ESAIM: COCV 26 (2020) 34
https://doi.org/10.1051/cocv/2019015

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

HYPERBOLIC MAXWELL VARIATIONAL INEQUALITIES OF THE SECOND KIND*

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Abstract. We analyze a class of hyperbolic Maxwell variational inequalities of the second kind. By means of a local boundedness assumption on the subdifferential of the underlying nonlinearity, we prove a well-posedness result, where the main tools for the proof are the semigroup theory for Maxwell's equations, the Yosida regularization and the subdifferential calculus. The second part of the paper focuses on a more general case omitting the local boundedness assumption. In this case, taking into account more regular initial data and test functions, we are able to prove a weaker existence result through the use of the minimal section operator associated with the Nemytskii operator of the governing subdifferential. Eventually, we transfer the developed well-posedness results to the case involving Faraday's law, which in particular allows us to improve the regularity property of the electric field in the weak existence result.

Mathematics Subject Classification. 35L85, 35Q60.

Received January 2, 2019. Accepted March 19, 2019.

1. Introduction

Physical phenomena in electromagnetism can lead to hyperbolic variational inequalities with a Maxwell structure. They include electromagnetic processes arising in polarizable media, nonlinear Ohm's law, and high-temperature superconductivity (HTS). The very first study of hyperbolic variational inequalities in electromagnetism goes back to Duvaut and Lions ([11], Chap. 7, Sect. 8). By means of parabolic regularization and penalization techniques, they proved a well-posedness result for an obstacle-type hyperbolic Maxwell variational inequality stemming from the modelling of polarizable media. Some years later, Milani [21, 22] extended their result to the case of time-dependent obstacle set. We also refer to Miranda and Santos [23] for the mathematical analysis of the non-Hilbertian counterpart to the antenna problem in Chapter 7 of [11]. The Bean critical-state model [7] for HTS governed by the eddy current equations [1, 9] leads to parabolic Maxwell variational and quasi-variational inequalities. See Prigozhin [26, 27] and Barrett and Prigozhin [4–6] for the corresponding mathematical analysis (*cf.* as well [17, 29]). More recently, the author [34] found out that the Bean critical-state model governed by the full Maxwell equations gives rise to a hyperbolic Maxwell variational inequality of the second kind involving an L^1-type nonlinearity. Last but not least, we mention the contributions by Rodrigues

*In the memory of my father Lie Yousept.

Keywords and phrases: Hyperbolic Maxwell variational inequality, well-posedness, Nemytskii operator, minimal section operator, regularity.

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Article published by EDP Sciences

and Santos *et al.* [2, 24, 30] regarding parabolic and elliptic p-**curl** systems and their related variational and quasi-variational inequalities.

This paper is devoted to the well-posedness analysis for a general class of hyperbolic Maxwell variational inequalities of the second kind with a proper, convex and lower semicontinuous nonlinearity $\varphi : \boldsymbol{X} \to \overline{\mathbb{R}}$; see Section 2 for the definition of the pivot Hilbert space $\boldsymbol{X}$. Based on a local boundedness assumption on the subdifferential $\partial\varphi$ (Assump. 3.1), we establish a well-posedness result (Thm. 3.3), where the main tools for the proof are the semigroup theory for Maxwell's equations, the Yosida regularization and the subdifferential calculus. While Assumption 3.1 is satisfied for various functionals φ, including L^1-type nonlinearities as encountered in Bean's critical state-model for HTS [34], it could fail to hold for some instances such as for indicator functionals (see Exam. 3.6). This motivates us to study a more general case without the local boundedness assumption on the subdifferential $\partial\varphi$. In this case, taking into account more regular initial data and test functions, we are able to prove a weaker existence result (Thm. 3.11) by employing the minimal section operator associated with the Nemytskii operator of $\partial\varphi$ acting in the Bochner space $L^2((0,T),\boldsymbol{X})$ (Defs. 3.7 and 3.9). In the final part of the paper, we focus on variational inequalities involving Faraday's law and transfer our well-posedness results to this particular case. Since the nonlinearity φ for Faraday's law is independent of the magnetic field, we are able to improve the weak existence result by achieving the $\boldsymbol{H}_0(\mathbf{curl})$-regularity in the electric field (Cor. 4.2).

The mathematical analysis of variational inequalities has a long history and goes back to Fichera [13, 14], Brézis and Stampacchia [10], and Lions and Stampacchia [19, 20]. See [8, 11, 16, 18, 28] for various applications in variational inequalities. To the best of the author's knowledge, the mathematical contributions in this paper are original, which we believe may help enrich the works on variational inequalities in electromagnetism. In particular, the developed results would open a way to study the optimal control governed by hyperbolic Maxwell variational inequalities (see [32, 33, 35] concerning the optimal control of Maxwell's equations). The paper is organized as follows. In the upcoming section, we introduce all the function spaces used in our analysis, including the definition of the Maxwell operator. Some well-known properties of the Maxwell operator are also mentioned in this section. In Section 3, we establish a well-posedness result for a class of hyperbolic Maxwell variational inequalities of the second kind under Assumption 3.1. In Section 3.1, we examine the case without Assumption 3.1, and the final section is devoted to the case of Faraday's law.

2. Preliminaries

Let $\Omega \subset \mathbb{R}^3$ be an open set. We introduce

$$\boldsymbol{H}(\mathbf{curl}) := \big\{\boldsymbol{q} \in \boldsymbol{L}^2(\Omega) \;\big|\; \mathbf{curl}\,\boldsymbol{q} \in \boldsymbol{L}^2(\Omega)\big\},$$

where the operator **curl** is understood in the sense of distributions. For a given Hilbert space V, we use the notation $\|\cdot\|_V$ and $(\cdot,\cdot)_V$ for a standard norm and a standard scalar product in V. A bold typeface is used to indicate a three-dimensional vector function or a Hilbert space of three-dimensional vector functions. As usual, $\boldsymbol{\mathcal{C}}_0^\infty(\Omega)$ stands for the space of all infinitely differentiable three-dimensional vector functions with compact support contained in Ω. We denote the closure of $\boldsymbol{\mathcal{C}}_0^\infty(\Omega)$ with respect to the $\boldsymbol{H}(\mathbf{curl})$-topology by

$$\boldsymbol{H}_0(\mathbf{curl}) := \overline{\boldsymbol{\mathcal{C}}_0^\infty(\Omega)}^{\|\cdot\|_{\boldsymbol{H}(\mathbf{curl})}}.$$

It is well known that the Hilbert space $\boldsymbol{H}_0(\mathbf{curl})$ admits the following characterization:

$$\boldsymbol{H}_0(\mathbf{curl}) = \big\{\boldsymbol{q} \in \boldsymbol{H}(\mathbf{curl}) \mid (\boldsymbol{q}, \mathbf{curl}\,\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} = (\mathbf{curl}\,\boldsymbol{q}, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl})\big\}. \tag{2.1}$$

For the convenience of the reader, a proof for (2.1) is given in Appendix A. Let $\epsilon, \mu : \Omega \to \mathbb{R}^{3\times 3}$ denote the electric permittivity and the magnetic permeability in the medium Ω. They are assumed to be of class $L^\infty(\Omega)^{3\times 3}$, symmetric and uniformly positive-definite in the sense that there exist constants $\underline{\epsilon}, \underline{\mu} > 0$ such that

$$\xi^T \epsilon(x)\xi \geq \underline{\epsilon}|\xi|^2 \quad \text{and} \quad \xi^T \mu(x)\xi \geq \underline{\mu}|\xi|^2 \quad \text{for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^3. \tag{2.2}$$

Note that, differently from Chapter 7, Section 8 of [11], we do not make the simplifying piecewise constant assumption on ϵ and μ. Given a symmetric and uniformly positive definite matrix-valued function $\alpha \in L^\infty(\Omega)^{3\times 3}$, let $\boldsymbol{L}^2_\alpha(\Omega)$ denote the weighted $\boldsymbol{L}^2(\Omega)$-space endowed with the weighted scalar product $(\alpha\cdot,\cdot)_{\boldsymbol{L}^2(\Omega)}$. Based on this notation, let us introduce the pivot Hilbert space used in our analysis:

$$\boldsymbol{X} := \boldsymbol{L}^2_\epsilon(\Omega) \times \boldsymbol{L}^2_\mu(\Omega),$$

equipped with the scalar product

$$((\boldsymbol{e},\boldsymbol{h}),(\boldsymbol{v},\boldsymbol{w}))_{\boldsymbol{X}} = (\epsilon\boldsymbol{e},\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} + (\mu\boldsymbol{h},\boldsymbol{w})_{\boldsymbol{L}^2(\Omega)} \quad \forall(\boldsymbol{e},\boldsymbol{h}),(\boldsymbol{v},\boldsymbol{w}) \in \boldsymbol{X}. \tag{2.3}$$

Now, we introduce the (unbounded) Maxwell operator

$$\mathcal{A} : D(\mathcal{A}) \subset \boldsymbol{X} \to \boldsymbol{X}, \quad \mathcal{A} := -\begin{pmatrix} \epsilon & 0 \\ 0 & \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & -\mathbf{curl} \\ \mathbf{curl} & 0 \end{pmatrix}, \tag{2.4}$$

with

$$D(\mathcal{A}) := \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{H}(\mathbf{curl}).$$

The choice of the domain $D(\mathcal{A})$ is motivated by the perfectly conducting electric boundary condition, which specifies that the tangential component of the electric field vanishes on the boundary. Obviously, $\mathcal{A} : D(\mathcal{A}) \subset \boldsymbol{X} \to \boldsymbol{X}$ is a densely defined and closed operator. Furthermore, due to the choice of the weighted Hilbert space $\boldsymbol{X}$ and (2.1), $\mathcal{A} : D(\mathcal{A}) \subset \boldsymbol{X} \to \boldsymbol{X}$ is skew-adjoint, *i.e.*, $D(\mathcal{A}^*) = D(\mathcal{A})$ and $\mathcal{A}^* = -\mathcal{A}$. Therefore, by virtue of Stone's theorem ([25], Thm. 10.8, p. 41), $\mathcal{A}$ generates a strongly continuous group $\{\mathbb{T}_t\}_{t\in\mathbb{R}}$ of unitary operators on $\boldsymbol{X}$. We close this section by recalling the energy balance equality result.

Lemma 2.1. *Let $\{\mathbb{S}_t\}_{t\in\mathbb{R}}$ be a strongly continuous group of unitary operators on $\boldsymbol{X}$. Furthermore, suppose that $(\boldsymbol{e},\boldsymbol{h}) \in \mathcal{C}([0,T],\boldsymbol{X})$, $(\boldsymbol{e}_0,\boldsymbol{h}_0) \in \boldsymbol{X}$ and $(\boldsymbol{w},\widetilde{\boldsymbol{w}}) \in L^1((0,T),\boldsymbol{X})$ satisfy the variation of constants formula*

$$(\boldsymbol{e},\boldsymbol{h})(t) = \mathbb{S}_t(\boldsymbol{e}_0,\boldsymbol{h}_0) + \int_0^t \mathbb{S}_{t-s}(\boldsymbol{w},\widetilde{\boldsymbol{w}})(s)\,\mathrm{d}s \quad \forall t \in [0,T].$$

Then, the following energy balance equality

$$\left\|(\boldsymbol{e},\boldsymbol{h})(t)\right\|^2_{\boldsymbol{X}} = \left\|(\boldsymbol{e}_0,\boldsymbol{h}_0)\right\|^2_{\boldsymbol{X}} + 2\int_0^t ((\boldsymbol{w},\widetilde{\boldsymbol{w}})(s),(\boldsymbol{e},\boldsymbol{h})(s))_{\boldsymbol{X}}\,\mathrm{d}s$$

holds true for all $t \in [0,T]$.

Proof. For the convenience of the reader, we include the proof in Appendix B. □

3. Well-posedness

In this section, we present well-posedness results for a general class of hyperbolic Maxwell variational inequalities of the second kind. In all what follows, let $\Omega \subset \mathbb{R}^3$ be an open set and $T \in \mathbb{R}^+$. Further, let

$$\varphi : \boldsymbol{X} \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$$

be a proper, convex and lower semicontinuous (l.s.c.) function. We recall that the subdifferential $\partial\varphi : \boldsymbol{X} \to 2^{\boldsymbol{X}}$ is a set-valued operator, where, for every $(\boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{X}$, $\partial\varphi(\boldsymbol{v}, \boldsymbol{w})$ contains all subgradients of φ at $(\boldsymbol{v}, \boldsymbol{w})$, *i.e.*,

$$\partial\varphi(\boldsymbol{v}, \boldsymbol{w}) = \big\{(\boldsymbol{y}, \boldsymbol{z}) \in \boldsymbol{X} \mid ((\boldsymbol{y}, \boldsymbol{z}), (\boldsymbol{p}, \boldsymbol{q}) - (\boldsymbol{v}, \boldsymbol{w}))_{\boldsymbol{X}} + \varphi(\boldsymbol{v}, \boldsymbol{w}) \leq \varphi(\boldsymbol{p}, \boldsymbol{q}) \quad \forall (\boldsymbol{p}, \boldsymbol{q}) \in \boldsymbol{X}\big\}. \tag{3.1}$$

For every $\lambda > 0$, let $J_\lambda : \boldsymbol{X} \to \boldsymbol{X}$ and $\Phi_\lambda : \boldsymbol{X} \to \boldsymbol{X}$ denote, respectively, the resolvent operator and the Yosida approximation of the subdifferential $\partial\varphi$, *i.e.*,

$$J_\lambda = (I_d + \lambda\partial\varphi)^{-1} \quad \text{and} \quad \Phi_\lambda = \lambda^{-1}\big(I_d - J_\lambda\big), \tag{3.2}$$

where $I_d : \boldsymbol{X} \to \boldsymbol{X}$ denotes the identity operator. Since φ is proper, convex, and l.s.c., the subdifferential $\partial\varphi : \boldsymbol{X} \to 2^{\boldsymbol{X}}$ is m-accretive. See, *e.g.*, Proposition 1.5, page 157 of [31] for this well-known result. As a consequence, the resolvent operator $J_\lambda : \boldsymbol{X} \to \boldsymbol{X}$ is non-expansive, and the Yosida approximation $\Phi_\lambda : \boldsymbol{X} \to \boldsymbol{X}$ is m-accretive and Lipschitz-continuous with the Lipschitz constant $L_\lambda = \lambda^{-1}$ (see [31], Thm. 1.1, p. 161). Note that, since our pivot space $\boldsymbol{X}$ is a Hilbert space, the notions of maximal montonicity and m-accretivity coincide ([15], Thm. 3.2.29, p. 317). Next, we define the domain of the subdifferential $\partial\varphi$ by

$$D(\partial\varphi) := \{(\boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{X} \mid \partial\varphi(\boldsymbol{v}, \boldsymbol{w}) \neq \emptyset\} \tag{3.3}$$

and denote by $\partial\varphi^0 : D(\partial\varphi) \to \boldsymbol{X}$ the minimal section operator of $\partial\varphi$ (*cf.* [31], p. 161), defined as

$$\|\partial\varphi^0(\boldsymbol{v}, \boldsymbol{w})\|_{\boldsymbol{X}} = \min_{(\boldsymbol{y}, \boldsymbol{z}) \in \partial\varphi(\boldsymbol{v}, \boldsymbol{w})} \|(\boldsymbol{y}, \boldsymbol{z})\|_{\boldsymbol{X}}. \tag{3.4}$$

For every $(\boldsymbol{v}, \boldsymbol{w}) \in D(\partial\varphi)$, the set $\partial\varphi(\boldsymbol{v}, \boldsymbol{w}) \subset \boldsymbol{X}$ is nonempty, closed, and convex (see [15], Prop. 3.2.7, p. 305). Therefore, by standard arguments, the minimization problem in (3.4) admits a unique solution, and so the minimal section operator $\partial\varphi^0 : D(\partial\varphi) \to \boldsymbol{X}$ is indeed well-defined. We summarize its well-known properties ([15], Thm. 3.2.38, p. 323) as follows:

$$\begin{aligned} &\|\Phi_\lambda(\boldsymbol{v}, \boldsymbol{w})\|_{\boldsymbol{X}} \leq \|\partial\varphi^0(\boldsymbol{v}, \boldsymbol{w})\|_{\boldsymbol{X}} && \forall \lambda > 0, \quad \forall (\boldsymbol{v}, \boldsymbol{w}) \in D(\partial\varphi), \\ &\lim_{\lambda \to 0} \Phi_\lambda(\boldsymbol{v}, \boldsymbol{w}) = \partial\varphi^0(\boldsymbol{v}, \boldsymbol{w}) \text{ in } \boldsymbol{X} && \qquad\quad \forall (\boldsymbol{v}, \boldsymbol{w}) \in D(\partial\varphi). \end{aligned} \tag{3.5}$$

Assumption 3.1. For every $M > 0$, there exists a constant $C(M) > 0$ such that

$$\|(\boldsymbol{y}, \boldsymbol{z})\|_{\boldsymbol{X}} \leq C(M) \quad \forall (\boldsymbol{y}, \boldsymbol{z}) \in \partial\varphi(\boldsymbol{v}, \boldsymbol{w}),$$

for all $(\boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{X}$ satisfying $\|(\boldsymbol{v}, \boldsymbol{w})\|_{\boldsymbol{X}} \leq M$.

Lemma 3.2. *Let $\varphi : \boldsymbol{X} \to \overline{\mathbb{R}}$ be a convex and l.s.c. function satisfying $\partial\varphi(0, 0) \neq \emptyset$ and Assumption 3.1. Furthermore, let $(\mathbf{f}, \mathbf{g}) \in L^1((0, T), \boldsymbol{X})$ and $(\boldsymbol{E}_0, \boldsymbol{H}_0) \in \boldsymbol{X}$. Then, there exist unique $(\boldsymbol{E}, \boldsymbol{H}) \in \mathcal{C}([0, T], \boldsymbol{X})$ and*

$(\widetilde{\boldsymbol{E}},\widetilde{\boldsymbol{H}}) \in L^\infty((0,T),\boldsymbol{X})$ *satisfying*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_\Omega \epsilon\boldsymbol{E}(t)\cdot\boldsymbol{v} + \mu\boldsymbol{H}(t)\cdot\boldsymbol{w}\,\mathrm{d}x + \int_\Omega \boldsymbol{E}(t)\cdot\mathbf{curl}\,\boldsymbol{w} - \boldsymbol{H}(t)\cdot\mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x + \int_\Omega \epsilon\widetilde{\boldsymbol{E}}(t)\cdot\boldsymbol{v} + \mu\widetilde{\boldsymbol{H}}(t)\cdot\boldsymbol{w}\,\mathrm{d}x \\ = \displaystyle\int_\Omega \mathbf{f}(t)\cdot\boldsymbol{v} + \mathbf{g}(t)\cdot\boldsymbol{w}\,\mathrm{d}x \quad \text{for a.e. } t\in(0,T) \text{ and all } (\boldsymbol{v},\boldsymbol{w})\in\boldsymbol{H}_0(\mathbf{curl})\times\boldsymbol{H}(\mathbf{curl}), \\ (\widetilde{\boldsymbol{E}},\widetilde{\boldsymbol{H}})(t)\in\partial\varphi((\boldsymbol{E},\boldsymbol{H})(t)) \quad \text{for a.e. } t\in(0,T), \\ (\boldsymbol{E},\boldsymbol{H})(0) = (\boldsymbol{E}_0,\boldsymbol{H}_0), \end{cases} \tag{3.6}$$

and, for every fixed $(\boldsymbol{v},\boldsymbol{w}) \in \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{H}(\mathbf{curl})$*, the mapping* $t \mapsto ((\boldsymbol{E},\boldsymbol{H})(t),(\boldsymbol{v},\boldsymbol{w}))_{\boldsymbol{X}}$ *is absolutely continuous from* $[0,T]$ *to* $\mathbb{R}$.

Proof. Let $\{\lambda_n\}_{n=1}^\infty$ be a sequence of positive real numbers converging to zero, and we consider the following integral equation: For every $n\in\mathbb{N}$, find $(\boldsymbol{E}_n,\boldsymbol{H}_n)\in\mathcal{C}([0,T],\boldsymbol{X})$ such that

$$(\boldsymbol{E}_n,\boldsymbol{H}_n)(t) = \mathbb{T}_t(\boldsymbol{E}_0,\boldsymbol{H}_0) + \int_0^t \mathbb{T}_{t-s}\big((\epsilon^{-1}\mathbf{f},\mu^{-1}\mathbf{g})(s) - \Phi_{\lambda_n}((\boldsymbol{E}_n,\boldsymbol{H}_n)(s))\big)\,\mathrm{d}s \quad \forall t\in[0,T]. \tag{3.7}$$

Since for every $\lambda>0$, the Yosida approximation $\Phi_\lambda : \boldsymbol{X}\to\boldsymbol{X}$ is Lipschitz-continous, a classical contraction argument ([25], Thm. 1.2, p. 184) implies that, for every $n\in\mathbb{N}$, the integral equation (3.7) admits a unique solution $(\boldsymbol{E}_n,\boldsymbol{H}_n)\in\mathcal{C}([0,T],\boldsymbol{X})$. Thanks to the Lipschitz continuity of $J_{\lambda_n} : \boldsymbol{X}\to\boldsymbol{X}$ and $\Phi_{\lambda_n} : \boldsymbol{X}\to\boldsymbol{X}$, we have that

$$J_{\lambda_n}(\boldsymbol{E}_n,\boldsymbol{H}_n), \Phi_{\lambda_n}(\boldsymbol{E}_n,\boldsymbol{H}_n) \in \mathcal{C}([0,T],\boldsymbol{X}) \quad \forall n\in\mathbb{N}.$$

Let us show that the sequences $\{(\boldsymbol{E}_n,\boldsymbol{H}_n)\}_{n=1}^\infty, \{J_{\lambda_n}(\boldsymbol{E}_n,\boldsymbol{H}_n)\}_{n=1}^\infty, \{\Phi_{\lambda_n}(\boldsymbol{E}_n,\boldsymbol{H}_n)\}_{n=1}^\infty$ are bounded in $\mathcal{C}([0,T],\boldsymbol{X})$.

Since $\{\mathbb{T}_t\}_{t\in\mathbb{R}}$ is a strongly continuous group of unitary operators on $\boldsymbol{X}$, we may apply the energy balance equality (Lem. 2.1) to (3.7) and obtain for all $t\in[0,T]$ and all $n\in\mathbb{N}$ that

$$\begin{aligned} &\big\|(\boldsymbol{E}_n,\boldsymbol{H}_n)(t)\big\|_{\boldsymbol{X}}^2 \\ &= \big\|(\boldsymbol{E}_0,\boldsymbol{H}_0)\big\|_{\boldsymbol{X}}^2 + 2\int_0^t \big((\epsilon^{-1}\mathbf{f},\mu^{-1}\mathbf{g})(s),(\boldsymbol{E}_n,\boldsymbol{H}_n)(s)\big)_{\boldsymbol{X}} - (\Phi_{\lambda_n}((\boldsymbol{E}_n,\boldsymbol{H}_n)(s)),(\boldsymbol{E}_n,\boldsymbol{H}_n)(s))_{\boldsymbol{X}}\,\mathrm{d}s \\ &= \big\|(\boldsymbol{E}_0,\boldsymbol{H}_0)\big\|_{\boldsymbol{X}}^2 + 2\int_0^t \big((\epsilon^{-1}\mathbf{f},\mu^{-1}\mathbf{g})(s) - \Phi_{\lambda_n}(0,0),(\boldsymbol{E}_n,\boldsymbol{H}_n)(s)\big)_{\boldsymbol{X}} \\ &\quad - (\Phi_{\lambda_n}((\boldsymbol{E}_n,\boldsymbol{H}_n)(s)) - \Phi_{\lambda_n}(0,0),(\boldsymbol{E}_n,\boldsymbol{H}_n)(s))_{\boldsymbol{X}}\,\mathrm{d}s \\ &\le \big\|(\boldsymbol{E}_0,\boldsymbol{H}_0)\big\|_{\boldsymbol{X}}^2 + 2\int_0^t \|(\epsilon^{-1}\mathbf{f},\mu^{-1}\mathbf{g})(s) - \Phi_{\lambda_n}(0,0)\|_{\boldsymbol{X}}\|(\boldsymbol{E}_n,\boldsymbol{H}_n)(s)\|_{\boldsymbol{X}}\,\mathrm{d}s \\ &\le \big\|(\boldsymbol{E}_0,\boldsymbol{H}_0)\big\|_{\boldsymbol{X}}^2 + 2\|(\epsilon^{-1}\mathbf{f},\mu^{-1}\mathbf{g}) - \Phi_{\lambda_n}(0,0)\|_{L^1((0,T),\boldsymbol{X})}^2 + \frac{1}{2}\|(\boldsymbol{E}_n,\boldsymbol{H}_n)\|_{\mathcal{C}([0,T],\boldsymbol{X})}^2, \end{aligned} \tag{3.8}$$

where we have also used the monotonicity of the Yosida approximation $\Phi_{\lambda_n} : \boldsymbol{X}\to\boldsymbol{X}$. As (3.8) is satisfied for all $t\in[0,T]$, it follows that

$$\big\|(\boldsymbol{E}_n,\boldsymbol{H}_n)\big\|_{\mathcal{C}([0,T],\boldsymbol{X})}^2 \le 2\big\|(\boldsymbol{E}_0,\boldsymbol{H}_0)\big\|_{\boldsymbol{X}}^2 + 4\|(\epsilon^{-1}\mathbf{f},\mu^{-1}\mathbf{g}) - \Phi_{\lambda_n}(0,0)\|_{L^1((0,T),\boldsymbol{X})}^2 \quad \forall n\in\mathbb{N}. \tag{3.9}$$

Due to $(0,0) \in D(\partial\varphi)$, (3.5) implies that $\{\Phi_{\lambda_n}(0,0)\}_{n=1}^\infty \subset \boldsymbol{X}$ is bounded, and so (3.9) yields the boundedness of $\{(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^\infty$ in $\mathcal{C}([0,T], \boldsymbol{X})$. Next, since $J_\lambda : \boldsymbol{X} \to \boldsymbol{X}$ is non-expansive,

$$\|J_{\lambda_n}((\boldsymbol{E}_n, \boldsymbol{H}_n)(t))\|_{\boldsymbol{X}} \leq \|(\boldsymbol{E}_n, \boldsymbol{H}_n)(t)\|_{\boldsymbol{X}} + \|J_{\lambda_n}(0,0)\|_{\boldsymbol{X}} \quad \forall t \in [0,T] \quad \forall n \in \mathbb{N}.$$

Thus, as $\lim_{n\to\infty} \|J_{\lambda_n}(0,0)\|_{\boldsymbol{X}} = 0$ (due to $\partial\varphi(0,0) \neq \emptyset$), the above inequality together with the boundedness of $\{(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^\infty \subset \mathcal{C}([0,T], \boldsymbol{X})$ implies that the sequence $\{J_{\lambda_n}(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^\infty \subset \mathcal{C}([0,T], \boldsymbol{X})$ is bounded. Moreover, by the definition of the resolvent operator and the Yosida approximation (3.2), it holds that

$$\Phi_{\lambda_n}((\boldsymbol{E}_n, \boldsymbol{H}_n)(t)) \in \partial\varphi\left(J_{\lambda_n}(\boldsymbol{E}_n, \boldsymbol{H}_n)(t)\right) \quad \forall t \in [0,T] \quad \forall n \in \mathbb{N}. \tag{3.10}$$

Thus, in view of Assumption 3.1 and the boundedness of $\{J_{\lambda_n}(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^\infty \subset \mathcal{C}([0,T], \boldsymbol{X})$, (3.10) implies that $\{\Phi_{\lambda_n}(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^\infty \subset \mathcal{C}([0,T], \boldsymbol{X})$ is bounded. In conclusion, the sequences $\{(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^\infty$, $\{J_{\lambda_n}(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^\infty$ and $\{\Phi_{\lambda_n}(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^\infty$ are bounded in $\mathcal{C}([0,T], \boldsymbol{X})$. Therefore, we can select a subsequence of $\{\lambda_n\}_{n=1}^\infty$, which we denote again by $\{\lambda_n\}_{n=1}^\infty$, such that

$$(\boldsymbol{E}_n, \boldsymbol{H}_n) \rightharpoonup (\boldsymbol{E}, \boldsymbol{H}) \quad \text{weakly star in } L^\infty((0,T), \boldsymbol{X}) \text{ as } n \to \infty, \tag{3.11}$$
$$\Phi_{\lambda_n}(\boldsymbol{E}_n, \boldsymbol{H}_n) \rightharpoonup (\widetilde{\boldsymbol{E}}, \widetilde{\boldsymbol{H}}) \quad \text{weakly star in } L^\infty((0,T), \boldsymbol{X}) \text{ as } n \to \infty, \tag{3.12}$$
$$(J_{\lambda_n} - I_d)(\boldsymbol{E}_n, \boldsymbol{H}_n) \to 0 \quad \text{strongly in } \mathcal{C}([0,T], \boldsymbol{X}) \text{ as } n \to \infty, \tag{3.13}$$
$$J_{\lambda_n}(\boldsymbol{E}_n, \boldsymbol{H}_n) \rightharpoonup (\boldsymbol{E}, \boldsymbol{H}) \quad \text{weakly star in } L^\infty((0,T), \boldsymbol{X}) \text{ as } n \to \infty, \tag{3.14}$$

for some $(\boldsymbol{E}, \boldsymbol{H}), (\widetilde{\boldsymbol{E}}, \widetilde{\boldsymbol{H}}) \in L^\infty((0,T), \boldsymbol{X})$. Note that (3.13) follows from the boundedness of $\{\Phi_{\lambda_n}(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^\infty \subset \mathcal{C}([0,T], \boldsymbol{X})$ and the definition of the Yosida approximation $\Phi_\lambda = \lambda^{-1}\big(I_d - J_\lambda\big)$. Moreover, (3.14) follows from (3.13) and (3.11). Passing to the limit $n \to \infty$ in (3.7), we obtain from (3.11) and (3.12) that

$$(\boldsymbol{E}, \boldsymbol{H})(t) = \mathbb{T}_t(\boldsymbol{E}_0, \boldsymbol{H}_0) + \int_0^t \mathbb{T}_{t-s}\left((\epsilon^{-1}\mathbf{f}, \mu^{-1}\mathbf{g})(s) - (\widetilde{\boldsymbol{E}}, \widetilde{\boldsymbol{H}})(s)\right) \mathrm{d}s \quad \forall t \in [0,T]. \tag{3.15}$$

and

$$(\boldsymbol{E}_n, \boldsymbol{H}_n)(t) \rightharpoonup (\boldsymbol{E}, \boldsymbol{H})(t) \quad \text{weakly in } \boldsymbol{X} \text{ as } n \to \infty \quad \forall t \in [0,T]. \tag{3.16}$$

Now, employing the classical result by Ball [3], the solution of (3.15) satisfies

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}((\boldsymbol{E}, \boldsymbol{H})(t), (\boldsymbol{v}, \boldsymbol{w}))_{\boldsymbol{X}} - ((\boldsymbol{E}, \boldsymbol{H})(t), \mathcal{A}^*(\boldsymbol{v}, \boldsymbol{w}))_{\boldsymbol{X}} = ((\epsilon^{-1}\mathbf{f}, \mu^{-1}\mathbf{g})(t) - (\widetilde{\boldsymbol{E}}, \widetilde{\boldsymbol{H}})(t), (\boldsymbol{v}, \boldsymbol{w}))_{\boldsymbol{X}} \\ \text{for a.e. } t \in (0,T) \text{ and all } (\boldsymbol{v}, \boldsymbol{w}) \in D(\mathcal{A}^*), \\ (\boldsymbol{E}, \boldsymbol{H})(0) = (\boldsymbol{E}_0, \boldsymbol{H}_0), \end{cases} \tag{3.17}$$

and, for every $(\boldsymbol{v}, \boldsymbol{w}) \in D(\mathcal{A}^*)$, the mapping $t \mapsto ((\boldsymbol{E}, \boldsymbol{H})(t), (\boldsymbol{v}, \boldsymbol{w}))_{\boldsymbol{X}}$ is absolutely continuous from $[0,T]$ to $\mathbb{R}$.

Since the Maxwell operator $\mathcal{A}$ is skew adjoint, *i.e.*, $D(\mathcal{A}^*) = D(\mathcal{A}) = \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{H}(\mathbf{curl})$ and $\mathcal{A}^* = -\mathcal{A}$ (see (2.4) for its definition), we see that (3.17) is nothing but the variational equality in (3.6). Thus, if we can prove that

$$(\widetilde{\boldsymbol{E}}, \widetilde{\boldsymbol{H}})(t) \in \partial\varphi((\boldsymbol{E}, \boldsymbol{H})(t)) \quad \text{for a.e. } t \in (0,T), \tag{3.18}$$

then $(\boldsymbol{E},\boldsymbol{H}) \in \mathcal{C}([0,T],\boldsymbol{X})$ is a solution to (3.6). To this aim, we introduce the set $B \subset L^2((0,T),\boldsymbol{X}) \times L^2((0,T),\boldsymbol{X})$ defined as follows:

$$((\boldsymbol{e},\boldsymbol{h}),(\boldsymbol{y},\boldsymbol{z})) \in B \quad \Leftrightarrow \quad (\boldsymbol{y},\boldsymbol{z})(t) \in \partial\varphi((\boldsymbol{e},\boldsymbol{h})(t)) \quad \text{for a.e. } t \in (0,T). \tag{3.19}$$

By definition, we see that (3.18) is nothing but

$$((\boldsymbol{E},\boldsymbol{H}),(\widetilde{\boldsymbol{E}},\widetilde{\boldsymbol{H}})) \in B. \tag{3.20}$$

Therefore, we have to show that (3.20) is satisfied.

Since $\partial\varphi : \boldsymbol{X} \to 2^{\boldsymbol{X}}$ is monotone, the set B is monotone, *i.e.*,

$$((\boldsymbol{y},\boldsymbol{z}) - (\widetilde{\boldsymbol{v}},\widetilde{\boldsymbol{w}}),(\boldsymbol{e},\boldsymbol{h}) - (\boldsymbol{v},\boldsymbol{w}))_{L^2((0,T),\boldsymbol{X})} \geq 0 \quad \forall ((\boldsymbol{e},\boldsymbol{h}),(\boldsymbol{y},\boldsymbol{z})),((\boldsymbol{v},\boldsymbol{w}),(\widetilde{\boldsymbol{v}},\widetilde{\boldsymbol{w}})) \in B.$$

Let us now show that B is m-accretive, *i.e.*, we have to show that, for every $\lambda > 0$, it holds that

$$Range(I_d + \lambda B) := \{(\boldsymbol{e},\boldsymbol{h}) + \lambda(\boldsymbol{y},\boldsymbol{z}) \mid ((\boldsymbol{e},\boldsymbol{h}),(\boldsymbol{y},\boldsymbol{z})) \in B\} = L^2((0,T),\boldsymbol{X}). \tag{3.21}$$

Let $\lambda > 0$. To show (3.21), we take an arbitrarily fixed function $(\boldsymbol{v},\boldsymbol{w}) \in L^2((0,T),\boldsymbol{X})$ and define

$$(\boldsymbol{e},\boldsymbol{h})(t) := J_\lambda((\boldsymbol{v},\boldsymbol{w})(t)) = (I_d + \lambda\partial\varphi)^{-1}(\boldsymbol{v},\boldsymbol{w})(t) \quad \text{for a.e. } t \in (0,T).$$

According to the definition (3.19), we see that (3.21) is valid, if $(\boldsymbol{e},\boldsymbol{h}) \in L^2((0,T),\boldsymbol{X})$. Indeed, as $J_\lambda : \boldsymbol{X} \to \boldsymbol{X}$ is non-expansive, we obtain that $(\boldsymbol{e},\boldsymbol{h}) = J_\lambda(\boldsymbol{v},\boldsymbol{w})$ is measurable and

$$\|(\boldsymbol{e},\boldsymbol{h})(t)\|_{\boldsymbol{X}} \leq \|J_\lambda((\boldsymbol{v},\boldsymbol{w})(t)) - J_\lambda(0,0)\|_{\boldsymbol{X}} + \|J_\lambda(0,0)\|_{\boldsymbol{X}} \leq \|(\boldsymbol{v},\boldsymbol{w})(t)\|_{\boldsymbol{X}} + \|J_\lambda(0,0)\|_{\boldsymbol{X}} \quad \text{for a.e. } t \in (0,T).$$

Since $(\boldsymbol{v},\boldsymbol{w}) \in L^2((0,T),\boldsymbol{X})$, it follows therefore that $(\boldsymbol{e},\boldsymbol{h}) \in L^2((0,T),\boldsymbol{X})$. In conclusion, B is m-accretive.

Now, making use of the energy balance equality (Lem. 2.1) in (3.15) and (3.7), we get

$$\|(\boldsymbol{E},\boldsymbol{H})(T)\|_{\boldsymbol{X}}^2 = \|(\boldsymbol{E}_0,\boldsymbol{H}_0)\|_{\boldsymbol{X}}^2 + 2\int_0^T ((\epsilon^{-1}\mathbf{f},\mu^{-1}\mathbf{g})(t) - (\widetilde{\boldsymbol{E}},\widetilde{\boldsymbol{H}})(t),(\boldsymbol{E},\boldsymbol{H})(t))_{\boldsymbol{X}}\,\mathrm{d}t,$$
$$\|(\boldsymbol{E}_n,\boldsymbol{H}_n)(T)\|_{\boldsymbol{X}}^2 = \|(\boldsymbol{E}_0,\boldsymbol{H}_0)\|_{\boldsymbol{X}}^2 + 2\int_0^T ((\epsilon^{-1}\mathbf{f},\mu^{-1}\mathbf{g})(t) - \Phi_{\lambda_n}((\boldsymbol{E}_n,\boldsymbol{H}_n)(t)),(\boldsymbol{E}_n,\boldsymbol{H}_n)(t))_{\boldsymbol{X}}\,\mathrm{d}t \quad \forall n \in \mathbb{N}.$$

Combining these two identities results in

$$\begin{aligned}
2\int_0^T (\Phi_{\lambda_n}((\boldsymbol{E}_n,\boldsymbol{H}_n)(t)),(\boldsymbol{E}_n,\boldsymbol{H}_n)(t))_{\boldsymbol{X}}\,\mathrm{d}t &= -\|(\boldsymbol{E}_n,\boldsymbol{H}_n)(T)\|_{\boldsymbol{X}}^2 + \|(\boldsymbol{E}_0,\boldsymbol{H}_0)\|_{\boldsymbol{X}}^2 \\
&\quad +2\int_0^T ((\epsilon^{-1}\mathbf{f},\mu^{-1}\mathbf{g})(t),(\boldsymbol{E}_n,\boldsymbol{H}_n)(t))_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t \\
&= -\|(\boldsymbol{E}_n,\boldsymbol{H}_n)(T)\|_{\boldsymbol{X}}^2 + \|(\boldsymbol{E},\boldsymbol{H})(T)\|_{\boldsymbol{X}}^2 + 2\int_0^T ((\widetilde{\boldsymbol{E}},\widetilde{\boldsymbol{H}})(t),(\boldsymbol{E},\boldsymbol{H})(t))_{\boldsymbol{X}}\,\mathrm{d}t \\
&\quad +2\int_0^T ((\epsilon^{-1}\mathbf{f},\mu^{-1}\mathbf{g})(t),(\boldsymbol{E}_n,\boldsymbol{H}_n)(t) - (\boldsymbol{E},\boldsymbol{H})(t))_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t.
\end{aligned}$$

Then, applying (3.13) to the above identity, we infer that

$$\begin{aligned} &2\liminf_{n\to\infty}\int_0^T(\Phi_{\lambda_n}((\boldsymbol{E}_n,\boldsymbol{H}_n)(t)),J_{\lambda_n}((\boldsymbol{E}_n,\boldsymbol{H}_n)(t)))_{\boldsymbol{X}}\,\mathrm{d}t\\ &\leq 2\limsup_{n\to\infty}\int_0^T(\Phi_{\lambda_n}((\boldsymbol{E}_n,\boldsymbol{H}_n)(t)),(\boldsymbol{E}_n,\boldsymbol{H}_n)(t))_{\boldsymbol{X}}\,\mathrm{d}t\\ &\leq -\liminf_{n\to\infty}\|(\boldsymbol{E}_n,\boldsymbol{H}_n)(T)\|^2_{\boldsymbol{X}}+\|(\boldsymbol{E},\boldsymbol{H})(T)\|^2_{\boldsymbol{X}}+2\int_0^T((\widetilde{\boldsymbol{E}},\widetilde{\boldsymbol{H}})(t),(\boldsymbol{E},\boldsymbol{H})(t))_{\boldsymbol{X}}\,\mathrm{d}t\\ &\quad+2\limsup_{n\to\infty}\int_0^T((\epsilon^{-1}\mathbf{f},\mu^{-1}\mathbf{g})(t),(\boldsymbol{E}_n,\boldsymbol{H}_n)(t)-(\boldsymbol{E},\boldsymbol{H})(t))_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t. \end{aligned}$$

It follows therefore from (3.11) and (3.16) with $t=T$ that

$$\liminf_{n\to\infty}(\Phi_{\lambda_n}(\boldsymbol{E}_n,\boldsymbol{H}_n),J_{\lambda_n}(\boldsymbol{E}_n,\boldsymbol{H}_n))_{L^2((0,T),\boldsymbol{X})}\leq((\widetilde{\boldsymbol{E}},\widetilde{\boldsymbol{H}}),(\boldsymbol{E},\boldsymbol{H}))_{L^2((0,T),\boldsymbol{X})}. \tag{3.22}$$

On the other hand, according to (3.10) and by the definition (3.19), it holds that

$$(J_{\lambda_n}(\boldsymbol{E}_n,\boldsymbol{H}_n),\Phi_{\lambda_n}(\boldsymbol{E}_n,\boldsymbol{H}_n))\in B\quad\forall n\in\mathbb{N}. \tag{3.23}$$

Concluding from (3.22)–(3.23) along with the weak convergence properties (3.12) and (3.14), the m-accretivity of B implies that (3.20) is satisfied ([31], Prop. 1.6(b), p. 159). Thus, $(\boldsymbol{E},\boldsymbol{H})\in\mathcal{C}([0,T],\boldsymbol{X})$ satisfies (3.18), and so it is a solution of (3.6).

Uniqueness: Suppose that $(\boldsymbol{E}^{(1)},\boldsymbol{H}^{(1)}),(\boldsymbol{E}^{(2)},\boldsymbol{H}^{(2)})\in\mathcal{C}([0,T],\boldsymbol{X})$ satisfy (3.6) and, for every $(\boldsymbol{v},\boldsymbol{w})\in\boldsymbol{H}_0(\mathbf{curl})\times\boldsymbol{H}(\mathbf{curl})$, the mappings $t\mapsto((\boldsymbol{E}^{(j)},\boldsymbol{H}^{(j)})(t),(\boldsymbol{v},\boldsymbol{w}))_{\boldsymbol{X}}$, $j=1,2$ are absolutely continuous from $[0,T]$ to $\mathbb{R}$. By definition, the difference $(\boldsymbol{e},\boldsymbol{h}):=(\boldsymbol{E}^{(1)}-\boldsymbol{E}^{(2)},\boldsymbol{H}^{(1)}-\boldsymbol{H}^{(2)})$ satisfies $(\boldsymbol{e},\boldsymbol{h})(0)=0$ and

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\epsilon\boldsymbol{e}(t)\cdot\boldsymbol{v}+\mu\boldsymbol{h}(t)\cdot\boldsymbol{w}\,\mathrm{d}x+\int_\Omega\boldsymbol{e}(t)\cdot\mathbf{curl}\,\boldsymbol{w}-\boldsymbol{h}(t)\cdot\mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x=\int_\Omega\epsilon\left(\widetilde{\boldsymbol{E}}^{(2)}(t)-\widetilde{\boldsymbol{E}}^{(1)}(t)\right)\cdot\boldsymbol{v}\\ &\qquad+\mu\left(\widetilde{\boldsymbol{H}}^{(2)}(t)-\widetilde{\boldsymbol{H}}^{(1)}(t)\right)\cdot\boldsymbol{w}\,\mathrm{d}x\quad\text{for a.e. in }(0,T)\text{ and all }(\boldsymbol{v},\boldsymbol{w})\in\boldsymbol{H}_0(\mathbf{curl})\times\boldsymbol{H}(\mathbf{curl}), \end{aligned} \tag{3.24}$$

for some $(\widetilde{\boldsymbol{E}}^{(j)},\widetilde{\boldsymbol{H}}^{(j)})\in L^\infty((0,T),\boldsymbol{X})$, $j=1,2$, satisfying

$$(\widetilde{\boldsymbol{E}}^{(j)},\widetilde{\boldsymbol{H}}^{(j)})(t)\in\partial\varphi((\boldsymbol{E}^{(j)},\boldsymbol{H}^{(j)})(t))\quad\text{for a.e. }t\in(0,T). \tag{3.25}$$

In view of (3.24), the classical result by Ball [3] implies that $(\boldsymbol{e},\boldsymbol{h})\in\mathcal{C}([0,T],\boldsymbol{X})$ satisfies the variation by constants formula

$$(\boldsymbol{e},\boldsymbol{h})(t)=\int_0^t\mathbb{T}_{t-s}((\widetilde{\boldsymbol{E}}^{(2)}-\widetilde{\boldsymbol{E}}^{(1)},\widetilde{\boldsymbol{H}}^{(2)}-\widetilde{\boldsymbol{H}}^{(1)})(s))\,\mathrm{d}s\quad\forall t\in[0,T].$$

Therefore, we obtain from the energy balance equality (Lem. 2.1) together with $(\boldsymbol{e},\boldsymbol{h})(0)=0$ that

$$\begin{aligned} \|(\boldsymbol{e},\boldsymbol{h})(t)\|^2_{\boldsymbol{X}}&=2\int_0^t((\widetilde{\boldsymbol{E}}^{(2)}-\widetilde{\boldsymbol{E}}^{(1)},\widetilde{\boldsymbol{H}}^{(2)}-\widetilde{\boldsymbol{H}}^{(1)})(s),(\boldsymbol{e},\boldsymbol{h})(s))_{\boldsymbol{X}}\mathrm{d}s\\ &=-2\int_0^t((\widetilde{\boldsymbol{E}}^{(2)}-\widetilde{\boldsymbol{E}}^{(1)},\widetilde{\boldsymbol{H}}^{(2)}-\widetilde{\boldsymbol{H}}^{(1)})(s),(\boldsymbol{E}^{(2)}-\boldsymbol{E}^{(1)},\boldsymbol{H}^{(2)}-\boldsymbol{H}^{(1)})(s))_{\boldsymbol{X}}\mathrm{d}s\leq 0\quad\forall t\in[0,T], \end{aligned}$$

where we have used the monotonicity property of the subdifferential $\partial\varphi$ to get the above inequality. In conclusion, $(\boldsymbol{E}^{(1)},\boldsymbol{H}^{(1)}) = (\boldsymbol{E}^{(2)},\boldsymbol{H}^{(2)})$. Finally, since $(\boldsymbol{e},\boldsymbol{h}) = 0$, (3.24) immediately implies that $(\widetilde{\boldsymbol{E}}^{(1)},\widetilde{\boldsymbol{H}}^{(1)}) = (\widetilde{\boldsymbol{E}}^{(2)},\widetilde{\boldsymbol{H}}^{(2)})$. □

Theorem 3.3. *Let $\varphi : \boldsymbol{X} \to \overline{\mathbb{R}}$ be a convex and l.s.c. function fulfilling $\partial\varphi(0,0) \neq \emptyset$ and Assumption 3.1. Furthermore, let $(\mathbf{f},\mathbf{g}) \in W^{1,\infty}((0,T),\boldsymbol{X})$ and $(\boldsymbol{E}_0,\boldsymbol{H}_0) \in D(\mathcal{A})$. Then, the variational inequality*

$$\begin{cases} \displaystyle\int_\Omega \epsilon\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t)) + \mu\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}(t)\cdot(\boldsymbol{w}-\boldsymbol{H}(t))\,\mathrm{d}x + \int_\Omega \mathbf{curl}\,\boldsymbol{E}(t)\cdot\boldsymbol{w} - \mathbf{curl}\,\boldsymbol{H}(t)\cdot\boldsymbol{v}\,\mathrm{d}x \\ \displaystyle + \varphi(\boldsymbol{v},\boldsymbol{w}) - \varphi((\boldsymbol{E},\boldsymbol{H})(t)) \geq \int_\Omega \mathbf{f}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t)) + \mathbf{g}(t)\cdot(\boldsymbol{w}-\boldsymbol{H}(t))\,\mathrm{d}x \\ \textit{for a.e. } t\in(0,T) \textit{ and all } (\boldsymbol{v},\boldsymbol{w})\in\boldsymbol{X}, \\ (\boldsymbol{E},\boldsymbol{H})(0) = (\boldsymbol{E}_0,\boldsymbol{H}_0) \end{cases} \tag{VI}$$

admits a unique solution $(\boldsymbol{E},\boldsymbol{H}) \in L^\infty((0,T),D(\mathcal{A})) \cap W^{1,\infty}((0,T),\boldsymbol{X})$.

Remark 3.4. According to (2.3)–(2.4) and (3.1), (VI) is nothing but

$$\begin{cases} -\left(\dfrac{\mathrm{d}}{\mathrm{d}t} - \mathcal{A}\right)(\boldsymbol{E},\boldsymbol{H})(t) + (\epsilon^{-1}\mathbf{f},\mu^{-1}\mathbf{g})(t) \in \partial\varphi((\boldsymbol{E},\boldsymbol{H})(t)) \quad \text{for a.e. } t\in(0,T), \\ (\boldsymbol{E},\boldsymbol{H})(0) = (\boldsymbol{E}_0,\boldsymbol{H}_0). \end{cases} \tag{3.26}$$

Proof.
Step 1: *Uniqueness.* Suppose that $(\boldsymbol{E}^{(j)},\boldsymbol{H}^{(j)}) \in L^\infty((0,T),\boldsymbol{H}_0(\mathbf{curl})\times\boldsymbol{H}(\mathbf{curl}))\cap W^{1,\infty}((0,T),\boldsymbol{X})$, $j=1,2$, are solutions to (VI). Setting the test function $(\boldsymbol{v},\boldsymbol{w}) = (\boldsymbol{E}^{(2)},\boldsymbol{H}^{(2)})(t)$ in the variational inequality for $(\boldsymbol{E}^{(1)},\boldsymbol{H}^{(1)})$ and the test function $(\boldsymbol{v},\boldsymbol{w}) = (\boldsymbol{E}^{(1)},\boldsymbol{H}^{(1)})(t)$ in the variational inequality for $(\boldsymbol{E}^{(2)},\boldsymbol{H}^{(2)})$, and then adding the resulting inequalities, we obtain for the difference $(\boldsymbol{e},\boldsymbol{h}) := (\boldsymbol{E}^{(1)}-\boldsymbol{E}^{(2)},\boldsymbol{H}^{(1)}-\boldsymbol{H}^{(2)})$ that

$$\begin{aligned} -\int_\Omega \epsilon\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{e}(t)\cdot\boldsymbol{e}(t) + \mu\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{h}(t)\cdot\boldsymbol{h}(t)\,\mathrm{d}x - \int_\Omega \mathbf{curl}\,\boldsymbol{H}^{(1)}(t)\cdot\boldsymbol{E}^{(2)}(t)\,\mathrm{d}x + \int_\Omega \mathbf{curl}\,\boldsymbol{E}^{(1)}(t)\cdot\boldsymbol{H}^{(2)}(t)\,\mathrm{d}x \\ - \int_\Omega \mathbf{curl}\,\boldsymbol{H}^{(2)}(t)\cdot\boldsymbol{E}^{(1)}(t)\,\mathrm{d}x + \int_\Omega \mathbf{curl}\,\boldsymbol{E}^{(2)}(t)\cdot\boldsymbol{H}^{(1)}(t)\,\mathrm{d}x \geq 0 \quad \text{for a.e. } t\in(0,T). \end{aligned}$$

On the other hand, we know that $\boldsymbol{E}^{(j)} \in \boldsymbol{H}_0(\mathbf{curl})$ and $\boldsymbol{H}^{(j)} \in \boldsymbol{H}(\mathbf{curl})$ for $j=1,2$ such that (2.1) implies

$$\begin{aligned} -\int_\Omega \mathbf{curl}\,\boldsymbol{H}^{(1)}(t)\cdot\boldsymbol{E}^{(2)}(t)\,\mathrm{d}x + \int_\Omega \mathbf{curl}\,\boldsymbol{E}^{(1)}(t)\cdot\boldsymbol{H}^{(2)}(t)\,\mathrm{d}x \\ - \int_\Omega \mathbf{curl}\,\boldsymbol{H}^{(2)}(t)\cdot\boldsymbol{E}^{(1)}(t)\,\mathrm{d}x + \int_\Omega \mathbf{curl}\,\boldsymbol{E}^{(2)}(t)\cdot\boldsymbol{H}^{(1)}(t)\,\mathrm{d}x = 0. \end{aligned}$$

It follows therefore that

$$0 \geq \int_\Omega \epsilon\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{e}(t)\cdot\boldsymbol{e}(t) + \mu\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{h}(t)\cdot\boldsymbol{h}(t)\,\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\boldsymbol{e},\boldsymbol{h})(t)\|_{\boldsymbol{X}}^2 \quad \text{for a.e. } t\in(0,T),$$

and so $0 \geq \|(\boldsymbol{e},\boldsymbol{h})(t)\|_{\boldsymbol{X}}^2 - \|(\boldsymbol{e},\boldsymbol{h})(0)\|_{\boldsymbol{X}}^2 = \|(\boldsymbol{e},\boldsymbol{h})(t)\|_{\boldsymbol{X}}^2$ for all $t\in[0,T]$.

Step 2: *Existence.* Let $\{\lambda_n\}_{n=1}^\infty$ be a sequence of positive real numbers converging to zero, and we consider again the integral equation (3.7). The regularity properties $(\mathbf{f},\mathbf{g}) \in W^{1,\infty}((0,T),\boldsymbol{X})$ and $(\boldsymbol{E}_0,\boldsymbol{H}_0) \in D(\mathcal{A})$ imply

$(\boldsymbol{E}_n, \boldsymbol{H}_n) \in \mathcal{C}^{0,1}([0,T], \boldsymbol{X})$ (Appendix C). Thus, since the Yosida approximation $\Phi_\lambda : \boldsymbol{X} \to \boldsymbol{X}$ is Lipschitz-continuous, we have that $\Phi_{\lambda_n}(\boldsymbol{E}_n, \boldsymbol{H}_n) \in \mathcal{C}^{0,1}([0,T], \boldsymbol{X})$ such that a classical result from the semigroup theory ([12], Cor. 7.6, p. 440) implies that the solution of (3.7) satisfies $(\boldsymbol{E}_n, \boldsymbol{H}_n) \in \mathcal{C}([0,T], D(\mathcal{A})) \cap \mathcal{C}^1([0,T], \boldsymbol{X})$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{E}_n, \boldsymbol{H}_n)(t) - \mathcal{A}(\boldsymbol{E}_n, \boldsymbol{H}_n)(t) = (\epsilon^{-1}\mathbf{f}, \mu^{-1}\mathbf{g})(t) - \Phi_{\lambda_n}((\boldsymbol{E}_n, \boldsymbol{H}_n)(t)) \quad \forall t \in [0,T]. \tag{3.27}$$

Moreover, the sequence $\{(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^\infty$ is bounded in $\mathcal{C}([0,T], D(\mathcal{A})) \cap \mathcal{C}^1([0,T], \boldsymbol{X})$, which we shall show in Step 3. Therefore, as readily proven in Lemma 3.2 and due to the boundedness of $\{(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^\infty \subset \mathcal{C}([0,T], D(\mathcal{A})) \cap \mathcal{C}^1([0,T], \boldsymbol{X})$, we can select a subsequence of $\{\lambda_n\}_{n=1}^\infty$ in (3.7), denoted again by the sequence itself, such that

$$\begin{aligned}
(\boldsymbol{E}_n, \boldsymbol{H}_n) &\rightharpoonup (\boldsymbol{E}, \boldsymbol{H}) \quad \text{weakly star in } L^\infty((0,T), \boldsymbol{X}) \text{ as } n \to \infty, \\
\Phi_{\lambda_n}(\boldsymbol{E}_n, \boldsymbol{H}_n) &\rightharpoonup (\widetilde{\boldsymbol{E}}, \widetilde{\boldsymbol{H}}) \quad \text{weakly star in } L^\infty((0,T), \boldsymbol{X}) \text{ as } n \to \infty,
\end{aligned}$$

where $(\boldsymbol{E}, \boldsymbol{H}) \in L^\infty((0,T), D(\mathcal{A})) \cap W^{1,\infty}((0,T), \boldsymbol{X})$ and $(\widetilde{\boldsymbol{E}}, \widetilde{\boldsymbol{H}}) \in L^\infty((0,T), \boldsymbol{X})$ satisfy (3.6). Then, applying the regularity property $(\boldsymbol{E}, \boldsymbol{H}) \in L^\infty((0,T), D(\mathcal{A})) \cap W^{1,\infty}((0,T), \boldsymbol{X})$ to (3.6), it follows that

$$\begin{aligned}
&\int_\Omega \epsilon \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t) \cdot \boldsymbol{v} + \mu \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}(t) \cdot \boldsymbol{w} \,\mathrm{d}x + \int_\Omega \mathbf{curl}\, \boldsymbol{E}(t) \cdot \boldsymbol{w} - \mathbf{curl}\, \boldsymbol{H}(t) \cdot \boldsymbol{v} \,\mathrm{d}x \\
&\quad + \int_\Omega \epsilon \widetilde{\boldsymbol{E}}(t) \cdot \boldsymbol{v} + \mu \widetilde{\boldsymbol{H}}(t) \cdot \boldsymbol{w} \,\mathrm{d}x = \int_\Omega \mathbf{f}(t) \cdot \boldsymbol{v} + \mathbf{g}(t) \cdot \boldsymbol{w} \,\mathrm{d}x \quad \text{for a.e. } t \in (0,T) \text{ and all } (\boldsymbol{v}, \boldsymbol{w}) \in D(\mathcal{A}).
\end{aligned} \tag{3.28}$$

Thus, since $D(\mathcal{A})$ is dense in $\boldsymbol{X}$, (3.28) implies that

$$-\left(\frac{\mathrm{d}}{\mathrm{d}t} - \mathcal{A}\right)(\boldsymbol{E}, \boldsymbol{H})(t) + (\epsilon^{-1}\mathbf{f}, \mu^{-1}\mathbf{g})(t) = (\widetilde{\boldsymbol{E}}, \widetilde{\boldsymbol{H}})(t) \in \partial\varphi((\boldsymbol{E}, \boldsymbol{H})(t)) \quad \text{for a.e. } t \in (0,T).$$

In conclusion, $(\boldsymbol{E}, \boldsymbol{H})$ is a solution to (VI).

Step 3: *Boundedness result for* $\{(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^\infty$. As readily shown in (3.9), (3.5) implies the boundedness of $\{(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^\infty \subset \mathcal{C}([0,T], \boldsymbol{X})$. Now, according to (3.27), we have that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\left((\boldsymbol{E}_n, \boldsymbol{H}_n)(s+h) - (\boldsymbol{E}_n, \boldsymbol{H}_n)(s)\right) - \mathcal{A}\left((\boldsymbol{E}_n, \boldsymbol{H}_n)(s+h) - (\boldsymbol{E}_n, \boldsymbol{H}_n)(s)\right) \\
&= (\epsilon^{-1}\mathbf{f}, \mu^{-1}\mathbf{g})(s+h) - (\epsilon^{-1}\mathbf{f}, \mu^{-1}\mathbf{g})(s) - \Phi_{\lambda_n}((\boldsymbol{E}_n, \boldsymbol{H}_n)(s+h)) + \Phi_{\lambda_n}((\boldsymbol{E}_n, \boldsymbol{H}_n)(s)).
\end{aligned}$$

Taking into account that $\mathcal{A}$ is skew-adjoint and Φ_{λ_n} is monotone, it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left\|(\boldsymbol{E}_n, \boldsymbol{H}_n)(s+h) - (\boldsymbol{E}_n, \boldsymbol{H}_n)(s)\right\|_{\boldsymbol{X}}^2 \\
&= \left((\epsilon^{-1}\mathbf{f}, \mu^{-1}\mathbf{g})(s+h) - (\epsilon^{-1}\mathbf{f}, \mu^{-1}\mathbf{g})(s), (\boldsymbol{E}_n, \boldsymbol{H}_n)(s+h) - (\boldsymbol{E}_n, \boldsymbol{H}_n)(s)\right)_{\boldsymbol{X}} \\
&\quad - \left(\Phi_{\lambda_n}((\boldsymbol{E}_n, \boldsymbol{H}_n)(s+h)) - \Phi_{\lambda_n}((\boldsymbol{E}_n, \boldsymbol{H}_n)(s)), (\boldsymbol{E}_n, \boldsymbol{H}_n)(s+h) - (\boldsymbol{E}_n, \boldsymbol{H}_n)(s)\right)_{\boldsymbol{X}} \\
&\le \left((\epsilon^{-1}\mathbf{f}, \mu^{-1}\mathbf{g})(s+h) - (\epsilon^{-1}\mathbf{f}, \mu^{-1}\mathbf{g})(s), (\boldsymbol{E}_n, \boldsymbol{H}_n)(s+h) - (\boldsymbol{E}_n, \boldsymbol{H}_n)(s)\right)_{\boldsymbol{X}}.
\end{aligned}$$

Consequently, it holds for all $t \in [0,T)$ and $h \in (0, T-t]$ that

$$\begin{aligned}
&\left\| \frac{(\boldsymbol{E}_n, \boldsymbol{H}_n)(t+h) - (\boldsymbol{E}_n, \boldsymbol{H}_n)(t)}{h} \right\|_{\boldsymbol{X}}^2 - \left\| \frac{(\boldsymbol{E}_n, \boldsymbol{H}_n)(h) - (\boldsymbol{E}_0, \boldsymbol{H}_0)}{h} \right\|_{\boldsymbol{X}}^2 \\
&\leq 2 \int_0^t \left(\frac{(\epsilon^{-1}\mathbf{f}, \mu^{-1}\mathbf{g})(s+h) - (\epsilon^{-1}\mathbf{f}, \mu^{-1}\mathbf{g})(s)}{h}, \frac{(\boldsymbol{E}_n, \boldsymbol{H}_n)(s+h) - (\boldsymbol{E}_n, \boldsymbol{H}_n)(s)}{h} \right)_{\boldsymbol{X}} \mathrm{d}s \\
&\leq 2T \left\| \frac{\mathrm{d}}{\mathrm{d}t} (\epsilon^{-1}\mathbf{f}, \mu^{-1}\mathbf{g}) \right\|_{L^\infty((0,T),\boldsymbol{X})} \left\| \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{E}_n, \boldsymbol{H}_n) \right\|_{\mathcal{C}([0,T],\boldsymbol{X})},
\end{aligned}$$

and passing to the limit $h \downarrow 0$ leads to

$$\left\| \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{E}_n, \boldsymbol{H}_n)(t) \right\|_{\boldsymbol{X}}^2 \leq \left\| \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{E}_n, \boldsymbol{H}_n)(0) \right\|_{\boldsymbol{X}}^2 + 2T \left\| \frac{\mathrm{d}}{\mathrm{d}t} (\epsilon^{-1}\mathbf{f}, \mu^{-1}\mathbf{g}) \right\|_{L^\infty((0,T),\boldsymbol{X})} \left\| \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{E}_n, \boldsymbol{H}_n) \right\|_{\mathcal{C}([0,T],\boldsymbol{X})},$$

from which it follows that

$$\frac{1}{2} \left\| \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{E}_n, \boldsymbol{H}_n) \right\|_{\mathcal{C}([0,T],\boldsymbol{X})}^2 \leq \left\| \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{E}_n, \boldsymbol{H}_n)(0) \right\|_{\boldsymbol{X}}^2 + 2T^2 \left\| \frac{\mathrm{d}}{\mathrm{d}t} (\epsilon^{-1}\mathbf{f}, \mu^{-1}\mathbf{g}) \right\|_{L^\infty((0,T),\boldsymbol{X})}^2 \quad \forall n \in \mathbb{N}. \tag{3.29}$$

In view of (3.27), we also have

$$\left\| \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{E}_n, \boldsymbol{H}_n)(0) \right\|_{\boldsymbol{X}} = \left\| (\epsilon^{-1}\mathbf{f}, \mu^{-1}\mathbf{g})(0) - \Phi_{\lambda_n}(\boldsymbol{E}_0, \boldsymbol{H}_0) + \mathcal{A}(\boldsymbol{E}_0, \boldsymbol{H}_0) \right\|_{\boldsymbol{X}} \quad \forall n \in \mathbb{N}. \tag{3.30}$$

On the other hand, we have shown in the proof of Lemma 3.2 that the sequence $\{\Phi_{\lambda_n}(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^\infty \subset \mathcal{C}([0,T], \boldsymbol{X})$ is bounded. For this reason, (3.29)–(3.30) implies that the sequence $\{\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^\infty \subset \mathcal{C}([0,T], \boldsymbol{X})$ is bounded. Next, (3.27) yields that

$$\|\mathcal{A}(\boldsymbol{E}_n, \boldsymbol{H}_n)\|_{\mathcal{C}([0,T],\boldsymbol{X})} = \left\| \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{E}_n, \boldsymbol{H}_n) - (\epsilon^{-1}\mathbf{f}, \mu^{-1}\mathbf{g}) + \Phi_{\lambda_n}(\boldsymbol{E}_n, \boldsymbol{H}_n) \right\|_{\mathcal{C}([0,T],\boldsymbol{X})} \quad \forall n \in \mathbb{N}.$$

Therefore, it follows from the boundedness of $\{\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^\infty, \{\Phi_{\lambda_n}(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^\infty \subset \mathcal{C}([0,T], \boldsymbol{X})$ that the sequence $\{\mathcal{A}(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^\infty \subset \mathcal{C}([0,T], \boldsymbol{X})$ is bounded. In conclusion, the sequence $\{(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^\infty$ is bounded in $\mathcal{C}([0,T], D(\mathcal{A})) \cap \mathcal{C}^1([0,T], \boldsymbol{X})$. ☐

Example 3.5. Suppose that $\Omega_s \subset \Omega$ is a bounded open subset and $j_c \in L^\infty(\Omega_s)$ is a nonnegative function. Then, the functional

$$\varphi : \boldsymbol{X} \to \mathbb{R}, \quad (\boldsymbol{v}, \boldsymbol{w}) \mapsto \int_{\Omega_s} j_c |\boldsymbol{v}| \, \mathrm{d}x$$

satisfies all assumptions of Theorem 3.3. In this case, the variational inequality (VI) describes the Bean critical-state model with Ω_s representing a high-temperature superconductor (see [34]).

Example 3.6. Let $\boldsymbol{K} \subset \boldsymbol{X}$ be a nonempty, closed and convex set and

$$\varphi(\boldsymbol{v}, \boldsymbol{w}) = \mathcal{I}_{\boldsymbol{K}}(\boldsymbol{v}, \boldsymbol{w}) := \begin{cases} 0 & \text{if } (\boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{K}, \\ \infty & \text{if } (\boldsymbol{v}, \boldsymbol{w}) \notin \boldsymbol{K}. \end{cases} \tag{3.31}$$

By the definition (3.1), for every $(\boldsymbol{v},\boldsymbol{w}) \in D(\partial\varphi) = \boldsymbol{K}$, the subdifferential $\partial\varphi(\boldsymbol{v},\boldsymbol{w})$ is given by the normal cone of $\boldsymbol{K}$ at $(\boldsymbol{v},\boldsymbol{w})$:

$$\mathcal{N}_{\boldsymbol{K}}(\boldsymbol{v},\boldsymbol{w}) := \{(\boldsymbol{y},\boldsymbol{z}) \in \boldsymbol{X} \mid ((\boldsymbol{y},\boldsymbol{z}),(\boldsymbol{p},\boldsymbol{q}) - (\boldsymbol{v},\boldsymbol{w}))_{\boldsymbol{X}} \leq 0 \quad \forall (\boldsymbol{p},\boldsymbol{q}) \in \boldsymbol{K}\}. \tag{3.32}$$

Thus, φ does not necessarily satisfy Assumption 3.1 so that Theorem 3.3 is in general not applicable to this case.

3.1. Existence analysis without Assumption 3.1

Motivated from Example 3.6, we study now the case where φ does not satisfy the local boundedness condition in Assumption 3.1. For this case, we shall establish a weaker existence result by making use of the minimal section operator associated with the Nemytskii operator of $\partial\varphi$ acting in the Bochner space $L^2((0,T),\boldsymbol{X})$.

Definition 3.7 (Nemytskii operator of $\partial\varphi$)**.** Let $\varphi : \boldsymbol{X} \to \overline{\mathbb{R}}$ be a proper, convex and l.s.c. function. We define the Nemytskii operator $\widehat{\partial\varphi} : L^2((0,T),\boldsymbol{X}) \to 2^{L^2((0,T),\boldsymbol{X})}$ of the subdifferential $\partial\varphi : \boldsymbol{X} \to 2^{\boldsymbol{X}}$ as follows:

$$\widehat{\partial\varphi}(\boldsymbol{v},\boldsymbol{w}) := \left\{(\boldsymbol{y},\boldsymbol{z}) \in L^2((0,T),\boldsymbol{X}) \,|\, (\boldsymbol{y},\boldsymbol{z})(t) \in \partial\varphi((\boldsymbol{v},\boldsymbol{w})(t)) \quad \text{for a.e. } t \in (0,T)\right\},$$

with the domain

$$D(\widehat{\partial\varphi}) := \{(\boldsymbol{v},\boldsymbol{w}) \in L^2((0,T),\boldsymbol{X}) \,|\, \widehat{\partial\varphi}(\boldsymbol{v},\boldsymbol{w}) \neq \emptyset\}.$$

Lemma 3.8. *Let $\varphi : \boldsymbol{X} \to \overline{\mathbb{R}}$ be a convex and l.s.c. function satisfying $(0,0) \in \partial\varphi(0,0)$. Then, $\widehat{\partial\varphi} : L^2((0,T),\boldsymbol{X}) \to 2^{L^2((0,T),\boldsymbol{X})}$ is m-accretive and, for every $\lambda > 0$, the corresponding resolvent operator $\widehat{J}_\lambda : L^2((0,T),\boldsymbol{X}) \to L^2((0,T),\boldsymbol{X})$ and Yosida approximation $\widehat{\Phi}_\lambda : L^2((0,T),\boldsymbol{X}) \to L^2((0,T),\boldsymbol{X})$, defined by*

$$\widehat{J}_\lambda := (I_d + \lambda\widehat{\partial\varphi})^{-1} \quad \text{and} \quad \widehat{\Phi}_\lambda := \lambda^{-1}\big(I_d - \widehat{J}_\lambda\big),$$

fulfill

$$(\widehat{J}_\lambda(\boldsymbol{v},\boldsymbol{w}))(t) = J_\lambda((\boldsymbol{v},\boldsymbol{w})(t)) \quad \text{and} \quad (\widehat{\Phi}_\lambda(\boldsymbol{v},\boldsymbol{w}))(t) = \Phi_\lambda((\boldsymbol{v},\boldsymbol{w})(t)), \tag{3.33}$$

for all $(\boldsymbol{v},\boldsymbol{w}) \in L^2((0,T),\boldsymbol{X})$ and a.e. $t \in (0,T)$. Here, we recall that $J_\lambda : \boldsymbol{X} \to \boldsymbol{X}$ and $\Phi_\lambda : \boldsymbol{X} \to \boldsymbol{X}$ denote, respectively, the resolvent operator and the Yosida approximation associated with $\partial\varphi$.

Proof. Since $\varphi : \boldsymbol{X} \to \overline{\mathbb{R}}$ is a convex and l.s.c. function satisfying $(0,0) \in \partial\varphi(0,0)$ and $\boldsymbol{X}$ is a separable Hilbert space, a well-known result ([15], Prop. 3.2.57, p. 339) implies that $\widehat{\partial\varphi} : L^2((0,T),\boldsymbol{X}) \to 2^{L^2((0,T),\boldsymbol{X})}$ is m-accretive, from which it follows that $\widehat{J}_\lambda : L^2((0,T),\boldsymbol{X}) \to L^2((0,T),\boldsymbol{X})$ is non-expansive and $\widehat{\Phi}_\lambda : L^2((0,T),\boldsymbol{X}) \to L^2((0,T),\boldsymbol{X})$ is m-accretive and Lipschitz continuous ([31], Thm. 1.1, p. 161). Let $(\boldsymbol{v},\boldsymbol{w}) \in L^2((0,T),\boldsymbol{X})$, $\lambda > 0$, and we set $(\boldsymbol{y},\boldsymbol{z}) := \widehat{J}_\lambda(\boldsymbol{v},\boldsymbol{w}) \in L^2((0,T),\boldsymbol{X})$. By definition, it holds that

$$\begin{aligned}\frac{1}{\lambda}((\boldsymbol{v},\boldsymbol{w}) - (\boldsymbol{y},\boldsymbol{z})) \in \widehat{\partial\varphi}(\boldsymbol{y},\boldsymbol{z}) &\Rightarrow \frac{1}{\lambda}((\boldsymbol{v},\boldsymbol{w})(t) - (\boldsymbol{y},\boldsymbol{z})(t)) \in \partial\varphi((\boldsymbol{y},\boldsymbol{z})(t)) \quad \text{for a.e. } t \in (0,T)\\ &\Rightarrow (\boldsymbol{y},\boldsymbol{z})(t) = J_\lambda((\boldsymbol{v},\boldsymbol{w})(t)) \quad \text{for a.e. } t \in (0,T).\end{aligned}$$

This implies (3.33). □

Definition 3.9 (Minimal section operator of $\widehat{\partial\varphi}$)**.** Let $\varphi : \boldsymbol{X} \to \overline{\mathbb{R}}$ be a convex and l.s.c. function satisfying $(0,0) \in \partial\varphi(0,0)$. Then, we denote by $\widehat{\partial\varphi}^0 : D(\widehat{\partial\varphi}) \to L^2((0,T),\boldsymbol{X})$ the minimal section operator of $\widehat{\partial\varphi}$, defined by

$$\left\| \widehat{\partial\varphi}^0(\boldsymbol{v},\boldsymbol{w}) \right\|_{L^2((0,T),\boldsymbol{X})} = \min_{(\boldsymbol{y},\boldsymbol{z}) \in \widehat{\partial\varphi}(\boldsymbol{v},\boldsymbol{w})} \|(\boldsymbol{y},\boldsymbol{z})\|_{L^2((0,T),\boldsymbol{X})}. \tag{3.34}$$

In the case of a convex and l.s.c. function $\varphi : \boldsymbol{X} \to \overline{\mathbb{R}}$ satisfying $(0,0) \in \partial\varphi(0,0)$, we already know that $\widehat{\partial\varphi}$ is m-accretive on the Hilbert space $\mathcal{H} := L^2((0,T),\boldsymbol{X})$. Thus, Proposition 3.2.7, p. 305 of [15] implies that for every $(\boldsymbol{v},\boldsymbol{w}) \in D(\widehat{\partial\varphi})$ the set $\widehat{\partial\varphi}(\boldsymbol{v},\boldsymbol{w}) \subset \mathcal{H}$ is nonempty, closed, and convex. Therefore, by standard arguments, the minimization problem (3.34) admits a unique solution. For this reason, $\widehat{\partial\varphi}^0 : D(\widehat{\partial\varphi}) \to \mathcal{H}$ is indeed a well-defined operator.

Lemma 3.10. *Let $\varphi : \boldsymbol{X} \to \overline{\mathbb{R}}$ be a convex and l.s.c. function satisfying $(0,0) \in \partial\varphi(0,0)$. Then, for every $(\boldsymbol{v},\boldsymbol{w}) \in D(\widehat{\partial\varphi})$, it holds that*

$$\widehat{\Phi}_\lambda(\boldsymbol{v},\boldsymbol{w}) \to \widehat{\partial\varphi}^0(\boldsymbol{v},\boldsymbol{w}) \quad \textit{strongly in } L^2((0,T),\boldsymbol{X}) \textit{ as } \lambda \to 0, \tag{3.35}$$

$$(\widehat{\partial\varphi}^0(\boldsymbol{v},\boldsymbol{w}))(t) = \partial\varphi^0((\boldsymbol{v},\boldsymbol{w})(t)) \quad \textit{for a.e. } t \in (0,T), \tag{3.36}$$

where $\partial\varphi^0 : D(\partial\varphi) \to \boldsymbol{X}$ denotes the minimal section operator of $\partial\varphi$ given by (3.4).

Proof. Let $(\boldsymbol{v},\boldsymbol{w}) \in D(\widehat{\partial\varphi})$. In view of Lemma 3.8, Theorem 1.1, p. 161 of [31] implies (3.35). This strong convergence yields the existence of a null sequence $\{\lambda_n\}_{n=1}^\infty$ of positive real numbers such that

$$\lim_{n\to\infty} (\widehat{\Phi}_{\lambda_n}(\boldsymbol{v},\boldsymbol{w}))(t) = (\widehat{\partial\varphi}^0(\boldsymbol{v},\boldsymbol{w}))(t) \quad \text{for a.e. } t \in (0,T). \tag{3.37}$$

On the other hand, (3.33) implies

$$\lim_{n\to\infty} (\widehat{\Phi}_{\lambda_n}(\boldsymbol{v},\boldsymbol{w}))(t) = \lim_{n\to\infty} \Phi_{\lambda_n}((\boldsymbol{v},\boldsymbol{w})(t)) \underbrace{=}_{(3.5)} \partial\varphi^0((\boldsymbol{v},\boldsymbol{w})(t)) \quad \text{for a.e. } t \in (0,T), \tag{3.38}$$

since $(\boldsymbol{v},\boldsymbol{w}) \in D(\widehat{\partial\varphi})$ implies that $(\boldsymbol{v},\boldsymbol{w})(t) \in D(\partial\varphi)$ for a.e. $t \in (0,T)$. Combining (3.37) and (3.38) together gives (3.36). □

Theorem 3.11. *Let $\varphi : \boldsymbol{X} \to \overline{\mathbb{R}}$ be a convex and l.s.c. function satisfying $(0,0) \in \partial\varphi(0,0)$. Furthermore, let $(\mathbf{f},\mathbf{g}) \in W^{1,\infty}((0,T),\boldsymbol{X})$ and $(\boldsymbol{E}_0,\boldsymbol{H}_0) \in D(\mathcal{A}) \cap D(\partial\varphi)$. Then, the variational inequality*

$$\begin{cases} \displaystyle\int_0^T \left(\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{E},\boldsymbol{H})(t), (\boldsymbol{v},\boldsymbol{w})(t) - (\boldsymbol{E},\boldsymbol{H})(t) \right)_{\boldsymbol{X}} + ((\boldsymbol{E},\boldsymbol{H})(t), \mathcal{A}(\boldsymbol{v},\boldsymbol{w})(t))_{\boldsymbol{X}} \,\mathrm{d}t \\ \displaystyle + \int_0^T (\partial\varphi^0((\boldsymbol{v},\boldsymbol{w})(t)), (\boldsymbol{v},\boldsymbol{w})(t) - (\boldsymbol{E},\boldsymbol{H})(t))_{\boldsymbol{X}} \,\mathrm{d}t \geq \int_0^T ((\epsilon^{-1}\mathbf{f}, \mu^{-1}\mathbf{g})(t), (\boldsymbol{v},\boldsymbol{w})(t) - (\boldsymbol{E},\boldsymbol{H})(t))_{\boldsymbol{X}} \,\mathrm{d}t, \\ \textit{for all } (\boldsymbol{v},\boldsymbol{w}) \in L^2((0,T), D(\mathcal{A})) \cap D(\widehat{\partial\varphi}), \\ (\boldsymbol{E},\boldsymbol{H})(0) = (\boldsymbol{E}_0,\boldsymbol{H}_0) \end{cases} \tag{$\widehat{\text{VI}}$}$$

admits a solution $(\boldsymbol{E},\boldsymbol{H}) \in W^{1,\infty}((0,T),\boldsymbol{X})$.

Proof. Let $\{\lambda_n\}_{n=1}^\infty$ be a sequence of positive real numbers converging to zero, and we consider the integral equation (3.7). For every $n \in \mathbb{N}$, we readily know that the solution of (3.7) satisfies $(\boldsymbol{E}_n, \boldsymbol{H}_n) \in \mathcal{C}([0,T], D(\mathcal{A})) \cap \mathcal{C}^1([0,T], \boldsymbol{X})$ and solves

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{E}_n, \boldsymbol{H}_n)(t) - \mathcal{A}(\boldsymbol{E}_n, \boldsymbol{H}_n)(t) = (\epsilon^{-1}\mathbf{f}, \mu^{-1}\mathbf{g})(t) - \Phi_{\lambda_n}((\boldsymbol{E}_n, \boldsymbol{H}_n)(t)) \quad \forall t \in [0,T] \\ (\boldsymbol{E}_n, \boldsymbol{H}_n)(0) = (\boldsymbol{E}_0, \boldsymbol{H}_0). \end{cases} \tag{3.39}$$

Also, as already shown in Lemma 3.2 (*cf.* (3.9) and (3.5)), the sequence $\{(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^\infty \subset \mathcal{C}([0,T], \boldsymbol{X})$ is bounded, due to $\partial\varphi(0,0) \neq \emptyset$. Now, since $(\boldsymbol{E}_0, \boldsymbol{H}_0) \in D(\varphi)$, (3.5) yields the boundedness of $\{\Phi_{\lambda_n}(\boldsymbol{E}_0, \boldsymbol{H}_0)\}_{n=1}^\infty \subset \boldsymbol{X}$, which together with (3.29)–(3.30) implies that $\{\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^\infty \subset \mathcal{C}([0,T], \boldsymbol{X})$ is bounded. In conclusion, $\{(\boldsymbol{E}_n, \boldsymbol{H}_n)\}_{n=1}^\infty$ is bounded in $\mathcal{C}^1([0,T], \boldsymbol{X})$, and so there exists a subsequence of $\{\lambda_n\}_{n=1}^\infty$, denoted again by $\{\lambda_n\}_{n=1}^\infty$, such that

$$(\boldsymbol{E}_n, \boldsymbol{H}_n) \rightharpoonup (\boldsymbol{E}, \boldsymbol{H}) \qquad \text{weakly star in } L^\infty((0,T), \boldsymbol{X}) \text{ as } n \to \infty, \tag{3.40}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{E}_n, \boldsymbol{H}_n) \rightharpoonup \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{E}, \boldsymbol{H}) \quad \text{weakly star in } L^\infty((0,T), \boldsymbol{X}) \text{ as } n \to \infty, \tag{3.41}$$

for some $(\boldsymbol{E}, \boldsymbol{H}) \in W^{1,\infty}((0,T), \boldsymbol{X})$. After a modification on a subset of $[0,T]$ with measure zero, it holds that $(\boldsymbol{E}, \boldsymbol{H}) \in \mathcal{C}([0,T], \boldsymbol{X})$. Now, we show

$$(\boldsymbol{E}_n, \boldsymbol{H}_n)(t) \rightharpoonup (\boldsymbol{E}, \boldsymbol{H})(t) \quad \text{weakly in } \boldsymbol{X} \text{ as } n \to \infty \quad \forall t \in [0,T], \tag{3.42}$$

$$(\boldsymbol{E}, \boldsymbol{H})(0) = (\boldsymbol{E}_0, \boldsymbol{H}_0). \tag{3.43}$$

To this aim, let $t \in (0,T]$, $(\boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{X}$, and $\phi \in \mathcal{C}^1[0,t]$. Then, integration by parts yields

$$\begin{aligned} \int^t \frac{\mathrm{d}}{\mathrm{d}s}\left((\boldsymbol{E}_n, \boldsymbol{H}_n)(s), (\boldsymbol{v}, \boldsymbol{w})\right)_{\boldsymbol{X}} \phi(s)\mathrm{d}s = {} & -\int_0^t ((\boldsymbol{E}_n, \boldsymbol{H}_n)(s), (\boldsymbol{v}, \boldsymbol{w}))_{\boldsymbol{X}} \phi'(s)\mathrm{d}s \\ & + ((\boldsymbol{E}_n, \boldsymbol{H}_n)(t), (\boldsymbol{v}, \boldsymbol{w}))_{\boldsymbol{X}} \phi(t) - ((\boldsymbol{E}_0, \boldsymbol{H}_0), (\boldsymbol{v}, \boldsymbol{w}))_{\boldsymbol{X}} \phi(0). \end{aligned} \tag{3.44}$$

On the other hand, (3.41) implies

$$\begin{aligned} \lim_{n\to\infty} \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\left((\boldsymbol{E}_n, \boldsymbol{H}_n)(s), (\boldsymbol{v}, \boldsymbol{w})\right)_{\boldsymbol{X}} \phi(s)\mathrm{d}s = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\left((\boldsymbol{E}, \boldsymbol{H})(s), (\boldsymbol{v}, \boldsymbol{w})\right)_{\boldsymbol{X}} \phi(s)\mathrm{d}s \\ = -\int_0^t ((\boldsymbol{E}, \boldsymbol{H})(s), (\boldsymbol{v}, \boldsymbol{w}))_{\boldsymbol{X}} \phi'(s)\mathrm{d}s + ((\boldsymbol{E}, \boldsymbol{H})(t), (\boldsymbol{v}, \boldsymbol{w}))_{\boldsymbol{X}} \phi(t) - ((\boldsymbol{E}, \boldsymbol{H})(0), (\boldsymbol{v}, \boldsymbol{w}))_{\boldsymbol{X}} \phi(0). \end{aligned} \tag{3.45}$$

From (3.40) and (3.44)–(3.45), we conclude that

$$\begin{aligned} \lim_{n\to\infty} ((\boldsymbol{E}_n, \boldsymbol{H}_n)(t), (\boldsymbol{v}, \boldsymbol{w}))_{\boldsymbol{X}} \phi(t) - ((\boldsymbol{E}_0, \boldsymbol{H}_0), (\boldsymbol{v}, \boldsymbol{w}))_{\boldsymbol{X}} \phi(0) \\ = ((\boldsymbol{E}, \boldsymbol{H})(t), (\boldsymbol{v}, \boldsymbol{w}))_{\boldsymbol{X}} \phi(t) - ((\boldsymbol{E}, \boldsymbol{H})(0), (\boldsymbol{v}, \boldsymbol{w}))_{\boldsymbol{X}} \phi(0). \end{aligned} \tag{3.46}$$

Choosing $\phi(0) \neq 0$ and $\phi(t) = 0$ in (3.46) leads to (3.43). Then, choosing $\phi(t) \neq 0$ implies (3.42).

Now, our goal is to prove that $(\boldsymbol{E},\boldsymbol{H}) \in W^{1,\infty}(0,T,\boldsymbol{X})$ is a solution to ($\widehat{\text{VI}}$). To this aim, let $(\boldsymbol{v},\boldsymbol{w}) \in L^2((0,T),D(\mathcal{A})) \cap D(\widehat{\partial\varphi})$. Using the monotonicity of Φ_{λ_n}, we obtain for a.e. $t \in (0,T)$ that

$$
\begin{aligned}
&(\Phi_{\lambda_n}((\boldsymbol{E}_n,\boldsymbol{H}_n)(t)),(\boldsymbol{v},\boldsymbol{w})(t))_{\boldsymbol{X}} = (\Phi_{\lambda_n}((\boldsymbol{E}_n,\boldsymbol{H}_n)(t)) - \Phi_{\lambda_n}((\boldsymbol{v},\boldsymbol{w})(t)),(\boldsymbol{v},\boldsymbol{w})(t)-(\boldsymbol{E}_n,\boldsymbol{H}_n)(t))_{\boldsymbol{X}} \\
&\quad + (\Phi_{\lambda_n}((\boldsymbol{v},\boldsymbol{w})(t)),(\boldsymbol{v},\boldsymbol{w})(t)-(\boldsymbol{E}_n,\boldsymbol{H}_n)(t))_{\boldsymbol{X}} + (\Phi_{\lambda_n}((\boldsymbol{E}_n,\boldsymbol{H}_n)(t)),(\boldsymbol{E}_n,\boldsymbol{H}_n)(t))_{\boldsymbol{X}} \\
&\le (\Phi_{\lambda_n}((\boldsymbol{v},\boldsymbol{w})(t)),(\boldsymbol{v},\boldsymbol{w})(t)-(\boldsymbol{E}_n,\boldsymbol{H}_n)(t))_{\boldsymbol{X}} + (\Phi_{\lambda_n}((\boldsymbol{E}_n,\boldsymbol{H}_n)(t)),(\boldsymbol{E}_n,\boldsymbol{H}_n)(t))_{\boldsymbol{X}} \\
&= (\Phi_{\lambda_n}((\boldsymbol{v},\boldsymbol{w})(t)),(\boldsymbol{v},\boldsymbol{w})(t)-(\boldsymbol{E}_n,\boldsymbol{H}_n)(t))_{\boldsymbol{X}} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\boldsymbol{E}_n,\boldsymbol{H}_n)(t)\|_{\boldsymbol{X}}^2 + ((\epsilon^{-1}\mathbf{f},\mu^{-1}\mathbf{g})(t),(\boldsymbol{E}_n,\boldsymbol{H}_n)(t))_{\boldsymbol{X}},
\end{aligned} \tag{3.47}
$$

where we have used (3.39) and the fact that $\mathcal{A}$ is skew-adjoint for the last identity. Applying the above inequality to (3.39), we obtain again by using the skew-adjoint property of $\mathcal{A}$ that

$$
\begin{aligned}
&\int_0^T \left(\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{E}_n,\boldsymbol{H}_n)(t),(\boldsymbol{v},\boldsymbol{w})(t)\right)_{\boldsymbol{X}} + ((\boldsymbol{E}_n,\boldsymbol{H}_n)(t),\mathcal{A}(\boldsymbol{v},\boldsymbol{w})(t))_{\boldsymbol{X}}\,\mathrm{d}t \\
&= \int_0^T ((\epsilon^{-1}\mathbf{f},\mu^{-1}\mathbf{g})(t),(\boldsymbol{v},\boldsymbol{w})(t))_{\boldsymbol{X}} - (\Phi_{\lambda_n}((\boldsymbol{E}_n,\boldsymbol{H}_n)(t)),(\boldsymbol{v},\boldsymbol{w})(t))_{\boldsymbol{X}}\,\mathrm{d}t \\
&\ge \int_0^T ((\epsilon^{-1}\mathbf{f},\mu^{-1}\mathbf{g})(t),(\boldsymbol{v},\boldsymbol{w})(t)-(\boldsymbol{E}_n,\boldsymbol{H}_n)(t))_{\boldsymbol{X}} - (\Phi_{\lambda_n}((\boldsymbol{v},\boldsymbol{w})(t)),(\boldsymbol{v},\boldsymbol{w})(t)-(\boldsymbol{E}_n,\boldsymbol{H}_n)(t))_{\boldsymbol{X}}\,\mathrm{d}t \\
&\quad + \frac{1}{2}\|(\boldsymbol{E}_n,\boldsymbol{H}_n)(T)\|_{\boldsymbol{X}}^2 - \frac{1}{2}\|(\boldsymbol{E}_0,\boldsymbol{H}_0)\|_{\boldsymbol{X}}^2,
\end{aligned} \tag{3.48}
$$

and consequently

$$
\begin{aligned}
&\int_0^T \left(\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{E},\boldsymbol{H})(t),(\boldsymbol{v},\boldsymbol{w})(t)\right)_{\boldsymbol{X}} + ((\boldsymbol{E},\boldsymbol{H})(t),\mathcal{A}(\boldsymbol{v},\boldsymbol{w})(t))_{\boldsymbol{X}}\mathrm{d}t \\
&\underbrace{=}_{(3.40)-(3.41)} \lim_{n\to\infty}\int_0^T \left(\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{E}_n,\boldsymbol{H}_n)(t),(\boldsymbol{v},\boldsymbol{w})(t)\right)_{\boldsymbol{X}} + ((\boldsymbol{E}_n,\boldsymbol{H}_n)(t),\mathcal{A}(\boldsymbol{v},\boldsymbol{w})(t))_{\boldsymbol{X}}\,\mathrm{d}t \\
&\ge \liminf_{n\to\infty}\int_0^T ((\epsilon^{-1}\mathbf{f},\mu^{-1}\mathbf{g})(t),(\boldsymbol{v},\boldsymbol{w})(t)-(\boldsymbol{E}_n,\boldsymbol{H}_n)(t))_{\boldsymbol{X}} - (\Phi_{\lambda_n}((\boldsymbol{v},\boldsymbol{w})(t)),(\boldsymbol{v},\boldsymbol{w})(t)-(\boldsymbol{E}_n,\boldsymbol{H}_n)(t))_{\boldsymbol{X}}\mathrm{d}t \\
&\quad + \liminf_{n\to\infty}\frac{1}{2}\|(\boldsymbol{E}_n,\boldsymbol{H}_n)(T)\|_{\boldsymbol{X}}^2 - \frac{1}{2}\|(\boldsymbol{E}_0,\boldsymbol{H}_0)\|_{\boldsymbol{X}}^2 \\
&\ge \int_0^T ((\epsilon^{-1}\mathbf{f},\mu^{-1}\mathbf{g})(t),(\boldsymbol{v},\boldsymbol{w})(t)-(\boldsymbol{E},\boldsymbol{H})(t))_{\boldsymbol{X}} - (\partial\varphi^0((\boldsymbol{v},\boldsymbol{w})(t)),(\boldsymbol{v},\boldsymbol{w})(t)-(\boldsymbol{E},\boldsymbol{H})(t))_{\boldsymbol{X}}\,\mathrm{d}t \\
&\quad + \underbrace{\frac{1}{2}\|(\boldsymbol{E},\boldsymbol{H})(T)\|_{\boldsymbol{X}}^2 - \frac{1}{2}\|(\boldsymbol{E}_0,\boldsymbol{H}_0)\|_{\boldsymbol{X}}^2}_{=\int_0^T(\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{E},\boldsymbol{H})(t),(\boldsymbol{E},\boldsymbol{H})(t))_{\boldsymbol{X}}\,\mathrm{d}t},
\end{aligned}
$$

where we have used Lemma 3.10, (3.40) and (3.42) with $t = T$ to achieve the last inequality. Thus, $(\boldsymbol{E},\boldsymbol{H})$ satisfies

$$
\begin{aligned}
&\int_0^T \left(\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{E},\boldsymbol{H})(t),(\boldsymbol{v},\boldsymbol{w})(t)-(\boldsymbol{E},\boldsymbol{H})(t)\right)_{\boldsymbol{X}} + ((\boldsymbol{E},\boldsymbol{H})(t),\mathcal{A}(\boldsymbol{v},\boldsymbol{w})(t))_{\boldsymbol{X}}\,\mathrm{d}t \\
&+ \int_0^T (\partial\varphi^0((\boldsymbol{v},\boldsymbol{w})(t)),(\boldsymbol{v},\boldsymbol{w})(t)-(\boldsymbol{E},\boldsymbol{H})(t))_{\boldsymbol{X}}\,\mathrm{d}t \ge \int_0^T ((\epsilon^{-1}\mathbf{f},\mu^{-1}\mathbf{g})(t),(\boldsymbol{v},\boldsymbol{w})(t)-(\boldsymbol{E},\boldsymbol{H})(t))_{\boldsymbol{X}}\,\mathrm{d}t,
\end{aligned}
$$

which concludes together with (3.43) that $(\boldsymbol{E},\boldsymbol{H}) \in W^{1,\infty}((0,T),\boldsymbol{X})$ is a solution to ($\widehat{\text{VI}}$). $\square$

Remark 3.12. We point out that the variational inequality ($\widehat{\text{VI}}$) can be seen as a generalization of (VI) in the sense that, if additionally Assumption 3.1 is satisfied, then the unique solution to (VI) satisfies ($\widehat{\text{VI}}$).

Corollary 3.13. *Suppose that all assumptions of Theorem 3.11 and Assumption 3.1 are satisfied. Then, the solution $(\boldsymbol{E},\boldsymbol{H}) \in L^\infty((0,T),D(\mathcal{A})) \cap W^{1,\infty}((0,T),\boldsymbol{X})$ to (VI) is a solution to ($\widehat{\text{VI}}$).*

Proof. As readily proven in Theorem 3.11, there exists a sequence $\{(\boldsymbol{E}_n,\boldsymbol{H}_n)\}_{n=1}^\infty$ of solutions to (3.27) converging weakly star in $L^\infty((0,T),\boldsymbol{X})$ towards a solution to ($\widehat{\text{VI}}$). But, in Theorem 3.3, it was shown that $\{(\boldsymbol{E}_n,\boldsymbol{H}_n)\}_{n=1}^\infty$ converges weakly star in $L^\infty((0,T),\boldsymbol{X})$ towards the unique solution to (VI). $\square$

Corollary 3.13 can also be also verified using the monotonicity property of the subdifferential: If $(\boldsymbol{E},\boldsymbol{H}) \in L^\infty((0,T),D(\mathcal{A})) \cap W^{1,\infty}((0,T),\boldsymbol{X})$ is the solution to (VI), then it holds for all $(\boldsymbol{v},\boldsymbol{w}) \in D(\partial\varphi)$ and all $(\boldsymbol{y},\boldsymbol{z})(t) \in \partial\varphi((\boldsymbol{E},\boldsymbol{H})(t))$ that

$$(\partial\varphi^0(\boldsymbol{v},\boldsymbol{w}),(\boldsymbol{v},\boldsymbol{w})-(\boldsymbol{E},\boldsymbol{H})(t))_{\boldsymbol{X}} \geq ((\boldsymbol{y},\boldsymbol{z})(t),(\boldsymbol{v},\boldsymbol{w})-(\boldsymbol{E},\boldsymbol{H})(t))_{\boldsymbol{X}} \quad \text{for a.e. } t \in (0,T),$$

and so by (3.26) it follows that

$$\begin{aligned}&\left(\left(\frac{\mathrm{d}}{\mathrm{d}t}-\mathcal{A}\right)(\boldsymbol{E},\boldsymbol{H})(t),(\boldsymbol{v},\boldsymbol{w})-(\boldsymbol{E},\boldsymbol{H})(t)\right)_{\boldsymbol{X}} + (\partial\varphi^0(\boldsymbol{v},\boldsymbol{w}),(\boldsymbol{v},\boldsymbol{w})-(\boldsymbol{E},\boldsymbol{H})(t))_{\boldsymbol{X}}\\ &\geq ((\epsilon^{-1}\mathbf{f},\mu^{-1}\mathbf{g})(t),(\boldsymbol{v},\boldsymbol{w})-(\boldsymbol{E},\boldsymbol{H})(t))_{\boldsymbol{X}} \quad \text{for a.e. } t\in(0,T) \text{ and all } (\boldsymbol{v},\boldsymbol{w}) \in D(\partial\varphi).\end{aligned}$$

In conclusion, the unique solution of (VI) is a solution to ($\widehat{\text{VI}}$).

Remark 3.14. Let $\boldsymbol{K} \subset \boldsymbol{X}$ be a closed and convex subset satisfying $(0,0) \in \boldsymbol{K}$. Then, the indicator functional $\varphi = \mathcal{I}_{\boldsymbol{K}}$ satisfies all the assumptions of Theorem 3.11. Also, in view of (3.32), we notice that

$$(0,0) \in \partial\varphi(\boldsymbol{v},\boldsymbol{w}) \quad \forall (\boldsymbol{v},\boldsymbol{w}) \in D(\partial\varphi) = \boldsymbol{K}, \tag{3.49}$$

from which and together with (3.4) it follows that $\partial\varphi^0(\boldsymbol{v},\boldsymbol{w}) = (0,0)$ for all $(\boldsymbol{v},\boldsymbol{w}) \in \boldsymbol{K}$. On the other hand, (3.49) along with Definition 3.7 yields that

$$D(\widehat{\partial\varphi}) = \big\{(\boldsymbol{v},\boldsymbol{w}) \in L^2((0,T),\boldsymbol{X}) \mid (\boldsymbol{v},\boldsymbol{w})(t) \in \boldsymbol{K} \quad \text{for a.e. } t \in (0,T)\big\}.$$

In conclusion, in the case of the indicator functional $\varphi = \mathcal{I}_{\boldsymbol{K}}$, ($\widehat{\text{VI}}$) admits the following obstacle-type form

$$\begin{cases}\displaystyle\int_0^T \left(\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{E},\boldsymbol{H})(t),(\boldsymbol{v},\boldsymbol{w})(t)-(\boldsymbol{E},\boldsymbol{H})(t)\right)_{\boldsymbol{X}} + ((\boldsymbol{E},\boldsymbol{H})(t),\mathcal{A}(\boldsymbol{v},\boldsymbol{w})(t))_{\boldsymbol{X}}\,\mathrm{d}t\\ \displaystyle\geq \int_0^T ((\epsilon^{-1}\mathbf{f},\mu^{-1}\mathbf{g})(t),(\boldsymbol{v},\boldsymbol{w})(t)-(\boldsymbol{E},\boldsymbol{H})(t))_{\boldsymbol{X}}\,\mathrm{d}t,\\ \text{for all } (\boldsymbol{v},\boldsymbol{w}) \in L^2((0,T),D(\mathcal{A})) \text{ with } (\boldsymbol{v},\boldsymbol{w})(t) \in \boldsymbol{K} \text{ a.e. in } (0,T),\\ (\boldsymbol{E},\boldsymbol{H})(0) = (\boldsymbol{E}_0,\boldsymbol{H}_0).\end{cases}$$

4. Faraday's law

This section is devoted to the case of variational inequalities with Faraday's law $\mu\partial_t \boldsymbol{H} + \mathbf{curl}\,\boldsymbol{E} = 0$, which is obtained by specifying the nonlinearity φ to be independent of the second variable, *i.e.*,

$$\varphi(\boldsymbol{v}, \boldsymbol{w}) = j(\boldsymbol{v}) \quad \forall(\boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{X}, \quad j : \boldsymbol{L}^2_\epsilon(\Omega) \to \overline{\mathbb{R}}. \tag{4.1}$$

Corollary 4.1. *Let all assumptions of* Theorem 3.3 *be satisfied with* $\mathbf{g} = 0$ *and* (4.1)*. Then,*

$$\begin{cases} \displaystyle\int_\Omega \epsilon \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t)) - \mathbf{curl}\,\boldsymbol{H}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t))\,\mathrm{d}x + j(\boldsymbol{v}) - j(\boldsymbol{E}(t)) \\ \displaystyle\geq \int_\Omega \mathbf{f}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t))\,\mathrm{d}x \quad \textit{for a.e. } t\in(0,T) \textit{ and all } \boldsymbol{v}\in \boldsymbol{L}^2_\epsilon(\Omega), \\ \displaystyle\mu\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}(t) + \mathbf{curl}\,\boldsymbol{E}(t) = 0 \quad \textit{for a.e. } t\in(0,T), \\ (\boldsymbol{E},\boldsymbol{H})(0) = (\boldsymbol{E}_0, \boldsymbol{H}_0) \end{cases} \tag{VI$_\mathrm{F}$}$$

admits a unique solution $(\boldsymbol{E},\boldsymbol{H}) \in L^\infty((0,T), \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{H}(\mathbf{curl})) \cap W^{1,\infty}((0,T), \boldsymbol{X})$.

Proof. As pointed out in Remark 3.4 and since $\mathbf{g} = 0$, (VI) is equivalent to

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{E},\boldsymbol{H})(t) + \mathcal{A}(\boldsymbol{E},\boldsymbol{H})(t) + (\epsilon^{-1}\mathbf{f}(t), 0) \in \partial\varphi((\boldsymbol{E},\boldsymbol{H})(t)) \quad \text{for a.e. } t\in(0,T), \\ (\boldsymbol{E},\boldsymbol{H})(0) = (\boldsymbol{E}_0,\boldsymbol{H}_0). \end{cases} \tag{4.2}$$

Furthermore, in view of (4.1), the definition of the subdifferential (3.1) yields for all $(\boldsymbol{v},\boldsymbol{w}) \in \boldsymbol{X}$ that

$$(\boldsymbol{y},\boldsymbol{z}) \in \partial\varphi(\boldsymbol{v},\boldsymbol{w}) \;\Leftrightarrow\; \boldsymbol{z} = 0 \text{ and } (\boldsymbol{y}, \boldsymbol{p}-\boldsymbol{v})_{\boldsymbol{L}^2_\epsilon(\Omega)} + j(\boldsymbol{v}) \leq j(\boldsymbol{p}) \quad \forall \boldsymbol{p}\in \boldsymbol{L}^2_\epsilon(\Omega). \tag{4.3}$$

Applying (4.3) to (4.2), we come to the conclusion that (4.2) is equivalent to (VI$_\mathrm{F}$), and so the assertion is valid. □

Analogously to Definition 3.7, we introduce the Nemytskii operator of $\partial j : \boldsymbol{L}^2_\epsilon(\Omega) \to 2^{\boldsymbol{L}^2_\epsilon(\Omega)}$ by $\widehat{\partial j} : L^2((0,T), \boldsymbol{L}^2_\epsilon(\Omega)) \to 2^{L^2((0,T),\boldsymbol{L}^2_\epsilon(\Omega))}$, defined as follows:

$$\begin{aligned} \widehat{\partial j}(\boldsymbol{v}) &:= \left\{\boldsymbol{y}\in L^2((0,T),\boldsymbol{L}^2_\epsilon(\Omega)) \,|\, \boldsymbol{y}(t) \in \partial j(\boldsymbol{v}(t)) \quad \text{for a.e. } t\in(0,T)\right\} \\ D(\widehat{\partial j}) &:= \{\boldsymbol{v}\in L^2((0,T),\boldsymbol{L}^2_\epsilon(\Omega)) \,|\, \widehat{\partial j}(\boldsymbol{v}) \neq \emptyset\}. \end{aligned} \tag{4.4}$$

Analogously, $\widehat{\partial j}^0 : D(\widehat{\partial j}) \to L^2((0,T), \boldsymbol{L}^2_\epsilon(\Omega))$ denotes the minimal section operator of $\widehat{\partial j}$, defined by

$$\left\|\widehat{\partial j}^0(\boldsymbol{v})\right\|_{L^2((0,T),\boldsymbol{L}^2_\epsilon(\Omega))} = \min_{\boldsymbol{y}\in\widehat{\partial j}(\boldsymbol{v})} \|\boldsymbol{y}\|_{L^2((0,T),\boldsymbol{L}^2_\epsilon(\Omega))}. \tag{4.5}$$

Considering Faraday's law as in (VI$_\mathrm{F}$) allows us to make a regularity improvement in Theorem 3.11, which leads to the following result:

Corollary 4.2. *Let* $j : \boldsymbol{L}^2_\epsilon(\Omega) \to \overline{\mathbb{R}}$ *be a convex and l.s.c. function satisfying* $0 \in \partial j(0)$*. Furthermore, let* $\mathbf{f} \in W^{1,\infty}((0,T), \boldsymbol{L}^2_\epsilon(\Omega))$, $\boldsymbol{E}_0 \in \boldsymbol{H}_0(\mathbf{curl}) \cap D(\partial j)$*, and* $\boldsymbol{H}_0 \in \boldsymbol{H}(\mathbf{curl})$*. Then,*

$$\begin{cases} \int_0^T\int_\Omega \epsilon\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t)\cdot(\boldsymbol{v}(t)-\boldsymbol{E}(t)) - \boldsymbol{H}(t)\cdot\mathbf{curl}\,(\boldsymbol{v}(t)-\boldsymbol{E}(t)) + \epsilon\partial j^0(\boldsymbol{v}(t))\cdot(\boldsymbol{v}(t)-\boldsymbol{E}(t))\,\mathrm{d}x\,\mathrm{d}t \\ \geq \int_0^T\int_\Omega \mathbf{f}(t)\cdot(\boldsymbol{v}(t)-\boldsymbol{E}(t))\,\mathrm{d}x\,\mathrm{d}t \quad \forall \boldsymbol{v}\in L^2((0,T),\boldsymbol{H}_0(\mathbf{curl}))\cap D(\widehat{\partial j}), \\ \mu\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}(t) + \mathbf{curl}\,\boldsymbol{E}(t) = 0 \textit{ for a.e. } t\in(0,T), \\ (\boldsymbol{E},\boldsymbol{H})(0) = (\boldsymbol{E}_0,\boldsymbol{H}_0) \end{cases} \tag{$\widehat{\mathrm{VI}_\mathrm{F}}$}$$

admits a solution $(\boldsymbol{E},\boldsymbol{H})\in L^\infty((0,T),\boldsymbol{H}_0(\mathbf{curl})\times \boldsymbol{L}^2_\mu(\Omega)))\cap W^{1,\infty}((0,T),\boldsymbol{X})$.

Proof. We set $\mathbf{g}=0$ and $\varphi:\boldsymbol{X}\to\overline{\mathbb{R}}$ as in (4.1). Then, according to (4.3), we have that

$$D(\widehat{\partial\varphi}) = \{(\boldsymbol{v},\boldsymbol{w})\in L^2((0,T),\boldsymbol{X})\,|\,\widehat{\partial\varphi}(\boldsymbol{v},\boldsymbol{w})\neq\emptyset\} = \{(\boldsymbol{v},\boldsymbol{w})\in L^2((0,T),\boldsymbol{X})\,|\,\boldsymbol{v}\in D(\widehat{\partial j})\}. \tag{4.6}$$

For every $(\boldsymbol{v},\boldsymbol{w})\in\boldsymbol{X}$ and $\lambda>0$, we set $J_\lambda(\boldsymbol{v},\boldsymbol{w}) = (J^1_\lambda(\boldsymbol{v},\boldsymbol{w}), J^2_\lambda(\boldsymbol{v},\boldsymbol{w}))$. Then, due to (4.1), the definition of the resolvent operator (3.2) yields for all $(\boldsymbol{v},\boldsymbol{w})\in\boldsymbol{X}$ and $\lambda>0$ that

$$\frac{1}{\lambda}\left((\boldsymbol{v},\boldsymbol{w}) - J_\lambda(\boldsymbol{v},\boldsymbol{w})\right)\in\partial\varphi(J_\lambda(\boldsymbol{v},\boldsymbol{w})) \quad \underbrace{\Rightarrow}_{(4.3)} \quad \boldsymbol{w} = J^2_\lambda(\boldsymbol{v},\boldsymbol{w}),$$

from which it follows that

$$\Phi_\lambda(\boldsymbol{v},\boldsymbol{w}) = \frac{1}{\lambda}(I_d - J_\lambda)(\boldsymbol{v},\boldsymbol{w}) = \frac{1}{\lambda}(\boldsymbol{v} - J^1_\lambda(\boldsymbol{v},\boldsymbol{w}),0) \quad \forall(\boldsymbol{v},\boldsymbol{w})\in\boldsymbol{X}, \quad \forall\lambda>0. \tag{4.7}$$

In the proof of Theorem 3.11, we have shown that there exists a null sequence $\{\lambda_n\}_{n=1}^\infty\subset\mathbb{R}^+$ such that the corresponding sequence $\{(\boldsymbol{E}_n,\boldsymbol{H}_n)\}_{n=1}^\infty$ of solutions to (3.7) is bounded in $\mathcal{C}^1([0,T],\boldsymbol{X})$ and

$$(\boldsymbol{E}_n,\boldsymbol{H}_n)\rightharpoonup(\boldsymbol{E},\boldsymbol{H}) \qquad \text{weakly star in } L^\infty((0,T),\boldsymbol{X}) \text{ as } n\to\infty, \tag{4.8}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{E}_n,\boldsymbol{H}_n)\rightharpoonup\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{E},\boldsymbol{H}) \quad \text{weakly star in } L^\infty((0,T),\boldsymbol{X}) \text{ as } n\to\infty, \tag{4.9}$$

where $(\boldsymbol{E},\boldsymbol{H})\in W^{1,\infty}((0,T),\boldsymbol{X})$ is a solution to ($\widehat{\mathrm{VI}}$). On the other hand, since $(\boldsymbol{E}_n,\boldsymbol{H}_n)\in\mathcal{C}([0,T],D(\mathcal{A}))\cap\mathcal{C}^1([0,T],\boldsymbol{X})$ is the solution to the Cauchy problem (3.39), we obtain from (4.7) that

$$\mu\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}_n(t) + \mathbf{curl}\,\boldsymbol{E}_n(t) = 0 \quad \text{for all } t\in[0,T] \text{ and all } n\in\mathbb{N}. \tag{4.10}$$

Therefore, thanks to the boundedness of $\{(\boldsymbol{E}_n,\boldsymbol{H}_n)\}_{n=1}^\infty\subset\mathcal{C}^1([0,T],\boldsymbol{X})$, it follows that $\{\boldsymbol{E}_n\}_{n=1}^\infty$ is bounded in $\mathcal{C}([0,T],\boldsymbol{H}_0(\mathbf{curl}))$. For this reason, there exists a subsequence of $\{\lambda_n\}_{n=1}^\infty$, denoted again by $\{\lambda_n\}_{n=1}^\infty$, such that

$$\boldsymbol{E}_n\rightharpoonup\widetilde{\boldsymbol{E}} \quad \text{weakly star in } L^\infty((0,T),\boldsymbol{H}_0(\mathbf{curl})) \text{ as } n\to\infty. \tag{4.11}$$

for some $\widetilde{\boldsymbol{E}} \in L^\infty((0,T), \boldsymbol{H}_0(\mathbf{curl}))$. Combining (4.8)–(4.11) together, we obtain

$$\boldsymbol{E} = \widetilde{\boldsymbol{E}} \in L^\infty((0,T), \boldsymbol{H}_0(\mathbf{curl})) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}(t) + \mathbf{curl}\,\boldsymbol{E}(t) = 0 \text{ for a.e. } t \in (0,T). \tag{4.12}$$

In particular, (4.12) implies for all $\boldsymbol{w} \in \boldsymbol{H}(\mathbf{curl})$ and a.e. $t \in (0,T)$ that

$$\begin{aligned} \int_\Omega \mu \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}(t)\cdot(\boldsymbol{w} - \boldsymbol{H}(t))\,\mathrm{d}x &= -\int_\Omega \mathbf{curl}\,\boldsymbol{E}(t)\cdot\boldsymbol{w} - \mathbf{curl}\,\boldsymbol{E}(t)\cdot\boldsymbol{H}(t)\,\mathrm{d}x \\ &= -\int_\Omega \boldsymbol{E}(t)\cdot\mathbf{curl}\,\boldsymbol{w} - \mathbf{curl}\,\boldsymbol{E}(t)\cdot\boldsymbol{H}(t)\,\mathrm{d}x. \end{aligned} \tag{4.13}$$

Finally, applying (4.1), (4.6) and (4.13) to ($\widehat{\text{VI}}$) along with (4.12), we come to the conclusion that $(\boldsymbol{E},\boldsymbol{H}) \in L^\infty((0,T), \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{L}^2_\mu(\Omega))) \cap W^{1,\infty}((0,T), \boldsymbol{X})$ is a solution to ($\widehat{\text{VI}_\text{F}}$). □

Remark 4.3. As in Remark 3.14, in the case of the indicator functional $j = \mathcal{I}_\mathbf{C} : \boldsymbol{L}^2_\epsilon(\Omega) \to \overline{\mathbb{R}}$ with a closed and convex subset $\mathbf{C} \subset \boldsymbol{L}^2_\epsilon(\Omega)$ satisfying $0 \in \mathbf{C}$, ($\widehat{\text{VI}_\text{F}}$) admits the following obstacle-type form:

$$\begin{cases} \int_0^T\int_\Omega \epsilon \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t)\cdot(\boldsymbol{v}(t) - \boldsymbol{E}(t)) - \boldsymbol{H}(t)\cdot\mathbf{curl}\,(\boldsymbol{v}(t) - \boldsymbol{E}(t))\,\mathrm{d}x\,\mathrm{d}t \geq \int_0^T\int_\Omega \mathbf{f}(t)\cdot(\boldsymbol{v}(t) - \boldsymbol{E}(t))\,\mathrm{d}x\,\mathrm{d}t \\ \text{for all } \boldsymbol{v} \in L^2((0,T), \boldsymbol{H}_0(\mathbf{curl})) \text{ with } \boldsymbol{v}(t) \in \mathbf{C} \text{ a.e. in } (0,T), \\ \mu\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}(t) + \mathbf{curl}\,\boldsymbol{E}(t) = 0 \text{ for a.e. } t \in (0,T), \\ (\boldsymbol{E},\boldsymbol{H})(0) = (\boldsymbol{E}_0,\boldsymbol{H}_0). \end{cases} \tag{$\widehat{\text{VI}_\text{F}}$}$$

Appendix A

Proof of (2.1). We set

$$\boldsymbol{Z} := \big\{\boldsymbol{q} \in \boldsymbol{H}(\mathbf{curl}) \mid (\boldsymbol{q}, \mathbf{curl}\,\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} = (\mathbf{curl}\,\boldsymbol{q}, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)}\ \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl})\big\}.$$

By definition, $\boldsymbol{\mathcal{C}}_0^\infty(\Omega) \subset \boldsymbol{Z}$, and $\boldsymbol{Z} \subset \boldsymbol{H}(\mathbf{curl})$ is a closed subspace. Therefore, the closure of $\boldsymbol{\mathcal{C}}_0^\infty(\Omega)$ w.r.t. the $\boldsymbol{H}(\mathbf{curl})$-topology is contained in $\boldsymbol{Z}$, *i.e.*, $\boldsymbol{H}_0(\mathbf{curl}) \subset \boldsymbol{Z}$. As a consequence, the Hilbert projection theorem yields that

$$\boldsymbol{Z} = \boldsymbol{H}_0(\mathbf{curl}) \oplus \boldsymbol{H}_0(\mathbf{curl})^\perp$$

with $\boldsymbol{H}_0(\mathbf{curl})^\perp = \{\boldsymbol{q} \in \boldsymbol{Z} \mid (\boldsymbol{q},\boldsymbol{v})_{\boldsymbol{H}(\mathbf{curl})} = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl})\}$. We show that $\boldsymbol{H}_0(\mathbf{curl})^\perp = \{\mathbf{0}\}$, which implies (2.1). To this aim, let $\boldsymbol{q} \in \boldsymbol{H}_0(\mathbf{curl})^\perp$ be arbitrarily fixed. By definition, it follows that

$$(\mathbf{curl}\,\boldsymbol{q}, \mathbf{curl}\,\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} = -(\boldsymbol{q},\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{C}}_0^\infty(\Omega),$$

and so $\mathbf{curl}\,\boldsymbol{q} \in \boldsymbol{H}(\mathbf{curl})$ with $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{q} = -\boldsymbol{q}$. This together with $\boldsymbol{q} \in \boldsymbol{Z}$ implies

$$(\mathbf{curl}\,\boldsymbol{q}, \mathbf{curl}\,\boldsymbol{q})_{\boldsymbol{L}^2(\Omega)} = (\boldsymbol{q}, \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{q})_{\boldsymbol{L}^2(\Omega)} = -(\boldsymbol{q},\boldsymbol{q})_{\boldsymbol{L}^2(\Omega)} \quad \Rightarrow \quad \|\boldsymbol{q}\|_{\boldsymbol{H}(\mathbf{curl})} = 0.$$

□

Appendix B

Proof of Lemma 2.1. By $\mathcal{B}: D(\mathcal{B}) \subset \boldsymbol{X} \to \boldsymbol{X}$, we denote the infinitesimal generator of $\{\mathbb{S}_t\}_{t\in\mathbb{R}}$. Since $\{\mathbb{S}_t\}_{t\in\mathbb{R}}$ is a strongly continuous group of unitary operators, Stone's theorem ([25], Thm. 10.8, p. 41) implies that $\mathcal{B}: D(\mathcal{B}) \subset \boldsymbol{X} \to \boldsymbol{X}$ is skew-adjoint. Now, since $D(\mathcal{B}) \subset \boldsymbol{X}$ and $\mathcal{C}_0^\infty((0,T),\boldsymbol{X}) \subset L^1((0,T),\boldsymbol{X})$ are dense, there exist $\{(\boldsymbol{e}_{n,0},\boldsymbol{h}_{n,0})\}_{n=1}^\infty \subset D(\mathcal{B})$ and $\{(\boldsymbol{w}_n,\widetilde{\boldsymbol{w}}_n)\}_{n=1}^\infty \subset \mathcal{C}_0^\infty((0,T),\boldsymbol{X})$ such that

$$\lim_{n\to\infty} \|(\boldsymbol{e}_{n,0}-\boldsymbol{e}_0,\boldsymbol{h}_{n,0}-\boldsymbol{h}_0)\|_{\boldsymbol{X}} = 0 \text{ and } \lim_{n\to\infty} \|(\boldsymbol{w}_n-\boldsymbol{w},\widetilde{\boldsymbol{w}}_n-\widetilde{\boldsymbol{w}})\|_{L^1((0,T),\boldsymbol{X})} = 0. \tag{B.1}$$

For every $n \in \mathbb{N}$, we define

$$(\boldsymbol{e}_n,\boldsymbol{h}_n)(t) := \mathbb{S}_t(\boldsymbol{e}_{n,0},\boldsymbol{h}_{n,0}) + \int_0^t \mathbb{S}_{t-s}(\boldsymbol{w}_n,\widetilde{\boldsymbol{w}}_n)(s)\,\mathrm{d}s \quad \forall t\in[0,T].$$

By definition and since $\{\mathbb{S}_t\}_{t\in\mathbb{R}}$ is unitary, we infer that

$$\begin{aligned}\|(\boldsymbol{e}_n-\boldsymbol{e},\boldsymbol{h}_n-\boldsymbol{h})(t)\|_{\boldsymbol{X}} &= \left\|\mathbb{S}_t(\boldsymbol{e}_{n,0}-\boldsymbol{e}_0,\boldsymbol{h}_{n,0}-\boldsymbol{h}_0) + \int_0^t \mathbb{S}_{t-s}(\boldsymbol{w}_n-\boldsymbol{w},\widetilde{\boldsymbol{w}}_n-\widetilde{\boldsymbol{w}})(s)\,\mathrm{d}s\right\|_{\boldsymbol{X}}\\ &\le \|(\boldsymbol{e}_{n,0}-\boldsymbol{e}_0,\boldsymbol{h}_{n,0}-\boldsymbol{h}_0)\|_{\boldsymbol{X}} + \|(\boldsymbol{w}_n-\boldsymbol{w},\widetilde{\boldsymbol{w}}_n-\widetilde{\boldsymbol{w}})\|_{L^1((0,t),\boldsymbol{X})} \quad \forall t\in[0,T], \quad \forall n\in\mathbb{N}.\end{aligned}$$

It follows therefore for every $n \in \mathbb{N}$ that

$$\|(\boldsymbol{e}_n-\boldsymbol{e},\boldsymbol{h}_n-\boldsymbol{h})\|_{\mathcal{C}([0,T],\boldsymbol{X})} \le \|(\boldsymbol{e}_{n,0}-\boldsymbol{e}_0,\boldsymbol{h}_{n,0}-\boldsymbol{h}_0)\|_{\boldsymbol{X}} + \|(\boldsymbol{w}_n-\boldsymbol{w},\widetilde{\boldsymbol{w}}_n-\widetilde{\boldsymbol{w}})\|_{L^1((0,T),\boldsymbol{X})},$$

and so (B.1) implies

$$\lim_{n\to\infty} \|(\boldsymbol{e}_n-\boldsymbol{e},\boldsymbol{h}_n-\boldsymbol{h})\|_{\mathcal{C}([0,T],\boldsymbol{X})} = 0. \tag{B.2}$$

On the other hand, since $(\boldsymbol{e}_{n,0},\boldsymbol{h}_{n,0}) \in D(\mathcal{B})$ and $(\boldsymbol{w}_n,\widetilde{\boldsymbol{w}}_n) \in \mathcal{C}_0^\infty((0,T),\boldsymbol{X})$, it holds for every $n\in\mathbb{N}$ that $(\boldsymbol{e}_n,\boldsymbol{h}_n) \in \mathcal{C}([0,T],D(\mathcal{B})) \cap \mathcal{C}^1([0,T],\boldsymbol{X})$, and it is exactly the solution of

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{e}_n,\boldsymbol{h}_n)(t) = \mathcal{B}(\boldsymbol{e}_n,\boldsymbol{h}_n)(t) + (\boldsymbol{w}_n,\widetilde{\boldsymbol{w}}_n)(t) \quad \forall t\in[0,T],\\ (\boldsymbol{e}_n,\boldsymbol{h}_n)(0) = (\boldsymbol{e}_{n,0},\boldsymbol{h}_{n,0}).\end{cases}$$

See ([25], Cor. 2.5, p. 107) for this classical result. Thus, for every $t\in[0,T]$ and $n\in\mathbb{N}$, it follows that

$$\begin{aligned}\int_0^t \left(\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{e}_n,\boldsymbol{h}_n)(s),(\boldsymbol{e}_n,\boldsymbol{h}_n)(s)\right)_{\boldsymbol{X}}\mathrm{d}s &= \int_0^t (\mathcal{B}(\boldsymbol{e}_n,\boldsymbol{h}_n)(s),(\boldsymbol{e}_n,\boldsymbol{h}_n)(s))_{\boldsymbol{X}}\,\mathrm{d}s\\ +\int_0^t ((\boldsymbol{w}_n,\widetilde{\boldsymbol{w}}_n)(s),(\boldsymbol{e}_n,\boldsymbol{h}_n)(s))_{\boldsymbol{X}}\,\mathrm{d}s &= \int_0^t ((\boldsymbol{w}_n,\widetilde{\boldsymbol{w}}_n)(s),(\boldsymbol{e}_n,\boldsymbol{h}_n)(s))_{\boldsymbol{X}}\,\mathrm{d}s,\end{aligned}$$

since $\mathcal{B}$ is skew-adjoint. In conclusion, we obtain for every $t \in [0,T]$ and $n \in \mathbb{N}$ that

$$\frac{1}{2}\|(\boldsymbol{e}_n, \boldsymbol{h}_n)(t)\|_{\boldsymbol{X}}^2 - \frac{1}{2}\|(\boldsymbol{e}_{n,0}, \boldsymbol{h}_{n,0})\|_{\boldsymbol{X}}^2 = \int_0^t ((\boldsymbol{w}_n, \widetilde{\boldsymbol{w}}_n)(s), (\boldsymbol{e}_n, \boldsymbol{h}_n)(s))_{\boldsymbol{X}}\, \mathrm{d}s.$$

Passing to the limit $n \to \infty$, (B.1) and (B.2) yield the energy balance equality:

$$\frac{1}{2}\big\|(\boldsymbol{e}, \boldsymbol{h})(t)\big\|_{\boldsymbol{X}}^2 = \frac{1}{2}\big\|(\boldsymbol{e}_0, \boldsymbol{h}_0)\big\|_{\boldsymbol{X}}^2 + \int_0^t ((\boldsymbol{w}, \widetilde{\boldsymbol{w}})(s), (\boldsymbol{e}, \boldsymbol{h})(s))_{\boldsymbol{X}}\, \mathrm{d}s \quad \forall t \in [0,T].$$

□

Appendix C

Let $n \in \mathbb{N}$ and let $(\boldsymbol{E}_n, \boldsymbol{H}_n) \in \mathcal{C}([0,T], \boldsymbol{X})$ denote the unique solution of (3.7). Furthermore, let $t \in [0,T)$ and $h \in (0, T-t]$. By definition,

$$\begin{aligned}
(\boldsymbol{E}_n, \boldsymbol{H}_n)(t+h) &= \mathbb{T}_{t+h}(\boldsymbol{E}_0, \boldsymbol{H}_0) + \int_0^h \mathbb{T}_{t+h-s}\left((\epsilon^{-1}\mathbf{f}, \mu^{-1}\mathbf{g})(s) - \Phi_{\lambda_n}((\boldsymbol{E}_n, \boldsymbol{H}_n)(s))\right) \mathrm{d}s \\
&\quad + \int_h^{t+h} \mathbb{T}_{t+h-s}\left((\epsilon^{-1}\mathbf{f}, \mu^{-1}\mathbf{g})(s) - \Phi_{\lambda_n}((\boldsymbol{E}_n, \boldsymbol{H}_n)(s))\right) \mathrm{d}s, \\
&= \mathbb{T}_t\bigg(\mathbb{T}_h(\boldsymbol{E}_0, \boldsymbol{H}_0) + \int_0^h \mathbb{T}_{h-s}\left((\epsilon^{-1}\mathbf{f}, \mu^{-1}\mathbf{g})(s) - \Phi_{\lambda_n}((\boldsymbol{E}_n, \boldsymbol{H}_n)(s))\right) \mathrm{d}s\bigg) \\
&\quad + \int_0^t \mathbb{T}_{t-s}\left((\epsilon^{-1}\mathbf{f}, \mu^{-1}\mathbf{g})(s+h) - \Phi_{\lambda_n}((\boldsymbol{E}_n, \boldsymbol{H}_n)(s+h))\right) \mathrm{d}s.
\end{aligned}$$

Subtracting (3.7) from the above expression, it follows that

$$\begin{aligned}
&(\boldsymbol{E}_n, \boldsymbol{H}_n)(t+h) - (\boldsymbol{E}_n, \boldsymbol{H}_n)(t) \\
&= h\mathbb{T}_t\bigg(\frac{\mathbb{T}_h(\boldsymbol{E}_0, \boldsymbol{H}_0) - (\boldsymbol{E}_0, \boldsymbol{H}_0)}{h} + \frac{1}{h}\int_0^h \mathbb{T}_{h-s}((\epsilon^{-1}\mathbf{f}, \mu^{-1}\mathbf{g})(s) - \Phi_{\lambda_n}((\boldsymbol{E}_n, \boldsymbol{H}_n)(s)))\, \mathrm{d}s\bigg) \\
&\quad + \int_0^t \mathbb{T}_{t-s}\bigg((\epsilon^{-1}\mathbf{f}, \mu^{-1}\mathbf{g})(s+h) - (\epsilon^{-1}\mathbf{f}, \mu^{-1}\mathbf{g})(s) - \Phi_{\lambda_n}((\boldsymbol{E}_n, \boldsymbol{H}_n)(s+h)) + \Phi_{\lambda_n}((\boldsymbol{E}_n, \boldsymbol{H}_n)(s))\bigg) \mathrm{d}s.
\end{aligned}$$

Making use of the Lipschitz continuity of $\Phi_{\lambda_n} : \boldsymbol{X} \to \boldsymbol{X}$ and the regularity properties $(\mathbf{f}, \mathbf{g}) \in W^{1,\infty}((0,T), \boldsymbol{X})$ and $(\boldsymbol{E}_0, \boldsymbol{H}_0) \in D(\mathcal{A})$, we obtain that

$$\begin{aligned}
&\|(\boldsymbol{E}_n, \boldsymbol{H}_n)(t+h) - (\boldsymbol{E}_n, \boldsymbol{H}_n)(t)\|_{\boldsymbol{X}} \\
&\leq h\left(\left\|\frac{\mathbb{T}_h(\boldsymbol{E}_0, \boldsymbol{H}_0) - (\boldsymbol{E}_0, \boldsymbol{H}_0)}{h}\right\|_{\boldsymbol{X}} + \|(\epsilon^{-1}\mathbf{f}, \mu^{-1}\mathbf{g})\|_{\mathcal{C}([0,T],\boldsymbol{X})} + \|\Phi_{\lambda_n}(\boldsymbol{E}_n, \boldsymbol{H}_n)\|_{\mathcal{C}([0,T],\boldsymbol{X})}\right) \\
&\quad + \int_0^t h\left\|\frac{(\epsilon^{-1}\mathbf{f}, \mu^{-1}\mathbf{g})(s+h) - (\epsilon^{-1}\mathbf{f}, \mu^{-1}\mathbf{g})(s)}{h}\right\|_{\boldsymbol{X}} \lambda_n^{-1} \left\|(\boldsymbol{E}_n, \boldsymbol{H}_n)(s+h) - (\boldsymbol{E}_n, \boldsymbol{H}_n)(s)\right\|_{\boldsymbol{X}} \mathrm{d}s \\
&\leq hc + h\lambda_n^{-1}\left\|\frac{\mathrm{d}}{\mathrm{d}t}(\epsilon^{-1}\mathbf{f}, \mu^{-1}\mathbf{g})\right\|_{L^\infty((0,T),\boldsymbol{X})} \int_0^t \|(\boldsymbol{E}_n, \boldsymbol{H}_n)(s+h) - (\boldsymbol{E}_n, \boldsymbol{H}_n)(s)\|_{\boldsymbol{X}}\, \mathrm{d}s
\end{aligned}$$

with a constant $c > 0$, independent of t, h and n. In conclusion, the Gronwall lemma yields that $(\boldsymbol{E}_n, \boldsymbol{H}_n) \in \mathcal{C}^{0,1}([0,T], \boldsymbol{X})$.

Acknowledgements. This work was supported by the German Research Foundation Priority Programm DFG SPP 1962 “Non-smooth and Complementarity-based Distributed Parameter Systems: Simulation and Hierarchical Optimization”, Project YO 159/2-2.

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