Point-missing s-resolvable t-designs: Infinite series of 4-designs with constant index

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Abstract

The paper deals with t-designs that can be partitioned into s-designs, each missing a point of the underlying set, called point-missing s-resolvable t-designs, with emphasis on their applications in constructing t-designs. The problem considered may be viewed as a generalization of overlarge sets which are defined as a partition of all the $\binom{v+1}{k}$ k-sets chosen from a $(v+1)$-set X into $(v+1)$ mutually disjoint s-(v,k,δ) designs, each missing a different point of X. Among others, it is shown that the existence of a point-missing $(t-1)$-resolvable t-(v,k,λ) design leads to the existence of a t-$(v,k+1,\lambda')$ design. As a result, we derive various infinite series of 4-designs with constant index using overlarge sets of disjoint Steiner quadruple systems. These have parameters 4-$(3^n,5,5)$, 4-$(3^n+2,5,5)$ and 4-$(2^n+1,5,5)$, for $n \geq 2$, and were unknown until now. We also include a recursive construction of point-missing s-resolvable t-designs and its application.

Mathematics Subject Classification: 05B05

Keywords: point-missing s-resolvable t-design, overlarge set of s-designs.

1 Introduction

The paper is concerned with point-missing s-resolutions of t-designs and applications thereof. In general, a partition of a t-(v,k,λ) design $(X,\mathcal{B})$ into mutually disjoint s-(w,k,δ) designs, $w \leq v$, $s < t$, is termed an s-resolution. If $w = v$, then $(X,\mathcal{B})$ is called s-resolvable; in particular, if $(X,\mathcal{B})$ is the complete k-$(v,k,1)$ design, then an s-resolution of $(X,\mathcal{B})$ is called a *large set* of s-designs. If $w = v-1$, then $(X,\mathcal{B})$ is called point-missing s-resolvable. A point-missing s-resolution of the complete k-$(v,k,1)$ design is called an *overlarge set* of s-designs. Point-missing s-resolvability remains still sparsely investigated; however, several computational and theoretical works on the subject can be found in the literature [9, 13, 15, 16, 19, 20, 23]. Point-missing s-resolvablity is complementarily related to what we call pencil-like s-resolvablity for

t-designs, and vice versa. As far as we know the first example of infinite series of non-trivial point-missing s-resolvable t-designs for $t \geq 4$ can be found in a paper of Alltop in 1972 [2], in which the author constructed a series of 4-$(2^n+1, 2^{n-1}, (2^{n-1}-3)(2^{n-2}-1))$ designs for $n \geq 4$ as the union of $2^n + 1$ mutually disjoint 3-$(2^n, 2^{n-1}, 2^{n-2} - 1)$ designs. We prove theorems for constructring new t-designs from point-missing and pencil-like s-resolvable t-designs. By using these theorems for overlarge sets of disjoint Steiner quadruple systems with $v = 3^n - 1$ and $v = 3^n + 1$ points constructed by Teirlinck [23], including the already known case with $v = 2^n$, we derive various infinite series of 4-$(v+1, 5, 5)$ designs, which were unknown until now. It is worthy of note that no large sets of Steiner quadruple systems are constructed to date; however, large sets of Steiner 2-designs for $k = 4$ with $v = 13, 16$ points are known to exist [10, 12, 14]. We also show a recursive construction of point-missing s-resolvable t-designs and its application.

For the sake of clarity we include a few basic definitions. A t-design, denoted by t-(v, k, λ), is a pair $(X, \mathcal{B})$, where X is a v-set of *points* and $\mathcal{B}$ is a collection of k-subsets of X, called *blocks*, such that every t-subset of X is a subset of exactly λ blocks, and λ is called the *index* of the design. A t-design is called *simple* if no two blocks are identical, otherwise, it is called *non-simple*. A t-$(v, k, 1)$ design is called a *Steiner t-design*. For any point $x \in X$, let $\mathcal{B}_x = \{B \setminus \{x\} : x \in B \in \mathcal{B}\}$. Then $(X \setminus \{x\}, \mathcal{B}_x)$ is a $(t-1)$-$(v-1, k-1, \lambda)$ design, called a *derived design* of $(X, \mathcal{B})$. It can be shown by simple counting that a t-(v, k, λ) design is an s-(v, k, λ_s) design for $0 \leq s \leq t$, where $\lambda_s = \lambda\binom{v-s}{t-s}/\binom{k-s}{t-s}$. Since λ_s is an integer, necessary conditions for the parameters of a t-design are $\binom{k-s}{t-s} | \lambda\binom{v-s}{t-s}$ for $0 \leq s \leq t$. The smallest positive integer λ for which these necessary conditions are satisfied is denoted by $\lambda_{\min}(t, k, v)$ or simply $\lambda_{\min}$. If $\mathcal{B}$ is the set of all k-subsets of X, then $(X, \mathcal{B})$ is a t-$(v, k, \lambda_{\max})$ design, called the *complete* design, where $\lambda_{\max} = \binom{v-t}{k-t}$. If we take δ copies of the complete design, we obtain a t-$(v, k, \delta\binom{v-t}{k-t})$ design, which is refered to as a *trivial t-design*; otherwise, it is called a *non-trivial t-design*.

2 Point-missing s-resolvable t-designs

A t-(v, k, λ) design $(X, \mathcal{B})$ is said to be s-resolvable, for $0 < s < t$, if its block set $\mathcal{B}$ can be partitioned into $N \geq 2$ classes $\mathcal{B}_1, \ldots, \mathcal{B}_N$ such that each $(X, \mathcal{B}_i)$ is an s-(v, k, δ) design for $i = 1, \ldots, N$. Such a partition is called an *s-resolution* of $(X, \mathcal{B})$ and each $\mathcal{B}_i$ is called an s-resolution class or simply a resolution class, see e.g. [25, 26].

If the complete k-$(v, k, 1)$ design can be partitioned into N disjoint t-(v, k, λ) designs, where $N = \binom{v-t}{k-t}/\lambda$, then we say that there exists a *large set* of t-designs denoted by $LS[N](t, k, v)$ or by $LS_\lambda(t, k, v)$ to emphasize the value λ.

In the most general form, the concept of point-missing s-resolvability of a t-(v, k, λ) design can be defined as follows.

Definition 2.1 *Let $(X, \mathcal{B})$ be a t-(v, k, λ) design and let $1 \leq s \leq t-1$. $(X, \mathcal{B})$ is called point-missing s-resolvable, if the block set $\mathcal{B}$ can be partitioned into mutually disjoint s-$(v-1, k, \delta)$ designs, each missing a point of X.*

However, Definition 2.1 is equivalent to a definition that describes point-missing resolutions with more exact details. We now give an explanation.

Let $X = \{x_1, \dots, x_v\}$ and let $X_i = X \setminus \{x_i\}$, $i = 1, \dots, v$. Let m_i denote the number of s-$(v-1, k, \delta)$ designs $(X_i, \mathcal{B}_i)$ missing x_i in the resolution. First we show that any $x_i \in X$ is a missing point of an s-design $(X_i, \mathcal{B}_i)$. More precisely, let $Y \subseteq X$ be the subset of X such that there is no design $(X_i, \mathcal{B}_i)$ missing point x_i, when $x_i \in Y$. Assume that $Y \neq \emptyset$. Then the blocks of $\mathcal{B}$ can be written as follows.

$$\mathcal{B} = \bigcup_{x_h \in X \setminus Y} m_h \mathcal{B}_h, \text{ where } \quad m_h \mathcal{B}_h := \underbrace{\mathcal{B}_h \cup \cdots \cup \mathcal{B}_h}_{m_h \text{ times}}.$$

Consider two given points $x_i \in Y$ and $x_j \in X \setminus Y$. Since $x_i \in Y$, there is no s-design $(X_i, \mathcal{B}_i)$ missing x_i. Thus x_i appears in each design $(X_h, \mathcal{B}_h)$, where $x_h \in X \setminus Y$, therefore x_i appears in $\sum_{x_h \in X \setminus Y} m_h \delta_1$ times in the blocks of $\mathcal{B}$, where $\delta_1 = \delta \frac{\binom{v-2}{s-1}}{\binom{k-1}{s-1}}$. Whereas the point $x_j \in X \setminus Y$ appears in $\sum_{x_h \in X \setminus \{Y \cup \{x_j\}\}} m_h \delta_1$ times in the blocks of $\mathcal{B}$, which is a contradiction if $Y \neq \emptyset$. Further, we show that $m_1 = \cdots = m_v$. W.l.o.g., assume by contradiction that $m_1 \neq m_2$. Then the number of blocks containing x_1 (resp. x_2) is then $\sum_{x \in X \setminus \{x_1\}} m_x \delta_1 = m_2 \delta_1 + \sum_{i=3}^{v} m_i \delta_1$ (resp. $\sum_{x \in X \setminus \{x_2\}} m_x \delta_1 = m_1 \delta_1 + \sum_{i=3}^{v} m_i \delta_1$). Since $m_2 \delta_1 + \sum_{i=3}^{v} m_i \delta_1 = m_1 \delta_1 + \sum_{i=3}^{v} m_i \delta_1$, we have $m_2 \delta_1 = m_1 \delta_1$, or equivalently $m_2 = m_1$, contradicting the assumption. Thus we must have $m_1 = \cdots = m_v$.

The discussion above suggests an equivalent formulation of Definition 2.1 as follows.

Definition 2.2 *Let $(X, \mathcal{B})$ be a t-(v, k, λ) design and let $1 \leq s < t$ be an integer. $(X, \mathcal{B})$ is said to be point-missing s-resolvable, if there is an integer $m \geq 1$ such that the following hold.*

1. *$\mathcal{B} = \mathcal{B}_{x_1} \cup \cdots \cup \mathcal{B}_{x_v}$, where $X = \{x_1, \dots, x_v\}$,*
2. *$\mathcal{B}_x = \mathcal{B}_x^1 \cup \cdots \cup \mathcal{B}_x^m$, each $(X \setminus \{x\}, \mathcal{B}_x^j)$ is an s-$(v-1, k, \delta)$ design, $j = 1, \dots, m$, and m is called the multiplicity of the point x.*

If $m = 1$, $(X, \mathcal{B})$ is simply called point-missing s-resolvable. Moreover, if $m > 1$, then $(X \setminus \{x\}, \mathcal{B}_x)$ is an s-$(v-1, k, m\delta)$ design. Therefore, $(X, \mathcal{B})$ again is a union of v mutually disjoint s-$(v-1, k, m\delta)$ design, each missing a different point of X. Hence, in general, when we speak of point-missing s-resolvable t-designs we mean $m = 1$.

If the complete k-$(v, k, 1)$ design can be partitioned into v mutually disjoint s-$(v-1, k, \delta)$ designs (i.e. point-missing s-resolvable), then we have an *overlarge set* of s-$(v-1, k, \delta)$ designs.

Lemma 2.1 *Let $(X, \mathcal{B})$ be a point-missing s-resolvable t-(v, k, λ) design and assume that each point in the resolution has multiplicity m. Then*

$$\delta = \lambda \binom{v-s}{t-s} \Big/ \binom{k-s}{t-s} m(v-s).$$

In particular, if the complete t-$(v,t,1)$ design is point-missing $(t-1)$-resolvable, then the designs in the resolution are Steiner $(t-1)$-$(v-1,t,1)$ designs.

Proof. By assumption, we have

$$\mathcal{B} = \bigcup_{x \in X} \{\mathcal{B}_x^1 \cup \cdots \cup \mathcal{B}_x^m\},$$

where $(X \setminus \{x\}, \mathcal{B}_x^i)$ is an s-$(v-1,k,\delta)$ design. Let $S = \{x_1, \ldots, x_s\} \subseteq X$. Then S does not appear in any block of $\mathcal{B}_{x_j}^i$, for $j = 1, \ldots, s$ and $i = 1, \ldots, m$. Further, S appears in each $\mathcal{B}_{x_j}^i$ with $j \neq 1, \ldots, s$, exactly δ times. Thus S appears $m(v-s)\delta$ times in the blocks of $\mathcal{B}$. On the other hand, the number of blocks in $\mathcal{B}$ containing S is $\lambda_s = \frac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}}\lambda$. Therefore $\lambda_s = m(v-s)\delta$ and thus $\delta = \frac{\lambda_s}{m(v-s)}$, as desired. □

Recall that the complement of an s-resolvable t-design is again s-resolvable. However, it is not true with a point-missing s-resolvable t-design. Let $X := \{x_1, \ldots, x_v\}$ and let $X_i := X \setminus \{x_i\}$, $i = 1, \ldots, v$. To simplify the typing we write: if $Y \subseteq X$, then $\overline{Y} := X \setminus Y$, whereas if $Y \subseteq X_i$, then $\widetilde{Y} := X_i \setminus Y$. Let $(X, \mathcal{D})$ be a point-missing s-resolvable t-design with parameters t-(v,k,λ) and let $(X, \overline{\mathcal{D}})$ be its complement which has parameters t-$(v, v-k, \overline{\lambda})$, where $\overline{\lambda} = \lambda\binom{v-k}{t}/\binom{k}{t}$. Let $\mathcal{D} = \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_v$ be a partition of $\mathcal{D}$ into v point-missing s-resolution classes, where $(X_i, \mathcal{D}_i)$ is an s-$(v-1,k,\delta)$ design, for $i = 1, \ldots, v$. The complement of $(X_i, \mathcal{D}_i)$ (within X_i) is an s-$(v-1, v-1-k, \widetilde{\delta})$ design $(X_i, \widetilde{\mathcal{D}}_i)$ with $\widetilde{\delta} = \delta\binom{v-1-k}{s}/\binom{k}{s}$. So, we have $\overline{\mathcal{D}} = \overline{\mathcal{D}}_1 \cup \cdots \cup \overline{\mathcal{D}}_v = (\{x_1\} \cup \widetilde{\mathcal{D}}_1) \cup \cdots \cup (\{x_v\} \cup \widetilde{\mathcal{D}}_v)$, where $\{x_i\} \cup \widetilde{\mathcal{D}}_i = \{\{x_i\} \cup \widetilde{D} \mid \widetilde{D} \in \widetilde{\mathcal{D}}_i\}$. Thus, $\overline{\mathcal{D}}_i = (\{x_i\} \cup \widetilde{\mathcal{D}}_i)$ is not an s-design, but rather a "pencil". Hence, the decomposition of $(X, \overline{\mathcal{D}})$ suggests the following definition.

Definition 2.3 *Let $X = \{x_1, \ldots, x_v\}$ and denote $X_i := X \setminus \{x_i\}$, $i = 1, \ldots, v$. Let $(X, \mathcal{B})$ be a t-(v,k,λ) design. If for some $x_i \in X$ there exists an s-$(v-1,k-1,\delta)$ design $(X_i, \mathcal{B}_i)$ for $1 \leq s < t$, then we call $\{x_i\} \cup \mathcal{B}_i = \{\{x_i\} \cup \widetilde{B} \mid \widetilde{B} \in \widetilde{\mathcal{B}}_i\} \subseteq \widetilde{\mathcal{B}}$ an s-pencil of $(X, \mathcal{B})$. If $\mathcal{B} = (\{x_1\} \cup \mathcal{B}_1) \cup \cdots \cup (\{x_v\} \cup \mathcal{B}_v)$, where $(X_i, \mathcal{B}_i)$ is an s-$(v-1,k-1,\delta)$ design, then $(X, \mathcal{B})$ is said to be pencil-like s-resolvable.*

As observed above, the complement of a point-missing s-resolvable t-design is a pencil-like s-resolvable t-design. Conversely, it is straightfoward to check that the complement of a pencil-like s-resolvable t-design is a point-missing s-resolvable t-design. Hence the notion of point-missing s-resolvability and that of pencil-like s-resolvability are complementary equivalent. We record this fact in the following lemma.

Lemma 2.2 *A t-design is point-missing s-resolvable if and only if its complement is pencil-like s-resolvable.*

The next theorem shows a relation between certain classes of t-designs and point-missing $(t-1)$-resolvable t-designs, in terms of derived designs.

Theorem 2.3 *Let* $(X, \mathcal{B})$ *be a simple* t-(v, k, λ) *design with* $|B \cap B'| \leq k - 2$ *for any two distinct blocks* $B, B' \in \mathcal{B}$. *Then there exists a simple point-missing* $(t-1)$-*resolvable* t-$(v, k-1, (k-t)\lambda)$ *design* $(X, \mathcal{D})$. *In particular, if* $(X, \mathcal{B})$ *is a Steiner* t-$(v, t+1, 1)$ *design, then there exists an overlarge set of Steiner* $(t-1)$-$(v-1, t, 1)$ *designs.*

Proof. For a given point $x \in X$ consider the derived design $(X \setminus \{x\}, \mathcal{B}_x)$ at x with parameters $(t-1)$-$(v-1, k-1, \lambda)$. Here $\mathcal{B}_x = \{B \setminus \{x\} \mid x \in B,\ B \in \mathcal{B}\}$. Define $\mathcal{D} = \bigcup_{x \in X} \mathcal{B}_x$. We claim that $(X, \mathcal{D})$ is a t-$(v, k-1, (k-t)\lambda)$ design. Let $T = \{x_1, \ldots, x_t\} \subseteq X$. Then there are λ blocks of $\mathcal{B}$, say, $B_1, \ldots, B_\lambda$ containing T. Each B_i, $i = 1, \ldots, \lambda$, gives rise to a set $\mathbb{D}_i = \{D = B_i \setminus \{x\} \ \mid x \in B_i \setminus T\} \subseteq \mathcal{D}$ having $(k-t)$ blocks D containing T. Thus there are $(k-t)\lambda$ blocks $D \in \mathcal{D}$ containing T in total, as desired. The simplicity of $(X, \mathcal{D})$ is a consequence of the property: $|B \cap B'| \leq k-2$, $B, B' \in \mathcal{B}$, $B \neq B'$, which can be seen as follows. Let D, D' be two blocks of $\mathcal{D}$. If $D, D' \in \mathcal{B}_x$ for some $x \in X$, then $D \neq D'$, since $(X \setminus \{x\}, \mathcal{B}_x)$ is the derived design at x. If $D \in \mathcal{B}_x$ and $D' \in \mathcal{B}_y$ with $x \neq y$, then again $D \neq D'$. This is because if $D = D'$, then the two blocks $B = D \cup \{x\}$ and $B' = D' \cup \{y\}$ of $\mathcal{B}$ would have $|B \cap B'| = k-1$, a contradiction. In addition, if $(X, \mathcal{B})$ is a Steiner t-$(v, t+1, 1)$ design, then $(X, \mathcal{D})$ becomes the complete t-$(v, t, 1)$ design. In other words, the set of v distinct $(t-1)$-$(v-1, t, 1)$ derived designs of $(X, \mathcal{B})$ forms an overlarge set. □

Remark 2.1

1. The proof of Theorem 2.3 shows that the constructed t-$(v, k-1, (k-t)\lambda)$ design is not simple, if there are two blocks $B, B' \in \mathcal{B}$ with $|B \cap B'| = k-1$.

2. It should be stressed that the set of v distinct derived designs of a Steiner t-$(v, k, 1)$ design with $k > t+1$ in Theorem 2.3 will not form an overlarge set of $(t-1)$-$(v-1, k-1, 1)$ designs, but rather a point-missing $(t-1)$-resolution of a t-$(v, k-1, (k-t))$ design.

The following corollary is an immediate consequence of Theorem 2.3.

Corollary 2.4 *Assume that there exists a Steiner* t-$(v, k, 1)$ *design. Then there exists a point-missing* $(t-1)$-*resolvable* t-$(v, k-1, k-t)$ *design.*

The case $k = t+1$ of Corollary 2.4 is known as examples of overlarge sets of Steiner designs, see [23]. Thus, if there exists a Steiner t-$(v, t+1, 1)$ design, then there exists a point-missing $(t-1)$-resolvable t-$(v, t, 1)$ design, i.e. an overlarge set of Steiner $(t-1)$-$(v-1, t, 1)$ designs. Note that the converse of this statement is not true, i.e. if there exists an overlarge set of Steiner $(t-1)$-$(v-1, t, 1)$ designs, it is not necessarily true that a Steiner t-$(v, t+1, 1)$ design exists. For example, Östergård and Pottonen [17] have shown that a Steiner 4-$(17, 5, 1)$ design does not exist. Nevertheless, there exists an overlarge set of Steiner 3-$(16, 4, 1)$ designs, see [23]. And crucially, Teirlinck [23] has shown that there are overlarge sets of Steiner 3-$(v, 4, 1)$ designs for $v = 3^n - 1$, $n \geq 2$ and $v = 3^n + 1$, $n \geq 1$, despite the fact that only a finite number of Steiner 4-$(v, 5, 1)$ designs are hitherto known.

The general case $k \geq t+2$ is interesting, since Theorem 2.3 provides a point-missing $(t-1)$-resolvable t-$(v, k-1, k-t)$ design, which is not a complete design. Examples about this case can be seen, for instance, from Steiner 5-$(24, 8, 1)$ and 5-$(28, 7, 1)$ designs. Here we obtain point-missing 4-resolvable 5-$(24, 7, 3)$ and 5-$(28, 6, 2)$ designs, where designs in the resolution are Steiner 4-$(23, 7, 1)$ and 4-$(27, 6, 1)$ designs, respectively. Similarly, there are point-missing 3-resolvable 4-$(23, 6, 3)$ and 4-$(27, 5, 2)$ designs having Steiner 3-$(22, 6, 1)$ and 3-$(26, 5, 1)$ designs in the resolution, respectively.

As a further application of Theorem 2.3, we consider the infinite series of 4-$(q+1, 6, 10)$ designs with $q = 2^n$, $n \geq 5$ and $\gcd(n, 6) = 1$, [8], having the property that any two blocks of the designs intersect in at most 4 points. Thus we have the following result.

Corollary 2.5 *Let $q = 2^n$, $n \geq 5$ and $\gcd(n, 6) = 1$. Then there exists a point-missing 3-resolvable 4-$(q+1, 5, 20)$ design having a 3-$(q, 5, 10)$ design in the resolution.*

Corollary 2.5 shows an interesting example of 4-designs that are 3-resolvable, and point-missing 3-resolvable as well.

3 Constructions of t-designs from point-missing $(t-1)$-resolvable t-designs

Recall that Lemma 2.2 shows a natural connection between point-missing and pencil-like s-resolvability via the complement action. However, we observe that point-missing $(t-1)$-resolvable t-designs may be used to construct pencil-like $(t-1)$-resolvable t-designs which are not related to the complementary connection, as shown in the following theorem.

Theorem 3.1 *Let $(X, \mathcal{B})$ be a point-missing $(t-1)$-resolvable t-(v, k, λ) design with $(t-1)$-$(v-1, k, \delta)$ designs in the resolution. Then there is a pencil-like $(t-1)$-resolvable t-$(v, k+1, t\delta+\lambda)$ design $(X, \mathcal{B}^*)$. If $|B \cap B'| \leq k-2$ for any two distinct blocks $B, B' \in \mathcal{B}$, then $(X, \mathcal{B}^*)$ is simple. Further, if there are two blocks $B, B' \in \mathcal{B}$ with $|B \cap B'| = k-1$, then the simplicity of $(X, \mathcal{B}^*)$ depends on the structure of the resolution.*

Proof. Let $X = \{1, \ldots, v\}$. For $i \in X$ denote $(X \setminus \{i\}, \mathcal{B}_i)$ the $(t-1)$-$(v-1, k, \delta)$ design in the point-missing $(t-1)$-resolution. Define $\mathcal{B}_i^* = \{i\} \cup \mathcal{B}_i = \{\{i\} \cup B \mid B \in \mathcal{B}_i\}$, for $i = 1, \ldots, v$, and $\mathcal{B}^* = \bigcup_{i \in X} \mathcal{B}_i^*$. We claim that $(X, \mathcal{B}^*)$ is a pencil-like $(t-1)$-resolvable t-$(v, k+1, t\delta+\lambda)$ design. Let $T = \{i_1, \ldots, i_t\} \subseteq X$. Consider a resolution class $\mathcal{B}_j$ with $j \in T$. Since $(X \setminus \{j\}, \mathcal{B}_j)$ is a $(t-1)$-$(v-1, k, \delta)$ design, it follows that $\{i_1, \ldots, i_t\} \setminus \{j\}$ is contained in δ blocks of $\mathcal{B}_j$. Therefore $\{j\} \cup \{i_1, \ldots, i_t\} \setminus \{j\} = \{i_1, \ldots, i_t\}$ is contained in δ blocks of $\mathcal{B}_j^*$. Thus $\mathcal{B}_{i_1}^*, \ldots, \mathcal{B}_{i_t}^*$ together have $t\delta$ blocks containing T. Further, the $(v-t)$ resolution classes $\mathcal{B}_j$ with $j \notin T$ have λ blocks containing T. Therefore the $(v-t)$ classes $\mathcal{B}_j^*$ with $j \notin T$ together have λ blocks containing T. It follows that $(X, \mathcal{B}^*)$ is a t-$(v, k+1, t\delta+\lambda)$ design. Assume that

$|B \cap B'| \le k-2$ for any two distinct blocks $B, B' \in \mathcal{B}$. Let $B^*, B'^* \in \mathcal{B}^*$ be the two corresponding blocks of B and B'. If $B^*, B'^* \in \mathcal{B}_i^*$, then $B^* = \{i\} \cup B$ and $B'^* = \{i\} \cup B'$, so $B^* \neq B'^*$, since $B \neq B'$. The other case is that $B^* \in \mathcal{B}_i^*$ and $B'^* \in \mathcal{B}_j^*$ for $i \neq j$, thus $B^* = \{i\} \cup B$, $B'^* = \{j\} \cup B'$, where $B \in \mathcal{B}_i$ and $B' \in \mathcal{B}'_j$. Since $|B \cap B'| \le k-2$, we have $B^* \neq B'^*$. Thus $(X, \mathcal{B}^*)$ is simple. □

The next theorem may be viewed as the reverse of Theorem 3.1.

Theorem 3.2 *Let $(X, \mathcal{B})$ be a pencil-like $(t-1)$-resolvable t-(v,k,λ) design with $(t-1)$-$(v-1,k-1,\delta)$ designs in the resolution. Then there is a point-missing $(t-1)$-resolvable t-$(v,k-1,\lambda-t\delta)$ design $(X, \mathcal{B}^*)$. If $|B \cap B'| \le k-2$ for any two distinct blocks $B, B' \in \mathcal{B}$, then $(X, \mathcal{B}^*)$ is simple. Further, if there are two blocks $B, B' \in \mathcal{B}$ with $|B \cap B'| = k-1$, then the simplicity of $(X, \mathcal{B}^*)$ depends on the structure of the pencil-like $(t-1)$-resolution.*

Proof. Let $X = \{1, \ldots, v\}$. For $i \in X$ denote $(X \setminus \{i\}, \mathcal{B}_i)$ the $(t-1)$-$(v-1,k-1,\delta)$ design in the pencil-like $(t-1)$-resolution of $(X, \mathcal{B})$. We have $\mathcal{B} = (\{1\} \cup \mathcal{B}_1) \cup \cdots \cup (\{v\} \cup \mathcal{B}_v)$ Define $\mathcal{B}^* = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_v$. We claim that $(X, \mathcal{B}^*)$ is a t-$(v,k-1,\lambda-t\delta)$ design, which is point-missing $(t-1)$-resolvable. Let $T = \{i_1, \ldots, i_t\} \subseteq X$. Then T is contained in λ blocks of $(X, \mathcal{B})$, which are distributed in v classes of the pencil-like $(t-1)$-resolution. Note that T is contained in δ blocks of $(\{i_j\} \cup \mathcal{B}_{i_j})$, for $i_j \in T$, so T is contained in $t\delta$ blocks of $(\{i_1\} \cup \mathcal{B}_{i_1}) \cup \cdots \cup (\{i_t\} \cup \mathcal{B}_{i_t})$ (i.e., T is not contained in any block of $\mathcal{B}_{i_1} \cup \cdots \cup \mathcal{B}_{i_t}$). The remaining $(v-t)$ classes $\{(\{1\} \cup \mathcal{B}_1) \cup \cdots \cup (\{v\} \cup \mathcal{B}_v)\} \setminus \{(\{i_1\} \cup \mathcal{B}_{i_1}) \cup \cdots \cup (\{i_t\} \cup \mathcal{B}_{i_t})\}$ of $(X, \mathcal{B})$ will have $\lambda - t\delta$ blocks containing T. Moreover, if T is contained in a block $\{j\} \cup B \in (\{j\} \cup \mathcal{B}_j)$, $j \in \{1, \ldots, v\} \setminus T$, then T is contained in $B \in \mathcal{B}_j$. Hence, $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_v$ will have $\lambda - t\delta$ blocks containing T and $(X, \mathcal{B}^*)$ is point-missing $(t-1)$-resolvable. Assume that $|B \cap B'| \le k-2$ for any two distinct blocks $B, B' \in \mathcal{B}$. Obviously, the two corresponding blocks $B^*, B'^* \in \mathcal{B}^*$ are distinct. Thus $(X, \mathcal{B}^*)$ is simple. □

The simplicity of $(X, \mathcal{B}^*)$ in Theorem 3.1 in the case that there are two blocks $B, B' \in \mathcal{B}$ with $|B \cap B'| = k-1$ remains a main open question. In fact, examples for simple as well as non-simple $(X, \mathcal{B}^*)$ do exist in this case. We illustrate the situation with two explicit examples. First, consider the unique Steiner 3-$(8,4,1)$ design $(X, \mathcal{B})$. By applying Lemma 2.2 we have

$$\begin{array}{rcl}
\mathcal{B}_0 & = & 123\ \ 345\ \ 256\ \ 136\ \ 467\ \ 157\ \ 237 \\
\mathcal{B}_1 & = & 024\ \ 235\ \ 456\ \ 036\ \ 057\ \ 267\ \ 347 \\
\mathcal{B}_2 & = & 014\ \ 135\ \ 346\ \ 056\ \ 167\ \ 037\ \ 457 \\
\mathcal{B}_3 & = & 125\ \ 246\ \ 045\ \ 016\ \ 567\ \ 027\ \ 147 \\
\mathcal{B}_4 & = & 012\ \ 236\ \ 035\ \ 156\ \ 067\ \ 137\ \ 257 \\
\mathcal{B}_5 & = & 123\ \ 034\ \ 146\ \ 026\ \ 367\ \ 017\ \ 247 \\
\mathcal{B}_6 & = & 234\ \ 145\ \ 025\ \ 013\ \ 357\ \ 047\ \ 127 \\
\mathcal{B}_7 & = & 356\ \ 046\ \ 015\ \ 126\ \ 023\ \ 134\ \ 245
\end{array}$$

Thus the block set $\mathcal{D} = \bigcup_{x \in X} \mathcal{B}_x$ is the union of derived designs of $(X, \mathcal{B})$ at all points of $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$. Here $\mathcal{B}_0, \ldots, \mathcal{B}_7$ form an overlarge set of Steiner 2-$(7, 3, 1)$ designs. It is easy to check that the resulting 3-$(8, 4, 4)$ design $(X, \mathcal{B}^*)$ is not simple, more precisely each block is repeated 4 times. The second example is chosen from the set of 11 non-isomorphic of overlarge sets for 2-$(7, 3, 1)$ designs [18]. The following representation is taken from[15].

$$\begin{array}{rcl}
\mathcal{B}'_0 &=& 123\ \ 145\ \ 167\ \ 247\ \ 256\ \ 346\ \ 357 \\
\mathcal{B}'_1 &=& 026\ \ 035\ \ 047\ \ 234\ \ 257\ \ 367\ \ 456 \\
\mathcal{B}'_2 &=& 015\ \ 037\ \ 046\ \ 136\ \ 147\ \ 345\ \ 567 \\
\mathcal{B}'_3 &=& 014\ \ 025\ \ 067\ \ 127\ \ 156\ \ 246\ \ 457 \\
\mathcal{B}'_4 &=& 016\ \ 023\ \ 057\ \ 125\ \ 137\ \ 267\ \ 356 \\
\mathcal{B}'_5 &=& 017\ \ 024\ \ 036\ \ 126\ \ 134\ \ 237\ \ 467 \\
\mathcal{B}'_6 &=& 013\ \ 027\ \ 045\ \ 124\ \ 157\ \ 235\ \ 347 \\
\mathcal{B}'_7 &=& 012\ \ 034\ \ 056\ \ 135\ \ 146\ \ 236\ \ 245
\end{array}$$

It is straightforward to check that $(X, \mathcal{B}'^*)$ forms a simple 3-$(8, 4, 4)$ design.

The examples indicate an involved problem of deciding the simplicity of $(X, \mathcal{B}^*)$, when $(X, \mathcal{B})$ has two blocks B and B' with $|B \cap B'| = k - 1$. The most interesting case for this situation, as mentioned in Theorem 2.3, is overlarge sets of disjoint Steiner $(t-1)$-$(v, t, 1)$ designs, i.e. the complete t-$(v+1, t, 1)$ design is point-missing $(t-1)$-resolvable having Steiner $(t-1)$-$(v, t, 1)$ designs in the resolution classes. Teirlinck [23] has shown that overlarge sets for Steiner 3-$(3^n - 1, 4, 1)$ and 3-$(3^n + 1, 4, 1)$ designs for $n \geq 2$ exist, including the known overlarge sets of Steiner 3-$(2^n, 4, 1)$ designs. By using these results we obtain the following infinite series of 4-designs with constant index as a corollary of Theorem 3.1.

Corollary 3.3 *There exist infinite series of pencil-like* 3*-resolvable* 4*-designs with the following parameters:*

1. 4-$(2^n + 1, 5, 5)$ *for* $n \geq 2$,
2. 4-$(3^n, 5, 5)$ *for* $n \geq 2$,
3. 4-$(3^n + 2, 5, 5)$ *for* $n \geq 2$.

Remark 3.1 It should be remarked that for all the designs in Corollary 3.3 we have $\lambda_{\min} = 1$ or 5. More precisely,

$$\lambda_{\min} = 5 \begin{cases} \text{for } v = 2^n + 1, & \text{and } n \equiv 3 \pmod 4, \\ \text{for } v = 3^n, & \text{and } n \equiv 2 \pmod 4, \\ \text{for } v = 3^n + 2, & \text{and } n \equiv 3 \pmod 4. \end{cases}$$

Note that Alltop [1] has constructed infinite series of simple 4-$(2^n + 1, 5, 5)$ designs for n odd and $n \geq 5$; thus the first series extends the point number to all possible values of n.

It is very likely that many non-isomorphic series of 4-designs with parameters given in Corollary 3.3 will exist, which are simple as well as non-simple, due to the fact that the number of non-isomorphic overlarge sets of 3-$(v, 4, 1)$ will strongly increase as v is getting large. In particular, it is important to decide whether the 4-designs in Corollary 3.3 are simple or not. As an observation we take a close look at the first design in each of the 4-$(3^n, 5, 5)$ and 4-$(3^n + 2, 5, 5)$ series. These are 4-$(9, 5, 5)$ and 4-$(11, 5, 5)$ designs, corresponding to $n = 2$. Note that each 4-$(9, 5, 5)$ design is simple, since its complement is the complete 4-$(9, 4, 1)$ design (otherwise, we would have a non-simple 4-$(9, 4, 1)$ design, which is impossible). In fact, this can also be verified directly by checking the two non-isomorphic overlarge sets of 3-$(8, 4, 1)$ designs given in [9], yielding 4-$(9, 5, 5)$ designs. Note also that 4-$(9, 5, 5)$ is the parameters of the second design in the 4-$(2^n + 1, 5, 5)$ series. The case of 4-$(11, 5, 5)$ designs is quite different. We have inspected the complete list of 21 non-isomorphic overlage sets of 3-$(10, 4, 1)$ designs as shown in [20] and found that they all yield non-simple 4-$(11, 5, 5)$ designs.

For the ease of the reader, we include a table of known infinite series of t-designs with constant index for $t \geq 4$.

Table 1: Known infinite series of t-designs with constant index for $t \geq 4$

No.	t-(v, k, λ)	Conditions	(Non-)Simplicity	References
1	4-$(2^n + 1, 5, 5)$	$n \geq 5$ odd	simple	[1]
2	4-$(4^n + 1, 5, 2)$	$n \geq 2$	non-simple	[3]
3	4-$(2^n + 1, 5, 5)$	$n \geq 4$	?	Cor.3.3
4	4-$(3^n, 5, 5)$	$n \geq 3$	?	Cor.3.3
5	4-$(3^n + 2, 5, 5)$	$n \geq 3$	?	Cor.3.3
6	4-$(2^n + 1, 5, \lambda)$	$\lambda \in \{20, 25\}$, $\gcd(n, 6) = 1$	simple	Cor.2.5, [8]
7	4-$(60u + 4, 5, 60)$	$\gcd(u, 60) = 1$ or 2	simple	[22]
8	4-$(2^n + 1, 6, 10)$	$n \geq 5$ odd	simple	[5]
9	4-$(2^n + 1, 6, \lambda)$	$\lambda \in \{60, 70, 90, 100, 150, 160\}$, $\gcd(n, 6) = 1$	simple	[4]
10	4-$(2^n + 1, 8, 35)$	$\gcd(n, 6) = 1$	simple	[4]
11	4-$(2^n + 1, 9, \lambda)$	$\lambda \in \{84, 63, 147\}$, $\gcd(n, 6) = 1$	simple	[6, 4]
12	5-$(2^n + 2, 6, 15)$	$n \geq 3$	non-simple	[11]
13	5-$(2^n, 6, 3)$	$n \geq 3$	non-simple	[7]
14	7-$(2^n, 8, 45)$	$n \geq 6$	non-simple	[7]
15	t-$(v, t + 1, (t + 1)!^{2t+1})$	$v \equiv t \pmod{(t + 1)!^{2t+1}}$, $v \geq t + 1$	simple	[21]

Theorem 3.4 *There exists a pencil-like 3-resolvable* 4-$(2^n + 1, 7, \frac{70}{3}(2^n - 5))$ *design for* $n \geq 5$ *and* $\gcd(n, 6) = 1$.

Proof. Each 4-$(2^n+1, 6, 10)$ design $(X, \mathcal{B})$ with $n \geq 5$ and $\gcd(n, 6) = 1$ in [8] has the property that $|B \cap B'| \leq 4$ for any two distinct blocks $B, B' \in \mathcal{B}$. Its complement is a 4-$(2^n+1, 2^n-5, \frac{2}{3}\binom{2^n-5}{4})$ design $(X, \bar{\mathcal{B}})$ having block intersections at most (2^n-3). By Theorem 2.3 there is a point-missing 3-resolvable 4-$(2^n+1, 2^n-6, (2^n-9)\frac{2}{3}\binom{2^n-5}{4})$ design $(X, \bar{\mathcal{D}})$. Again, the complement of $(X, \bar{\mathcal{D}})$ is pencil-like 3-resolvable 4-$(2^n+1, 7, \frac{70}{3}(2^n-5))$ design, as desired. □

By applying Theorem 3.2 to the point-missing 3-resolvable 4-$(2^n+1, 2^{n-1}, (2^{n-1}-3)(2^{n-2}-1))$ design $(X, \mathcal{B})$ of Alltop [2], we obtain an interesting result. Namely, we prove that there is a point-missing 3-resolvable design $(X, \mathcal{B}^*)$ with the same parameters as $(X, \mathcal{B})$ and disjoint from $(X, \mathcal{B})$ (recall that any two distinct blocks $B, B' \in \mathcal{B}$ have $|B \cap B'| \leq 2^{n-1}-2$). Let $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_v$ be a partition of $\mathcal{B}$ into point-missing 3-resolution classes, i.e. each $(X_i, \mathcal{B}_i)$ is a 3-$(2^n, 2^{n-1}, 2^{n-2}-1)$ design with $X_i = X \setminus \{i\}$. Consider $(X, \bar{\mathcal{B}})$ as the complement of $(X, \mathcal{B})$. So, $(X, \bar{\mathcal{B}})$ has parameters 4-$(2^n+1, 2^{n-1}+1, (2^{n-1}+1)(2^{n-2}-1))$ and is pencil-like 3-resolvable. Here, $\bar{\mathcal{B}} = (\{1\} \cup \tilde{\mathcal{B}}_1) \cup \cdots \cup (\{v\} \cup \tilde{\mathcal{B}}_v)$, where $\tilde{\mathcal{B}}_j$ is the complement of $\mathcal{B}_j$ in X_j, and $(X_j, \tilde{\mathcal{B}}_j)$ is a 3-$(2^n, 2^{n-1}, 2^{n-2}-1)$ design, for $j = 1, \ldots, v$. The proof of Theorem 3.2 shows that $(X, \tilde{\mathcal{B}}^*)$ with $\tilde{\mathcal{B}}^* = \tilde{\mathcal{B}}_1 \cup \cdots \cup \tilde{\mathcal{B}}_v$, is point-missing 3-resolvable with $(X_j, \tilde{\mathcal{B}}_j)$ as the design in the resolution. Clearly, $(X, \mathcal{B})$ and $(X, \tilde{\mathcal{B}}^*)$ are disjoint and they have the same parameters. Further, the 4-design $(X, \mathcal{B} \cup \tilde{\mathcal{B}}^*)$ can be extended to a 5-design. Thus we have the following theorem.

Theorem 3.5 *Let $n \geq 4$. Then*

1. *there exists a simple point-missing* 3*-resovable* 4-$(2^n+1, 2^{n-1}, 2(2^{n-1}-3)(2^{n-2}-1))$ *design,*
2. *there exists a simple* 5-$(2^n+2, 2^{n-1}+1, 2(2^{n-1}-3)(2^{n-2}-1))$ *design.*

4 A construction of point-missing s-resolvable t-designs

In this section we show that the recursive construction of t-designs in [24] can be extended to a construction of point-missing s-resolvable t-designs. More precisely, we prove the following theorem.

Theorem 4.1 *Assume that there exists a point-missing s-resolvable t-(v, k, λ) design having s-$(v-1, k, \delta)$ designs in its resolution. If $v\lambda_0(\lambda_0 - \lambda_1) < \binom{v}{k}$, then there exists a point-missing s-resolvable t-$(v+1, k, (v+1-t)\lambda)$ design having s-$(v, k, (v-s)\delta)$ designs in its resolution.*

Proof. Assume that $(Y, \mathcal{D})$ is a point-missing s-resolvable t-(v, k, λ) design. Let $X = \{1, \ldots, v+1\}$ and denote $X_j = X \setminus \{j\}$ for $j = 1, \ldots, v+1$. Let $(X_j, \mathcal{B}^{(j)})$ be a copy of $(Y, \mathcal{D})$ defined on X_j. If $v\lambda_0(\lambda_0 - \lambda_1) < \binom{v}{k}$, then by Theorem A in [24] there are $(v+1)$ mutually disjoint $\mathcal{B}^{(1)}, \ldots, \mathcal{B}^{(v+1)}$ and they form a t-$(v+1, k, (v+1-t)\lambda)$ design $(X, \mathcal{B})$, where

$$\mathcal{B} = \bigcup_{j=1}^{v+1} \mathcal{B}^{(j)}.$$

We prove that $(X, \mathcal{B})$ is point-missing s-resolvable. Denote the partition of $(X_j, \mathcal{B}^{(j)})$ into point-missing s-resolution classes by

$$\mathcal{B}^{(j)} = \overbrace{\mathcal{C}_1^{(j)} \cup \cdots \cup \mathcal{C}_{j-1}^{(j)} \cup \mathcal{C}_{j+1}^{(j)} \cup \cdots \cup \mathcal{C}_{v+1}^{(j)}}^{v},$$

with $(X_{i,j}, \mathcal{C}_i^{(j)})$ as an s-$(v-1, k, \delta)$ design, where $X_{i,j} = X_j \setminus \{i\}$ and $i \in X_j$. For each point $j \in X$ define

$$\mathcal{C}_j = \overbrace{\mathcal{C}_j^{(1)} \cup \mathcal{C}_j^{(2)} \cup \cdots \cup \mathcal{C}_j^{(j-1)} \cup \mathcal{C}_j^{(j+1)} \cup \cdots \cup \mathcal{C}_j^{(v+1)}}^{v}.$$

We claim that $(X_j, \mathcal{C}_j)$ is an s-$(v, k, (v-s)\delta)$ design. Let $S = \{j_1, \ldots, j_s\} \subseteq X_j$. Then S will not appear in the blocks of $\mathcal{C}_j^{(j_1)}, \mathcal{C}_j^{(j_2)}, \ldots, \mathcal{C}_j^{(j_s)}$. Hence S appears in $(v-s)$ block sets $\mathcal{C}_j^{(i)}$, for $i \neq j_1, \ldots, j_s$. In other words, S is contained in the blocks of $\mathcal{C}_j$ exactly $(v-s)\delta$ times, which proves the claim. Further, since

$$\mathcal{B} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_{v+1},$$

$(X, \mathcal{B})$ is point-missing s-resolvable with $\mathcal{C}_1, \ldots, \mathcal{C}_{v+1}$ as resolution classes. Note that the value of δ can be computed in terms of $t,\ v,\ k,\ \lambda$ by using Lemma 2.1. □

As an application of Theorem 4.1 consider the infinite series of 4-designs $(X, \mathcal{D})$ constructed by Alltop in [2]. $(X, \mathcal{D})$ has parameters 4-$(2^n+1, 2^{n-1}, (2^{n-1}-3)(2^{n-2}-1))$, $n \geq 4$, and is point-missing 3-resolvable with 3-$(2^n, 2^{n-1}, 2^{n-2}-1)$ designs in its resolution. For $n \geq 5$ the condition $v\lambda_0(\lambda_0 - \lambda_1) < \binom{v}{k}$ is satisfied, therefore Theorem 4.1 gives the following corollary.

Corollary 4.2 *For $n \geq 5$, there exists an infinite series of simple point-missing 3-resolvable* 4-$(2^n+2, 2^{n-1}, (2^n-2)(2^{n-1}-3)(2^{n-2}-1))$ *designs. The parameters of the 3-designs in the resolution are* 3-$(2^n+1, 2^{n-1}, (2^n-2)(2^{n-2}-1))$.

5 Conclusion

The paper deals with point-missing s-resolvable t-designs with emphasis on their use in constructing t-designs. Among others, we show the existence of infinite series of 4-$(v, 5, 5)$ designs with $v = 2^n+1,\ 3^n,\ 3^n+2$ for $n \geq 2$. It remains an open question about the simplicity of the designs in these series. We also present a recursive construction of point-missing s-resolvable t-designs including an application.

References

[1] Alltop W.O.: An infinite class of 4-designs, J. Combin. Theory A **6** 320–322 (1969).

[2] Alltop W.O.: An infinite class of 5-designs, J. Combin. Theory A **12** (1972) 390–395 (1972).

[3] Baartmans A.H., Bluskov I. and Tonchev V.D.: The Preparata codes and a class of 4-designs, J. Combin. Des. **2** 167–170 (1994).

[4] Bierbrauer J.: Some friends of Alltop's designs 4-$(2^f + 1, 5, 5)$, J. Combin. Math. Combin. Comput. **36** 43–53 (2001).

[5] Bierbrauer J.: A new family of 4-designs, Graphs Combin. **11**, 209–211 (1995).

[6] Bierbrauer J.: A family of 4-designs with block size 9, Discret. Math. **138**, 113–117 (1995).

[7] Bierbrauer J.: An infinite family of 7-designs, Discret. Math. **240**, 1–11 (2001).

[8] Bierbrauer J. and van Trung Tran: Shadow and shade of designs 4-$(2^f + 1, 6, 10)$ (unpublished),
https://www.uni-due.de/∼hx0026/ShadowAndShade.pdf

[9] Breach D.R. and Street A.P.: Partitioning sets of quadruples into designs II, J. Combin. Math. Combin. Comput. **3**, 41–48 (1988).

[10] Chouinard L.G.: Partitions of the 4-subsets of a 13-set into disjoint projective planes, Discret. Math. **45**, 396–407 (1983).

[11] Jungnickel D. and Vanstone S.A.: Hyperfactorizations of graphs and 5-designs, J. Univ. Kuwait(Sci.) **14**, 213–223 (1987).

[12] Kalotoğlu E., Magliveras S.S.: On large sets of projective planes of order 3 and 4, Discret. Math. **313**, 2247–2252 (2013).

[13] Liebler R.A., Magliveras S.S., and Tsaranov S.V.: Block transitive resolutions of t-designs and room rectangles, J. Statist. Plann. and Inference **58**, 119–133 (1997).

[14] Mathon R.: Searching for spreads and packings, in Geometry, Combinatorial Designs and Related Structures, J. W. P. Hirschfeld, S. S. Magliveras, and M. J. de Resmini, eds., Cambridge Univ. Press, Cambridge 161–176 (1997).

[15] Mathon R., Street A.P.: Partitions of sets of designs on seven, eight and nine points, J. Statist. Plann. and Inference **58**, 135–150 (1997).

[16] Mathon R., Street A.P.: Overlarge sets of 2-(11,5,2) designs and related configurations, Discret. Math. **255**, 275–286 (2002).

[17] Östergård P.R.J. and Pottonen O.: There exists no Steiner system $S(4, 5, 17)$, J. Combin. Theory A **115**, 1570–1573 (2008).

[18] Sharry M.J., Street A.P.: Partitioning sets of triples into designs, Ars Combin. **26B**, 51–66 (1988).

[19] Sharry M.J., Street A.P.: Partitions of sets of blocks into designs, Australas. J. Combin. **3**, 111–140 (1991).

[20] Sharry M.J., Street A.P.: Partitions of sets of quadruples into designs III, Discret. Math. **92**, 341–359 (1991).

[21] Teirlinck L.: Non-trivial t-designs without repeated blocks exist for all t, Discrete Math. **65**, 301–311 (1987).

[22] Teirlinck L.: Locally trivial t-designs and t-designs without repeated blocks, Discrete Math. **77**, 345–356 (1989).

[23] Teirlinck L.: Overlarge sets of disjoint Steiner quadruple systems, J. Combin. Designs **7**, 311–315 (1999).

[24] van Trung Tran: On the existence of an infinite family of simple 5-designs, Math. Z. **187**, 285–287 (1984).

[25] van Trung Tran: A recursive construction for simple t-designs using resolutions, Des. Codes Cryptogr. **86**, 1185–1200 (2018).

[26] van Trung Tran: Recursive construction for s-resolvable t-designs, Des. Codes Cryptogr. **87**, 2835–2845 (2019).