S}) &= \mathring{\mathbf{R}}_0(\Omega, \mathbb{S}) \cap \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) = N(\mathring{\text{Rot}}_{\mathbb{S}}) \cap R(\text{sym Rot}_{\mathbb{T}}) = \{0\}, \\ \mathring{\mathbf{D}}_0(\Omega, \mathbb{T}) \cap \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}) &= \mathring{\mathbf{D}}_0(\Omega, \mathbb{T}) \cap \text{dev Grad } \mathbf{H}^1(\Omega) = N(\mathring{\text{Div}}_{\mathbb{T}}) \cap R(\text{dev Grad}) = \{0\}, \end{aligned}$$

i.e., the cohomology groups are trivial. By Lemma 3.10 we get improved Helmholtz decompositions

$$(3.9) \quad \begin{aligned} \mathbf{L}^2(\Omega, \mathbb{S}) &= \mathring{\mathbf{R}}_0(\Omega, \mathbb{S}) \oplus_{\mathbf{L}^2(\Omega, \mathbb{S})} \text{DD}_0(\Omega, \mathbb{S}), & \mathbf{L}^2(\Omega, \mathbb{T}) &= \mathring{\mathbf{D}}_0(\Omega, \mathbb{T}) \oplus_{\mathbf{L}^2(\Omega, \mathbb{T})} \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}), \\ \mathring{\mathbf{R}}_0(\Omega, \mathbb{S}) &= \text{Grad grad } \mathring{\mathbf{H}}^2(\Omega), & \mathring{\mathbf{D}}_0(\Omega, \mathbb{T}) &= \text{Rot } \mathring{\mathbf{R}}(\Omega, \mathbb{S}), \\ \text{DD}_0(\Omega, \mathbb{S}) &= \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}), & \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}) &= \text{dev Grad } \mathbf{H}^1(\Omega). \end{aligned}$$

Let us summarize our main results of this section.

Theorem 3.14. *Let Ω be additionally topologically trivial. Then the cohomology groups are trivial, i.e.,*

$$\mathring{\mathbf{R}}_0(\Omega, \mathbb{S}) \cap \text{DD}_0(\Omega, \mathbb{S}) = \{0\}, \quad \mathring{\mathbf{D}}_0(\Omega, \mathbb{T}) \cap \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}) = \{0\},$$

and the Helmholtz type decompositions (3.9), i.e.,

$$\mathbf{L}^2(\Omega, \mathbb{S}) = \mathring{\mathbf{R}}_0(\Omega, \mathbb{S}) \oplus_{\mathbf{L}^2(\Omega, \mathbb{S})} \text{DD}_0(\Omega, \mathbb{S}), \quad \mathbf{L}^2(\Omega, \mathbb{T}) = \mathring{\mathbf{D}}_0(\Omega, \mathbb{T}) \oplus_{\mathbf{L}^2(\Omega, \mathbb{T})} \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}),$$

hold. The ranges

$$\begin{aligned} \mathring{\mathbf{R}}_0(\Omega, \mathbb{S}) &= \text{Grad grad } \mathring{\mathbf{H}}^2(\Omega), \\ \mathbf{L}^2(\Omega) &= \text{div Div } \text{DD}(\Omega, \mathbb{S}) = \text{div Div} (\text{DD}(\Omega, \mathbb{S}) \cap \mathring{\mathbf{R}}_0(\Omega, \mathbb{S})), \\ \mathring{\mathbf{D}}_0(\Omega, \mathbb{T}) &= \text{Rot } \mathring{\mathbf{R}}(\Omega, \mathbb{S}) = \text{Rot} (\mathring{\mathbf{R}}(\Omega, \mathbb{S}) \cap \text{DD}_0(\Omega, \mathbb{S})), \\ \text{DD}_0(\Omega, \mathbb{S}) &= \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) = \text{sym Rot} (\mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega, \mathbb{T})), \\ \text{RT}_0^{\perp \mathbf{L}^2(\Omega)} &= \text{Div } \mathring{\mathbf{D}}(\Omega, \mathbb{T}) = \text{Div} (\mathring{\mathbf{D}}(\Omega, \mathbb{T}) \cap \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T})), \\ \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}) &= \text{dev Grad } \mathbf{H}^1(\Omega) = \text{dev Grad} (\mathbf{H}^1(\Omega) \cap \text{RT}_0^{\perp \mathbf{L}^2(\Omega)}). \end{aligned}$$

are closed. The more regular potentials on the right hand sides are uniquely determined and depend linearly and continuously on the data. Especially, there exist positive constants $c_{\mathbb{Gg}}$, $c_{\mathbb{D}}$, $c_{\mathbb{R}}$ such that the Friedrichs/Poincaré type estimates

$$\begin{aligned} \forall u \in \mathring{\mathbf{H}}^2(\Omega) & & |u|_{\mathbf{L}^2(\Omega)} &\leq c_{\mathbb{Gg}} |\text{Grad grad } u|_{\mathbf{L}^2(\Omega)}, \\ \forall \mathbf{M} \in \text{DD}(\Omega, \mathbb{S}) \cap \mathring{\mathbf{R}}_0(\Omega, \mathbb{S}) & & |\mathbf{M}|_{\mathbf{L}^2(\Omega)} &\leq c_{\mathbb{Gg}} |\text{div Div } \mathbf{M}|_{\mathbf{L}^2(\Omega)}, \\ \forall \mathbf{N} \in \mathring{\mathbf{D}}_0(\Omega, \mathbb{T}) \cap \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}) & & |\mathbf{N}|_{\mathbf{L}^2(\Omega)} &\leq c_{\mathbb{D}} |\text{Div } \mathbf{N}|_{\mathbf{L}^2(\Omega)}, \\ \forall E \in \mathbf{H}^1(\Omega) \cap \text{RT}_0^{\perp \mathbf{L}^2(\Omega)} & & |E|_{\mathbf{L}^2(\Omega)} &\leq c_{\mathbb{D}} |\text{dev Grad } E|_{\mathbf{L}^2(\Omega)}, \\ \forall \mathbf{M} \in \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \cap \text{DD}_0(\Omega, \mathbb{S}) & & |\mathbf{M}|_{\mathbf{L}^2(\Omega)} &\leq c_{\mathbb{R}} |\text{Rot } \mathbf{M}|_{\mathbf{L}^2(\Omega)}, \\ \forall \mathbf{N} \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega, \mathbb{T}) & & |\mathbf{N}|_{\mathbf{L}^2(\Omega)} &\leq c_{\mathbb{R}} |\text{sym Rot } \mathbf{N}|_{\mathbf{L}^2(\Omega)} \end{aligned}$$

hold. Moreover, the reduced versions of the operators

$$\mathring{\text{Grad}} \mathring{\text{grad}}, \quad \text{div Div}_{\mathbb{S}}, \quad \mathring{\text{Div}}_{\mathbb{T}}, \quad \text{dev Grad}, \quad \mathring{\text{Rot}}_{\mathbb{S}}, \quad \text{sym Rot}_{\mathbb{T}}$$

have continuous inverse operators

$$\begin{aligned} (\mathring{\text{Grad}} \mathring{\text{grad}})^{-1} &: \mathring{\mathbf{R}}_0(\Omega, \mathbb{S}) \longrightarrow \mathring{\mathbf{H}}^2(\Omega), \\ (\text{div Div}_{\mathbb{S}})^{-1} &: \mathbf{L}^2(\Omega) \longrightarrow \mathbf{DD}(\Omega, \mathbb{S}) \cap \mathring{\mathbf{R}}_0(\Omega, \mathbb{S}), \\ (\mathring{\text{Div}}_{\mathbb{T}})^{-1} &: \mathbf{RT}_0^{\perp \mathbf{L}^2(\Omega)} \longrightarrow \mathring{\mathbf{D}}(\Omega, \mathbb{T}) \cap \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}), \\ (\text{dev Grad})^{-1} &: \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}) \longrightarrow \mathbf{H}^1(\Omega) \cap \mathbf{RT}_0^{\perp \mathbf{L}^2(\Omega)}, \\ (\mathring{\text{Rot}}_{\mathbb{S}})^{-1} &: \mathring{\mathbf{D}}_0(\Omega, \mathbb{T}) \longrightarrow \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \cap \mathbf{DD}_0(\Omega, \mathbb{S}), \\ (\text{sym Rot}_{\mathbb{T}})^{-1} &: \mathbf{DD}_0(\Omega, \mathbb{S}) \longrightarrow \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega, \mathbb{T}) \end{aligned}$$

with norms $(1 + c_{\mathbb{G}\mathbb{g}}^2)^{1/2}$ resp. $(1 + c_{\mathbb{D}}^2)^{1/2}$, resp. $(1 + c_{\mathbb{R}}^2)^{1/2}$.

Recalling Remark 3.9 we have the following result.

Remark 3.15. *Let Ω be additionally topologically trivial. Theorem 3.14 easily leads to the following equivalent results in terms of complexes: The sequence*

$$\{0\} \xrightarrow{0} \mathring{\mathbf{H}}^2(\Omega) \xrightarrow{\mathring{\text{Grad}} \mathring{\text{grad}}} \mathring{\mathbf{R}}(\Omega; \mathbb{S}) \xrightarrow{\mathring{\text{Rot}}_{\mathbb{S}}} \mathring{\mathbf{D}}(\Omega, \mathbb{T}) \xrightarrow{\mathring{\text{Div}}_{\mathbb{T}}} \mathbf{L}^2(\Omega) \xrightarrow{\pi_{\mathbf{RT}_0}} \mathbf{RT}_0$$

and thus also its dual or adjoint sequence

$$\{0\} \xleftarrow{0} \mathbf{L}^2(\Omega) \xleftarrow{\text{div Div}_{\mathbb{S}}} \mathbf{DD}(\Omega, \mathbb{S}) \xleftarrow{\text{sym Rot}_{\mathbb{T}}} \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \xleftarrow{-\text{dev Grad}} \mathbf{H}^1(\Omega) \xleftarrow{\iota_{\mathbf{RT}_0}} \mathbf{RT}_0$$

are closed and exact Hilbert complexes. The first complex might be called Grad grad-complex and the second one div Div-complex.

Remark 3.16. *The part*

$$\{0\} \xrightarrow{0} \mathring{\mathbf{H}}^2(\Omega) \xrightarrow{\mathring{\text{Grad}} \mathring{\text{grad}}} \mathring{\mathbf{R}}(\Omega; \mathbb{S}) \xrightarrow{\mathring{\text{Rot}}_{\mathbb{S}}} \mathbf{L}^2(\Omega)$$

of the Hilbert complex from above and the related adjoint complex

$$\{0\} \xleftarrow{0} \mathbf{L}^2(\Omega) \xleftarrow{\text{div Div}_{\mathbb{S}}} \mathbf{DD}(\Omega, \mathbb{S}) \xleftarrow{\text{sym Rot}_{\mathbb{T}}} \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T})$$

have been discussed in [24] for problems in general relativity.

Remark 3.17. *In 2D and under similar assumptions we obtain by completely analogous but much simpler arguments that the Hilbert complexes*

$$\begin{aligned} \{0\} &\xrightarrow{0} \mathring{\mathbf{H}}^2(\Omega) \xrightarrow{\mathring{\text{Grad}} \mathring{\text{grad}}} \mathring{\mathbf{R}}(\Omega; \mathbb{S}) \xrightarrow{\mathring{\text{Rot}}_{\mathbb{S}}} \mathbf{L}^2(\Omega) \xrightarrow{\pi_{\mathbf{RT}_0}} \mathbf{RT}_0, \\ \{0\} &\xleftarrow{0} \mathbf{L}^2(\Omega) \xleftarrow{\text{div Div}_{\mathbb{S}}} \mathbf{DD}(\Omega, \mathbb{S}) \xleftarrow{\text{sym Rot}} \mathbf{H}^1(\Omega) \xleftarrow{\iota_{\mathbf{RT}_0}} \mathbf{RT}_0 \end{aligned}$$

are dual to each other, closed and exact. Contrary to the 3D case, the operator $\mathring{\text{Rot}}_{\mathbb{S}}$ maps a tensor field to a vector field and the operator sym Rot is applied row-wise to a vector field and maps this vector field to a tensor field. The associated Helmholtz decomposition is

$$\mathbf{L}^2(\Omega, \mathbb{S}) = \mathring{\mathbf{R}}_0(\Omega, \mathbb{S}) \oplus_{\mathbf{L}^2(\Omega, \mathbb{S})} \mathbf{DD}_0(\Omega, \mathbb{S})$$

with

$$\mathring{\mathbf{R}}_0(\Omega, \mathbb{S}) = \mathring{\text{Grad}} \mathring{\text{grad}} \mathring{\mathbf{H}}^2(\Omega), \quad \mathbf{DD}_0(\Omega, \mathbb{S}) = \text{sym Rot} \mathbf{H}^1(\Omega).$$

4. COMPACT EMBEDDINGS FOR SYMMETRIC ROTATIONS

We will show that indeed the embeddings (3.6) and (3.7) are compact.

Lemma 4.1. *Let Ω be additionally topologically trivial. Then the embeddings (3.7), i.e.,*

$$\mathring{\mathbf{R}}(\Omega, \mathbb{S}) \cap \mathbf{DD}(\Omega, \mathbb{S}) \hookrightarrow \mathbf{L}^2(\Omega, \mathbb{S}), \quad \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}(\Omega, \mathbb{T}) \hookrightarrow \mathbf{L}^2(\Omega, \mathbb{T}),$$

and (3.6) are compact.

Proof. Let (\mathbf{M}_n) be a bounded sequence in $\mathring{\mathbf{R}}(\Omega, \mathbb{S}) \cap \mathbf{DD}(\Omega, \mathbb{S})$. By Theorem 3.14 and Lemma 3.13 we have

$$\begin{aligned} \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \cap \mathbf{DD}(\Omega, \mathbb{S}) &= (\mathring{\mathbf{R}}_0(\Omega, \mathbb{S}) \cap \mathbf{DD}(\Omega, \mathbb{S})) \oplus_{\mathbf{L}^2(\Omega, \mathbb{S})} (\mathring{\mathbf{R}}(\Omega, \mathbb{S}) \cap \mathbf{DD}_0(\Omega, \mathbb{S})), \\ \mathring{\mathbf{R}}_0(\Omega, \mathbb{S}) &= \text{Grad grad } \mathring{\mathbf{H}}^2(\Omega), \\ \mathbf{DD}_0(\Omega, \mathbb{S}) &= \text{sym Rot } \mathbf{H}^1(\Omega) \end{aligned}$$

with linear and continuous potential operators. Therefore, we can decompose

$$\mathring{\mathbf{R}}(\Omega, \mathbb{S}) \cap \mathbf{DD}(\Omega, \mathbb{S}) \ni \mathbf{M}_n = \mathbf{M}_{n,r} + \mathbf{M}_{n,d} \in (\mathring{\mathbf{R}}_0(\Omega, \mathbb{S}) \cap \mathbf{DD}(\Omega, \mathbb{S})) \oplus_{\mathbf{L}^2(\Omega, \mathbb{S})} (\mathring{\mathbf{R}}(\Omega, \mathbb{S}) \cap \mathbf{DD}_0(\Omega, \mathbb{S}))$$

with $\mathbf{M}_{n,r} \in \text{Grad grad } \mathring{\mathbf{H}}^2(\Omega) \cap \mathbf{DD}(\Omega, \mathbb{S})$, $\text{Rot } \mathbf{M}_{n,d} = \text{Rot } \mathbf{M}_n$, and $\mathbf{M}_{n,r} = \text{Grad grad } u_n$, $u_n \in \mathring{\mathbf{H}}^2(\Omega)$, as well as $\mathbf{M}_{n,d} \in \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \cap \text{sym Rot } \mathbf{H}^1(\Omega)$, $\text{div Div } \mathbf{M}_{n,r} = \text{div Div } \mathbf{M}_n$, and $\mathbf{M}_{n,d} = \text{sym Rot } \mathbf{N}_n$, $\mathbf{N}_n \in \mathbf{H}^1(\Omega)$, and both u_n and \mathbf{N}_n depend continuously on \mathbf{M}_n , i.e.,

$$|u_n|_{\mathring{\mathbf{H}}^2(\Omega)} \leq c |\mathbf{M}_{n,r}|_{\mathbf{L}^2(\Omega)} \leq c |\mathbf{M}_n|_{\mathbf{L}^2(\Omega)}, \quad |\mathbf{N}_n|_{\mathbf{H}^1(\Omega)} \leq c |\mathbf{M}_{n,d}|_{\mathbf{L}^2(\Omega)} \leq c |\mathbf{M}_n|_{\mathbf{L}^2(\Omega)}.$$

By Rellich's selection theorem, there exist subsequences, again denoted by (u_n) and (\mathbf{N}_n) , such that (u_n) converges in $\mathring{\mathbf{H}}^1(\Omega)$ and (\mathbf{N}_n) converges in $\mathbf{L}^2(\Omega)$. Thus with $\mathbf{M}_{n,m} := \mathbf{M}_n - \mathbf{M}_m$, and similarly for $\mathbf{M}_{n,m,r}$, $\mathbf{M}_{n,m,d}$, $u_{n,m}$, $\mathbf{N}_{n,m}$, we see

$$\begin{aligned} |\mathbf{M}_{n,m,r}|_{\mathbf{L}^2(\Omega)}^2 &= \langle \mathbf{M}_{n,m,r}, \text{Grad grad } u_{n,m} \rangle_{\mathbf{L}^2(\Omega)} = \langle \text{div Div } \mathbf{M}_{n,m,r}, u_{n,m} \rangle_{\mathbf{L}^2(\Omega)} \\ &= \langle \text{div Div } \mathbf{M}_{n,m}, u_{n,m} \rangle_{\mathbf{L}^2(\Omega)} \leq c |u_{n,m}|_{\mathbf{L}^2(\Omega)}, \\ |\mathbf{M}_{n,m,d}|_{\mathbf{L}^2(\Omega)}^2 &= \langle \mathbf{M}_{n,m,d}, \text{sym Rot } \mathbf{N}_{n,m} \rangle_{\mathbf{L}^2(\Omega)} = \langle \text{Rot } \mathbf{M}_{n,m,d}, \mathbf{N}_{n,m} \rangle_{\mathbf{L}^2(\Omega)} \\ &= \langle \text{Rot } \mathbf{M}_{n,m}, \mathbf{N}_{n,m} \rangle_{\mathbf{L}^2(\Omega)} \leq c |\mathbf{N}_{n,m}|_{\mathbf{L}^2(\Omega)}. \end{aligned}$$

Hence, (\mathbf{M}_n) is a Cauchy sequence in $\mathbf{L}^2(\Omega, \mathbb{S})$. So

$$(4.1) \quad \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \cap \mathbf{DD}(\Omega, \mathbb{S}) \hookrightarrow \mathbf{L}^2(\Omega, \mathbb{S})$$

is compact. To show the second compact embedding, let $(\mathbf{N}_n) \subset \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}(\Omega, \mathbb{T})$ be a bounded sequence. By Theorem 3.14 and Lemma 3.13 we have

$$\begin{aligned} \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}(\Omega, \mathbb{T}) &= (\mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}(\Omega, \mathbb{T})) \oplus_{\mathbf{L}^2(\Omega, \mathbb{T})} (\mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega, \mathbb{T})), \\ \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}) &= \text{dev Grad } \mathbf{H}^1(\Omega), \\ \mathring{\mathbf{D}}_0(\Omega, \mathbb{T}) &= \text{Rot } (\mathring{\mathbf{R}}(\Omega, \mathbb{S}) \cap \mathbf{DD}_0(\Omega, \mathbb{S})) \end{aligned}$$

with linear and continuous potential operators. Therefore, we can decompose

$$\mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}(\Omega, \mathbb{T}) \ni \mathbf{N}_n = \mathbf{N}_{n,r} + \mathbf{N}_{n,d} \in (\mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}(\Omega, \mathbb{T})) \oplus_{\mathbf{L}^2(\Omega, \mathbb{T})} (\mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega, \mathbb{T}))$$

with $\mathbf{N}_{n,r} \in \text{dev Grad } H^1(\Omega) \cap \mathring{\mathbf{D}}(\Omega, \mathbb{T})$, $\text{sym Rot } \mathbf{N}_{n,d} = \text{sym Rot } \mathbf{N}_n$, $\mathbf{N}_{n,r} = \text{dev Grad } E_n$, $E_n \in H^1(\Omega)$, as well as $\mathbf{N}_{n,d} \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \text{Rot}(\mathring{\mathbf{R}}(\Omega, \mathbb{S}) \cap \mathbf{DD}_0(\Omega, \mathbb{S}))$, $\text{Div } \mathbf{N}_{n,r} = \text{Div } \mathbf{N}_n$, and $\mathbf{N}_{n,d} = \text{Rot } \mathbf{M}_n$, $\mathbf{M}_n \in \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \cap \mathbf{DD}_0(\Omega, \mathbb{S})$, and both E_n and \mathbf{M}_n depend continuously on \mathbf{N}_n , i.e.,

$$|E_n|_{H^1(\Omega)} \leq c |\mathbf{N}_{n,r}|_{L^2(\Omega)} \leq c |\mathbf{N}_n|_{L^2(\Omega)}, \quad |\mathbf{M}_n|_{\mathbf{R}(\Omega)} \leq c |\mathbf{N}_{n,d}|_{L^2(\Omega)} \leq c |\mathbf{N}_n|_{L^2(\Omega)}.$$

By Rellich's selection theorem and the previously proved compact embedding (4.1), there exist subsequences, again denoted by (E_n) and (\mathbf{M}_n) , such that (E_n) converges in $L^2(\Omega)$ and (\mathbf{M}_n) converges in $L^2(\Omega)$. Thus with $\mathbf{N}_{n,m} := \mathbf{N}_n - \mathbf{N}_m$, and similarly for $\mathbf{N}_{n,m,r}$, $\mathbf{N}_{n,m,d}$, $E_{n,m}$, $\mathbf{M}_{n,m}$, we see

$$\begin{aligned} |\mathbf{N}_{n,m,r}|_{L^2(\Omega)}^2 &= \langle \mathbf{N}_{n,m,r}, \text{dev Grad } E_{n,m} \rangle_{L^2(\Omega)} = -\langle \text{Div } \mathbf{N}_{n,m,r}, E_{n,m} \rangle_{L^2(\Omega)} \\ &= -\langle \text{Div } \mathbf{N}_{n,m}, E_{n,m} \rangle_{L^2(\Omega)} \leq c |E_{n,m}|_{L^2(\Omega)}, \\ |\mathbf{N}_{n,m,d}|_{L^2(\Omega)}^2 &= \langle \mathbf{N}_{n,m,d}, \text{Rot } \mathbf{M}_{n,m} \rangle_{L^2(\Omega)} = \langle \text{sym Rot } \mathbf{N}_{n,m,d}, \mathbf{M}_{n,m} \rangle_{L^2(\Omega)} \\ &= \langle \text{sym Rot } \mathbf{N}_{n,m}, \mathbf{M}_{n,m} \rangle_{L^2(\Omega)} \leq c |\mathbf{M}_{n,m}|_{L^2(\Omega)}. \end{aligned}$$

Note, that here the symmetry of $\mathbf{M}_{n,m}$ is crucial. Finally, (\mathbf{N}_n) is a Cauchy sequence in $L^2(\Omega, \mathbb{T})$. So

$$(4.2) \quad \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}(\Omega, \mathbb{T}) \hookrightarrow L^2(\Omega, \mathbb{T})$$

is compact. \square

In the Appendix we will prove the following lemma.

Lemma 4.2. *Let $\varphi \in \mathring{\mathbf{C}}^\infty(\mathbb{R}^3)$.*

(i) *If $\mathbf{M} \in \mathring{\mathbf{R}}(\Omega)$ resp. $\mathring{\mathbf{R}}(\Omega, \mathbb{S})$ resp. $\mathring{\mathbf{R}}(\Omega, \mathbb{T})$, then $\varphi \mathbf{M} \in \mathring{\mathbf{R}}(\Omega)$ resp. $\mathring{\mathbf{R}}(\Omega, \mathbb{S})$ resp. $\mathring{\mathbf{R}}(\Omega, \mathbb{T})$ and*

$$(4.3) \quad \text{Rot}(\varphi \mathbf{M}) = \varphi \text{Rot } \mathbf{M} + \text{grad } \varphi \times \mathbf{M}.$$

(ii) *If $\mathbf{M} \in \mathbf{R}(\Omega)$ resp. $\mathbf{R}(\Omega, \mathbb{S})$ resp. $\mathbf{R}(\Omega, \mathbb{T})$, then $\varphi \mathbf{M} \in \mathbf{R}(\Omega)$ resp. $\mathbf{R}(\Omega, \mathbb{S})$ resp. $\mathbf{R}(\Omega, \mathbb{T})$ and (4.3) holds.*

(iii) *If $\mathbf{N} \in \mathring{\mathbf{D}}(\Omega)$ resp. $\mathring{\mathbf{D}}(\Omega, \mathbb{T})$ resp. $\mathring{\mathbf{D}}(\Omega, \mathbb{S})$, then $\varphi \mathbf{N} \in \mathring{\mathbf{D}}(\Omega)$ resp. $\mathring{\mathbf{D}}(\Omega, \mathbb{T})$ resp. $\mathring{\mathbf{D}}(\Omega, \mathbb{S})$ and*

$$(4.4) \quad \text{Div}(\varphi \mathbf{N}) = \varphi \text{Div } \mathbf{N} + \text{grad } \varphi \cdot \mathbf{N}.$$

(iv) *If $\mathbf{N} \in \mathbf{D}(\Omega)$ resp. $\mathbf{D}(\Omega, \mathbb{T})$ resp. $\mathbf{D}(\Omega, \mathbb{S})$, then $\varphi \mathbf{N} \in \mathbf{D}(\Omega)$ resp. $\mathbf{D}(\Omega, \mathbb{T})$ resp. $\mathbf{D}(\Omega, \mathbb{S})$ and (4.4) holds.*

(v) *If $\mathbf{N} \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T})$, then $\varphi \mathbf{N} \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T})$ and*

$$\text{sym Rot}(\varphi \mathbf{N}) = \varphi \text{sym Rot } \mathbf{N} + \text{sym}(\text{grad } \varphi \times \mathbf{N}).$$

(vi) *If $\mathbf{M} \in \mathbf{DD}(\Omega, \mathbb{S})$, then $\varphi \mathbf{M} \in \mathbf{DD}^{0,-1}(\Omega, \mathbb{S})$ and*

$$\text{div Div}(\varphi \mathbf{M}) = \varphi \text{div Div } \mathbf{M} + 2 \text{grad } \varphi \cdot \text{Div } \mathbf{M} + \text{tr}(\text{Grad grad } \varphi \cdot \mathbf{M}).$$

By mollifying these formulas extend to $\varphi \in \mathring{\mathbf{C}}^{0,1}(\mathbb{R}^3)$.

Here $\text{grad } \varphi \times$ resp. $\text{grad } \varphi \cdot$ is applied rows-wise to a tensor \mathbf{M} and we see $\text{grad } \varphi \cdot \mathbf{M} = \mathbf{M} \text{grad } \varphi$. Moreover, we introduce

$$\mathbf{DD}^{0,-1}(\Omega, \mathbb{S}) = \{\mathbf{M} \in L^2(\Omega, \mathbb{S}) : \text{div Div } \mathbf{M} \in H^{-1}(\Omega)\}.$$

Lemma 4.3. *The Helmholtz type decomposition*

$$\mathbf{DD}^{0,-1}(\Omega, \mathbb{S}) = \mathring{\mathbf{H}}^1(\Omega) \cdot \mathbf{I} \dot{+} \mathbf{DD}_0(\Omega, \mathbb{S})$$

holds, where $\dot{+}$ denotes the direct sum. More precisely, for each $\mathbf{M} \in \mathbf{DD}^{0,-1}(\Omega, \mathbb{S})$ there are unique $u \in \mathring{\mathbf{H}}^1(\Omega)$ and $\mathbf{M}_0 \in \mathbf{DD}_0(\Omega, \mathbb{S})$ such that $\mathbf{M} = u\mathbf{I} + \mathbf{M}_0$. The scalar function $u \in \mathring{\mathbf{H}}^1(\Omega)$ is given as the unique solution of the Dirichlet Poisson problem

$$\langle \text{grad } u, \text{grad } \varphi \rangle_{\mathbf{L}^2(\Omega)} = -\langle \text{div Div } \mathbf{M}, \varphi \rangle_{\mathbf{H}^{-1}(\Omega)} \quad \text{for all } \varphi \in \mathring{\mathbf{H}}^1(\Omega),$$

and the decomposition is continuous, more precisely there exists $c > 0$, such that

$$|u|_{\mathbf{H}^1(\Omega)} \leq c |\text{div Div } \mathbf{M}|_{\mathbf{H}^{-1}(\Omega)}, \quad |\mathbf{M} - u\mathbf{I}|_{\mathbf{L}^2(\Omega)} \leq c |\mathbf{M}|_{\mathbf{DD}^{0,-1}(\Omega, \mathbb{S})}.$$

Proof. The unique solution $u \in \mathring{\mathbf{H}}^1(\Omega)$ satisfies

$$\mathbf{H}^{-1}(\Omega) \ni \text{div Div } u\mathbf{I} = \text{div grad } u = \text{div Div } \mathbf{M},$$

i.e., $\mathbf{M}_0 := \mathbf{M} - u\mathbf{I} \in \mathbf{DD}_0(\Omega, \mathbb{S})$, which shows the decomposition. Moreover,

$$|u|_{\mathbf{H}^1(\Omega)} \leq (1 + c_{\mathbf{g}}^2) |\text{div Div } \mathbf{M}|_{\mathbf{H}^{-1}(\Omega)}$$

shows, that u depends continuously on \mathbf{M} and hence also \mathbf{M}_0 since

$$|\mathbf{M}_0|_{\mathbf{L}^2(\Omega)} \leq |\mathbf{M}|_{\mathbf{L}^2(\Omega)} + |u|_{\mathbf{L}^2(\Omega)} \leq \sqrt{2}(1 + c_{\mathbf{g}}^2) |\mathbf{M}|_{\mathbf{DD}^{0,-1}(\Omega, \mathbb{S})}.$$

Let $u\mathbf{I} \in \mathbf{DD}_0(\Omega, \mathbb{S})$ with $u \in \mathring{\mathbf{H}}^1(\Omega)$. Then $0 = \text{div Div } u\mathbf{I} = \text{div grad } u = \Delta u$, yielding $u = 0$. Hence, the decomposition is direct, completing the proof. \square

Lemma 4.4. *The embeddings (3.7), i.e.,*

$$\mathring{\mathbf{R}}(\Omega, \mathbb{S}) \cap \mathbf{DD}(\Omega, \mathbb{S}) \hookrightarrow \mathbf{L}^2(\Omega, \mathbb{S}), \quad \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}(\Omega, \mathbb{T}) \hookrightarrow \mathbf{L}^2(\Omega, \mathbb{T}),$$

and (3.6) are compact.

Proof. Let (U_i) be an open covering of $\bar{\Omega}$, such that $\Omega_i := \Omega \cap U_i$ is topologically trivial for all i . As $\bar{\Omega}$ is compact, there is a finite subcovering denoted by $(U_i)_{i=1, \dots, I}$ with $I \in \mathbb{N}$. Let (φ_i) with $\varphi_i \in \mathring{C}^\infty(U_i)$ be a partition of unity subordinate to (U_i) . Suppose $(\mathbf{N}_n) \subset \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}(\Omega, \mathbb{T})$ is a bounded sequence. Then $\mathbf{N}_n = \sum_{i=1}^I \varphi_i \mathbf{N}_n$ and $(\varphi_i \mathbf{N}_n) \subset \mathbf{R}_{\text{sym}}(\Omega_i, \mathbb{T}) \cap \mathring{\mathbf{D}}(\Omega_i, \mathbb{T})$ is a bounded sequence for all i by Lemma 4.2. As Ω_i is topologically trivial, there exists a subsequence again denoted by $(\varphi_i \mathbf{N}_n)$, which is a Cauchy sequence in $\mathbf{L}^2(\Omega_i)$ by Lemma 4.1. Picking successively subsequences yields that $(\varphi_i \mathbf{N}_n)$ is a Cauchy sequence in $\mathbf{L}^2(\Omega_i)$ for all i . Hence (\mathbf{N}_n) is a Cauchy sequence in $\mathbf{L}^2(\Omega)$. So the second embedding of the lemma is compact. Let $(\mathbf{M}_n) \subset \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \cap \mathbf{DD}(\Omega, \mathbb{S})$ be a bounded sequence. Then $\mathbf{M}_n = \sum_{i=1}^I \varphi_i \mathbf{M}_n$ and $(\varphi_i \mathbf{M}_n) \subset \mathring{\mathbf{R}}(\Omega_i, \mathbb{S}) \cap \mathbf{DD}^{0,-1}(\Omega_i, \mathbb{S})$ is a bounded sequence for all i by Lemma 4.2 as $|\text{Div } \mathbf{M}_n|_{\mathbf{H}^{-1}(\Omega)} \leq |\mathbf{M}_n|_{\mathbf{L}^2(\Omega)}$. Using Lemma 4.3 we decompose

$$\varphi_i \mathbf{M}_n = u_{i,n} \mathbf{I} + \mathbf{M}_{0,i,n} \in \mathring{\mathbf{H}}^1(\Omega_i) \cdot \mathbf{I} \dot{+} (\mathring{\mathbf{R}}(\Omega_i, \mathbb{S}) \cap \mathbf{DD}_0(\Omega_i, \mathbb{S})).$$

Moreover, $(u_{i,n})$ is bounded in $\mathring{\mathbf{H}}^1(\Omega_i)$ and $(\mathbf{M}_{0,i,n})$ is bounded in $(\mathring{\mathbf{R}}(\Omega_i, \mathbb{S}) \cap \mathbf{DD}_0(\Omega_i, \mathbb{S}))$. By Rellich's selection theorem and Lemma 4.1 as well as picking successively subsequences we get that $(\varphi_i \mathbf{M}_n)$ is a Cauchy sequence in $\mathbf{L}^2(\Omega_i)$ for all i . Hence (\mathbf{M}_n) is a Cauchy sequence in $\mathbf{L}^2(\Omega)$, showing that the first embedding of the lemma is also compact, finishing the proof. \square

5. THE Grad grad- AND div Div-COMPLEXES REVISITED

Utilizing the crucial compact embeddings of Lemma 4.4, we can apply the functional analysis toolbox Section 2.1 to the operators $\text{Rot}_{\mathbb{S}}$ and $(\text{Rot}_{\mathbb{S}})^* = \text{sym Rot}_{\mathbb{T}}$ as well, and obtain the following theorem.

Theorem 5.1. *The ranges*

$$R(\mathring{\text{Rot}}_{\mathbb{S}}) = \text{Rot } \mathring{\mathbf{R}}(\Omega, \mathbb{S}), \quad R(\text{sym Rot}_{\mathbb{T}}) = \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T})$$

are closed and the cohomology groups

$$\mathcal{H}_{\text{DD}}(\Omega, \mathbb{S}) := \mathring{\mathbf{R}}_0(\Omega, \mathbb{S}) \cap \text{DD}_0(\Omega, \mathbb{S}), \quad \mathcal{H}_{\text{sym}}(\Omega, \mathbb{T}) := \mathring{\mathbf{D}}_0(\Omega, \mathbb{T}) \cap \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T})$$

are finite dimensional. Moreover, Lemma 3.8 and Lemma 3.10 hold and all occurring ranges are closed. The sequences in Remark 3.9, i.e.,

$$\{0\} \xrightarrow{0} \mathring{\mathbf{H}}^2(\Omega) \xrightarrow{\text{Grad grad}} \mathring{\mathbf{R}}(\Omega; \mathbb{S}) \xrightarrow{\mathring{\text{Rot}}_{\mathbb{S}}} \mathring{\mathbf{D}}(\Omega, \mathbb{T}) \xrightarrow{\mathring{\text{Div}}_{\mathbb{T}}} \mathbf{L}^2(\Omega) \xrightarrow{\pi_{\text{RT}_0}} \text{RT}_0$$

and its adjoint

$$\{0\} \xleftarrow{0} \mathbf{L}^2(\Omega) \xleftarrow{\text{div Div}_{\mathbb{S}}} \text{DD}(\Omega, \mathbb{S}) \xleftarrow{\text{sym Rot}_{\mathbb{T}}} \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \xleftarrow{-\text{dev Grad}} \mathbf{H}^1(\Omega) \xleftarrow{\iota_{\text{RT}_0}} \text{RT}_0,$$

are closed. They are also exact and $\mathcal{H}_{\text{DD}}(\Omega, \mathbb{S}) = \{0\}$, $\mathcal{H}_{\text{sym}}(\Omega, \mathbb{T}) = \{0\}$ hold, if Ω is topologically trivial. Especially, the Helmholtz type decompositions

$$\begin{aligned} \mathbf{L}^2(\Omega, \mathbb{S}) &= \text{Grad grad } \mathring{\mathbf{H}}^2(\Omega) \oplus_{\mathbf{L}^2(\Omega, \mathbb{S})} \text{DD}_0(\Omega, \mathbb{S}) \\ &= \mathring{\mathbf{R}}_0(\Omega, \mathbb{S}) \oplus_{\mathbf{L}^2(\Omega, \mathbb{S})} \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \\ &= \text{Grad grad } \mathring{\mathbf{H}}^2(\Omega) \oplus_{\mathbf{L}^2(\Omega, \mathbb{S})} \mathcal{H}_{\text{DD}}(\Omega, \mathbb{S}) \oplus_{\mathbf{L}^2(\Omega, \mathbb{S})} \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}), \\ \mathbf{L}^2(\Omega, \mathbb{T}) &= \text{Rot } \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \oplus_{\mathbf{L}^2(\Omega, \mathbb{T})} \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}) \\ &= \mathring{\mathbf{D}}_0(\Omega, \mathbb{T}) \oplus_{\mathbf{L}^2(\Omega, \mathbb{T})} \text{dev Grad } \mathbf{H}^1(\Omega) \\ &= \text{Rot } \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \oplus_{\mathbf{L}^2(\Omega, \mathbb{T})} \mathcal{H}_{\text{sym}}(\Omega, \mathbb{T}) \oplus_{\mathbf{L}^2(\Omega, \mathbb{T})} \text{dev Grad } \mathbf{H}^1(\Omega) \end{aligned}$$

are valid. The ranges

$$\begin{aligned} \mathring{\mathbf{R}}_0(\Omega, \mathbb{S}) \ominus_{\mathbf{L}^2(\Omega, \mathbb{S})} \mathcal{H}_{\text{DD}}(\Omega, \mathbb{S}) &= \text{Grad grad } \mathring{\mathbf{H}}^2(\Omega), \\ \mathbf{L}^2(\Omega) &= \text{div Div } \text{DD}(\Omega, \mathbb{S}) = \text{div Div } (\text{DD}(\Omega, \mathbb{S}) \cap \text{Grad grad } \mathring{\mathbf{H}}^2(\Omega)), \\ \mathring{\mathbf{D}}_0(\Omega, \mathbb{T}) \ominus_{\mathbf{L}^2(\Omega, \mathbb{T})} \mathcal{H}_{\text{sym}}(\Omega, \mathbb{T}) &= \text{Rot } \mathring{\mathbf{R}}(\Omega, \mathbb{S}) = \text{Rot } (\mathring{\mathbf{R}}(\Omega, \mathbb{S}) \cap \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T})), \\ \text{DD}_0(\Omega, \mathbb{S}) \ominus_{\mathbf{L}^2(\Omega, \mathbb{S})} \mathcal{H}_{\text{DD}}(\Omega, \mathbb{S}) &= \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) = \text{sym Rot } (\mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \text{Rot } \mathring{\mathbf{R}}(\Omega, \mathbb{S})), \\ \text{RT}_0^{\perp \mathbf{L}^2(\Omega)} &= \text{Div } \mathring{\mathbf{D}}(\Omega, \mathbb{T}) = \text{Div } (\mathring{\mathbf{D}}(\Omega, \mathbb{T}) \cap \text{dev Grad } \mathbf{H}^1(\Omega)), \\ \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}) \ominus_{\mathbf{L}^2(\Omega, \mathbb{T})} \mathcal{H}_{\text{sym}}(\Omega, \mathbb{T}) &= \text{dev Grad } \mathbf{H}^1(\Omega) = \text{dev Grad } (\mathbf{H}^1(\Omega) \cap \text{RT}_0^{\perp \mathbf{L}^2(\Omega)}) \end{aligned}$$

are closed. The more regular potentials on the right hand sides are uniquely determined and depend linearly and continuously on the data. Especially, there exist positive constants c_{Gg} , c_{D} , c_{R} such that the Friedrichs/Poincaré type estimates

$$\begin{aligned} \forall u \in \mathring{\mathbf{H}}^2(\Omega) & \quad |u|_{\mathbf{L}^2(\Omega)} \leq c_{\text{Gg}} | \text{Grad grad } u |_{\mathbf{L}^2(\Omega)}, \\ \forall \mathbf{M} \in \text{DD}(\Omega, \mathbb{S}) \cap \text{Grad grad } \mathring{\mathbf{H}}^2(\Omega) & \quad |\mathbf{M}|_{\mathbf{L}^2(\Omega)} \leq c_{\text{Gg}} | \text{div Div } \mathbf{M} |_{\mathbf{L}^2(\Omega)}, \\ \forall \mathbf{N} \in \mathring{\mathbf{D}}(\Omega, \mathbb{T}) \cap \text{dev Grad } \mathbf{H}^1(\Omega) & \quad |\mathbf{N}|_{\mathbf{L}^2(\Omega)} \leq c_{\text{D}} | \text{Div } \mathbf{N} |_{\mathbf{L}^2(\Omega)}, \\ \forall E \in \mathbf{H}^1(\Omega) \cap \text{RT}_0^{\perp \mathbf{L}^2(\Omega)} & \quad |E|_{\mathbf{L}^2(\Omega)} \leq c_{\text{D}} | \text{dev Grad } E |_{\mathbf{L}^2(\Omega)}, \end{aligned}$$

$$\begin{aligned} \forall \mathbf{M} \in \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \cap \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) & \quad |\mathbf{M}|_{\mathbf{L}^2(\Omega)} \leq c_{\mathbb{R}} |\text{Rot } \mathbf{M}|_{\mathbf{L}^2(\Omega)}, \\ \forall \mathbf{N} \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \text{Rot } \mathring{\mathbf{R}}(\Omega, \mathbb{S}) & \quad |\mathbf{N}|_{\mathbf{L}^2(\Omega)} \leq c_{\mathbb{R}} |\text{sym Rot } \mathbf{N}|_{\mathbf{L}^2(\Omega)} \end{aligned}$$

hold. Moreover, the inverse operators

$$\begin{aligned} (\text{Grad } \mathring{\text{grad}})^{-1} : \text{Grad grad } \mathring{\mathbf{H}}^2(\Omega) & \longrightarrow \mathring{\mathbf{H}}^2(\Omega), \\ (\text{div Div}_{\mathbb{S}})^{-1} : \mathbf{L}^2(\Omega) & \longrightarrow \mathbf{DD}(\Omega, \mathbb{S}) \cap \text{Grad grad } \mathring{\mathbf{H}}^2(\Omega), \\ (\text{Div}_{\mathbb{T}})^{-1} : \text{RT}_0^{\perp \mathbf{L}^2(\Omega)} & \longrightarrow \mathring{\mathbf{D}}(\Omega, \mathbb{T}) \cap \text{dev Grad } \mathbf{H}^1(\Omega), \\ (\text{dev Grad})^{-1} : \text{dev Grad } \mathbf{H}^1(\Omega) & \longrightarrow \mathbf{H}^1(\Omega) \cap \text{RT}_0^{\perp \mathbf{L}^2(\Omega)}, \\ (\mathring{\text{Rot}}_{\mathbb{S}})^{-1} : \text{Rot } \mathring{\mathbf{R}}(\Omega, \mathbb{S}) & \longrightarrow \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \cap \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}), \\ (\text{sym Rot}_{\mathbb{T}})^{-1} : \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) & \longrightarrow \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \text{Rot } \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \end{aligned}$$

are continuous with norms $(1 + c_{\mathbb{G}\mathbb{g}}^2)^{1/2}$ resp. $(1 + c_{\mathbb{D}}^2)^{1/2}$, resp. $(1 + c_{\mathbb{R}}^2)^{1/2}$ and their modifications

$$\begin{aligned} (\text{Grad } \mathring{\text{grad}})^{-1} : \text{Grad grad } \mathring{\mathbf{H}}^2(\Omega) & \longrightarrow \mathring{\mathbf{H}}^1(\Omega) \subset \mathbf{L}^2(\Omega), \\ (\text{div Div}_{\mathbb{S}})^{-1} : \mathbf{L}^2(\Omega) & \longrightarrow \text{Grad grad } \mathring{\mathbf{H}}^2(\Omega) \subset \mathbf{L}^2(\Omega, \mathbb{S}), \\ (\text{Div}_{\mathbb{T}})^{-1} : \text{RT}_0^{\perp \mathbf{L}^2(\Omega)} & \longrightarrow \text{dev Grad } \mathbf{H}^1(\Omega) \subset \mathbf{L}^2(\Omega, \mathbb{T}), \\ (\text{dev Grad})^{-1} : \text{dev Grad } \mathbf{H}^1(\Omega) & \longrightarrow \text{RT}_0^{\perp \mathbf{L}^2(\Omega)} \subset \mathbf{L}^2(\Omega), \\ (\mathring{\text{Rot}}_{\mathbb{S}})^{-1} : \text{Rot } \mathring{\mathbf{R}}(\Omega, \mathbb{S}) & \longrightarrow \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \subset \mathbf{L}^2(\Omega, \mathbb{S}), \\ (\text{sym Rot}_{\mathbb{T}})^{-1} : \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) & \longrightarrow \text{Rot } \mathring{\mathbf{R}}(\Omega, \mathbb{S}) \subset \mathbf{L}^2(\Omega, \mathbb{T}) \end{aligned}$$

are compact.

6. APPLICATION TO BIHARMONIC PROBLEMS

By $\Delta^2 = \text{div Div Grad grad}$, a standard (primal) variational formulation of (1.1) in \mathbb{R}^3 reads as follows: For given $f \in \mathbf{H}^{-2}(\Omega)$, find $u \in \mathring{\mathbf{H}}^2(\Omega)$ such that

$$(6.1) \quad \langle \text{Grad grad } u, \text{Grad grad } \phi \rangle_{\mathbf{L}^2(\Omega)} = \langle f, \phi \rangle_{\mathbf{H}^{-2}(\Omega)} \quad \text{for all } \phi \in \mathring{\mathbf{H}}^2(\Omega).$$

Existence, uniqueness, and continuous dependence on f of a solution to (6.1) is guaranteed by the theorem of Lax-Milgram, see, e.g., [16, 15] or Lemma 3.3. Note that then

$$\mathbf{M} := \text{Grad grad } u \in \mathring{\mathbf{R}}_0(\Omega, \mathbb{S}) \ominus_{\mathbf{L}^2(\Omega, \mathbb{S})} \mathcal{H}_{\mathbf{DD}}(\Omega, \mathbb{S}) \subset \mathbf{L}^2(\Omega, \mathbb{S})$$

with $\text{div Div } \mathbf{M} = f \in \mathbf{H}^{-2}(\Omega)$. In other words the operator

$$(6.2) \quad \text{div Div} : \mathbf{L}^2(\Omega, \mathbb{S}) \rightarrow \mathbf{H}^{-2}(\Omega)$$

is surjective and

$$(6.3) \quad \text{div Div} : \mathring{\mathbf{R}}_0(\Omega, \mathbb{S}) \ominus_{\mathbf{L}^2(\Omega, \mathbb{S})} \mathcal{H}_{\mathbf{DD}}(\Omega, \mathbb{S}) \rightarrow \mathbf{H}^{-2}(\Omega)$$

is bijective and even a topological isomorphism by the bounded inverse theorem. For our decomposition result we need the following variant of the Hilbert complex from Theorem 5.1.

$$\text{RT}_0 \xrightarrow{\iota_{\text{RT}_0}} \mathbf{H}^1(\Omega) \xrightarrow{-\text{dev Grad}} \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \xrightarrow{\text{sym Rot}_{\mathbb{T}}} \mathbf{DD}^{0,-1}(\Omega, \mathbb{S}) \xrightarrow{\text{div Div}_{\mathbb{S}}} \mathbf{H}^{-1}(\Omega) \xrightarrow{0} \{0\},$$

where we recall $\mathbf{DD}^{0,-1}(\Omega, \mathbb{S})$ from Lemma 4.3. This is obviously also a closed Hilbert complex as $\operatorname{div} \operatorname{Div} : \mathbf{DD}^{0,-1}(\Omega, \mathbb{S}) \rightarrow \mathbf{H}^{-1}(\Omega)$ is surjective as well by (6.2). Observe that

$$\mathbf{H}^1(\Omega, \mathbb{S}) \subset \mathbf{DD}^{0,-1}(\Omega, \mathbb{S}) \subset \mathbf{L}^2(\Omega, \mathbb{S}).$$

For right-hand sides $f \in \mathbf{H}^{-1}(\Omega)$ we consider the following mixed variational problem for u and the Hessian \mathbf{M} of u : Find $\mathbf{M} \in \mathbf{DD}^{0,-1}(\Omega, \mathbb{S})$ and $u \in \mathring{\mathbf{H}}^1(\Omega)$ such that

$$(6.4) \quad \langle \mathbf{M}, \boldsymbol{\Psi} \rangle_{\mathbf{L}^2(\Omega)} + \langle u, \operatorname{div} \operatorname{Div} \boldsymbol{\Psi} \rangle_{\mathbf{H}^{-1}(\Omega)} = 0 \quad \text{for all} \quad \boldsymbol{\Psi} \in \mathbf{DD}^{0,-1}(\Omega, \mathbb{S}),$$

$$(6.5) \quad \langle \operatorname{div} \operatorname{Div} \mathbf{M}, \psi \rangle_{\mathbf{H}^{-1}(\Omega)} = -\langle f, \psi \rangle_{\mathbf{H}^{-1}(\Omega)} \quad \text{for all} \quad \psi \in \mathring{\mathbf{H}}^1(\Omega).$$

The first row and the second row of this mixed problem are variational formulations of (1.2) and (1.3), respectively. We recall the following two results related to these mixed problems from [14].

Theorem 6.1. *Let $f \in \mathbf{H}^{-1}(\Omega)$. Then:*

(i) *Problem (6.4)-(6.5) is a well-posed saddle point problem.*

(ii) *The variational problems (6.1) and (6.4)-(6.5) are equivalent, i.e., if $u \in \mathring{\mathbf{H}}^2(\Omega)$ solves (6.1), then $\mathbf{M} = -\operatorname{Grad} \operatorname{grad} u$ lies in $\mathbf{DD}^{0,-1}(\Omega, \mathbb{S})$ with $\operatorname{div} \operatorname{Div} \mathbf{M} = -f$ and (\mathbf{M}, u) solves (6.4)-(6.5). And, vice versa, if $(\mathbf{M}, u) \in \mathbf{DD}^{0,-1}(\Omega, \mathbb{S}) \times \mathring{\mathbf{H}}^1(\Omega)$ solves (6.4)-(6.5), then $u \in \mathring{\mathbf{H}}^2(\Omega)$ with $\operatorname{Grad} \operatorname{grad} u = -\mathbf{M}$ and u solves (6.1).*

Although only two-dimensional biharmonic problems were considered in [14], the proof of the latter theorem is completely identical for the three-dimensional case. The same holds for Lemma 4.3.

Proof. To show (i), we first note that $(\boldsymbol{\Phi}, \boldsymbol{\Psi}) \mapsto \langle \boldsymbol{\Phi}, \boldsymbol{\Psi} \rangle_{\mathbf{L}^2(\Omega)}$ is coercive over the kernel of (6.5), i.e., for $\boldsymbol{\Phi} \in \mathbf{DD}_0(\Omega, \mathbb{S})$ we have $\langle \boldsymbol{\Phi}, \boldsymbol{\Phi} \rangle_{\mathbf{L}^2(\Omega)} = |\boldsymbol{\Phi}|_{\mathbf{L}^2(\Omega)}^2 = |\boldsymbol{\Phi}|_{\mathbf{DD}^{0,-1}(\Omega, \mathbb{S})}^2$. Moreover, the inf-sup-condition holds, as

$$\begin{aligned} & \inf_{0 \neq \varphi \in \mathring{\mathbf{H}}^1(\Omega)} \sup_{0 \neq \boldsymbol{\Phi} \in \mathbf{DD}^{0,-1}(\Omega, \mathbb{S})} \frac{\langle \varphi, \operatorname{div} \operatorname{Div} \boldsymbol{\Phi} \rangle_{\mathbf{H}^{-1}(\Omega)}}{|\operatorname{grad} \varphi|_{\mathbf{L}^2(\Omega)} |\boldsymbol{\Phi}|_{\mathbf{DD}^{0,-1}(\Omega, \mathbb{S})}} \\ & \geq \inf_{0 \neq \varphi \in \mathring{\mathbf{H}}^1(\Omega)} \frac{-\langle \varphi, \operatorname{div} \operatorname{Div} \varphi \mathbf{I} \rangle_{\mathbf{H}^{-1}(\Omega)}}{|\operatorname{grad} \varphi|_{\mathbf{L}^2(\Omega)} |\varphi \mathbf{I}|_{\mathbf{DD}^{0,-1}(\Omega, \mathbb{S})}} = \inf_{0 \neq \varphi \in \mathring{\mathbf{H}}^1(\Omega)} \frac{|\operatorname{grad} \varphi|_{\mathbf{L}^2(\Omega)}}{\left(|\varphi \mathbf{I}|_{\mathbf{L}^2(\Omega)}^2 + |\operatorname{div} \operatorname{Div}(\varphi \mathbf{I})|_{\mathbf{H}^{-1}(\Omega)}^2 \right)^{1/2}} \\ & = \inf_{0 \neq \varphi \in \mathring{\mathbf{H}}^1(\Omega)} \frac{|\operatorname{grad} \varphi|_{\mathbf{L}^2(\Omega)}}{\left(3|\varphi|_{\mathbf{L}^2(\Omega)}^2 + |\operatorname{grad} \varphi|_{\mathbf{L}^2(\Omega)}^2 \right)^{1/2}} \geq (3c_g^2 + 1)^{-1/2} \end{aligned}$$

by choosing $\boldsymbol{\Phi} := -\varphi \mathbf{I} \in \mathring{\mathbf{H}}^1(\Omega) \cdot \mathbf{I} \subset \mathbf{DD}^{0,-1}(\Omega, \mathbb{S})$ and observing

$$\begin{aligned} -\langle \varphi, \operatorname{div} \operatorname{Div}(\varphi \mathbf{I}) \rangle_{\mathbf{H}^{-1}(\Omega)} &= -\langle \varphi, \operatorname{div} \operatorname{grad} \varphi \rangle_{\mathbf{H}^{-1}(\Omega)} = |\operatorname{grad} \varphi|_{\mathbf{L}^2(\Omega)}^2, \\ |\operatorname{div} \operatorname{Div}(\varphi \mathbf{I})|_{\mathbf{H}^{-1}(\Omega)} &= \sup_{0 \neq \phi \in \mathring{\mathbf{H}}^1(\Omega)} \frac{\langle \phi, \operatorname{div} \operatorname{grad} \varphi \rangle_{\mathbf{H}^{-1}(\Omega)}}{|\operatorname{grad} \phi|_{\mathbf{L}^2(\Omega)}} \\ &= \sup_{0 \neq \phi \in \mathring{\mathbf{H}}^1(\Omega)} \frac{\langle \operatorname{grad} \phi, \operatorname{grad} \varphi \rangle_{\mathbf{L}^2(\Omega)}}{|\operatorname{grad} \phi|_{\mathbf{L}^2(\Omega)}} = |\operatorname{grad} \varphi|_{\mathbf{L}^2(\Omega)}. \end{aligned}$$

To show (ii), let $u \in \mathring{\mathbf{H}}^2(\Omega)$ solve (6.1). Then $\mathbf{M} := -\operatorname{Grad} \operatorname{grad} u \in \mathbf{DD}^{0,-1}(\Omega, \mathbb{S})$ with $\operatorname{div} \operatorname{Div} \mathbf{M} = -f$ in $\mathbf{H}^{-2}(\Omega)$ and hence in $\mathbf{H}^{-1}(\Omega)$. Thus (6.5) holds. Moreover, for $\boldsymbol{\Psi} \in \mathbf{DD}^{0,-1}(\Omega, \mathbb{S})$ we see

$$\langle \mathbf{M}, \boldsymbol{\Psi} \rangle_{\mathbf{L}^2(\Omega)} = -\langle \operatorname{Grad} \operatorname{grad} u, \boldsymbol{\Psi} \rangle_{\mathbf{L}^2(\Omega)} = -\langle u, \operatorname{div} \operatorname{Div} \boldsymbol{\Psi} \rangle_{\mathbf{H}^{-2}(\Omega)} = -\langle u, \operatorname{div} \operatorname{Div} \boldsymbol{\Psi} \rangle_{\mathbf{H}^{-1}(\Omega)}$$

and hence (6.4) is true. Therefore, (\mathbf{M}, u) solves (6.4)-(6.5). On the other hand, if the solution (\mathbf{M}, u) in $\mathbf{DD}^{0,-1}(\Omega, \mathbb{S}) \times \mathring{\mathbf{H}}^1(\Omega)$ solves (6.4)-(6.5), then $\operatorname{div} \operatorname{Div} \mathbf{M} = -f$ in $\mathbf{H}^{-1}(\Omega)$ and (6.4) holds. Especially, (6.4) holds for $\Psi \in \mathbf{H}^2(\Omega, \mathbb{S}) \subset \mathbf{H}^1(\Omega, \mathbb{S}) \subset \mathbf{DD}^{0,-1}(\Omega, \mathbb{S})$, i.e.,

$$(6.6) \quad -\langle \mathbf{M}, \Psi \rangle_{\mathbf{L}^2(\Omega)} = \langle u, \operatorname{div} \operatorname{Div} \Psi \rangle_{\mathbf{H}^{-1}(\Omega)} = \langle u, \operatorname{div} \operatorname{Div} \Psi \rangle_{\mathbf{L}^2(\Omega)}.$$

But then (6.6) holds for all $\Psi \in \mathbf{H}^2(\Omega)$ as $\operatorname{sym} \Psi \in \mathbf{H}^2(\Omega, \mathbb{S})$ and

$$(6.7) \quad -\langle \mathbf{M}, \Psi \rangle_{\mathbf{L}^2(\Omega)} = -\langle \mathbf{M}, \operatorname{sym} \Psi \rangle_{\mathbf{L}^2(\Omega)} = \langle u, \operatorname{div} \operatorname{Div} \operatorname{sym} \Psi \rangle_{\mathbf{L}^2(\Omega)} = \langle u, \operatorname{div} \operatorname{Div} \Psi \rangle_{\mathbf{L}^2(\Omega)},$$

since $\operatorname{div} \operatorname{Div} \operatorname{skw} \Psi = 0$ by

$$\langle \operatorname{div} \operatorname{Div} \operatorname{skw} \Psi, \phi \rangle_{\mathbf{L}^2(\Omega)} = \langle \operatorname{skw} \Psi, \operatorname{Grad} \operatorname{grad} \phi \rangle_{\mathbf{L}^2(\Omega)} = 0$$

for all $\phi \in \mathring{\mathbf{C}}^\infty(\Omega)$. (6.7) yields that $u \in \mathring{\mathbf{H}}^2(\Omega)$ with $\operatorname{Grad} \operatorname{grad} u = -\mathbf{M}$. Finally, for all $\phi \in \mathring{\mathbf{H}}^2(\Omega)$

$$\langle \operatorname{Grad} \operatorname{grad} u, \operatorname{Grad} \operatorname{grad} \phi \rangle_{\mathbf{L}^2(\Omega)} = -\langle \mathbf{M}, \operatorname{Grad} \operatorname{grad} \phi \rangle_{\mathbf{L}^2(\Omega)} = -\langle \operatorname{div} \operatorname{Div} \mathbf{M}, \phi \rangle_{\mathbf{H}^{-2}(\Omega)} = \langle f, \phi \rangle_{\mathbf{H}^{-2}(\Omega)},$$

showing that $u \in \mathring{\mathbf{H}}^2(\Omega)$ solves (6.1). \square

We note that the decomposition of $\mathbf{DD}^{0,-1}(\Omega, \mathbb{S})$ in Lemma 4.3 is different to the Helmholtz type decomposition of the larger space $\mathbf{L}^2(\Omega, \mathbb{S})$ in Theorem 3.14 and Theorem 5.1 and does not involve the Hessian of scalar functions in $\mathring{\mathbf{H}}^2(\Omega)$. Using the decomposition of $\mathbf{DD}^{0,-1}(\Omega, \mathbb{S})$ in Lemma 4.3, we have the following decomposition result for the biharmonic problem. Let $(\mathbf{M}, u) \in \mathbf{DD}^{0,-1}(\Omega, \mathbb{S}) \times \mathring{\mathbf{H}}^1(\Omega)$ be the unique solution^{vi} of (6.4)-(6.5). Using Lemma 4.3 we have the following direct decompositions for $\mathbf{M}, \Psi \in \mathbf{DD}^{0,-1}(\Omega, \mathbb{S})$

$$\mathbf{M} = p \mathbf{I} + \mathbf{M}_0, \quad \Psi = \varphi \mathbf{I} + \Psi_0, \quad p, \varphi \in \mathring{\mathbf{H}}^1(\Omega), \quad \mathbf{M}_0, \Psi_0 \in \mathbf{DD}_0(\Omega, \mathbb{S}).$$

This allows to rewrite (6.4)-(6.5) equivalently in terms of (p, \mathbf{M}_0, u) and for all (φ, Ψ_0, ψ) , i.e.,

$$\begin{aligned} \langle p \mathbf{I}, \varphi \mathbf{I} \rangle_{\mathbf{L}^2(\Omega)} + \langle \mathbf{M}_0, \Psi_0 \rangle_{\mathbf{L}^2(\Omega)} + \langle p \mathbf{I}, \Psi_0 \rangle_{\mathbf{L}^2(\Omega)} + \langle \mathbf{M}_0, \varphi \mathbf{I} \rangle_{\mathbf{L}^2(\Omega)} + \langle u, \operatorname{div} \operatorname{Div}(\varphi \mathbf{I}) \rangle_{\mathbf{H}^{-1}(\Omega)} &= 0, \\ \langle \operatorname{div} \operatorname{Div}(p \mathbf{I}), \psi \rangle_{\mathbf{H}^{-1}(\Omega)} &= -\langle f, \psi \rangle_{\mathbf{H}^{-1}(\Omega)} \end{aligned}$$

or equivalently

$$\begin{aligned} \langle \operatorname{grad} u, \operatorname{grad} \varphi \rangle_{\mathbf{L}^2(\Omega)} + 3\langle p, \varphi \rangle_{\mathbf{L}^2(\Omega)} + \langle \mathbf{M}_0, \Psi_0 \rangle_{\mathbf{L}^2(\Omega)} + \langle p, \operatorname{tr} \Psi_0 \rangle_{\mathbf{L}^2(\Omega)} + \langle \operatorname{tr} \mathbf{M}_0, \varphi \rangle_{\mathbf{L}^2(\Omega)} &= 0, \\ \langle \operatorname{grad} p, \operatorname{grad} \psi \rangle_{\mathbf{L}^2(\Omega)} &= -\langle f, \psi \rangle_{\mathbf{H}^{-1}(\Omega)}, \end{aligned}$$

which leads to the equivalent system

$$\begin{aligned} \langle \operatorname{grad} u, \operatorname{grad} \varphi \rangle_{\mathbf{L}^2(\Omega)} + 3\langle p, \varphi \rangle_{\mathbf{L}^2(\Omega)} + \langle \operatorname{tr} \mathbf{M}_0, \varphi \rangle_{\mathbf{L}^2(\Omega)} &= 0, \\ + \langle \mathbf{M}_0, \Psi_0 \rangle_{\mathbf{L}^2(\Omega)} + \langle p, \operatorname{tr} \Psi_0 \rangle_{\mathbf{L}^2(\Omega)} &= 0, \\ \langle \operatorname{grad} p, \operatorname{grad} \psi \rangle_{\mathbf{L}^2(\Omega)} &= -\langle f, \psi \rangle_{\mathbf{H}^{-1}(\Omega)}. \end{aligned}$$

Theorem 6.2. *The variational problem (6.4)-(6.5) is equivalent to the following well-posed and uniquely solvable variational problem. For $f \in \mathbf{H}^{-1}(\Omega)$ find $p \in \mathring{\mathbf{H}}^1(\Omega)$, $\mathbf{M}_0 \in \mathbf{DD}_0(\Omega, \mathbb{S})$, and $u \in \mathring{\mathbf{H}}^1(\Omega)$ such that*

$$(6.8) \quad \langle \operatorname{grad} u, \operatorname{grad} \varphi \rangle_{\mathbf{L}^2(\Omega)} + \langle \operatorname{tr} \mathbf{M}_0, \varphi \rangle_{\mathbf{L}^2(\Omega)} + 3\langle p, \varphi \rangle_{\mathbf{L}^2(\Omega)} = 0,$$

$$(6.9) \quad \langle \mathbf{M}_0, \Psi_0 \rangle_{\mathbf{L}^2(\Omega)} + \langle p, \operatorname{tr} \Psi_0 \rangle_{\mathbf{L}^2(\Omega)} = 0,$$

^{vi}By Theorem 6.1 we even have $u \in \mathring{\mathbf{H}}^2(\Omega)$ and

$$\begin{aligned} \mathbf{M} &= -\operatorname{Grad} \operatorname{grad} u \in \mathbf{DD}^{0,-1}(\Omega, \mathbb{S}) \cap \operatorname{Grad} \operatorname{grad} \mathring{\mathbf{H}}^2(\Omega) \\ &= (\mathbf{DD}^{0,-1}(\Omega, \mathbb{S}) \cap \mathring{\mathbf{R}}_0(\Omega, \mathbb{S})) \ominus_{\mathbf{L}^2(\Omega, \mathbb{S})} \mathcal{H}_{\mathbf{DD}}(\Omega, \mathbb{S}) \subset \mathbf{L}^2(\Omega, \mathbb{S}) \end{aligned}$$

as $\operatorname{div} \operatorname{Div} \mathbf{M} = -f$.

$$(6.10) \quad \langle \text{grad } p, \text{grad } \psi \rangle_{\mathbf{L}^2(\Omega)} = -\langle f, \psi \rangle_{\mathbf{H}^{-1}(\Omega)}$$

for all $\psi \in \mathring{\mathbf{H}}^1(\Omega)$, $\Psi_0 \in \mathbf{DD}_0(\Omega, \mathbb{S})$, and $\varphi \in \mathring{\mathbf{H}}^1(\Omega)$. Moreover, the unique solution (\mathbf{M}, u) of (6.4)-(6.5) is given by $\mathbf{M} := p\mathbf{I} + \mathbf{M}_0$ and u for the unique solution (p, \mathbf{M}_0, u) of (6.11)-(6.13).

If Ω is additionally topologically trivial, then by Theorem 3.14 or Theorem 5.1

$$\mathbf{DD}_0(\Omega, \mathbb{S}) = \text{sym Rot } \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) = \text{sym Rot } (\mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega, \mathbb{T}))$$

and we obtain the following result.

Theorem 6.3. *Let Ω be additionally topologically trivial. The variational problem (6.4)-(6.5) is equivalent to the following well-posed and uniquely solvable variational problem. For $f \in \mathbf{H}^{-1}(\Omega)$ find $p \in \mathring{\mathbf{H}}^1(\Omega)$, $\mathbf{N} \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega, \mathbb{T})$, and $u \in \mathring{\mathbf{H}}^1(\Omega)$ such that*

$$(6.11) \quad \langle \text{grad } u, \text{grad } \varphi \rangle_{\mathbf{L}^2(\Omega)} + \langle \text{tr sym Rot } \mathbf{N}, \varphi \rangle_{\mathbf{L}^2(\Omega)} + 3\langle p, \varphi \rangle_{\mathbf{L}^2(\Omega)} = 0,$$

$$(6.12) \quad \langle \text{sym Rot } \mathbf{N}, \text{sym Rot } \Phi \rangle_{\mathbf{L}^2(\Omega)} + \langle p, \text{tr sym Rot } \Phi \rangle_{\mathbf{L}^2(\Omega)} = 0,$$

$$(6.13) \quad \langle \text{grad } p, \text{grad } \psi \rangle_{\mathbf{L}^2(\Omega)} = -\langle f, \psi \rangle_{\mathbf{H}^{-1}(\Omega)}$$

for all $\psi \in \mathring{\mathbf{H}}^1(\Omega)$, $\Phi \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega, \mathbb{T})$, and $\varphi \in \mathring{\mathbf{H}}^1(\Omega)$. Moreover, the unique solution (\mathbf{M}, u) of (6.4)-(6.5) is given by $\mathbf{M} := p\mathbf{I} + \text{sym Rot } \mathbf{N}$ and u for the unique solution (p, \mathbf{N}, u) of (6.11)-(6.13).

Note that, e.g., $\langle \text{tr sym Rot } \mathbf{N}, \varphi \rangle_{\mathbf{L}^2(\Omega)} = \langle \text{sym Rot } \mathbf{N}, \varphi \mathbf{I} \rangle_{\mathbf{L}^2(\Omega)}$ and $3\langle p, \varphi \rangle_{\mathbf{L}^2(\Omega)} = \langle p\mathbf{I}, \varphi \mathbf{I} \rangle_{\mathbf{L}^2(\Omega)}$.

Proof. (6.4)-(6.5) is equivalent to (6.8)-(6.10) and hence also to (6.11)-(6.13), if the latter system is well-posed. By Theorem 3.14 or Theorem 5.1 the bilinear form $\langle \text{sym Rot } \cdot, \text{sym Rot } \cdot \rangle_{\mathbf{L}^2(\Omega)}$ is coercive over $\mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega, \mathbb{T})$, which shows the consecutive unique solvability of (6.11)-(6.13). \square

The three problems in the previous theorem are weak formulations of the following three second-order problems in strong form. A homogeneous Dirichlet Poisson problem for the auxiliary scalar function p

$$\Delta p = f \quad \text{in } \Omega, \quad p = 0 \quad \text{on } \Gamma,$$

a second-order inhomogeneous Neumann type Rot sym Rot-Div-system for the auxiliary tensor field \mathbf{N}

$$\begin{aligned} \text{tr } \mathbf{N} = 0, \quad \text{Rot sym Rot } \mathbf{N} = -\text{Rot}(p\mathbf{I}) = \text{spn grad } p, \quad \text{Div } \mathbf{N} = 0 & \quad \text{in } \Omega, \\ n \times \text{sym Rot } \mathbf{N} = -n \times p\mathbf{I} = p \text{spn } n = \mathbf{0}, \quad \mathbf{N}n = 0 & \quad \text{on } \Gamma, \end{aligned}$$

and, finally, a homogeneous Dirichlet Poisson problem for the original scalar function u

$$\Delta u = 3p + \text{tr sym Rot } \mathbf{N} = \text{tr}(p\mathbf{I} + \text{sym Rot } \mathbf{N}) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Gamma.$$

In other words, the system (6.11)-(6.13) has triangular structure

$$\begin{bmatrix} \mathring{\Delta} & -\text{tr sym Rot } \mathbb{T} & -3 \\ 0 & \overset{(\circ)}{\text{Rot}}_{\mathbb{S}} \text{sym Rot } \mathbb{T} & \overset{(\circ)}{\text{Rot}}_{\mathbb{S}}(\cdot)\mathbf{I} \\ 0 & 0 & \mathring{\Delta} \end{bmatrix} \begin{bmatrix} u \\ \mathbf{N} \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{0} \\ f \end{bmatrix}, \quad (\text{tr sym Rot } \mathbb{T})^* = \overset{\circ}{\text{Rot}}_{\mathbb{S}}(\cdot)\mathbf{I}.$$

Indeed we see that $\mathbf{N} \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega, \mathbb{T})$ with

$$\langle \text{sym Rot } \mathbf{N}, \text{sym Rot } \Phi \rangle_{\mathbf{L}^2(\Omega)} + \langle p, \text{tr sym Rot } \Phi \rangle_{\mathbf{L}^2(\Omega)} = 0$$

for all $\Phi \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega, \mathbb{T})$ is equivalent to $\mathbf{N} \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega, \mathbb{T})$ and

$$(6.14) \quad \langle \text{sym Rot } \mathbf{N} + p\mathbf{I}, \text{sym Rot } \Phi \rangle_{\mathbf{L}^2(\Omega)} = 0$$

for all $\Phi \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T})$ as by Theorem 3.14

$$(6.15) \quad \text{sym Rot} \left(\mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \overset{\circ}{\mathbf{D}}_0(\Omega, \mathbb{T}) \right) = \text{sym Rot} \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}).$$

Now (6.14) shows that

$$\text{sym Rot} \mathbf{N} + p \mathbf{I} \in D(\text{sym Rot}_{\mathbb{T}}^*) = D(\text{Rot}_{\mathbb{S}}) = \overset{\circ}{\mathbf{R}}(\Omega, \mathbb{S})$$

with $\text{Rot}(\text{sym Rot} \mathbf{N} + p \mathbf{I}) = 0$.

Finally, we want to get rid of the complicated space $\mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \overset{\circ}{\mathbf{D}}_0(\Omega, \mathbb{T})$ in the variational formulation in Theorem 6.3, which might be very complicated to implement in forthcoming numerical applications using finite elements due to the solenoidal and homogeneous normal boundary conditions. For given $p \in \overset{\circ}{\mathbf{H}}^1(\Omega)$ the part (6.12), i.e., find $\mathbf{N} \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \overset{\circ}{\mathbf{D}}_0(\Omega, \mathbb{T})$ such that

$$(6.16) \quad \langle \text{sym Rot} \mathbf{N}, \text{sym Rot} \Phi \rangle_{\mathbf{L}^2(\Omega)} + \langle p, \text{tr sym Rot} \Phi \rangle_{\mathbf{L}^2(\Omega)} = 0$$

for all $\Phi \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \cap \overset{\circ}{\mathbf{D}}_0(\Omega, \mathbb{T})$, of the variational system (6.11)-(6.13), has also a saddle point structure. By Theorem 3.14 we have (6.15) as well as

$$\overset{\circ}{\mathbf{D}}_0(\Omega, \mathbb{T}) = \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T})^{\perp_{\mathbf{L}^2(\Omega, \mathbb{T})}} = \text{dev Grad} \left(\mathbf{H}^1(\Omega) \cap \text{RT}_0^{\perp_{\mathbf{L}^2(\Omega)}} \right)^{\perp_{\mathbf{L}^2(\Omega)}}.$$

Hence (6.16) is equivalent to find $\mathbf{N} \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T})$ such that

$$(6.17) \quad \langle \text{sym Rot} \mathbf{N}, \text{sym Rot} \Phi \rangle_{\mathbf{L}^2(\Omega)} + \langle p, \text{tr sym Rot} \Phi \rangle_{\mathbf{L}^2(\Omega)} = 0,$$

$$(6.18) \quad \langle \mathbf{N}, \text{dev Grad} \Phi \rangle_{\mathbf{L}^2(\Omega)} = 0$$

for all $\Phi \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T})$ and $\Phi \in \mathbf{H}^1(\Omega) \cap \text{RT}_0^{\perp_{\mathbf{L}^2(\Omega)}}$. Observe that

$$(\mathbf{N}, E) := (\mathbf{N}, 0) \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \times \left(\mathbf{H}^1(\Omega) \cap \text{RT}_0^{\perp_{\mathbf{L}^2(\Omega)}} \right)$$

solves the modified variational system

$$(6.19) \quad \langle \text{sym Rot} \mathbf{N}, \text{sym Rot} \Phi \rangle_{\mathbf{L}^2(\Omega)} + \langle \Phi, \text{dev Grad} E \rangle_{\mathbf{L}^2(\Omega)} = -\langle p, \text{tr sym Rot} \Phi \rangle_{\mathbf{L}^2(\Omega)},$$

$$(6.20) \quad \langle \mathbf{N}, \text{dev Grad} \Phi \rangle_{\mathbf{L}^2(\Omega)} = 0$$

for all $\Phi \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T})$ and $\Phi \in \mathbf{H}^1(\Omega) \cap \text{RT}_0^{\perp_{\mathbf{L}^2(\Omega)}}$. On the other hand, any solution

$$(\mathbf{N}, E) \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \times \left(\mathbf{H}^1(\Omega) \cap \text{RT}_0^{\perp_{\mathbf{L}^2(\Omega)}} \right)$$

of (6.19)-(6.20) satisfies $E = 0$, as (6.19) tested with

$$\Phi := \text{dev Grad} E \in \text{dev Grad} \mathbf{H}^1(\Omega) = \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T})$$

shows $\text{dev Grad} E = 0$ and thus $E \in \text{RT}_0$ by Lemma 3.2 yielding $E = 0$. Note that (6.19)-(6.20) has the saddle point structure

$$\begin{bmatrix} \overset{\circ}{\text{Rot}}_{\mathbb{S}} \text{sym Rot}_{\mathbb{T}} & \text{dev Grad} \\ -\overset{\circ}{\text{Div}}_{\mathbb{T}} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{N} \\ E \end{bmatrix} = \begin{bmatrix} -\overset{\circ}{\text{Rot}}_{\mathbb{S}}(v \cdot \mathbf{I}) \\ 0 \end{bmatrix}, \quad (\text{dev Grad})^* = -\overset{\circ}{\text{Div}}_{\mathbb{T}}.$$

We obtain the following theorem.

Theorem 6.4. *Let Ω be additionally topologically trivial. The variational problem (6.11)-(6.13) is equivalent to the following well-posed and uniquely solvable variational system. For $f \in \mathbf{H}^{-1}(\Omega)$ find $p \in \overset{\circ}{\mathbf{H}}^1(\Omega)$, $\mathbf{N} \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T})$, $E \in \mathbf{H}^1(\Omega) \cap \text{RT}_0^{\perp_{\mathbf{L}^2(\Omega)}}$, and $u \in \overset{\circ}{\mathbf{H}}^1(\Omega)$ such that*

$$(6.21) \quad \langle \text{grad} u, \text{grad} \varphi \rangle_{\mathbf{L}^2(\Omega)} + \langle \text{tr sym Rot} \mathbf{N}, \varphi \rangle_{\mathbf{L}^2(\Omega)} + 3 \langle p, \varphi \rangle_{\mathbf{L}^2(\Omega)} = 0,$$

$$(6.22) \quad \langle \text{sym Rot} \mathbf{N}, \text{sym Rot} \Phi \rangle_{\mathbf{L}^2(\Omega)} + \langle \Phi, \text{dev Grad} E \rangle_{\mathbf{L}^2(\Omega)} + \langle p, \text{tr sym Rot} \Phi \rangle_{\mathbf{L}^2(\Omega)} = 0,$$

$$(6.23) \quad \langle \mathbf{N}, \operatorname{dev} \operatorname{Grad} \Phi \rangle_{\mathbf{L}^2(\Omega)} = 0,$$

$$(6.24) \quad \langle \operatorname{grad} p, \operatorname{grad} \psi \rangle_{\mathbf{L}^2(\Omega)} = -\langle f, \psi \rangle_{\mathbf{H}^{-1}(\Omega)}$$

for all $\psi \in \mathring{\mathbf{H}}^1(\Omega)$, $\Phi \in \mathbf{R}_{\operatorname{sym}}(\Omega, \mathbb{T})$, $\Phi \in \mathbf{H}^1(\Omega) \cap \operatorname{RT}_0^{\perp \mathbf{L}^2(\Omega)}$, and $\varphi \in \mathring{\mathbf{H}}^1(\Omega)$. Moreover, the unique solution (p, \mathbf{N}, E, u) of (6.21)-(6.24) satisfies $E = 0$ and (p, \mathbf{N}, u) is the unique solution of (6.11)-(6.13).

Note that the system (6.21)-(6.24) has the block triangular saddle point structure

$$\begin{bmatrix} \mathring{\Delta} & -\operatorname{tr} \operatorname{sym} \operatorname{Rot}_{\mathbb{T}} & 0 & -3 \\ 0 & \operatorname{Rot}_{\mathbb{S}} \operatorname{sym} \operatorname{Rot}_{\mathbb{T}} & \operatorname{dev} \operatorname{Grad} & \operatorname{Rot}_{\mathbb{S}}(\cdot) \mathbf{I} \\ 0 & -\operatorname{Div}_{\mathbb{T}} & 0 & 0 \\ 0 & 0 & 0 & \mathring{\Delta} \end{bmatrix} \begin{bmatrix} u \\ \mathbf{N} \\ E \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{0} \\ 0 \\ f \end{bmatrix}.$$

with $(\operatorname{tr} \operatorname{sym} \operatorname{Rot}_{\mathbb{T}})^* = \operatorname{Rot}_{\mathbb{S}}(\cdot) \mathbf{I}$ and $(\operatorname{dev} \operatorname{Grad})^* = -\operatorname{Div}_{\mathbb{T}}$.

Proof. We only have to show well-posedness of the partial system (6.22)-(6.23). First note that by Theorem 3.14 the bilinear form $\langle \operatorname{sym} \operatorname{Rot} \cdot, \operatorname{sym} \operatorname{Rot} \cdot \rangle_{\mathbf{L}^2(\Omega)}$ is coercive over $\mathbf{R}_{\operatorname{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega, \mathbb{T})$, which equals the kernel of (6.23). Indeed it follows from (6.23) that

$$\mathbf{N} \in \operatorname{dev} \operatorname{Grad} (\mathbf{H}^1(\Omega) \cap \operatorname{RT}_0^{\perp \mathbf{L}^2(\Omega)})^{\perp \mathbf{L}^2(\Omega)} = \mathring{\mathbf{D}}_0(\Omega, \mathbb{T}).$$

Moreover, the inf-sup-condition is satisfied as by picking for fixed $0 \neq \Phi \in \mathbf{H}^1(\Omega) \cap \operatorname{RT}_0^{\perp \mathbf{L}^2(\Omega)}$ the tensor $\Phi := \operatorname{dev} \operatorname{Grad} \Phi \in \operatorname{dev} \operatorname{Grad} \mathbf{H}^1(\Omega) = \mathbf{R}_{\operatorname{sym},0}(\Omega, \mathbb{T})$ we have

$$\inf_{\substack{0 \neq \Phi \in \mathbf{H}^1(\Omega), \\ \Phi \perp_{\mathbf{L}^2(\Omega)} \operatorname{RT}_0}} \sup_{\Phi \in \mathbf{R}_{\operatorname{sym}}(\Omega, \mathbb{T})} \frac{\langle \Phi, \operatorname{dev} \operatorname{Grad} \Phi \rangle_{\mathbf{L}^2(\Omega)}}{|\Phi|_{\mathbf{R}_{\operatorname{sym}}(\Omega, \mathbb{T})} |\Phi|_{\mathbf{H}^1(\Omega)}} \geq \inf_{\substack{0 \neq \Phi \in \mathbf{H}^1(\Omega), \\ \Phi \perp_{\mathbf{L}^2(\Omega)} \operatorname{RT}_0}} \frac{|\operatorname{dev} \operatorname{Grad} \Phi|_{\mathbf{L}^2(\Omega)}}{|\Phi|_{\mathbf{H}^1(\Omega)}} \geq \frac{1}{c}$$

by Lemma 3.2 (iv). □

Remark 6.5. *The corresponding result for the two-dimensional case is completely analogous with the exception that the tensor potential $\mathbf{N} \in \mathbf{R}_{\operatorname{sym}}(\Omega, \mathbb{T}) \cap \mathring{\mathbf{D}}_0(\Omega, \mathbb{T})$ is to be replaced by a much simpler vector potential $N \in \mathbf{H}^1(\Omega)$. Furthermore, observe that*

$$\langle \operatorname{sym} \operatorname{Rot} N, \operatorname{sym} \operatorname{Rot} \Phi \rangle_{\mathbf{L}^2(\Omega)} = \langle \operatorname{sym} \operatorname{Grad}^{\perp} N, \operatorname{sym} \operatorname{Grad}^{\perp} \Phi \rangle_{\mathbf{L}^2(\Omega)}$$

holds for vector fields $N, \Phi \in \mathbf{H}^1(\Omega)$. Here the superscript \perp denotes the rotation of a vector field by 90° . Note that the complicated second-order inhomogeneous Neumann type $\operatorname{Rot} \operatorname{sym} \operatorname{Rot}$ - Div -system for the auxiliary tensor field \mathbf{N} is replaced in 2D by a much simpler inhomogeneous Neumann linear elasticity problem, where the standard Sobolev space $\mathbf{H}^1(\Omega)$ resp. $\mathbf{H}^1(\Omega) \cap \operatorname{RM}^{\perp \mathbf{L}^2(\Omega)}$ can be used, where RM denotes the space of rigid motions. This yields the decomposition result in [14] for the two-dimensional case, which was shortly mentioned in the introduction.

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APPENDIX A. PROOFS OF SOME USEFUL IDENTITIES

Proof of Lemma 3.4 and Lemma 3.12. Note that for $a, b \in \mathbb{R}^3$ and $A \in \mathbb{R}^{3 \times 3}$

$$(A.1) \quad \operatorname{spn} a : \operatorname{spn} b = 2 a \cdot b, \quad \operatorname{skw} A = \frac{1}{2} \operatorname{spn} \begin{bmatrix} A_{32} - A_{23} \\ A_{13} - A_{31} \\ A_{21} - A_{12} \end{bmatrix}$$

hold and hence for skew-symmetric A

$$(A.2) \quad \operatorname{spn} a : A = \operatorname{spn} a : \operatorname{spn} \operatorname{spn}^{-1} A = 2 a \cdot \operatorname{spn}^{-1} A,$$

i.e., $\operatorname{spn}^* = 2 \operatorname{spn}^{-1}$. Moreover, we have for two matrices A, B

$$A^\top : B = \operatorname{tr}(AB) = \operatorname{tr}(BA) = B^\top : A = A : B^\top.$$

Lemma A.1. *For smooth functions, vector fields and tensor fields we have*

- (i) $\operatorname{skw} \operatorname{Grad} \operatorname{grad} u = 0$,
- (ii) $\operatorname{div} \operatorname{Div} \mathbf{M} = 0$, if \mathbf{M} is skew-symmetric,

- (iii) $\text{Rot}(u \mathbf{I}) = -\text{spn grad } u$,
- (iv) $\text{tr Rot } \mathbf{M} = 2 \text{div}(\text{spn}^{-1} \text{skw } \mathbf{M})$,
especially $\text{tr Rot } \mathbf{M} = 0$, *if* \mathbf{M} *is symmetric*,
- (v) $\text{Div}(u \mathbf{I}) = \text{grad } u$,
- (vi) $\text{tr Grad } E = \text{div } E$,
- (vii) $\text{Div}(\text{spn } E) = -\text{rot } E$,
especially $\text{Div}(\text{skw } \mathbf{M}) = -\text{rot } E$ *for* $E = \text{spn}^{-1} \text{skw } \mathbf{M}$,
- (viii) $\text{Rot}(\text{spn } E) = (\text{div } E) \mathbf{I} - (\text{Grad } E)^\top$,
especially $\text{Rot skw } \mathbf{M} = (\text{div } E) \mathbf{I} - (\text{Grad } E)^\top$ *for* $E = \text{spn}^{-1} \text{skw } \mathbf{M}$,
- (ix) $\text{skw Grad } E = \frac{1}{2} \text{spn rot } E$ *and* $\text{Rot}(\text{sym Grad } E) = -\text{Rot}(\text{skw Grad } E) = -\frac{1}{2} \text{Rot}(\text{spn rot } E)$,
- (x) $\text{skw Rot } \mathbf{M} = \text{spn } E$ *and* $\text{Div}(\text{sym Rot } \mathbf{M}) = -\text{Div}(\text{skw Rot } \mathbf{M}) = \text{rot } E$
with $E = \frac{1}{2}(\text{Div } \mathbf{M}^\top - \text{grad}(\text{tr } \mathbf{M}))$,
especially $\text{Div}(\text{sym Rot } \mathbf{M}) = -\text{Div}(\text{skw Rot } \mathbf{M}) = \frac{1}{2} \text{rot Div } \mathbf{M}^\top$, *if* $\text{tr } \mathbf{M} = 0$.

These formulas hold for distributions as well.

(i)-(ix) and the first identity in (x) follow by elementary calculations. For the second identity in (x) observe that $0 = \text{Div Rot } \mathbf{M} = \text{Div}(\text{sym Rot } \mathbf{M}) + \text{Div}(\text{skw Rot } \mathbf{M})$ for $\mathbf{M} \in \mathring{\mathbf{C}}^\infty(\mathbb{R}^3)$ and hence, using the first identity in (x) and (vii), we obtain

$$\text{Div}(\text{sym Rot } \mathbf{M}) = -\text{Div}(\text{skw Rot } \mathbf{M}) = -\text{Div}(\text{spn } E) = \text{rot } E.$$

Therefore, the stated formulas hold in the smooth case. By density these formulas extend to u , E , and \mathbf{M} in respective Sobolev spaces. Let us give proofs for distributions as well. For this, let $m \in \mathbb{N}_0$ and $u \in \mathbf{H}^{-m}(\Omega)$, $E \in \mathbf{H}^{-m}(\Omega)$, $\mathbf{M} \in \mathbf{H}^{-m}(\Omega)$ and $\varphi \in \mathring{\mathbf{C}}^\infty(\Omega)$, $\Phi \in \mathring{\mathbf{C}}^\infty(\Omega)$, and $\Phi \in \mathring{\mathbf{C}}^\infty(\Omega)$. By

$$\langle u, \partial_i \partial_j \varphi \rangle_{\mathbf{H}^{-m}(\Omega)} = \langle u, \partial_j \partial_i \varphi \rangle_{\mathbf{H}^{-m}(\Omega)}, \quad \text{or (with (ii))} \quad \langle u, \text{div Div skw } \Phi \rangle_{\mathbf{H}^{-m}(\Omega)} = 0$$

we see that $\text{Grad grad } u \in \mathbf{H}^{-m-2}(\Omega)$ is symmetric and hence (i). Note that the formal adjoint is $(\text{skw Grad grad})^* = \text{div Div skw}$. If \mathbf{M} is skew-symmetric we have $\langle \mathbf{M}, \text{Grad grad } \varphi \rangle_{\mathbf{H}^{-m}(\Omega)} = 0$, i.e., (ii). We compute with (iv)

$$\begin{aligned} \langle u \mathbf{I}, \text{Rot } \Phi \rangle_{\mathbf{H}^{-m}(\Omega)} &= \langle u, \text{tr}(\text{Rot } \Phi) \rangle_{\mathbf{H}^{-m}(\Omega)} = 2 \langle u, \text{div}(\text{spn}^{-1} \text{skw } \Phi) \rangle_{\mathbf{H}^{-m}(\Omega)} \\ &= -\langle \text{spn grad } u, \text{skw } \Phi \rangle_{\mathbf{H}^{-m-1}(\Omega)} = -\langle \text{spn grad } u, \Phi \rangle_{\mathbf{H}^{-m-1}(\Omega)}, \end{aligned}$$

showing (iii). Formally, $(\text{tr Rot})^* = \text{Rot}(\cdot \mathbf{I})$. Hence by (iii)

$$\begin{aligned} \langle \mathbf{M}, \text{Rot}(\varphi \mathbf{I}) \rangle_{\mathbf{H}^{-m}(\Omega)} &= -\langle \mathbf{M}, \text{spn grad } \varphi \rangle_{\mathbf{H}^{-m}(\Omega)} = -\langle \text{skw } \mathbf{M}, \text{spn grad } \varphi \rangle_{\mathbf{H}^{-m}(\Omega)} \\ &= -2 \langle \text{spn}^{-1} \text{skw } \mathbf{M}, \text{grad } \varphi \rangle_{\mathbf{H}^{-m}(\Omega)} = 2 \langle \text{div spn}^{-1} \text{skw } \mathbf{M}, \varphi \rangle_{\mathbf{H}^{-m-1}(\Omega)}, \end{aligned}$$

yielding (iv). (v) follows by

$$-\langle u \mathbf{I}, \text{Grad } \Phi \rangle_{\mathbf{H}^{-m}(\Omega)} = -\langle u, \text{tr}(\text{Grad } \Phi) \rangle_{\mathbf{H}^{-m}(\Omega)} = -\langle u, \text{div } \Phi \rangle_{\mathbf{H}^{-m}(\Omega)}.$$

Formally, $(\text{tr Grad})^* = -\text{Div}(\cdot \mathbf{I})$. Thus by (v)

$$-\langle E, \text{Div}(\varphi \mathbf{I}) \rangle_{\mathbf{H}^{-m}(\Omega)} = -\langle E, \text{grad } \varphi \rangle_{\mathbf{H}^{-m}(\Omega)} = \langle \text{div } E, \varphi \rangle_{\mathbf{H}^{-m-1}(\Omega)},$$

yielding (vi). We have the formal adjoint $(\text{Div spn})^* = (\text{Div skw spn})^* = -2 \text{spn}^{-1} \text{skw Grad}$, and by the formula $2 \text{skw Grad } \Phi = \text{spn rot } \Phi$ from (ix), we obtain (vii), i.e.,

$$-2 \langle E, \text{spn}^{-1} \text{skw Grad } \Phi \rangle_{\mathbf{H}^{-m}(\Omega)} = -\langle E, \text{rot } \Phi \rangle_{\mathbf{H}^{-m}(\Omega)}.$$

Using the formal adjoint $(\text{Rot spn})^* = 2 \text{spn}^{-1} \text{skw Rot}$ we calculate with (x)

$$\begin{aligned} 2 \langle E, \text{spn}^{-1} \text{skw Rot } \Phi \rangle_{\mathbf{H}^{-m}(\Omega)} &= \langle E, \text{Div } \Phi^\top - \text{grad}(\text{tr } \Phi) \rangle_{\mathbf{H}^{-m}(\Omega)} \\ &= -\langle \text{Grad } E, \Phi^\top \rangle_{\mathbf{H}^{-m-1}(\Omega)} + \langle \text{div } E, \text{tr } \Phi \rangle_{\mathbf{H}^{-m-1}(\Omega)}, \end{aligned}$$

i.e., (viii) holds. Formally $(\text{skw Grad})^* = -\text{Div skw}$. Using (vii) we see

$$-\langle E, \text{Div skw } \Phi \rangle_{\mathbf{H}^{-m}(\Omega)} = \langle E, \text{rot spn}^{-1} \text{skw } \Phi \rangle_{\mathbf{H}^{-m}(\Omega)} = \frac{1}{2} \langle \text{spn rot } E, \text{skw } \Phi \rangle_{\mathbf{H}^{-m-1}(\Omega)},$$

which proves (ix). We compute by (viii)

$$\begin{aligned} \langle \mathbf{M}, \text{Rot skw } \Phi \rangle_{\mathbf{H}^{-m}(\Omega)} &= \langle \text{tr } \mathbf{M}, \text{div}(\text{spn}^{-1} \text{skw } \Phi) \rangle_{\mathbf{H}^{-m}(\Omega)} - \langle \mathbf{M}^\top, \text{Grad}(\text{spn}^{-1} \text{skw } \Phi) \rangle_{\mathbf{H}^{-m}(\Omega)} \\ &= -\langle \text{grad}(\text{tr } \mathbf{M}), \text{spn}^{-1} \text{skw } \Phi \rangle_{\mathbf{H}^{-m-1}(\Omega)} + \langle \text{Div } \mathbf{M}^\top, \text{spn}^{-1} \text{skw } \Phi \rangle_{\mathbf{H}^{-m-1}(\Omega)} \\ &= -\frac{1}{2} \langle \text{spn}(\text{grad tr } \mathbf{M}), \text{skw } \Phi \rangle_{\mathbf{H}^{-m-1}(\Omega)} + \frac{1}{2} \langle \text{spn Div } \mathbf{M}^\top, \text{skw } \Phi \rangle_{\mathbf{H}^{-m-1}(\Omega)}, \end{aligned}$$

showing the first formula in (x) and the second one follows by $\text{Div Rot} = 0$ and (vii). \square

Proof of Lemma 4.2. For $\mathbf{M} \in \mathring{\mathbf{R}}(\Omega, \mathbb{S})$ there exists a sequence $(\Phi_n) \subset \mathring{\mathbf{C}}^\infty(\Omega) \cap \mathbf{L}^2(\Omega, \mathbb{S})$ with $\Phi_n \rightarrow \mathbf{M}$ in $\mathbf{R}(\Omega)$. But then $(\varphi \Phi_n) \subset \mathring{\mathbf{C}}^\infty(\Omega) \cap \mathbf{L}^2(\Omega, \mathbb{S})$ with $\varphi \Phi_n \rightarrow \varphi \mathbf{M}$ in $\mathbf{R}(\Omega)$, proving $\varphi \mathbf{M} \in \mathring{\mathbf{R}}(\Omega, \mathbb{S})$, as we have $\text{Rot}(\varphi \Phi_n) = \varphi \text{Rot } \Phi_n + \text{grad } \varphi \times \Phi_n$. This formula also shows for $\Psi \in \mathring{\mathbf{C}}^\infty(\Omega)$ (note $\varphi \Psi \in \mathring{\mathbf{C}}^\infty(\Omega)$)

$$\begin{aligned} \langle \varphi \mathbf{M}, \text{Rot } \Psi \rangle_{\mathbf{L}^2(\Omega)} &= \langle \mathbf{M}, \varphi \text{Rot } \Psi \rangle_{\mathbf{L}^2(\Omega)} = \langle \mathbf{M}, \text{Rot}(\varphi \Psi) \rangle_{\mathbf{L}^2(\Omega)} - \langle \mathbf{M}, \text{grad } \varphi \times \Psi \rangle_{\mathbf{L}^2(\Omega)} \\ \text{(A.3)} \quad &= \langle \text{Rot } \mathbf{M}, \varphi \Psi \rangle_{\mathbf{L}^2(\Omega)} + \langle \text{grad } \varphi \times \mathbf{M}, \Psi \rangle_{\mathbf{L}^2(\Omega)}, \end{aligned}$$

and thus $\text{Rot}(\varphi \mathbf{M}) = \varphi \text{Rot } \mathbf{M} + \text{grad } \varphi \times \mathbf{M}$. Analogously we prove the other cases of (i). Similarly we show (iii) using the formula $\text{Div}(\varphi \Phi_n) = \varphi \text{Div } \Phi_n + \text{grad } \varphi \cdot \Phi_n$. To show (ii), let $\mathbf{M} \in \mathbf{R}(\Omega, \mathbb{S})$. Then $\varphi \mathbf{M} \in \mathbf{L}^2(\Omega, \mathbb{S})$ and (A.3) shows $\varphi \mathbf{M} \in \mathbf{R}(\Omega, \mathbb{S})$ with the desired formula. Analogously the other cases of (ii) follow. Similarly we prove (iv). Let $\mathbf{N} \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T})$ and $\Phi \in \mathring{\mathbf{C}}^\infty(\Omega)$. Then $\varphi \mathbf{N} \in \mathbf{L}^2(\Omega, \mathbb{T})$ and with $\varphi \Phi \in \mathring{\mathbf{C}}^\infty(\Omega)$ we get

$$\begin{aligned} \langle \varphi \mathbf{N}, \text{Rot sym } \Phi \rangle_{\mathbf{L}^2(\Omega)} &= \langle \mathbf{N}, \varphi \text{Rot sym } \Phi \rangle_{\mathbf{L}^2(\Omega)} = \langle \mathbf{N}, \text{Rot sym}(\varphi \Phi) \rangle_{\mathbf{L}^2(\Omega)} - \langle \mathbf{N}, \text{grad } \varphi \times \text{sym } \Phi \rangle_{\mathbf{L}^2(\Omega)} \\ &= \langle \text{sym Rot } \mathbf{N}, \varphi \Phi \rangle_{\mathbf{L}^2(\Omega)} + \langle \text{grad } \varphi \times \mathbf{N}, \text{sym } \Phi \rangle_{\mathbf{L}^2(\Omega)}, \end{aligned}$$

which shows $\varphi \mathbf{N} \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T})$ and $\text{sym Rot}(\varphi \mathbf{N}) = \varphi \text{sym Rot } \mathbf{N} + \text{sym}(\text{grad } \varphi \times \mathbf{N})$ and hence (v). To prove (vi), let $\mathbf{M} \in \mathbf{DD}(\Omega, \mathbb{S})$ and $\phi \in \mathring{\mathbf{C}}^\infty(\Omega)$. Then $\varphi \mathbf{M} \in \mathbf{L}^2(\Omega, \mathbb{S})$ and we compute by

$$\begin{aligned} \text{Grad grad}(\varphi \phi) &= \varphi \text{Grad grad } \phi + \phi \text{Grad grad } \varphi + 2 \text{sym}((\text{grad } \varphi)(\text{grad } \phi)^\top), \\ (\text{grad } \varphi)(\text{grad } \phi)^\top &= \text{Grad}(\phi \text{grad } \varphi) - \phi \text{Grad grad } \varphi \end{aligned}$$

the identity

$$\text{Grad grad}(\varphi \phi) = \varphi \text{Grad grad } \phi - \phi \text{Grad grad } \varphi + 2 \text{sym}(\text{Grad}(\phi \text{grad } \varphi)).$$

Finally with $\varphi \phi \in \mathring{\mathbf{C}}^\infty(\Omega)$ we get

$$\begin{aligned} \langle \varphi \mathbf{M}, \text{Grad grad } \phi \rangle_{\mathbf{L}^2(\Omega)} &= \langle \mathbf{M}, \varphi \text{Grad grad } \phi \rangle_{\mathbf{L}^2(\Omega)} \\ &= \langle \mathbf{M}, \text{Grad grad}(\varphi \phi) \rangle_{\mathbf{L}^2(\Omega)} + \langle \mathbf{M}, \phi \text{Grad grad } \varphi \rangle_{\mathbf{L}^2(\Omega)} - 2 \langle \mathbf{M}, \text{sym}(\text{Grad}(\phi \text{grad } \varphi)) \rangle_{\mathbf{L}^2(\Omega)} \\ &= \langle \text{div Div } \mathbf{M}, \varphi \phi \rangle_{\mathbf{L}^2(\Omega)} + \langle \mathbf{M} : \text{Grad grad } \varphi, \phi \rangle_{\mathbf{L}^2(\Omega)} - 2 \langle \mathbf{M}, \text{Grad}(\phi \text{grad } \varphi) \rangle_{\mathbf{L}^2(\Omega)} \\ &= \langle \varphi \text{div Div } \mathbf{M}, \phi \rangle_{\mathbf{L}^2(\Omega)} + \langle \text{tr}(\mathbf{M} \cdot \text{Grad grad } \varphi), \phi \rangle_{\mathbf{L}^2(\Omega)} + 2 \underbrace{\langle \text{Div } \mathbf{M}, \phi \text{grad } \varphi \rangle_{\mathbf{H}^{-1}(\Omega)}}_{= \langle \text{Div } \mathbf{M} \cdot \text{grad } \varphi, \phi \rangle_{\mathbf{H}^{-1}(\Omega)}}, \end{aligned}$$

which shows (vi), i.e., $\varphi \mathbf{M} \in \mathbf{DD}^{0,-1}(\Omega, \mathbb{S})$ and

$$\text{div Div}(\varphi \mathbf{M}) = \varphi \text{div Div } \mathbf{M} + 2 \text{grad } \varphi \cdot \text{Div } \mathbf{M} + \text{tr}(\text{Grad grad } \varphi \cdot \mathbf{M}).$$

The proof is finished. \square

APPENDIX B. A REGULAR DECOMPOSITION FOR TENSOR FIELDS WITH SYMMETRIC ROTATION

Lemma B.1. *Let Ω be additionally topologically trivial. Then*

$$\mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) = \mathbf{H}^1(\Omega, \mathbb{T}) + \text{dev Grad } \mathbf{H}^1(\Omega)$$

with linear and continuous potential operators

$$P_{\mathbf{H}^1} : \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \longrightarrow \mathbf{H}^1(\Omega, \mathbb{T}), \quad P_{\mathbf{H}^1} : \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T}) \longrightarrow \mathbf{H}^1(\Omega).$$

Proof. Let $\mathbf{N} \in \mathbf{R}_{\text{sym}}(\Omega, \mathbb{T})$. Then

$$\text{sym Rot } \mathbf{N} \in \mathbf{DD}_0(\Omega, \mathbb{S}) = \text{sym Rot } \mathbf{H}^1(\Omega, \mathbb{T})$$

with linear and continuous potential operator $P_{\text{sym Rot}} : \mathbf{DD}_0(\Omega, \mathbb{S}) \longrightarrow \mathbf{H}^1(\Omega, \mathbb{T})$ by Lemma 3.13 (i). Thus, there is $\tilde{\mathbf{N}} \in \mathbf{H}^1(\Omega, \mathbb{T})$ depending linearly and continuously on \mathbf{N} with $\text{sym Rot } \tilde{\mathbf{N}} = \text{sym Rot } \mathbf{N}$. Hence,

$$\mathbf{N} - \tilde{\mathbf{N}} \in \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}) = \text{dev Grad } \mathbf{H}^1(\Omega)$$

with linear and continuous potential operator $P_{\text{dev Grad}} : \mathbf{R}_{\text{sym},0}(\Omega, \mathbb{T}) \longrightarrow \mathbf{H}^1(\Omega)$ by Lemma 3.13 (ii). Finally, there exists $E \in \mathbf{H}^1(\Omega)$ with $\mathbf{N} - \tilde{\mathbf{N}} = \text{dev Grad } E$ and E depends linearly and continuously on \mathbf{N} . \square

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