A Posteriori Estimates
for the Stationary Stokes Problem in Exterior Domains

by

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ABSTRACT. This paper is concerned with the analysis of the inf-sup condition arising in the stationary Stokes problem in exterior domains and applications to the derivation of computable bounds of the distance between the exact solution of the exterior Stokes problem and a certain approximation (which may be of rather general form). In the first part, we deduce guaranteed bounds of the constant in the stability lemma associated with the exterior domain. These bounds depend only on known constants and the stability constant related to bounded domains, which are truncations of the unbounded domains. Then we obtain computable estimates of the distance to the set of divergence free fields defined in exterior domains, which are crucial for the derivation of computable majorants of the difference between the exact solution of the Stokes problem in exterior domains and an approximation from the admissible (energy) class of functions satisfying the Dirichlet boundary condition but not necessarily divergence free (solenoidal). Estimates of this type are often called *a posteriori* estimates of the functional type. The constant in the stability lemma (or equivalently in the inf-sup or LBB condition) arises as a penalty factor at the term that controls violation of the divergence free condition. If this condition is violated in a bounded domain only, then the respective estimates have a simpler form and use the constant associated with a bounded domain. In the last part of the paper, we deduce analogous estimates of the distance to the exact solution for non-conforming approximations, i.e., for those that may violate some continuity and boundary conditions. The case where the dimensionality of the domain equals 2 requires a special consideration because the corresponding weighted spaces differs from those natural for the dimension 3 (or larger). At the end of the paper, we briefly discuss this special case and deduce analogous estimates of the distance to the exact solution of the exterior Stokes problem.

Dedicated to the 110th anniversary of Solomon Grigor'evich Mikhlin
(April 23, 1908 – August 29, 1990)

CONTENTS

1. INTRODUCTION

1.1. **Notation and nomenclature.** Throughout the paper we consider domains in $\mathbb{R}^d$, $d \geq 2$, with Lipschitz boundaries. The symbol ω is used for bounded domains and the boundary of such a domain is denoted by γ (typically, the latter is composed of two open and disjoint parts γ_D and γ_N associated with the Dirichlet and Neumann parts). Exterior domains (i.e., those having the form $\mathbb{R}^d \setminus \overline{\omega}$) are denoted by Ω. By the letter $\mathcal{D}$, we denote domains which may be bounded or unbounded depending on the context (if this property is not necessary to outline).

For Lebesgue and Sobolev spaces of functions (scalar, vector, or tensor valued) with generalised square integrale derivatives of the first order we use the standard notation $\mathsf{L}^2(\omega)$ and $\mathsf{H}^1(\omega)$ (or $\mathsf{L}^2(\Omega)$ and $\mathsf{H}^1(\Omega)$), respectively. The standard inner product, norm, and orthogonality in $\mathsf{L}^2(\omega)$ will be denoted by $\langle \cdot, \cdot \rangle_{0,\omega}$, $\| \cdot \|_{0,\omega}$, and $\perp_{0,\omega}$. If $\gamma_D \neq \emptyset$, then the homogeneous Dirichlet boundary conditions are encoded in $\mathsf{H}^1_{\gamma_D}(\mathcal{D})$ defined as the closure of compactly supported test functions. Also, for bounded domains we use spaces with vanishing mean values[i]

$$\mathsf{L}^2_\perp(\omega) := \mathsf{L}^2(\omega) \cap \mathbb{R}^{\perp_{0,\omega}} = \big\{ \phi \in \mathsf{L}^2(\omega) \, : \, \langle \phi, 1 \rangle_{0,\omega} = 0 \big\} = \big\{ \phi \in \mathsf{L}^2(\omega) \, : \, \int_\omega \phi = 0 \big\},$$

$$\mathsf{H}^1_\perp(\omega) := \mathsf{H}^1(\omega) \cap \mathsf{L}^2_\perp(\omega) = \big\{ \phi \in \mathsf{H}^1(\omega) \, : \, \int_\omega \phi = 0 \big\}.$$

To handle the special case $\gamma_D = \emptyset$ adequately we introduce

$$\mathsf{L}^2_{\gamma_D}(\omega) := \begin{cases} \mathsf{L}^2(\omega) & \text{if } \gamma_D \neq \gamma, \\ \mathsf{L}^2_\perp(\omega) & \text{if } \gamma_D = \gamma, \end{cases} \qquad \mathsf{H}^1_{\gamma_D}(\omega) := \begin{cases} \mathsf{H}^1_{\gamma_D}(\omega) & \text{if } \gamma_D \neq \emptyset, \\ \mathsf{H}^1_\perp(\omega) & \text{if } \gamma_D = \emptyset. \end{cases}$$

To be more precise with our definitions and to extend our spaces to exterior domains as well, let $\mathcal{D} \subset \mathbb{R}^d$, $d \geq 2$, be a domain (an open and connected set) with (strong) Lipschitz boundary $\mathcal{B}$, which consists of two relatively open and disjoint parts $\mathcal{B}_D, \mathcal{B}_N \subset \mathcal{B}$ such that $\overline{\mathcal{B}} = \overline{\mathcal{B}}_D \cup \overline{\mathcal{B}}_N$ associated with Dirichlet and Neumann boundary conditions (we recall that $\mathcal{D}$ can be bounded or unbounded). As before, we introduce the standard Lebesgue and Sobolev spaces by $\mathsf{L}^2(\mathcal{D})$ and $\mathsf{H}^1(\mathcal{D})$, respectively. If $\mathcal{B}_D \neq \emptyset$, we introduce homogeneous Dirichlet boundary conditions in $\mathsf{H}^1_{\mathcal{B}_D}(\mathcal{D})$ defined as the closure of

$$\mathsf{C}^\infty_{\mathcal{B}_D}(\mathcal{D}) := \big\{ \phi|_\mathcal{D} \, : \, u \in \mathsf{C}^\infty(\mathbb{R}^d), \ \mathrm{supp}\,\phi \text{ compact}, \ \mathrm{dist}(\mathrm{supp}\,\phi, \mathcal{B}_D) > 0 \big\}$$

in $\mathsf{H}^1(\mathcal{D})$. As above we utilise the notations $\mathsf{L}^2_\mathcal{B}(\mathcal{D}) = \mathsf{L}^2_\perp(\mathcal{D})$, $\mathsf{L}^2_{\mathcal{B}_D}(\mathcal{D})$, and $\mathsf{H}^1_\emptyset(\mathcal{D}) = \mathsf{H}^1_\perp(\mathcal{D})$ provided that $\mathcal{D}$ is bounded. Next, we introduce polynomially weighted spaces

$$\mathsf{L}^2_{\pm 1}(\mathcal{D}) := \big\{ \phi \in \mathsf{L}^2_{\mathsf{loc}}(\mathcal{D}) \, : \, \rho^{\pm 1} \phi \in \mathsf{L}^2(\mathcal{D}) \big\},$$

$$\mathsf{H}^1_{-1}(\mathcal{D}) := \big\{ \phi \in \mathsf{L}^2_{-1}(\mathcal{D}) \, : \, \nabla \phi \in \mathsf{L}^2(\mathcal{D}) \big\},$$

where the weight function ρ is defined by $\rho(r) := (1 + r^2)^{1/2}$, and $r(x) := |x|$. The inner product, norm, and orthogonality in $\mathsf{L}^2_{\pm 1}(\mathcal{D})$ are denoted by

$$\langle \cdot, \cdot \rangle_{\pm 1, \mathcal{D}} := \langle \rho^{\pm 2} \cdot, \cdot \rangle_{0,\mathcal{D}}, \quad \| \cdot \|_{\pm 1, \mathcal{D}}, \quad \text{and} \quad \perp_{\pm 1, \mathcal{D}},$$

respectively. In the case of a bounded domain, there is no difference between the unweighted and weighted spaces (if we mean that the spaces coincide as sets and possess different

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[i]Throughout this paper we do not express the respective measure in the notation of integrals, so that, e.g., with often use notations like

$$\int_\omega f = \int_\omega f d\lambda = \int_\omega f dx, \qquad \int_\gamma f = \int_\gamma f do = \int_\gamma f ds.$$

inner products). However, in analysis of problems in exterior domains a proper selection of weights is important (in Section 4.6 devoted to the case $d = 2$ we define the weighted spaces differently). As before, if $\mathcal{B}_D \neq \emptyset$, then the homogeneous Dirichlet boundary conditions are encoded in $\mathsf{H}^1_{-1,\mathcal{B}_D}(\mathcal{D})$ the closure of $\mathsf{C}^\infty_{\mathcal{B}_D}(\mathcal{D})$ in $\mathsf{H}^1_{-1}(\mathcal{D})$. Finally, for the Stokes equations, we introduce spaces of solenoidal fields

$$\mathsf{S}(\mathcal{D}) := \left\{ \varphi \in \mathsf{H}^1(\mathcal{D}) \,:\, \operatorname{div} \varphi = 0 \right\}, \qquad \mathsf{S}_{\mathcal{B}_D}(\mathcal{D}) := \mathsf{H}^1_{\mathcal{B}_D}(\mathcal{D}) \cap \mathsf{S}(\mathcal{D}),$$
$$\mathsf{S}_{-1}(\mathcal{D}) := \left\{ \varphi \in \mathsf{H}^1_{-1}(\mathcal{D}) \,:\, \operatorname{div} \varphi = 0 \right\}, \qquad \mathsf{S}_{-1,\mathcal{B}_D}(\mathcal{D}) := \mathsf{H}^1_{-1,\mathcal{B}_D}(\mathcal{D}) \cap \mathsf{S}_{-1}(\mathcal{D}).$$

1.2. **Stability Lemma and the Stokes problem in bounded domains.** The classical stationary Stokes problem consists of finding a vector field u (velocity) and a scalar valued function p (pressure) that solve the system

(1)	$-\nu \Delta u + \nabla p = f$	in ω,
(2)	$\operatorname{div} u = 0$	in ω,
(3)	$u = u_D$	on γ_D,
(4)	$\sigma n = 0$	on γ_N,

where $\sigma := \nu \nabla u - p\,\mathbb{I}$, ν (viscosity) is a positive constant or a positive function taking values in $[\nu_\ominus, \nu_\oplus]$, $\nu_\ominus, \nu_\oplus > 0$, and $f \in \mathsf{L}^2(\omega)$. The boundary conditions are defined by the vector valued function u_D. Henceforth, we assume that u_D is given by a solenoidal vector field u_D, i.e., the Dirichlet boundary condition is defined by $u_D \in \mathsf{S}(\omega)$ in the sense that $u = u_D$ on γ_D, i.e., $u - u_D \in \mathsf{H}^1_{\gamma_D}(\omega)$. If $\gamma = \gamma_D$, then we additionally assume that

$$(5) \qquad \int_\gamma n \cdot u_D = \int_\omega \operatorname{div} u_D = \langle u_D, 1 \rangle_{0,\omega} = 0.$$

Existence of the corresponding generalised solution follows from the well known solution theory for uniformly elliptic linear equations and the stability lemma, which plays an important role in the theory of incompressible flow.

Lemma 1.1 (stability lemma, [22, 1, 3, 13, 14]). *There exists $c > 0$ such that for any $g \in \mathsf{L}^2_{\gamma_D}(\omega)$ there is a vector field $u_g \in \mathsf{H}^1_{\gamma_D}(\omega)$ with*

$$(6) \qquad \operatorname{div} u_g = g \quad and \quad \|\nabla u_g\|_{0,\omega} \le c\|g\|_{0,\omega}.$$

Henceforth, the best constants in (6) and similar inequalities for unbounded domains are denoted by the letter κ, i.e., $\kappa(\omega, \gamma_D)$ is the smallest c in (6). For $u \in \mathsf{H}^1_{\gamma_D}(\omega)$ we also have the Friedrichs/Poincaré inequality

$$\|u\|_{0,\omega} \le c\|\nabla u\|_{0,\omega},$$

and $c_{\mathrm{FP}}(\omega, \gamma_D)$ denotes the best constant c. Hence from Lemma 1.1, we conclude that u_g satisfies the inequalities

$$\frac{1}{c_{\mathrm{FP}}(\omega, \gamma_D)}\|u_g\|_{0,\omega} \le \|\nabla u_g\|_{0,\omega} \le \kappa(\omega, \gamma_D)\|\operatorname{div} u_g\|_{0,\omega}.$$

We notice that in the theory of electrodynamics the function u_g is called a regular potential as it admits (for Maxwell's equations) an unphysical (high) regularity and boundary condition, which is much stronger than the usual normal boundary condition related to the divergence operator.

Lemma 1.1 generates several important corollaries. First, it guarantees the solvability of the stationary Stokes problem (in the velocity-pressure posing). By setting $g = \operatorname{div} u_g$, Lemma 1.1 immediately yields the well known inf-sup (or LBB) condition:

$$(7) \qquad \inf_{g \in \mathsf{L}^2_{\gamma_D}(\omega)} \sup_{u \in \mathsf{H}^1_{\gamma_D}(\omega)} \frac{\langle g, \operatorname{div} u \rangle_{0,\omega}}{\|g\|_{0,\omega}\|\nabla u\|_{0,\omega}} \ge \frac{1}{\kappa(\omega, \gamma_D)} =: c_{\mathrm{LBB}}$$

Another direct corollary of Lemma 1.1 is an estimate of the distance between a vector field $u \in \mathsf{H}^1_{\gamma_D}(\omega)$ and the set $\mathsf{S}_{\gamma_D}(\omega)$ (see [35, 36])

$$\operatorname{dist}\left(u, \mathsf{S}_{\gamma_D}(\omega)\right) := \inf_{v \in \mathsf{S}_{\gamma_D}(\omega)} \left\| \nabla(u - v) \right\|_{0,\omega}.$$

Corollary 1.2. *For any $u \in \mathsf{H}^1_{\gamma_D}(\omega)$ there exists $u_0 \in \mathsf{S}_{\gamma_D}(\omega)$ such that*

$$\operatorname{dist}\left(u, \mathsf{S}_{\gamma_D}(\omega)\right) \leq \left\| \nabla(u - u_0) \right\|_{0,\omega} \leq \kappa(\omega, \gamma_D) \| \operatorname{div} u \|_{0,\omega}.$$

Proof. For $u \in \mathsf{H}^1_{\gamma_D}(\omega)$ solve $\operatorname{div} \widetilde{u} = \operatorname{div} u \in \mathsf{L}^2_{\gamma_D}(\omega)$ with $\widetilde{u} \in \mathsf{H}^1_{\gamma_D}(\omega)$ and the stability estimate $\| \nabla \widetilde{u} \|_{0,\omega} \leq \kappa(\omega, \gamma_D) \| \operatorname{div} u \|_{0,\omega}$ by Lemma 1.1. Note that for $\gamma_D = \gamma$ it holds

$$\tag{8} \int_\gamma n \cdot u = \int_\omega \operatorname{div} u = \langle u, 1 \rangle_{0,\omega} = 0.$$

Then $u_0 := u - \widetilde{u} \in \mathsf{S}_{\gamma_D}(\omega)$ and $\left\| \nabla(u - u_0) \right\|_{0,\omega} = \left\| \nabla \widetilde{u} \right\|_{0,\omega} \leq \kappa(\omega, \gamma_D) \| \operatorname{div} u \|_{0,\omega}$. $\qquad \square$

In [38, 39, 40], this result was extended to vector fields satisfying non-homogeneous Dirichlet boundary conditions (and also for vector fields in $\mathsf{W}^{1,q}(\Omega)$ for $q \in (1, \infty)$) provided that such a vector field u satisfies $\operatorname{div} u \in \mathsf{L}^2_{\gamma_D}(\Omega)$, i.e., the mean value condition (8), if $\gamma_D = \gamma$. Moreover, it was shown that if the mean value conditions hold for a collection of subdomains whose union of closures coincides with the closure of ω, then estimates of the distance can be based on local constants associated with subdomains. In the case of non-homogeneous boundary conditions, a modified version of Lemma 1.2 reads as follows.

Corollary 1.3. *For any $u \in \mathsf{H}^1(\omega)$ with $\operatorname{div} u \in \mathsf{L}^2_{\gamma_D}(\omega)$ there exists a solenoidal $u_0 \in \mathsf{S}(\omega)$ such that $u_0 - u \in \mathsf{H}^1_{\gamma_D}(\omega)$, i.e., $u_0|_{\gamma_D} = u|_{\gamma_D}$, and*

$$\left\| \nabla(u_0 - u) \right\|_{0,\omega} \leq \kappa(\omega, \gamma_D) \| \operatorname{div} u \|_{0,\omega}.$$

It is worth noting that Corollary 1.3 can be also viewed as a lifting lemma, since a boundary datum $u|_{\gamma_D}$ is lifted to the domain ω. In this case lifting is performed with the help of a solenoidal representative.

Quantitative estimates of the constant $\kappa(\omega, \gamma_D)$ has been studied in [10, 25, 29, 44, 6] and some other publications. It is not difficult to see that the constant c_{LBB} in (7) is nonnegative and cannot exceed 1 so that $\kappa(\omega, \gamma_D) \geq 1$. It is known that $c_{\mathrm{LBB}} > 0$ for any bounded Lipschitz domain (e.g., $c_{\mathrm{LBB}} = 1/\sqrt{d}$ for a ball in $\mathbb{R}^d$). However, the exact values of this constant are unknown except of some very special cases (e.g., we do not know the constant even for a cube!). In [6], simply computable and sufficiently accurate estimates of the constant were obtained for domains in $\mathbb{R}^2$, which are contained in a ball of radius R and are star-shaped with respect to a concentric ball of radius ρ. It was shown that

$$\tag{9} \kappa(\omega, \gamma) \leq \frac{\sqrt{2}}{\zeta} \left(1 + \sqrt{1 - \zeta^2} \right)^{1/2},$$

where $\zeta = \rho/R$. For $d = 3$, estimates of c_{LBB} are known only for domains with sufficiently regular boundaries (see [29]). A systematic numerical analysis of constants in the inf-sup condition (7) was performed in [11], where approximate values of the constants has been computed for a wide collection of bounded domains. Computational approaches to the evaluation of the distance to the set of divergence free fields based on domain decomposition are suggested in [38, 39, 40]. In our subsequent analysis, we assume that using above mentioned results and methods we are able to find a majorant of the constant $\kappa(\omega, \gamma_D)$ for bounded domains ω, which arise as truncations of an unbounded domain Ω.

1.3. **A posteriori estimates.** Estimates of the distance to $\mathsf{S}_{\gamma_D}(\omega)$ have not only theoretical meaning. They are important for quantitative analysis of boundary value problems generated by incompressible media models (e.g., in the theory of viscous incompressible fluids). First of all estimates of this type are necessary for getting computable bounds of the difference between the exact solution of a boundary value problem and an approximation

obtained by some computational procedure. The term "computable" means that the corresponding estimates does not contain unknown functions and constants and indeed could be computed by means of an approximate solution only. Estimates of this type are often called *a posteriori error estimates* and nowadays are widely used in quantitative analysis of mathematical problems. Unlike the *a priori* (asymptotic) convergence estimates, a posteriori estimates provide explicit verification of the accuracy of a particular numerical solution. First methods of a posteriori error control for PDEs originates from the works of W. Prager and J. L. Synge [33] and S. G. Mikhlin [19]. The latter monograph contained a pioneering idea of a new approach to error estimation, which differs principally from asymptotic rate convergence estimates dominated at that time and several decades subsequently. For variational problems generated by quadratic type functionals

$$(10) \qquad J(v) = \frac{1}{2}a(v,v) - \langle f, v \rangle, \quad f \in V,$$

where V is a Hilbert space and $a : V \times V \to \mathbb{R}$ is a V–elliptic bilinear form, S. Mikhlin deduced the principal relation

$$(11) \qquad \frac{1}{2}a(u-v, u-v) = J(v) - J(u).$$

Here u is the minimizer that satisfies $J(u) = \min_{w \in V} J(w)$ and $v \in V$ is *any* function compared with u. Since the exact infimum is unknown, it is impossible to use (11) directly. In [19], it was suggested to estimate $J(u)$ from below using a dual variational problem and further apply the orthogonal projection method of H. Weyl [48]. Certainly these first estimates were derived for a rather limited set of problems and suffered from serious restrictions imposed on the set of functions that are admissible in the dual setting. For these reasons, they were rarely used in computational practice. Moreover, the methods developed in 1970-1980 years for measuring errors of finite element approximations (such as the "gradient averaging" and "residual" methods, see, e.g., [47] and the references therein) were based on different grounds. These methods strongly exploit properties of a particular approximation computed on a particular mesh. In essence, they provide certain error indicators (for mesh adaptive procedures) rather than guaranteed error bounds. Subsequent studies focused on the problem of guaranteed error control (performed in 1990th years) confirmed the idea (encompassed in (11)) that the corresponding methods should be justified on the functional level by means of the same mathematical tools that are used in analysis of PDEs without attracting specific features of approximations and numerical methods. If we have a general (universal) estimate of the distance between a function and the exact solution of a boundary value problem, then it can be used with any approximation and requires no changes if one approximation (mesh) is replaced by another. In the last two decades computable bounds of this type has been derived and tested for a wide spectrum of problems (see [34, 23, 37, 17] and many other publications cited in these monographs). By clear reasons, they are often called *a posteriori estimates of the functional type*. They differ from others due to two important properties: the estimates

(a) do not contain constants associated with a particular finite dimensional subspace (mesh) and a method used to solve the problem and
(b) are valid for any approximation in the energy space and do not use special conditions required for the exact solution (e.g., extra regularity) or its approximation (e.g., Galerkin orthogonality, quasi–uniformity of meshes).

A posteriori estimates of the functional type contain only global constants generated by functional inequalities such as various embedding estimates, trace inequalities, Poincare, Maxwell, Korn inequalities, etc. It should be noted that although constants of this type do not appear in (11) (and in the estimates derived in [33]), the importance of studying them was already understood by Mikhlin (see [20]). The reader can find an overview of the history of a posteriori error estimation methods and a consequent exposition of the functional approach to the problem in [37].

First estimates of the distance between a function in the energy space and the exact solution of the stationary Stokes problem in a bounded domain have been derived in [35] (by means of the variational duality method) and in [36] (by transformations of the integral identity that defines the corresponding weak solution). It is worth starting a short overview of these results with the error identity

$$(12) \qquad \nu \big\| \nabla (v - u) \big\|^2 + \nu^{-1} \| \tau_f - \sigma \|^2 = 2(I(v) - I^*(\tau_f)),$$

which can be viewed as an analog of (11) for the stationary Stokes problem. Here $v \in \mathsf{S}_\gamma(\omega)$ and

$$\tau_f \in \mathsf{L}_f^2(\omega) := \big\{ \tau \in \mathsf{L}^2(\omega) \,:\, \langle \tau, \nabla\, w \rangle_{0,\omega} = \langle f, w \rangle_{0,\omega} \quad \forall w \in \mathsf{S}_\gamma(\omega) \big\},$$

are considered as approximations of the exact velocity field u and exact stress fiels σ, respectively. The identity (12) holds for any $v \in \mathsf{S}_\gamma(\omega)$ and any $\tau_f \in \mathsf{L}_f^2(\omega)$. However, it is not very useful for practice by the same reasons as (11), namely, the functions in $\mathsf{S}_\gamma(\omega)$ and $\mathsf{L}_f^2(\omega)$ are subjected to differential relations. In [35] (see also [37]), it was shown a way to overcome these difficulties by using computable estimates of distances to the sets $\mathsf{S}_\gamma(\omega)$ and $\mathsf{L}_f^2(\omega)$. As a result, the following estimates for the velocity and pressure fields were derived:

$$
(13) \qquad \begin{aligned}
\nu \big\| \nabla (u - \widetilde{u}) \big\|_{0,\omega} &\le \| \tau + \widetilde{p}\,\mathbb{I} - \nu \nabla \widetilde{u} \|_{0,\omega} + c_{\mathrm{FP}}(\omega,\gamma) \| \operatorname{Div} \tau + f \|_{0,\omega} \\
&\quad + 2\nu\kappa(\omega,\gamma) \| \operatorname{div} \widetilde{u} \|_{0,\omega},
\end{aligned}
$$

$$
(14) \qquad \begin{aligned}
\frac{1}{2\kappa(\omega,\gamma)} \| p - \widetilde{p} \|_{0,\omega} &\le \| \tau + \widetilde{p}\,\mathbb{I} - \nu \nabla \widetilde{u} \|_{0,\omega} + c_{\mathrm{FP}}(\omega,\gamma) \| \operatorname{Div} \tau + f \|_{0,\omega} \\
&\quad + \nu\kappa(\omega,\gamma) \| \operatorname{div} \widetilde{u} \|_{0,\omega}
\end{aligned}
$$

Here ν is a positive constant and $\widetilde{u} \in \mathsf{H}^1(\omega)$ is a vector valued function satisfying the Dirichlet boundary conditions. The function $\widetilde{u}$ is considered as an approximation of the exact velocity u. Analogously, $\widetilde{p}$ is a square integrable function (with zero mean value if the Dirichlet conditions are imposed on the whole boundary γ) considered as an approximation of p and $\tau \in \mathsf{L}^2(\omega)$ is an approximation of the exact stress field σ. The right hand sides of (13) and (14) have a clear meaning: they contain three nonnegative terms that vanish if the approximations coincide with the exact velocity, pressure, and stress, respectively. In other cases, the terms can be viewed as penalties for possible violations of the three basic relations that form (1) and (2).

It is easy to see that the constant $\kappa(\omega,\gamma)$ plays an important role in (13) and (14) and, therefore, it is indeed necessary to have guaranteed majorants of this constant. These constants arise in a posteriori analysis of a numerical solution if it satisfies the divergence free condition only approximately. If the constant $\kappa(\omega,\gamma_D)$ is known, then by using Corollary 1.2 we can deduce guaranteed and fully computable error estimates. For problems in bounded Lipschitz domains the respective results are presented in [35, 36, 37, 39] and other publications cited therein. Also, it should be mentioned that explicit bounds of the constant $\kappa(\omega,\gamma)$ are required not only for Stokes type problems. They arise in other continuous media problems, which are not related to viscous fluids (e.g., see [42]).

Subsequently analogous estimates for the velocity and pressure fields were obtained for the Stokes problem in the velocity-vorticity-pressure formulation [18] and for the generalised Stokes problem [41]. In [7], the estimates were derived for a class of stationary problems associated with nonlinear viscous fluids and in [24] for the evolutionary Stokes problem. We note that the approach used in these publications and in the present paper differs essentially from the so-called *residual* method often used in the finite element community for getting indicators of approximation errors (e.g., see [46]).

1.4. **Outline of the paper.** In the first part of the paper, we recall some known results related to the analysis of boundary value problems in exterior domains paying a special attention to the stability Lemma 2.3 and the corresponding corollaries.

Section 3 is devoted to computable bounds $\kappa_\oplus(\Omega, \Gamma_D)$ for the stability constant $\kappa(\Omega, \Gamma_D)$. In Lemma 3.2, we obtain a desired estimate for $\kappa_\oplus(\Omega, \Gamma_D)$, which uses known constants and the constant $\kappa(\omega, \gamma_D)$ associated with a bounded domain ω (which is a suitable truncation

of Ω). Estimates of the latter constant have been derived in several above cited publications, so that we consider this problem as solvable (at least in the sense that a certain guaranteed bound of $\kappa(\omega, \gamma_D)$ can be derived). As a result, we obtain estimates of the distance between a vector field in Ω and the respective set of solenoidal fields defined in Ω and satisfying the same boundary conditions (Lemma 3.4). These estimates are used in Section 4, we derive a posteriori error estimates of the functional type, which are valid for a wide class of approximate solutions to the stationary Stokes problem in exterior domains.

Estimates for the velocity are obtained in three different forms. The first (and the simplest) form is valid for approximations in $\mathsf{S}(\Omega)$, i.e., for solenoidal vector valued functions that satisfy the Dirichlet boundary condition exactly (Theorem 4.1). These estimates do not contain the constant $\kappa(\Omega, \Gamma_D)$. Estimates of the second type are valid for approximations in $\mathsf{H}^1(\Omega)$ still satisfying the boundary condition exactly but admit possibly non–solenoidal functions (Theorem 4.2). They contain a term that penalises possible violation of the solenoidality condition and has the constant $\kappa(\Omega, \Gamma_D)$ as a penalty factor. Finally, the most general form of the estimate is applicable for non-conforming approximations, which even may not belong to the energy class $\mathsf{H}^1(\Omega)$ (Theorem 4.6). It contains one more term that can be viewed as a measure of the distance to the energy class natural for the velocity function. Also, we deduce estimates for approximations of the pressure (Theorem 4.4 and Theorem 4.6) and the stress field (Section 4.4). In Section 4.5 we consider lower bounds and in Section 4.6 we adapt our results to the special case of space dimension $d = 2$.

2. Preliminaries

2.1. **Exterior domain and main functional inequalities.** We consider an exterior domain $\Omega \subset \mathbb{R}^d$, where $d \geq 3$ (the special case $d = 2$ is studied in Section 4.6), with a (strong) Lipschitz boundary Γ, which is composed of two open and disjoint parts $\Gamma_D, \Gamma_N \subset \Gamma$ (Dirichlet and Neumann part) with $\overline{\Gamma} = \overline{\Gamma}_D \cup \overline{\Gamma}_N$. Moreover, we assume that there exist $0 < r_1 < r_2$ such that $\mathbb{R}^d \setminus \Omega \subset B_{r_1}$ and denote (see Fig.1)

$$(15) \qquad \omega := \Omega_{r_2} := \Omega \cap B_{r_2}, \quad \gamma = \Gamma \cup S_{r_2}, \quad \gamma_D := \Gamma_D \cup S_{r_2},$$

where B_r and S_r denote the open ball and the sphere of radius r centered at the origin in $\mathbb{R}^d$. By η we denote a Lipschitz continuous cut-off function, which vanishes in the ball[ii] $\overline{B}_{r_1}$, equals to 1 in $\mathbb{R}^d \setminus B_{r_2}$, and attains values in $[0, 1]$.

The two main ingredients for our proofs are Lemma 1.1 and a few elementary results from the theory of ∇-rot-div–systems in exterior domains and especially $\mathbb{R}^d$ (see, e.g., [16, 43] or [12, 26] and in particular [28] as well as references therein), which can be summarised in the two subsequent lemmas as follows:

Lemma 2.1 (Friedrichs/Poincaré lemma for exterior domains). *The following weighted Friedrichs/Poincaré estimates hold:*

   (i) *There exists $c > 0$ such that for all $v \in \mathsf{H}^1_{-1, \Gamma_D}(\Omega)$ it holds $\|v\|_{-1, \Omega} \leq c \|\nabla v\|_{0, \Omega}$. The best constant c is called the Friedrichs/Poincaré constant and we denote it by $c_{\mathrm{FP}}(\Omega, \Gamma_D)$.*
   (ii) *If $\Gamma_D = \Gamma$, then $c_{\mathrm{FP}}(\Omega, \Gamma) = c_{\mathrm{F}}(\Omega)$ (the Friedrichs constant) and*

$$c_{\mathrm{F}}(\Omega) \leq c_d := \frac{2}{d-2}.$$

   Hence for all $v \in \mathsf{H}^1_{-1, \Gamma}(\Omega)$ the Friedrichs estimate $\|v\|_{-1, \Omega} \leq c_d \|\nabla v\|_{0, \Omega}$ holds. If $\Gamma_D = \emptyset$, then $c_{\mathrm{FP}}(\Omega, \emptyset)$ is replaced by the Poincaré constant $c_{\mathrm{P}}(\Omega)$. In this case the Poincaré estimate $\|v\|_{-1, \Omega} \leq c_{\mathrm{P}}(\Omega) \|\nabla v\|_{0, \Omega}$ holds for all $v \in \mathsf{H}^1_{-1}(\Omega)$.
   (iii) *If $\Omega = \mathbb{R}^d$, then the Friedrichs and Poincaré constants coincide and, moreover,*

$$c_{\mathrm{FP}}(\mathbb{R}^d) = c_{\mathrm{F}}(\mathbb{R}^d) = c_{\mathrm{P}}(\mathbb{R}^d) \leq c_d.$$

[ii]For the sake of simplicity we henceforth operate with the balls B_{r_1} and B_{r_2}. However, if necessary B_{r_1} can be replaced by a Lipschitz domain containing $\mathbb{R}^2 \setminus \overline{\Omega}$ and B_{r_2} by another Lipschitz domain containing B_{r_1}.
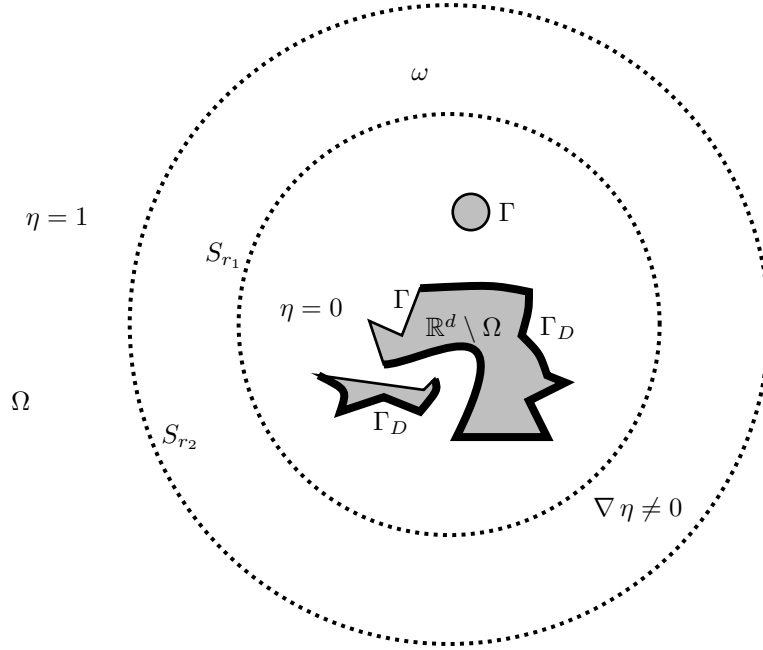
FIGURE 1. $\mathbb{R}^d \setminus \overline{\Omega}$ (gray) surrounded by the boundary Γ (thin black lines), the boundary part Γ_D (thick black lines), and the artificial boundary spheres (dashed lines)

Hence for all $v \in \mathsf{H}^1_{-1}(\mathbb{R}^d)$ it holds $\|v\|_{-1,\mathbb{R}^d} \leq c_d \|\nabla v\|_{0,\mathbb{R}^d}$.

Note that no boundary or mean value conditions are needed in Lemma 2.1 (because constant functions are not integrable in $\mathsf{L}^2_{-1}(\Omega)$).

Lemma 2.2 (∇-rot-div lemma for exterior domains)**.** *Let $\Omega = \mathbb{R}^d$ and $h \in \mathsf{L}^2(\mathbb{R}^d)$. Then there exists a unique $v \in \mathsf{H}^1_{-1}(\mathbb{R}^d)$ such that $\operatorname{rot} v = 0$ and $\operatorname{div} v = h$. Moreover,*

$$\frac{1}{c_d} \|v\|_{-1,\mathbb{R}^d} \leq \|\nabla v\|_{0,\mathbb{R}^d} = \|\operatorname{div} v\|_{0,\mathbb{R}^d} = \|h\|_{0,\mathbb{R}^d}.$$

The results of Lemma 2.2 are well known and follow directly by Fourier analysis, see, e.g., in [30, 31, 32, 16] or [12, 26]. Notice that the well known equation $-\Delta = \operatorname{rot}^* \operatorname{rot} - \nabla \operatorname{div}$ implies the identity

$$(16) \qquad\qquad \|\nabla v\|^2_{0,\mathbb{R}^d} = \|\operatorname{rot} v\|^2_{0,\mathbb{R}^d} + \|\operatorname{div} v\|^2_{0,\mathbb{R}^d}$$

for all $v \in \mathsf{C}^\infty(\mathbb{R}^d)$ having compact support. By density and continuity arguments it can be extended to all $v \in \mathsf{H}^1_{-1}(\mathbb{R}^d)$. Hence the equality $\|\nabla v\|_{0,\mathbb{R}^d} = \|\operatorname{div} v\|_{0,\mathbb{R}^d}$ in Lemma 2.2 follows immediately.

2.2. **Stability Lemma and Stokes problem in exterior domains.** For exterior domains we have a result analogous to Lemma 1.1 (see [9]).

Lemma 2.3 (stability lemma for exterior domains)**.** *There exists $c > 0$ such that for any $h \in \mathsf{L}^2(\Omega)$ there is a vector field $u_h \in \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$ with*

$$\operatorname{div} u_h = h \quad \text{and} \quad \|\nabla u_h\|_{0,\Omega} \leq c\|h\|_{0,\Omega}.$$

The best constant is denoted by $\kappa(\Omega, \Gamma_D)$.

In the subsequent Lemma 3.2 we present computable upper bounds for $\kappa(\Omega, \Gamma_D)$.

Remark 2.4. *In [9] only the case $\Gamma_D = \Gamma$ is considered, but it is trivial that then the stability lemma holds for general Γ_D as well. It is worth noting that unlike in the case of a bounded domain, no mean value condition is imposed on h even if $\Gamma_D = \Gamma$.*

Lemma 2.3 immediately implies the inf-sup condition.

Corollary 2.5 (inf-sup lemma for exterior domains). *It holds*

$$\inf_{h \in \mathsf{L}^2(\Omega)} \sup_{u \in \mathsf{H}^1_{-1,\Gamma_D}(\Omega)} \frac{\langle h, \operatorname{div} u \rangle_{0,\Omega}}{\|h\|_{0,\Omega} \|\nabla u\|_{0,\Omega}} \geq \frac{1}{\kappa(\Omega, \Gamma_D)}.$$

From now on, let $f \in \mathsf{L}^2_1(\Omega)$, $u_D \in \mathsf{S}_{-1}(\Omega)$ be given data. The classical Stokes problem in an exterior domain Ω consists of finding a vector valued function u (velocity) and a scalar valued function p (pressure) satysfying the system

$$(17) \qquad -\operatorname{Div} \sigma = f \qquad\qquad \text{in } \Omega,$$
$$(18) \qquad \sigma = \nu \nabla u - p\, \mathbb{I} \qquad\qquad \text{in } \Omega,$$
$$(19) \qquad \operatorname{div} u = 0 \qquad\qquad \text{in } \Omega,$$
$$(20) \qquad u = u_D \qquad\qquad \text{on } \Gamma_D,$$
$$(21) \qquad \sigma n = 0 \qquad\qquad \text{on } \Gamma_N,$$

where u additionally satisfies a proper decay conditions at infinity, so that $u \in \mathsf{S}_{-1}(\Omega)$ or $u \in \mathsf{H}^1_{-1}(\Omega)$, i.e.,

$$u \in \mathsf{L}^2_{-1}(\Omega) \quad \text{and} \quad \nabla u \in \mathsf{L}^2(\Omega).$$

The classical (point wise) condition reads

$$(22) \qquad u(x) \xrightarrow{|x| \to \infty} 0.$$

The corresponding generalised solution $u \in u_D + \mathsf{S}_{-1,\Gamma_D}(\Omega)$ is defined by the integral identity (see, e.g., [13, 9])

$$(23) \qquad \langle \nu \nabla u, \nabla \varphi \rangle_{0,\Omega} = \langle f, \varphi \rangle_{0,\Omega} \qquad \forall \varphi \in \mathsf{S}_{-1,\Gamma_D}(\Omega).$$

Note that by $\langle f, \varphi \rangle_{0,\Omega} = \langle \rho\, f, \rho^{-1}\varphi \rangle_{0,\Omega}$ the right hand side of (23) is well defined. Using the ansatz $u = u_D + \widehat{u}$ with $\widehat{u} \in \mathsf{S}_{-1,\Gamma_D}(\Omega)$ we reduce this formulation to find $\widehat{u} \in \mathsf{S}_{-1,\Gamma_D}(\Omega)$ such that for all $\varphi \in \mathsf{S}_{-1,\Gamma_D}(\Omega)$

$$\langle \nu \nabla \widehat{u}, \nabla \varphi \rangle_{0,\Omega} = \langle f, \varphi \rangle_{0,\Omega} - \langle \nu \nabla u_D, \nabla \varphi \rangle_{0,\Omega}.$$

Another formulation taking the pressure into account leads to the following saddle point formulation: Find

$$(u, p) \in \left(u_D + \mathsf{H}^1_{-1,\Gamma_D}(\Omega)\right) \times \mathsf{L}^2(\Omega)$$

such that for all $(\varphi, \phi) \in \mathsf{H}^1_{-1,\Gamma_D}(\Omega) \times \mathsf{L}^2(\Omega)$

$$(24) \qquad \langle \nu \nabla u, \nabla \varphi \rangle_{0,\Omega} - \langle p, \operatorname{div} \varphi \rangle_{0,\Omega} = \langle f, \varphi \rangle_{0,\Omega},$$
$$(25) \qquad \langle \operatorname{div} \widehat{u}, \phi \rangle_{0,\Omega} = 0.$$

By standard arguments (see, e.g., [8, 9, 13]) and the inf-sup lemma, we know that for $f \in \mathsf{L}^2_1(\Omega)$ and $u_D \in \mathsf{S}_{-1}(\Omega)$ the Stokes system is uniquely solvable with $u \in u_D + \mathsf{S}_{-1,\Gamma_D}(\Omega)$ and $p \in \mathsf{L}^2(\Omega)$. Moreover, the solution meets the estimates

$$\nu \|\nabla \widehat{u}\|_{0,\Omega} \leq c_{\mathrm{FP}}(\Omega, \Gamma_D) \|f\|_{1,\Omega} + \nu \|\nabla u_D\|_{0,\Omega},$$
$$\nu \|\nabla u\|_{0,\Omega} \leq c_{\mathrm{FP}}(\Omega, \Gamma_D) \|f\|_{1,\Omega} + 2\nu \|\nabla u_D\|_{0,\Omega},$$
$$\|p\|_{0,\Omega} \leq 2\kappa(\Omega, \Gamma_D)\left(c_{\mathrm{FP}}(\Omega, \Gamma_D) \|f\|_{1,\Omega} + \nu \|\nabla u_D\|_{0,\Omega}\right).$$

3. Estimates of $\kappa(\Omega, \Gamma_D)$

Our goal is to deduce majorants for the constant in Lemma 2.3 that contain only known constants (such as c_d) and stability constants for a bounded domain ω generated by a certain truncation of Ω.

First we discuss the simplest case of a compactly supported right hand side.

Lemma 3.1. *There exists $c > 0$ such that for all $h \in \mathsf{L}^2(\Omega)$ with supp $h \subset \overline{\omega}$ and $h \in \mathsf{L}^2_{\gamma_D}(\omega)$, i.e., $\int_\Omega h = \int_\omega h = 0$ if $\Gamma_D = \Gamma$, there exists a vector field $u_h \in \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$ such that*

$$\operatorname{div} u_h = h \quad and \quad \|\nabla u_h\|_{0,\Omega} \leq c\|h\|_{0,\Omega}.$$

u_h *can be chosen with compact support in $\overline{\omega}$, in particular, $u_h \in \mathsf{H}^1_{\gamma_D}(\omega) \subset \mathsf{H}^1_{\Gamma_D}(\Omega)$. In this case, $\kappa(\Omega, \Gamma_D) \leq \kappa(\omega, \gamma_D)$.*

Hence we arrive at the rather obvious conclusion that for a compactly supported h the constant $\kappa(\Omega, \Gamma_D)$ can be estimated by $\kappa(\omega, \gamma_D)$.

Proof of Lemma 3.1. Set $g := h|_\omega \in \mathsf{L}^2_{\gamma_D}(\omega)$. By Lemma 1.1 there exist $\kappa(\omega, \gamma_D) > 0$ and $u_g \in \mathsf{H}^1_{\gamma_D}(\omega)$ such that $\operatorname{div} u_g = g$ and $\|\nabla u_g\|_{0,\omega} \leq \kappa(\omega, \gamma_D)\|g\|_{0,\omega}$. Let u_h be the extension by zero of u_g to Ω. Then $u_h \in \mathsf{H}^1_{\Gamma_D}(\Omega)$ and supp $u_h = \operatorname{supp} u_g \subset \overline{\omega}$. Moreover, $\operatorname{div} u_h = h$ and

$$\|\nabla u_h\|_{0,\Omega} = \|\nabla u_g\|_{0,\omega} \leq \kappa(\omega, \gamma_D)\|g\|_{0,\omega} = \kappa(\omega, \gamma_D)\|h\|_{0,\Omega},$$

completing the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now we present the main result, which provides an upper bound for the best constant $\kappa(\Omega, \Gamma_D)$ in the stability Lemma 2.3 for exterior domains as well as an alternative proof of this lemma.

Lemma 3.2. *Let Ω and ω be defined by (15). Then*

$$\kappa(\Omega, \Gamma_D) \leq \kappa_\oplus(\Omega, \Gamma_D) := (1 + \kappa)\big(1 + c_d\,\alpha\,\rho(r_2)\big)$$

with $\kappa(\Omega, \Gamma_D)$ from Lemma 2.3, where α and κ are defined in (29) and (31), i.e.,

$$\alpha = \alpha(r_1, r_2, \eta) = \max_{x \in \overline{B}_{r_2} \setminus B_{r_1}} \big|\nabla\,\eta(x)\big|, \qquad \kappa = \min\big\{\kappa(\omega, \gamma_D), \kappa(\omega, \gamma)\big\}.$$

Remark 3.3. *It is with noting that $\kappa \leq \kappa(\omega, \gamma_D)$ and $\kappa \leq \kappa(\omega, \gamma)$ holds, where the second bound means that κ can be estimated independently of the boundary part Γ_D of Γ. Moreover, the constant $\kappa_\oplus(\Omega, \Gamma_D)$ depends on the dimension d, the radii r_1, r_2, and the Lipschitz continuous cut-off function η. Optimal values of these parameters (which minimise the constant) are not known a priori and should be defined by solving an additional algebraic problem. If η is constructed, e.g., by a simple affine function with $r_2 = r_1 + 1$, then $\alpha = 1$ and we have one possible upper bound*

$$\kappa_\oplus(\Omega, \Gamma_D) \leq (1 + \kappa)\big(1 + c_d\sqrt{1 + r_2^2}\big) \leq \big(1 + \kappa(\omega, \gamma)\big)\big(1 + \frac{2\sqrt{2}}{d-2}r_2\big).$$

In some cases, this bound may be rather coarse, but anyway it presents a guaranteed upper bound of the stability constant $\kappa_\oplus(\Omega, \Gamma_D)$ associated with the exterior domain Ω.

Proof of Lemma 3.2. We extend h by 0 to $\mathbb{R}^d \setminus \overline{\Omega}$ and identify the extended function with $h \in \mathsf{L}^2(\mathbb{R}^d)$. By Lemma 2.2 we have a vector field $v \in \mathsf{H}^1_{-1}(\mathbb{R}^d)$ with rot $v = 0$ and div $v = h$ in $\mathbb{R}^d$ as well as

$$(26) \qquad\qquad \frac{1}{c_d}\|v\|_{-1,\mathbb{R}^d} \leq \|\nabla v\|_{0,\mathbb{R}^d} = \|\operatorname{div} v\|_{0,\mathbb{R}^d} = \|h\|_{0,\Omega}.$$

We recall that our cut-off function η satisfies $\eta|_{B_{r_1}} = 0$ and $\eta|_{\mathbb{R}^d \setminus B_{r_2}} = 1$. Therefore, $\eta v \in \mathsf{H}^1_{-1,\Gamma}(\Omega)$ and $\operatorname{supp}(\eta v) \subset \mathbb{R}^d \setminus B_{r_1}$. In view of Lemma 2.3, our goal is to find a vector valued function $v_h \in \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$ such that $\operatorname{div} v_h = h$ in Ω and the stability inequality holds. We suggest to construct it in the form

$$(27) \qquad\qquad\qquad\qquad v_h := \eta v + v_\omega,$$

where $v_\omega \in \mathsf{H}^1_{\Gamma_D}(\Omega)$ with supp $v_\omega \subset \overline{\omega}$ is the extension by zero to Ω of some vector valued function u_ω, which is supported in $\overline{\omega}$ and belongs to $\mathsf{H}^1_{\gamma_D}(\omega)$. The relation

$$h = \operatorname{div} v_h = \eta h + \nabla\,\eta \cdot v + \operatorname{div} v_\omega \quad \text{in } \Omega$$

imposes the condition on v_ω and u_ω. Let

$$g := (1 - \eta)h - \nabla \eta \cdot v \in \mathsf{L}^2(\omega) \quad \text{in } \omega.$$

Notice that $\operatorname{supp}(1 - \eta) \subset \overline{B}_{r_2}$, $\operatorname{supp} \nabla \eta \subset \overline{B}_{r_2} \setminus B_{r_1}$ and hence $\operatorname{supp} g \subset \overline{\omega}$ (if we consider g as a function in Ω). As

$$g = (1 - \eta)h + \nabla(1 - \eta) \cdot v = \operatorname{div}\big((1 - \eta)v\big)$$

and by the properties of η we have by Gauss' or Stokes theorem

$$\int_\omega g = \int_\gamma (1 - \eta)\, n \cdot v = \int_\Gamma n \cdot v = -\int_{\mathbb{R}^d \setminus \overline{\Omega}} \operatorname{div} v = -\int_{\mathbb{R}^d \setminus \overline{\Omega}} h = 0.$$

Hence g has zero mean value independent of the boundary part $\gamma = \gamma_D$, i.e., it always holds

$$\text{(28)} \qquad\qquad \int_\omega g = 0.$$

Now, the existence of $u_\omega = u_{g,\gamma_D} \in \mathsf{H}^1_{\gamma_D}(\omega)$ resp. $u_\omega = u_{g,\gamma} \in \mathsf{H}^1_\gamma(\omega)$ such that $\operatorname{div} u_\omega = g$ together with the stability estimate

$$\text{(29)} \qquad\qquad \|\nabla u_\omega\|_{0,\omega} \le \kappa \|g\|_{0,\omega}, \quad \kappa := \min\big\{\kappa(\omega, \gamma_D), \kappa(\omega, \gamma)\big\}$$

is provided by Lemma 1.1. Note that $u_\omega \in \mathsf{H}^1_\gamma(\omega)$ can also be considered as a suitable one if the homogeneous Dirichlet boundary conditions are imposed only on a part Γ_D of Γ, so that we can operate with one and the same constant $\kappa(\omega, \gamma)$. Since $u_\omega = 0$ on S_{r_2},

$$v_\omega := \begin{cases} u_\omega & \text{in } \omega \\ 0 & \text{in } \mathbb{R}^d \setminus \overline{B}_{r_2} \end{cases}$$

meets our needs. Then $v_h = \eta v + v_\omega \in \mathsf{H}^1_{-1, \Gamma_D}(\Omega)$ and

$$\operatorname{div} v_h = \eta h + \nabla \eta \cdot v + g = h \quad \text{in } \Omega.$$

It remains to estimate $\|\nabla v_h\|_{0,\Omega}$. By the properties of η we have

$$\text{(30)} \qquad\qquad \|\nabla \eta \otimes v\|_{0,\mathbb{R}^d},\ \|\nabla \eta \cdot v\|_{0,\mathbb{R}^d} \le \alpha \|v\|_{0, B_{r_2} \setminus \overline{B}_{r_1}},$$

where

$$\text{(31)} \qquad\qquad \alpha := \alpha(r_1, r_2, \eta) := \max_{x \in \mathbb{R}^d} \big|\nabla \eta(x)\big| = \max_{x \in \overline{B}_{r_2} \setminus B_{r_1}} \big|\nabla \eta(x)\big|.$$

Since

$$\|g\|_{0,\omega} \le \big\|(1 - \eta)h\big\|_{0,\omega} + \|\nabla \eta \cdot v\|_{0,\omega},$$

we find by (30)

$$\|\nabla v_\omega\|_{0,\Omega} = \|\nabla u_\omega\|_{0,\omega} \le \kappa \|g\|_{0,\omega} \le \kappa\big(\big\|(1-\eta)h\big\|_{0,\omega} + \alpha \|v\|_{0, B_{r_2}\setminus\overline{B}_{r_1}}\big)$$

$$\text{(32)} \qquad\qquad \le \kappa\big(\|h\|_{0,\Omega} + \alpha \|v\|_{0, B_{r_2}\setminus\overline{B}_{r_1}}\big),$$

$$\big\|\nabla(\eta v)\big\|_{0,\Omega} \le \|h\|_{0,\Omega} + \alpha \|v\|_{0, B_{r_2}\setminus\overline{B}_{r_1}}.$$

By (26), the second terms in the right hand sides can be estimated as follows:

$$\text{(33)} \qquad \|v\|_{0, B_{r_2}\setminus\overline{B}_{r_1}} \le \rho(r_2)\|v\|_{-1,\mathbb{R}^d} \le c_d\, \rho(r_2)\|\nabla v\|_{0,\mathbb{R}^d} = c_d\, \rho(r_2)\|h\|_{0,\Omega}.$$

Finally, by (27), (32), and (33), we conclude that

$$\|\nabla v_h\|_{0,\Omega} \le (1 + \kappa)\big(\|h\|_{0,\Omega} + c_d\, \alpha\, \rho(r_2)\|h\|_{0,\Omega}\big) = \kappa_\oplus(\Omega, \Gamma_D)\|h\|_{0,\Omega},$$

finishing the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\ \square$

3.1. **Estimates of the distance to the set $\mathsf{S}_{-1,\Gamma_D}(\Omega)$.** As in the case of bounded domains, the stability Lemma 2.3 implies estimates of the quantity

$$\operatorname{dist}\big(u,\mathsf{S}_{-1,\Gamma_D}(\Omega)\big) = \inf_{\varphi\in\mathsf{S}_{-1,\Gamma_D}(\Omega)} \big\|\nabla(u-\varphi)\big\|_{0,\Omega},$$

which is a measure of the distance between $u \in \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$ and the corresponding set of divergence free vector fields.

Lemma 3.4. *For any $u \in \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$ there exists a solenoidal $u_0 \in \mathsf{S}_{-1,\Gamma_D}(\Omega)$ such that*

$$\operatorname{dist}\big(u,\mathsf{S}_{-1,\Gamma_D}(\Omega)\big) \leq \big\|\nabla(u-u_0)\big\|_{0,\Omega} \leq \kappa(\Omega,\Gamma_D)\|\operatorname{div} u\|_{0,\Omega}.$$

Proof. Let $h = \operatorname{div} u \in \mathsf{L}^2(\Omega)$. By Lemma 2.3 there exists $u_h \in \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$ such that $\operatorname{div} u_h = h$ and

$$\|\nabla u_h\|_{0,\Omega} \leq \kappa(\Omega,\Gamma_D)\|h\|_{0,\Omega}.$$

Then $u_0 := u - u_h \in \mathsf{S}_{-1,\Gamma_D}(\Omega)$ and we have

$$\big\|\nabla(u-u_0)\big\|_{0,\Omega} = \big\|\nabla u_h\big\|_{0,\Omega} \leq \kappa(\Omega,\Gamma_D)\|\operatorname{div} u\|_{0,\Omega}.$$

$\square$

It is easy to extend this result to the case where v satisfies inhomogeneous Dirichlet boundary conditions on a part of the boundary.

Corollary 3.5. *For any $u \in \mathsf{H}^1_{-1}(\Omega)$ there exists a solenoidal $u_0 \in \mathsf{S}_{-1}(\Omega)$ such that $u - u_0 \in \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$, i.e., $u_0|_{\Gamma_D} = u|_{\Gamma_D}$, and*

$$\big\|\nabla(u-u_0)\big\|_{0,\Omega} \leq \kappa(\Omega,\Gamma_D)\|\operatorname{div} u\|_{0,\Omega}.$$

The latter assertion shows existence of a continuous lifting operator that lifts of the boundary datum $u|_{\Gamma_D}$ to the domain Ω with a solenoidal representative.

4. A posteriori estimates

In this section, we derive estimates of the difference between the exact solution of the exterior Stokes problem (17)–(21) resp. (24)–(25) (presented by the fields of velocity u, pressure p and stress σ) and respective approximations $\widetilde{u}$, $\widetilde{p}$, and $\widetilde{\sigma}$.

First, we introduce two more weighted spaces for tensor valued functions

$$\mathbb{D}(\Omega) := \big\{\tau \in \mathsf{L}^2(\Omega) \,:\, \operatorname{Div}\tau \in \mathsf{L}^2_1(\Omega)\big\},$$

where Div is the divergence operator for tensor fields acting as usual row-wise divergence. $\mathbb{D}(\Omega)$ is a Hilbert space with norm defined by

$$\|\tau\|^2_{\mathbb{D}} := \|\tau\|^2_{0,\Omega} + \|\operatorname{Div}\tau\|^2_{1,\Omega}.$$

By $\mathbb{D}_{\Gamma_N}(\Omega)$ we denote the closure of $\mathsf{C}^\infty_{\Gamma_N}(\Omega)$-tensor fields in the norm of $\mathbb{D}(\Omega)$. Then we observe for all $\varphi \in \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$ and all $\tau \in \mathbb{D}_{\Gamma_N}(\Omega)$

(34) $\langle\tau, \nabla\varphi\rangle_{0,\Omega} + \langle\operatorname{Div}\tau, \varphi\rangle_{0,\Omega} = 0.$

Notice that $\langle\operatorname{Div}\tau, \varphi\rangle_{0,\Omega} = \langle\rho\operatorname{Div}\tau, \rho^{-1}\varphi\rangle_{0,\Omega}$, so that the second term in the above relation is well defined.

4.1. **Estimates for the velocity.** From now on

$$\widetilde{u} \in \mathsf{L}^2_{-1}(\Omega) \quad\text{and}\quad \widetilde{p} \in \mathsf{L}^2(\Omega)$$

are considered as approximations of our exact solutions

$$u = u_D + \widehat{u} \in u_D + \mathsf{S}_{-1,\Gamma_D}(\Omega) \subset \mathsf{S}_{-1}(\Omega) \quad\text{and}\quad p \in \mathsf{L}^2(\Omega)$$

to our exterior Stokes problem (24)–(25), respectively. By

$$\widetilde{\sigma} \in \mathsf{L}^2(\Omega)$$

we denote an approximation of the tensor field $\sigma = \nu\nabla u - p\mathbb{I}$, which can be constructed by a certain reconstruction of $\widetilde{u}$ and $\widetilde{p}$ or computed independently.

First, we consider the simplest case where

$$\widetilde{u} \in u_D + \mathsf{S}_{-1,\Gamma_D}(\Omega) \subset \mathsf{S}_{-1}(\Omega).$$

Then by (24) we have for all solenoidal $\varphi \in \mathsf{S}_{-1,\Gamma_D}(\Omega)$

$$\langle \nu \nabla(u - \widetilde{u}), \nabla \varphi \rangle_{0,\Omega} = \langle f, \varphi \rangle_{0,\Omega} - \langle \nu \nabla \widetilde{u}, \nabla \varphi \rangle_{0,\Omega}.$$

Let $\tau \in \mathbb{D}_{\Gamma_N}(\Omega)$ and $q \in \mathsf{L}^2(\Omega)$. Using Lemma 2.1, the identity (34), and the relation $\langle q\,\mathbb{I}, \nabla \varphi \rangle_{0,\Omega} = 0$ (which holds for all $\varphi \in \mathsf{S}_{-1}(\Omega)$ since $\mathbb{I} : \nabla \varphi = \operatorname{div} \varphi$), we find that

$$\langle \nu \nabla(u - \widetilde{u}), \nabla \varphi \rangle_{0,\Omega} = \langle \operatorname{Div} \tau + f, \varphi \rangle_{0,\Omega} + \langle \tau + q\,\mathbb{I} - \nu \nabla \widetilde{u}, \nabla \varphi \rangle_{0,\Omega}$$

$$(35) \quad \leq \| \operatorname{Div} \tau + f \|_{1,\Omega} \|\varphi\|_{-1,\Omega} + \left\| \nu^{-1/2}(\tau + q\,\mathbb{I} - \nu \nabla \widetilde{u}) \right\|_{0,\Omega} \|\nu^{1/2} \nabla \varphi\|_{0,\Omega}$$

$$\leq \left(\nu_{\ominus}^{-1/2} c_{\mathrm{FP}}(\Omega, \Gamma_D) \| \operatorname{Div} \tau + f \|_{1,\Omega} + \left\| \nu^{-1/2}(\tau + q\,\mathbb{I} - \nu \nabla \widetilde{u}) \right\|_{0,\Omega} \right) \|\nu^{1/2} \nabla \varphi\|_{0,\Omega}.$$

Choosing $\varphi = u - \widetilde{u} \in \mathsf{S}_{-1,\Gamma_D}(\Omega)$, we arrive at the following estimate:

Theorem 4.1. *Let* $\widetilde{u} \in u_D + \mathsf{S}_{-1,\Gamma_D}(\Omega)$. *Then for all* $\tau \in \mathbb{D}_{\Gamma_N}(\Omega)$ *and all* $q \in \mathsf{L}^2(\Omega)$ *it holds*

$$\left\| \nu^{1/2} \nabla(u - \widetilde{u}) \right\|_{0,\Omega} \leq \nu_{\ominus}^{-1/2} c_{\mathrm{FP}}(\Omega, \Gamma_D) \| \operatorname{Div} \tau + f \|_{1,\Omega} + \left\| \nu^{-1/2}(\tau + q\,\mathbb{I} - \nu \nabla \widetilde{u}) \right\|_{0,\Omega}.$$

The upper bound coincides with the norm of the error on the left hand side, if $\tau = \sigma$ (i.e., τ coincides with the exact stress tensor) and $q = p$ (i.e., q represents the exact pressure p). In other words, we have the principal error identity

$$\left\| \nu^{1/2} \nabla(u - \widetilde{u}) \right\|_{0,\Omega}$$

$$= \min_{\substack{\tau \in \mathbb{D}_{\Gamma_N}(\Omega), \\ q \in \mathsf{L}^2(\Omega)}} \left(\nu_{\ominus}^{-1/2} c_{\mathrm{FP}}(\Omega, \Gamma_D) \| \operatorname{Div} \tau + f \|_{1,\Omega} + \left\| \nu^{-1/2}(\tau + q\,\mathbb{I} - \nu \nabla \widetilde{u}) \right\|_{0,\Omega} \right)$$

and the minimum is attained at $\tau = \sigma$ and $q = p$. Similar identities hold for many other linear elliptic problems. However, Theorem 4.1 has a drawback: The estimate is valid only for those approximate vector fields $\widetilde{u}$, which exactly satisfy the solenoidal condition and the boundary condition. In practice, the solenoidal requirement is difficult to fulfil and approximations arising in 'real life' computations often satisfy the solenoidal condition only approximately. Therefore, our next goal is to extend the estimate to a wider class of non-solenoidal vector fields. Below we extend the latter estimate to a wider class including non-solenoidal vector functions and assume only that

$$\widetilde{u} \in u_D + \mathsf{H}^1_{-1,\Gamma_D}(\Omega) \subset \mathsf{H}^1_{-1}(\Omega),$$

i.e., $\widetilde{u}$ is possibly non-solenoidal but satisfies the boundary condition exactly. Corollary 3.5 guarantees the existence of $u_0 \in \mathsf{S}_{-1}(\Omega)$ such that $u_0 - \widetilde{u} \in \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$ and

$$(36) \qquad \qquad \| \nabla(u_0 - \widetilde{u})\|_{0,\Omega} \leq \kappa(\Omega, \Gamma_D) \| \operatorname{div} \widetilde{u} \|_{0,\Omega}.$$

Hence $u_0 = \widetilde{u} + u_0 - \widetilde{u} \in u_D + \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$, i.e., $u_0 \in u_D + \mathsf{S}_{-1,\Gamma_D}(\Omega)$, and by Theorem 4.1

$$\left\| \nu^{1/2} \nabla(u - \widetilde{u}) \right\|_{0,\Omega} \leq \left\| \nu^{1/2} \nabla(u - u_0) \right\|_{0,\Omega} + \left\| \nu^{1/2} \nabla(u_0 - \widetilde{u}) \right\|_{0,\Omega}$$

$$\leq \nu_{\ominus}^{-1/2} c_{\mathrm{FP}}(\Omega, \Gamma_D) \| \operatorname{Div} \tau + f \|_{1,\Omega} + \left\| \nu^{-1/2}(\tau + q\,\mathbb{I} - \nu \nabla u_0) \right\|_{0,\Omega}$$

$$(37) \qquad \qquad + \left\| \nu^{1/2} \nabla(u_0 - \widetilde{u}) \right\|_{0,\Omega}$$

$$\leq \nu_{\ominus}^{-1/2} c_{\mathrm{FP}}(\Omega, \Gamma_D) \| \operatorname{Div} \tau + f \|_{1,\Omega} + \left\| \nu^{-1/2}(\tau + q\,\mathbb{I} - \nu \nabla \widetilde{u}) \right\|_{0,\Omega}$$

$$+ 2\nu_{\oplus}^{1/2} \left\| \nabla(u_0 - \widetilde{u}) \right\|_{0,\Omega}.$$

In view of (36), we obtain the following result.

Theorem 4.2. *Let* $\widetilde{u} \in u_D + \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$. *Then for all* $\tau \in \mathbb{D}_{\Gamma_N}(\Omega)$ *and all* $q \in \mathsf{L}^2(\Omega)$ *it holds*

$$\big\|\nu^{1/2}\,\nabla(u-\widetilde{u})\big\|_{0,\Omega} \le \nu_{\ominus}^{-1/2} c_{\mathrm{FP}}(\Omega,\Gamma_D)\|\operatorname{Div}\tau + f\|_{1,\Omega} + \big\|\nu^{-1/2}(\tau + q\,\mathbb{I} - \nu\,\nabla\,\widetilde{u})\big\|_{0,\Omega}$$
$$+\, 2\nu_{\oplus}^{1/2}\kappa(\Omega,\Gamma_D)\|\operatorname{div}\widetilde{u}\|_{0,\Omega}.$$

If the approximation $\widetilde{u}$ is solenoidal we get back Theorem 4.1 and, again, the upper bound coincides with the norm of the error on the left hand side if $\tau = \sigma$, $q = p$. If the approximation $\widetilde{u}$ is solenoidal just in, e.g., $\mathbb{R}^d \setminus \overline{B}_{r_2}$ then we get trivially an estimate by Theorem 4.2, replacing the term $\|\operatorname{div}\widetilde{u}\|_{0,\Omega}$ by $\|\operatorname{div}\widetilde{u}\|_{0,\omega}$. But with a moderate additional assumption on the decay of the approximation we can even do better in this case, replacing the constant $\kappa(\Omega,\Gamma_D)$ by a stability constant $\kappa(\omega,\gamma_D)$ of the bounded domain ω.

For this let $\widetilde{u} = u_D + w \in u_D + \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$ with $\operatorname{div}\widetilde{u} = \operatorname{div} w = 0$ in $\mathbb{R}^d \setminus \overline{B}_{r_2}$ and if $\gamma_D = \gamma$ (i.e., $\Gamma_D = \Gamma$) then $\widetilde{u}$ satisfies $\operatorname{div}\widetilde{u} \in \mathsf{L}^2_\perp(\omega)$. To meet the latter condition, we additionally assume for the case $\Gamma_D = \Gamma$ that

$$(38) \qquad\qquad\qquad |w| \le c\,r^{-m}, \qquad m > d - 1$$

for $r \to \infty$ with some $c > 0$ independent of r (notice that $r^{-m} \in \mathsf{L}^2_{-1}(\mathbb{R}^d \setminus \overline{B}_1)$ if $m > d/2 - 1$). Indeed, it is easy to see that for $r > r_2$

$$\Big|\int_\omega \operatorname{div}\widetilde{u}\Big| = \Big|\int_\omega \operatorname{div} w\Big| = \Big|\int_{S_r} n \cdot w\Big| \le c\,r^{d-1-m} \xrightarrow{r\to\infty} 0$$

holds. Now we consider the ansatz

$$u_0 := \widetilde{u} + \begin{cases} u_\omega & \text{in } \omega, \\ 0 & \text{in } \mathbb{R}^d \setminus \overline{B}_{r_2}, \end{cases}$$

where $u_\omega \in \mathsf{H}^1_{\gamma_D}(\omega)$. Utilising Lemma 1.1, we find $u_\omega \in \mathsf{H}^1_{\gamma_D}(\omega)$ such that

$$\operatorname{div} u_\omega = -\operatorname{div}\widetilde{u} \quad \text{in } \omega$$

together with the stability estimate $\|\nabla u_\omega\|_{0,\omega} \le \kappa(\omega,\gamma_D)\|\operatorname{div}\widetilde{u}\|_{0,\omega}$. The so constructed u_0 satisfies the boundary condition on Γ_D and it is a solenoidal field, i.e., $u_0 \in u_D + \mathsf{S}_{-1,\Gamma_D}(\Omega)$. Using u_0 as an admissible vector field in (37) yields the estimate

$$\big\|\nu^{1/2}\,\nabla(u-\widetilde{u})\big\|_{0,\Omega} \le \nu_{\ominus}^{-1/2} c_{\mathrm{FP}}(\Omega,\Gamma_D)\|\operatorname{Div}\tau + f\|_{1,\Omega} + \big\|\nu^{-1/2}(\tau + q\,\mathbb{I} - \nu\,\nabla\,\widetilde{u})\big\|_{0,\Omega}$$
$$+\, 2\nu_{\oplus}^{1/2} \underbrace{\big\|\nabla(u_0-\widetilde{u})\big\|_{0,\Omega}}_{=\|\nabla u_\omega\|_{0,\omega}}.$$

Hence we have the following improved estimate for the case of a partially solenoidal approximation.

Corollary 4.3. *Let* $\widetilde{u} \in u_D + \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$ *and*

$$\operatorname{div}\widetilde{u} = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{B}_{r_2}.$$

If $\Gamma_D = \Gamma$, *then we additionally impose the condition* (38). *Then for all* $\tau \in \mathbb{D}_{\Gamma_N}(\Omega)$ *and all* $q \in \mathsf{L}^2(\Omega)$ *we have*

$$\big\|\nu^{1/2}\,\nabla(u-\widetilde{u})\big\|_{0,\Omega} \le \nu_{\ominus}^{-1/2} c_{\mathrm{FP}}(\Omega,\Gamma_D)\|\operatorname{Div}\tau + f\|_{1,\Omega}$$
$$+ \big\|\nu^{-1/2}(\tau + q\,\mathbb{I} - \nu\,\nabla\,\widetilde{u})\big\|_{0,\Omega} + 2\nu_{\oplus}^{1/2}\kappa(\omega,\gamma_D)\|\operatorname{div}\widetilde{u}\|_{0,\omega}.$$

Here the last term on the right hand side is a penalty for possible violation of the solenoidal condition in ω.

4.2. Estimates for the pressure. By Lemma 2.3 there exists a vector field $\psi \in \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$ such that

$$(39) \qquad \operatorname{div} \psi = p - \widetilde{p} \quad \text{and} \quad \|\nabla \psi\|_{0,\Omega} \leq \kappa(\Omega, \Gamma_D)\|p - \widetilde{p}\|_{0,\Omega}.$$

For all $\widetilde{u} \in u_D + \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$ and all $\tau \in \mathbb{D}_{\Gamma_N}(\Omega)$ we have

$$\|p - \widetilde{p}\|^2_{0,\Omega} = \langle p - \widetilde{p}, \operatorname{div} \psi \rangle_{0,\Omega}$$
$$= \langle \nu \nabla(u - \widetilde{u}), \nabla \psi \rangle_{0,\Omega} - \langle \operatorname{Div}\tau + f, \psi \rangle_{0,\Omega} + \langle \nu \nabla \widetilde{u} - \widetilde{p}\,\mathbb{I} - \tau, \nabla \psi \rangle_{0,\Omega}$$
$$\leq \Big(\|\nu \nabla(u - \widetilde{u})\|_{0,\Omega} + c_{\mathrm{FP}}(\Omega, \Gamma_D)\|\operatorname{Div}\tau + f\|_{1,\Omega} + \|\nu \nabla \widetilde{u} - \widetilde{p}\,\mathbb{I} - \tau\|_{0,\Omega} \Big) \|\nabla \psi\|_{0,\Omega},$$

where we have used Lemma 2.1 for ψ and the relation $\operatorname{div} \psi = \mathbb{I} : \nabla \psi$. By (39) we obtain

$$\|p - \widetilde{p}\|_{0,\Omega} \leq \kappa(\Omega, \Gamma_D)\Big(\nu_\oplus^{1/2}\|\nu^{1/2} \nabla(u - \widetilde{u})\|_{0,\Omega} + c_{\mathrm{FP}}(\Omega, \Gamma_D)\|\operatorname{Div}\tau + f\|_{1,\Omega}$$
$$+ \nu_\oplus^{1/2}\|\nu^{-1/2}(\tau + \widetilde{p}\,\mathbb{I} - \nu \nabla \widetilde{u})\|_{0,\Omega} \Big).$$

In order to estimate the first term in the right hand side, we use Theorem 4.2 with $q = \widetilde{p}$ and arrive at the desired estimate for the pressure field.

Theorem 4.4. *Let $\widetilde{p} \in \mathsf{L}^2(\Omega)$. Then for all $\tau \in \mathbb{D}_{\Gamma_N}(\Omega)$ and all $\widetilde{u} \in u_D + \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$ it holds*

$$(40) \qquad \begin{aligned} \|p - \widetilde{p}\|_{0,\Omega} \leq \kappa(\Omega, \Gamma_D)\Big(&(\nu_\ominus^{-1/2}\nu_\oplus^{1/2} + 1)c_{\mathrm{FP}}(\Omega, \Gamma_D)\|\operatorname{Div}\tau + f\|_{1,\Omega} \\ &+ 2\nu_\oplus^{1/2}\|\nu^{-1/2}(\tau + \widetilde{p}\,\mathbb{I} - \nu \nabla \widetilde{u})\|_{0,\Omega} + 2\nu_\oplus \kappa(\Omega, \Gamma_D)\|\operatorname{div}\widetilde{u}\|_{0,\Omega}\Big). \end{aligned}$$

Remark 4.5. *The upper bound (40) consists of the same terms as the upper bound of Theorem 4.2 and vanishes if $\widetilde{u} = u$, $\tau = \sigma$, $\widetilde{p} = p$. However, in this case, the estimate stronger depends on the stability constant $\kappa(\Omega, \Gamma_D)$. Similar effect has been observed in the estimates related to bounded domains (see [35, 36]).*

4.3. Estimates for non-conforming approximations. The term non-conforming is usually applied to approximations that belong to a functional class wider than the natural energy class of the problem in question. For example, non–conformity of approximations may arise due to violation of continuity conditions or main boundary conditions. Nowadays such type approximations are widely used in computational practice (e.g., mortar, finite volume, and discontinuous Galerkin approximations) because they offer more freedom for various mesh adaptive procedures. Application of functional type a posteriori estimates to non-conforming approximations of elliptic problems has been earlier studied in [4, 15, 37, 45]. In this section, we briefly discuss this question in the context of the exterior Stokes problem.

Let us now assume that we have a non-conforming approximation

$$\widetilde{\Upsilon} \in \mathsf{L}^2(\Omega)$$

of the exact strain tensor field

$$\Upsilon := \nabla u, \qquad u = u_D + \widehat{u} \in u_D + \mathsf{S}_{-1,\Gamma_D}(\Omega) \subset \mathsf{S}_{-1}(\Omega).$$

E.g., $\widetilde{\Upsilon}$ could be a "broken gradient" tensor field as output of some discontinuous Galerkin method. By the triangle inequality, we estimate the difference between these tensor fields using a certain conforming approximation $\widetilde{u} \in u_D + \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$ and obtain

$$\|\nu^{1/2}(\Upsilon - \widetilde{\Upsilon})\|_{0,\Omega} \leq \|\nu^{1/2} \nabla(u - \widetilde{u})\|_{0,\Omega} + \|\nu^{1/2}(\nabla \widetilde{u} - \widetilde{\Upsilon})\|_{0,\Omega}.$$

Theorem 4.2 implies the estimate

$$\|\nu^{1/2}(\Upsilon - \widetilde{\Upsilon})\|_{0,\Omega} \leq \nu_\ominus^{-1/2}c_{\mathrm{FP}}(\Omega, \Gamma_D)\|\operatorname{Div}\tau + f\|_{1,\Omega} + \|\nu^{-1/2}(\tau + q\,\mathbb{I} - \nu \nabla \widetilde{u})\|_{0,\Omega}$$
$$+ 2\nu_\oplus^{1/2}\kappa(\Omega, \Gamma_D)\|\operatorname{div}\widetilde{u}\|_{0,\Omega} + \|\nu^{1/2}(\nabla \widetilde{u} - \widetilde{\Upsilon})\|_{0,\Omega}.$$

Using the triangle inequality and Theorem 4.4 we obtain a posteriori error estimates for non-conforming approximations.

Theorem 4.6. *Let* $\widetilde{\Upsilon} \in \mathsf{L}^2(\Omega)$ *and* $\widetilde{p} \in \mathsf{L}^2(\Omega)$. *Then for all* $\tau \in \mathbb{D}_{\Gamma_N}(\Omega)$, $q \in \mathsf{L}^2(\Omega)$, *and* $\widetilde{u} \in u_D + \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$ *it holds*

$$\left\| \nu^{1/2}(\Upsilon - \widetilde{\Upsilon}) \right\|_{0,\Omega} \leq \nu_\ominus^{-1/2} c_{\mathrm{FP}}(\Omega, \Gamma_D) \| \operatorname{Div} \tau + f \|_{1,\Omega} + \left\| \nu^{-1/2}(\tau + q\,\mathbb{I} - \nu\widetilde{\Upsilon}) \right\|_{0,\Omega}$$
$$+ 2\nu_\oplus^{1/2} \kappa(\Omega, \Gamma_D) \| \operatorname{div} \widetilde{u} \|_{0,\Omega} + 2 \left\| \nu^{1/2}(\nabla \widetilde{u} - \widetilde{\Upsilon}) \right\|_{0,\Omega}$$

and

$$\| p - \widetilde{p} \|_{0,\Omega} \leq \kappa(\Omega, \Gamma_D) \Big((\nu_\ominus^{-1/2} \nu_\oplus^{1/2} + 1) c_{\mathrm{FP}}(\Omega, \Gamma_D) \| \operatorname{Div} \tau + f \|_{1,\Omega}$$
$$+ 2\nu_\oplus^{1/2} \left\| \nu^{-1/2}(\tau + \widetilde{p}\,\mathbb{I} - \nu\widetilde{\Upsilon}) \right\|_{0,\Omega} + 2\nu_\oplus \kappa(\Omega, \Gamma_D) \| \operatorname{div} \widetilde{u} \|_{0,\Omega}$$
$$+ 2\nu_\oplus^{1/2} \left\| \nu^{1/2}(\nabla \widetilde{u} - \widetilde{\Upsilon}) \right\|_{0,\Omega} \Big).$$

It is easy to see that in the case where $\widetilde{\Upsilon} = \nabla \widetilde{u}$ is generated by the conforming approximation $\widetilde{u} \in u_D + \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$ the last term vanishes and we get back Theorem 4.2 and Theorem 4.4.

4.4. **Estimates for the stress field.** Error estimates for the stress tensor field follow directly from the above derived estimates for the velocity vector field and the pressure function. Indeed, let $\widetilde{\sigma} \in \mathsf{L}^2(\Omega)$ be an approximation of the exact stress tensor

$$\sigma = \nu \nabla u - p\,\mathbb{I} = \nu\Upsilon - p\,\mathbb{I}.$$

Moreover, let $\widetilde{\Upsilon} \in \mathsf{L}^2(\Omega)$ and $\widetilde{p} \in \mathsf{L}^2(\Omega)$. Then, the respective error estimate follows from the triangle inequality

$$\| \widetilde{\sigma} - \sigma \|_{0,\Omega} \leq \| \widetilde{\sigma} - \nu\widetilde{\Upsilon} + \widetilde{p}\,\mathbb{I} \|_{0,\Omega} + \nu_\oplus^{1/2} \left\| \nu^{1/2}(\Upsilon - \widetilde{\Upsilon}) \right\|_{0,\Omega} + d^{1/2} \| p - \widetilde{p} \|_{0,\Omega}.$$

In particular, we can set $\widetilde{\Upsilon} = \nabla \widetilde{u}$, where $\widetilde{u} \in u_D + \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$. The first term on the right hand side contains only known tensor fields and the second and third ones are estimated by, e.g., Theorem 4.2, Theorem 4.4, and Theorem 4.6.

4.5. **Lower bounds of the error.** Let $\widetilde{u} \in u_D + \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$, i.e., $u - \widetilde{u} \in \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$. Obviously, (as the subsequent max-property holds for any Hilbert[iii] space) we have by (24)

$$\left\| \nu^{1/2} \nabla(u - \widetilde{u}) \right\|_{0,\Omega}^2 = \max_{\varphi \in \mathsf{H}^1_{-1,\Gamma_D}(\Omega)} \left(2\langle \nu \nabla(u - \widetilde{u}), \nabla \varphi \rangle_{0,\Omega} - \| \nu^{1/2} \nabla \varphi \|_{0,\Omega}^2 \right)$$
$$\geq 2\langle \nu \nabla u, \nabla \varphi \rangle_{0,\Omega} - 2\langle \nu \nabla \widetilde{u}, \nabla \varphi \rangle_{0,\Omega} - \| \nu^{1/2} \nabla \varphi \|_{0,\Omega}^2$$
$$= 2\langle f, \varphi \rangle_{0,\Omega} + 2\langle q, \operatorname{div} \varphi \rangle_{0,\Omega} - \langle \nu \nabla(2\widetilde{u} + \varphi), \nabla \varphi \rangle_{0,\Omega}$$
$$+ 2\langle p - q, \operatorname{div} \varphi \rangle_{0,\Omega}$$

and the maximum is attained at $\varphi = u - \widetilde{u} \in \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$. The last term can be simply (but rather coarsly) estimated by Theorem 4.4 ($\widetilde{p} = q$) what yields the estimate presented below.

Theorem 4.7. *For all* $\widetilde{u}, v \in u_D + \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$ *and all* $\varphi \in \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$, $\tau \in \mathbb{D}_{\Gamma_N}(\Omega)$, *and* $q \in \mathsf{L}^2(\Omega)$ *it holds*

$$\left\| \nu^{1/2} \nabla(u - \widetilde{u}) \right\|_{0,\Omega}^2 \geq 2\langle f, \varphi \rangle_{0,\Omega} + 2\langle q, \operatorname{div} \varphi \rangle_{0,\Omega} - \langle \nu \nabla(2\widetilde{u} + \varphi), \nabla \varphi \rangle_{0,\Omega}$$
$$- 2\kappa(\Omega, \Gamma_D) \| \operatorname{div} \varphi \|_{0,\Omega} \Big((\nu_\ominus^{-1/2} \nu_\oplus^{1/2} + 1) c_{\mathrm{FP}}(\Omega, \Gamma_D) \| \operatorname{Div} \tau + f \|_{1,\Omega}$$
$$+ 2\nu_\oplus^{1/2} \left\| \nu^{-1/2}(\tau + q\,\mathbb{I} - \nu \nabla v) \right\|_{0,\Omega} + 2\nu_\oplus \kappa(\Omega, \Gamma_D) \| \operatorname{div} v \|_{0,\Omega} \Big).$$

In particular $v = \widetilde{u}$ *is possible. If* $\varphi \in \mathsf{S}_{-1,\Gamma_D}(\Omega)$ *then the simple lower bound*

$$\left\| \nu^{1/2} \nabla(u - \widetilde{u}) \right\|_{0,\Omega}^2 \geq 2\langle f, \varphi \rangle_{0,\Omega} - \langle \nu \nabla(2\widetilde{u} + \varphi), \nabla \varphi \rangle_{0,\Omega}$$

holds. Moreover, equality is given for $\varphi = u - \widetilde{u}$, *provided that the approximation* $\widetilde{u}$ *is also solenoidal, i.e.,* $\widetilde{u} \in u_D + \mathsf{S}_{-1,\Gamma_D}(\Omega)$.

[iii]In any Hilbert space H it holds $|x|^2 = \max_{y \in \mathsf{H}} \left(2\langle x, y \rangle - |y|^2 \right)$. Here $\mathsf{H} = \nabla \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$.

To handle a non-conforming approximation $\widetilde{\Upsilon} \in \mathsf{L}^2(\Omega)$ we can simply utilise the triangle inequality and apply Theorem 4.7.

Theorem 4.8. *Let* $\widetilde{\Upsilon} \in \mathsf{L}^2(\Omega)$. *Then for all* $\varphi \in \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$, $\tau \in \mathbb{D}_{\Gamma_N}(\Omega)$, $q \in \mathsf{L}^2(\Omega)$, *and* $\widetilde{u}, v \in u_D + \mathsf{H}^1_{-1,\Gamma_D}(\Omega)$, *it holds*

$$\left\| \nu^{1/2}(\Upsilon - \widetilde{\Upsilon}) \right\|_{0,\Omega} \geq \left\| \nu^{1/2} \nabla(u - \widetilde{u}) \right\|_{0,\Omega} - \left\| \nu^{1/2}(\nabla \widetilde{u} - \widetilde{\Upsilon}) \right\|_{0,\Omega},$$

$$\left\| \nu^{1/2} \nabla(u - \widetilde{u}) \right\|_{0,\Omega}^2 \geq 2\langle f, \varphi \rangle_{0,\Omega} + 2\langle q, \operatorname{div}\varphi \rangle_{0,\Omega} - \langle \nu \nabla(2\widetilde{u} + \varphi), \nabla \varphi \rangle_{0,\Omega}$$

$$- 2\kappa(\Omega, \Gamma_D) \| \operatorname{div}\varphi \|_{0,\Omega} \Big((\nu_\ominus^{-1/2} \nu_\oplus^{1/2} + 1) c_{\mathrm{FP}}(\Omega, \Gamma_D) \| \operatorname{Div}\tau + f \|_{1,\Omega}$$

$$+ 2\nu_\oplus^{1/2} \| \nu^{-1/2}(\tau + q\,\mathbb{I} - \nu \nabla v) \|_{0,\Omega} + 2\nu_\oplus \kappa(\Omega, \Gamma_D) \| \operatorname{div} v \|_{0,\Omega} \Big).$$

4.6. Special case: 2D exterior domains.

For a Lipschitz domain $\mathcal{D} \subset \mathbb{R}^2$ we introduce somewhat different weighted spaces using logarithmic weights, namely,

$$\mathsf{L}^2_{\pm 1,\ln}(\mathcal{D}) := \left\{ \phi \in \mathsf{L}^2_{\mathrm{loc}}(\mathcal{D}) \,:\, \big(\rho \ln(e + \rho)\big)^{\pm 1} \phi \in \mathsf{L}^2(\mathcal{D}) \right\},$$

$$\mathsf{H}^1_{-1,\ln}(\mathcal{D}) := \left\{ \phi \in \mathsf{L}^2_{-1,\ln}(\mathcal{D}) \,:\, \nabla \phi \in \mathsf{L}^2(\mathcal{D}) \right\},$$

where e is the Euler number, see, e.g., [16, 43, 9]. We notice that at infinity $\big(\rho \ln(e + \rho)\big)^{\pm 1}$ behaves like $(r \ln r)^{\pm 1}$. The inner product in $\mathsf{L}^2_{\pm 1,\ln}(\mathcal{D})$ is defined by the relation

$$\langle \phi, \psi \rangle_{\pm 1,\ln,\mathcal{D}} := \big\langle \big(\rho \ln(e + \rho)\big)^{\pm 2} \phi, \psi \big\rangle_{0,\mathcal{D}}.$$

All other weighted spaces and norms are modified and defined in a similar way.

The sets $\Omega \subset \mathbb{R}^2$ and $\omega \subset \mathbb{R}^2$ are defined as in Section 2, i.e., $\Omega \subset \mathbb{R}^2$ is an exterior Lipschitz domain and ω is a certain truncation of Ω. The situation is now different from the case $d \geq 3$ as the constants[iv] will be integrable in our weighted spaces, i.e., $1 \in \mathsf{L}^2_{-1,\ln}(\Omega)$. Introducing additionally

$$\mathsf{H}^1_{-1,\ln,\emptyset}(\Omega) := \mathsf{H}^1_{-1,\ln}(\Omega) \cap \mathbb{R}^{\perp_{-1,\ln,\Omega}}$$

we have the following Friedrichs/Poincare estimate for exterior domains:

Lemma 4.9. *There exists* $c > 0$ *such that for all* $v \in \mathsf{H}^1_{-1,\ln,\Gamma_D}(\Omega)$

$$\| v \|_{-1,\ln,\Omega} \leq c \| \nabla v \|_{0,\Omega}.$$

The best constant c *is denoted by* $c_{\mathrm{FP}}(\Omega, \Gamma_D)$. *In the special case* $B_e \subset \mathbb{R}^d \setminus \overline{\Omega}$ *and* $\Gamma_D = \Gamma$ *we have* $c_{\mathrm{FP}}(\Omega, \Gamma) \leq 2$.

This theorem follows from [27, Appendix 4.2, Lemma 4.1, Corollary 4.2, Remark 4.3], see also [43, Lemma 4.1] and [28, 16]. It is worth noting that in this case we need boundary or mean value conditions as in the case of a bounded domain. Now, all results from the sections for $d \geq 3$ follow with the obvious modifications. In particular, Lemma 3.2 reads as follows.

Lemma 4.10 (stability lemma for 2D exterior domains). *There exists* $c > 0$ *such that for any* $h \in \mathsf{L}^2(\Omega)$ *there exists* $u_h \in \mathsf{H}^1_{-1,\ln,\Gamma_D}(\Omega)$ *with*

$$\operatorname{div} u_h = h \quad and \quad \| \nabla u_h \|_{0,\Omega} \leq c \| h \|_{0,\Omega}.$$

The best constant is denoted by $\kappa(\Omega, \Gamma_D)$ *and satisfies the estimate*

$$\kappa(\Omega, \Gamma_D) \leq \kappa_\oplus(\Omega, \Gamma_D) := (1 + \kappa)\big(1 + c_{\mathrm{P}}(\mathbb{R}^2)\, \alpha\, \rho(r_2) \ln(e + \rho(r_2))\big)$$

with

$$\alpha = \alpha(r_1, r_2, \eta) = \max_{x \in \overline{B}_{r_2} \setminus B_{r_1}} \big| \nabla \eta(x) \big|, \qquad \kappa = \min \big\{ \kappa(\omega, \gamma_D), \kappa(\omega, \gamma) \big\}.$$

[iv]More precisely, for $0 < \epsilon < 1$ we have

$$(r \ln r)^{-1} \in \mathsf{L}^2(B_\epsilon), \qquad (r \ln r)^{-1} \notin \mathsf{L}^2(B_{1+\epsilon} \setminus \overline{B}_{1-\epsilon}), \qquad (r \ln r)^{-1} \in \mathsf{L}^2(\mathbb{R}^2 \setminus \overline{B}_e).$$

If h has compact support in $\overline{\omega}$ and (if $\Gamma_D = \Gamma$) additionally $\int_\Omega h = \int_\omega h = 0$, then u_h can be chosen with a compact support in $\overline{\omega}$ as well, in particular $u_h \in \mathsf{H}^1_{\gamma_D}(\omega) \subset \mathsf{H}^1_{\Gamma_D}(\Omega)$. In this case, $\kappa(\Omega, \Gamma_D) \leq \kappa(\omega, \gamma_D)$.

Estimates of the distance to the set of solenoidal fields are derived quite similarly.

Corollary 4.11. *For any $u \in \mathsf{H}^1_{-1,\ln,\Gamma_D}(\Omega)$ there exists a solenoidal $u_0 \in \mathsf{S}_{-1,\ln,\Gamma_D}(\Omega)$ such that*

$$\operatorname{dist}\big(u, \mathsf{S}_{-1,\ln,\Gamma_D}(\Omega)\big) \leq \big\| \nabla(u - u_0) \big\|_{0,\Omega} \leq \kappa(\Omega, \Gamma_D) \| \operatorname{div} u \|_{0,\Omega}.$$

For any $u \in \mathsf{H}^1_{-1,\ln}(\Omega)$ there exists a solenoidal $u_0 \in \mathsf{S}_{-1,\ln}(\Omega)$ such that $u - u_0 \in \mathsf{H}^1_{-1,\ln,\Gamma_D}(\Omega)$, i.e., $u_0|_{\Gamma_D} = u|_{\Gamma_D}$, and

$$\big\| \nabla(u - u_0) \big\|_{0,\Omega} \leq \kappa(\Omega, \Gamma_D) \| \operatorname{div} u \|_{0,\Omega}.$$

Another obvious corollary is the inf-sup lemma for 2D exterior domains.

Corollary 4.12. *It holds*

$$\inf_{h \in \mathsf{L}^2(\Omega)} \sup_{u \in \mathsf{H}^1_{-1,\ln,\Gamma_D}(\Omega)} \frac{\langle h, \operatorname{div} u \rangle_{0,\Omega}}{\|h\|_{0,\Omega} \| \nabla u \|_{0,\Omega}} \geq \frac{1}{\kappa(\Omega, \Gamma_D)}.$$

As in the case of $d \geq 3$, the solvability of the Stokes problem and respective energy estimates follow. Below we remind these results. Let

$$\mathsf{L}^2_{1,\ln,\Gamma_N}(\Omega) := \begin{cases} \mathsf{L}^2_{1,\ln}(\Omega) & \text{, if } \Gamma_D \neq \emptyset, \\ \mathsf{L}^2_{1,\ln,\perp}(\Omega) & \text{, if } \Gamma_D = \emptyset, \end{cases}$$

and

$$\mathsf{L}^2_{1,\ln,\perp}(\Omega) := \mathsf{L}^2_{1,\ln}(\Omega) \cap (\mathbb{R}^2)^{\perp_{0,\Omega}} = \big\{ \phi \in \mathsf{L}^2_{1,\ln}(\Omega) \ : \ \int_\Omega \phi_i = 0 \big\}.$$

Corollary 4.13. *For ν, $f \in \mathsf{L}^2_{1,\ln,\Gamma_N}(\Omega)$, and $u_D \in \mathsf{S}_{-1,\ln}(\Omega)$ the 2D Stokes system is uniquely solvable with a solenoidal vector field $u = u_D + \widehat{u} \in u_D + \mathsf{S}_{-1,\ln,\Gamma_D}(\Omega) \subset \mathsf{S}_{-1,\ln}(\Omega)$ and $p \in \mathsf{L}^2(\Omega)$. Moreover,*

$$\nu \| \nabla \widehat{u} \|_{0,\Omega} \leq c_{\mathrm{FP}}(\Omega, \Gamma_D) \|f\|_{1,\ln,\Omega} + \nu \| \nabla u_D \|_{0,\Omega},$$
$$\nu \| \nabla u \|_{0,\Omega} \leq c_{\mathrm{FP}}(\Omega, \Gamma_D) \|f\|_{1,\ln,\Omega} + 2\nu \| \nabla u_D \|_{0,\Omega},$$
$$\|p\|_{0,\Omega} \leq 2\kappa(\Omega, \Gamma_D)\big(c_{\mathrm{FP}}(\Omega, \Gamma_D)\|f\|_{1,\ln,\Omega} + \nu \| \nabla u_D \|_{0,\Omega}\big).$$

Now we address the subject of a posteriori error estimation and introduce the sets

$$\mathbb{D}(\Omega) := \big\{ \tau \in \mathsf{L}^2(\Omega) \ : \ \operatorname{Div} \tau \in \mathsf{L}^2_{1,\ln}(\Omega) \big\}$$

and $\mathbb{D}_{\Gamma_N}(\Omega)$ as closure of $\mathsf{C}^\infty_{\Gamma_N}(\Omega)$-tensor fields in the norm of $\mathbb{D}(\Omega)$. For errors encompassed in approximation of the velocity field we have the following results which repeat (with some modifications) those derived for $d \geq 3$. First, we present an analog of Theorem 4.2.

Theorem 4.14. *Let $\widetilde{u} \in u_D + \mathsf{H}^1_{-1,\ln,\Gamma_D}(\Omega)$. Then for all $\tau \in \mathbb{D}_{\Gamma_N}(\Omega)$ and $q \in \mathsf{L}^2(\Omega)$*

$$\big\| \nu^{1/2} \nabla(u - \widetilde{u}) \big\|_{0,\Omega} \leq \nu_\ominus^{-1/2} c_{\mathrm{FP}}(\Omega, \Gamma_D) \| \operatorname{Div} \tau + f \|_{1,\ln,\Omega} + \big\| \nu^{-1/2}(\tau + q\,\mathbb{I} - \nu \nabla \widetilde{u}) \big\|_{0,\Omega}$$
$$+ 2\nu_\oplus^{1/2} \kappa(\Omega, \Gamma_D) \| \operatorname{div} \widetilde{u} \|_{0,\Omega}.$$

As in the case $d \geq 3$, the estimate is simplified if $\operatorname{div} \widetilde{v} = 0$ in $\mathbb{R}^2 \setminus \overline{B}_{r_2}$ and additionally (if $\Gamma_D = \Gamma$) the condition (38) is satisfied. Then, in Theorem 4.14 we can replace the constant $\kappa(\Omega, \Gamma_D)$ by $\kappa(\omega, \gamma_D)$. If the approximation $\widetilde{u}$ is solenoidal in Ω, then the last term vanishes and we arrive at estimates analogous to Theorem 4.1. It possesses the same property: the upper bound coincides with the norm of the error on the left hand side if $\tau = \sigma$ and $p = q$ (i.e., the estimate is sharp in the sense that there is no "gap" between its left and right hand sides).

For the approximation of the pressure function, Theorem 4.4 is modified as follows.

Theorem 4.15. *Let $\widetilde{p} \in \mathsf{L}^2(\Omega)$. Then for all $\tau \in \mathbb{D}_{\Gamma_N}(\Omega)$ and all $\widetilde{u} \in u_D + \mathsf{H}^1_{-1,\ln,\Gamma_D}(\Omega)$ it holds*

$$\|p - \widetilde{p}\|_{0,\Omega} \leq \kappa(\Omega, \Gamma_D)\Big((\nu_{\ominus}^{-1/2}\nu_{\oplus}^{1/2} + 1)c_{\mathrm{FP}}(\Omega, \Gamma_D)\|\operatorname{Div}\tau + f\|_{1,\ln,\Omega}$$
$$+ 2\nu_{\oplus}^{1/2}\|\nu^{-1/2}(\tau + \widetilde{p}\,\mathbb{I} - \nu\,\nabla\,\widetilde{u})\|_{0,\Omega} + 2\nu_{\oplus}\kappa(\Omega, \Gamma_D)\|\operatorname{div}\widetilde{u}\|_{0,\Omega}\Big).$$

Finally, we consider a non-conforming approximation $\widetilde{u}$ and obtain analogs of the theorems exposed in Section 4.3.

Theorem 4.16. *Let $\widetilde{\Upsilon} \in \mathsf{L}^2(\Omega)$ and $\widetilde{p} \in \mathsf{L}^2(\Omega)$. Then for all $\widetilde{u} \in u_D + \mathsf{H}^1_{-1,\ln,\Gamma_D}(\Omega)$, $\tau \in \mathbb{D}_{\Gamma_N}(\Omega)$, and $q \in \mathsf{L}^2(\Omega)$ it holds*

$$\big\|\nu^{1/2}(\Upsilon - \widetilde{\Upsilon})\big\|_{0,\Omega} \leq \nu_{\ominus}^{-1/2}c_{\mathrm{FP}}(\Omega, \Gamma_D)\|\operatorname{Div}\tau + f\|_{1,\ln,\Omega} + \big\|\nu^{-1/2}(\tau + q\,\mathbb{I} - \nu\,\nabla\,\widetilde{u})\big\|_{0,\Omega}$$
$$+ 2\nu_{\oplus}^{1/2}\kappa(\Omega, \Gamma_D)\|\operatorname{div}\widetilde{u}\|_{0,\Omega} + \big\|\nu^{1/2}(\nabla\,\widetilde{u} - \widetilde{\Upsilon})\big\|_{0,\Omega}$$
$$\leq \nu_{\ominus}^{-1/2}c_{\mathrm{FP}}(\Omega, \Gamma_D)\|\operatorname{Div}\tau + f\|_{1,\ln,\Omega} + \big\|\nu^{-1/2}(\tau + q\,\mathbb{I} - \nu\widetilde{\Upsilon})\big\|_{0,\Omega}$$
$$+ 2\nu_{\oplus}^{1/2}\kappa(\Omega, \Gamma_D)\|\operatorname{div}\widetilde{u}\|_{0,\Omega} + 2\big\|\nu^{1/2}(\nabla\,\widetilde{u} - \widetilde{\Upsilon})\big\|_{0,\Omega}$$

and

$$\|p - \widetilde{p}\|_{0,\Omega} \leq \kappa(\Omega, \Gamma_D)\Big((\nu_{\ominus}^{-1/2}\nu_{\oplus}^{1/2} + 1)c_{\mathrm{FP}}(\Omega, \Gamma_D)\|\operatorname{Div}\tau + f\|_{1,\ln,\Omega}$$
$$+ 2\nu_{\oplus}^{1/2}\big\|\nu^{-1/2}(\tau + \widetilde{p}\,\mathbb{I} - \nu\widetilde{\Upsilon})\big\|_{0,\Omega} + 2\nu_{\oplus}\kappa(\Omega, \Gamma_D)\|\operatorname{div}\widetilde{u}\|_{0,\Omega}$$
$$+ 2\nu_{\oplus}^{1/2}\|\nu^{1/2}(\nabla\,\widetilde{u} - \widetilde{\Upsilon})\|_{0,\Omega}\Big).$$

For $\widetilde{\Upsilon} = \nabla\,\widetilde{u}$, where $\widetilde{u} \in u_D + \mathsf{H}^1_{-1,\ln,\Gamma_D}(\Omega)$ we get back Theorem 4.14 and Theorem 4.15. As in Section 4.4, error estimates for the stress tensor field σ follow immediately by the triangle inequality.

Finally, we briefly present lower bounds of the error derived in the spirit of Section 4.5.

Theorem 4.17. *For all $\widetilde{u}, v \in u_D + \mathsf{H}^1_{-1,\ln,\Gamma_D}(\Omega)$ and all $\varphi \in \mathsf{H}^1_{-1,\ln,\Gamma_D}(\Omega)$, $\tau \in \mathbb{D}_{\Gamma_N}(\Omega)$, and $q \in \mathsf{L}^2(\Omega)$ it holds*

$$\big\|\nu^{1/2}\,\nabla(u - \widetilde{u})\big\|_{0,\Omega}^2 \geq 2\langle f, \varphi\rangle_{0,\Omega} + 2\langle q, \operatorname{div}\varphi\rangle_{0,\Omega} - \langle\nu\,\nabla(2\widetilde{u} + \varphi), \nabla\,\varphi\rangle_{0,\Omega}$$
$$- 2\kappa(\Omega, \Gamma_D)\|\operatorname{div}\varphi\|_{0,\Omega}\Big((\nu_{\ominus}^{-1/2}\nu_{\oplus}^{1/2} + 1)c_{\mathrm{FP}}(\Omega, \Gamma_D)\|\operatorname{Div}\tau + f\|_{1,\ln,\Omega}$$
$$+ 2\nu_{\oplus}^{1/2}\big\|\nu^{-1/2}(\tau + q\,\mathbb{I} - \nu\,\nabla v)\big\|_{0,\Omega} + 2\nu_{\oplus}\kappa(\Omega, \Gamma_D)\|\operatorname{div}v\|_{0,\Omega}\Big).$$

In particular, $v = \widetilde{u}$ is possible. If $\varphi \in \mathsf{S}_{-1,\ln,\Gamma_D}(\Omega)$ the estimate simplifies to

$$\big\|\nu^{1/2}\,\nabla(u - \widetilde{u})\big\|_{0,\Omega}^2 \geq 2\langle f, \varphi\rangle_{0,\Omega} - \langle\nu\,\nabla(2\widetilde{u} + \varphi), \nabla\,\varphi\rangle_{0,\Omega}$$

and equality holds for $\varphi = u - \widetilde{u}$, provided that $\widetilde{u} \in u_D + \mathsf{S}_{-1,\ln,\Gamma_D}(\Omega)$.

Finally, to handle non-conforming approximations $\widetilde{\Upsilon} \in \mathsf{L}^2(\Omega)$ we use again the triangle inequality to estimate

$$\big\|\nu^{1/2}(\Upsilon - \widetilde{\Upsilon})\big\|_{0,\Omega} \geq \big\|\nu^{1/2}\,\nabla(u - \widetilde{u})\big\|_{0,\Omega} - \big\|\nu^{1/2}(\nabla\,\widetilde{u} - \widetilde{\Upsilon})\big\|_{0,\Omega},$$

valid for any $\widetilde{u} \in u_D + \mathsf{H}^1_{-1,\ln,\Gamma_D}(\Omega)$. Theorem 4.17 applied to the first term on the right hand side yields the estimate analogous to Theorem 4.8.

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