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Friedrichs/Poincaré Type Constants for Gradient, Rotation, and Divergence: Theory and Numerical Experiments

by

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FRIEDRICHS/POINCARÉ TYPE CONSTANTS FOR GRADIENT, ROTATION, AND DIVERGENCE: THEORY AND NUMERICAL EXPERIMENTS

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Abstract. We give some theoretical as well as computational results on Laplace and Maxwell constants, i.e., on the smallest constants $c_n > 0$ arising in estimates of the form

$$|u|_{\mathsf{L}^2(\Omega)} \leq c_0 |\operatorname{grad} u|_{\mathsf{L}^2(\Omega)}, \qquad |E|_{\mathsf{L}^2(\Omega)} \leq c_1 |\operatorname{rot} E|_{\mathsf{L}^2(\Omega)}, \qquad |H|_{\mathsf{L}^2(\Omega)} \leq c_2 |\operatorname{div} H|_{\mathsf{L}^2(\Omega)}.$$

We consider mixed boundary conditions and bounded Lipschitz domains of arbitrary topology.

Contents

1. Introduction

We present some theoretical results as well as some computations on Laplace and Maxwell constants for bounded Lipschitz domains Ω with mixed boundary conditions defined on boundary parts Γ_τ and Γ_ν of the boundary Γ. While a lot of our theoretical findings hold for domains Ω in arbitrary dimensions, we restrict numerical experiments to the 2D and 3D cases. Moreover, we verify various theoretical results established in the last few years in [37, 38, 39, 41]. There is a recent interest in these eigenvalues, see, e.g., [17, 18, 29] and related contributions [19, 20, 26, 27, 28, 30, 45], but little results for mixed boundary

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conditions are known in the literature, except for, e.g., [39, 41]. In 3D these constants are the best possible real numbers c_{0,Γ_τ}, c_{1,Γ_τ}, $c_{2,\Gamma_\tau} > 0$ in the estimates

$$\begin{aligned}
&\forall\, u \in D(\operatorname{grad}_{\Gamma_\tau}) \cap R(\operatorname{div}_{\Gamma_\nu}) && |u|_{\mathsf{L}^2(\Omega)} \leq c_{0,\Gamma_\tau} |\operatorname{grad} u|_{\mathsf{L}^2(\Omega)},\\
&\forall\, E \in D(\operatorname{rot}_{\Gamma_\tau}) \cap R(\operatorname{rot}_{\Gamma_\nu}) && |E|_{\mathsf{L}^2(\Omega)} \leq c_{1,\Gamma_\tau} |\operatorname{rot} E|_{\mathsf{L}^2(\Omega)},\\
&\forall\, H \in D(\operatorname{div}_{\Gamma_\tau}) \cap R(\operatorname{grad}_{\Gamma_\nu}) && |H|_{\mathsf{L}^2(\Omega)} \leq c_{2,\Gamma_\tau} |\operatorname{div} H|_{\mathsf{L}^2(\Omega)},
\end{aligned}$$

which are often called Friedrichs/Poincaré type constants. We also point out the strong connection to the well known de Rham complex with mixed boundary conditions.

It turns out that in <u>3D</u>, cf. Theorem 2.20, always

$$c_{0,\Gamma} \leq c_{0,\Gamma_\tau} = c_{2,\Gamma_\nu}, \qquad c_{1,\Gamma_\tau} = c_{1,\Gamma_\nu}, \qquad c_{0,\Gamma} \leq \min\big\{c_{0,\emptyset}, \frac{\operatorname{diam}(\Omega)}{\pi}\big\}$$

hold and that in convex domains we even have

$$c_{1,\Gamma} = c_{1,\emptyset} \leq c_{0,\emptyset} \leq \frac{\operatorname{diam}(\Omega)}{\pi}.$$

Here, $c_{0,\Gamma}$ and $c_{0,\emptyset}$ are the classical Friedrichs and Poincaré constants, respectively. The constants $c_{1,\Gamma}$ and $c_{1,\emptyset}$ are often called tangential (electric) and normal (magnetic) Maxwell constants. All these constants relate to minimal positive eigenvalues of certain Laplace and Maxwell operators. More precisely,

$$\lambda_{0,\Gamma_\tau} = \frac{1}{c_{0,\Gamma_\tau}}, \qquad \lambda_{1,\Gamma_\tau} = \frac{1}{c_{1,\Gamma_\tau}}, \qquad \lambda_{2,\Gamma_\tau} = \frac{1}{c_{2,\Gamma_\tau}}$$

are the smallest positive eigenvalues of the first order matrix operators

$$\begin{bmatrix} 0 & -\operatorname{div}_{\Gamma_\nu} \\ \operatorname{grad}_{\Gamma_\tau} & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & \operatorname{rot}_{\Gamma_\nu} \\ \operatorname{rot}_{\Gamma_\tau} & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & -\operatorname{grad}_{\Gamma_\nu} \\ \operatorname{div}_{\Gamma_\tau} & 0 \end{bmatrix},$$

respectively, and

$$\lambda^2_{0,\Gamma_\tau}, \qquad \lambda^2_{1,\Gamma_\tau}, \qquad \lambda^2_{2,\Gamma_\tau} \tag{1}$$

are the smallest positive eigenvalues of the second order operators

$$-\operatorname{div}_{\Gamma_\nu} \operatorname{grad}_{\Gamma_\tau}, \qquad \operatorname{rot}_{\Gamma_\nu} \operatorname{rot}_{\Gamma_\tau}, \qquad -\operatorname{grad}_{\Gamma_\nu} \operatorname{div}_{\Gamma_\tau}, \tag{2}$$

respectively. In <u>2D</u>, cf. Corollary 2.23, we will see that

$$c_{0,\Gamma} \leq c_{0,\Gamma_\tau} = c_{1,\Gamma_\nu} = c_{2,\Gamma_\nu}, \qquad c_{0,\Gamma} \leq \min\big\{c_{0,\emptyset}, \frac{\operatorname{diam}(\Omega)}{\pi}\big\}.$$

Generally, in <u>ND</u>, cf. Theorem 2.25, we have

$$c_{0,\Gamma} \leq c_{0,\Gamma_\tau} = c_{N-1,\Gamma_\nu}, \qquad c_{q,\Gamma_\tau} = c_{N-q-1,\Gamma_\nu}, \qquad c_{0,\Gamma} \leq \min\big\{c_{0,\emptyset}, \frac{\operatorname{diam}(\Omega)}{\pi}\big\}$$

and in convex domains

$$c_{q,\Gamma} = c_{N-q-1,\emptyset} \leq c_{0,\emptyset} \leq \frac{\operatorname{diam}(\Omega)}{\pi}.$$

Here, $q = 0, \dots, N-1$ and the differential operators grad, rot, and div are simply replaced by the exterior derivative d_q acting on the different grades q of the respective differential forms. So far, all findings are related to the ND de Rham complex. We will present more examples and results for the 3D elasticity complex as well as for the 3D biharmonic complex.

In a series of numerical tests we discretize the operators (2) by the finite element method and compute upper bounds for the eigenvalues (1) from generalized eigenvalue systems

$$Ku = \lambda^2 Mu$$

with discretized stiffness and mass matrices K and M, respectively. There are also recent interests in guaranteed lower bounds, cf. [22, 21, 56]. In a search for the smallest positive eigenvalue λ^2 we exploit a projection to the range of K for smaller size problems or the nested iteration technique for large size problems. The latter theoretical results are confirmed by computations in 2D for the unit square and the L-shape domain as well as in 3D for the unit cube and the Fichera corner domain. Moreover, we performed some monotonicity tests which are just partially guaranteed by our theoretical findings. To our surprise we found (numerically) much stronger inequalities in (25), see also the related Figure 7.

2. Theoretical Results

We shall summarise some basic results from functional analysis and apply those to the classical operators of vector analysis.

2.1. **Functional Analysis ToolBox.** We start with collecting and citing some results from [42, 40, 43, 39, 41] about the so-called functional analysis toolbox (fa-toolbox).

2.1.1. *Preliminaries.* Let $\mathrm{A} : D(\mathrm{A}) \subset \mathsf{H}_0 \to \mathsf{H}_1$ be a densely defined and closed linear operator with domain of definition $D(\mathrm{A})$ on two Hilbert spaces H_0 and H_1. Then the adjoint $\mathrm{A}^* : D(\mathrm{A}^*) \subset \mathsf{H}_1 \to \mathsf{H}_0$ is well defined and characterised by

$$\forall\, x \in D(\mathrm{A}) \quad \forall\, y \in D(\mathrm{A}^*) \qquad \langle \mathrm{A}\, x, y\rangle_{\mathsf{H}_1} = \langle x, \mathrm{A}^*\, y\rangle_{\mathsf{H}_0}.$$

A and A^* are both densely defined and closed, but typically unbounded. Often $(\mathrm{A}, \mathrm{A}^*)$ is called a dual pair as $(\mathrm{A}^*)^* = \overline{\mathrm{A}} = \mathrm{A}$. The projection theorem shows

$$\mathsf{H}_0 = N(\mathrm{A}) \oplus_{\mathsf{H}_0} \overline{R(\mathrm{A}^*)}, \quad \mathsf{H}_1 = N(\mathrm{A}^*) \oplus_{\mathsf{H}_1} \overline{R(\mathrm{A})}, \tag{3}$$

often called Helmholtz/Hodge/Weyl decompositions, where we introduce the notation N for the kernel (or null space) and R for the range of a linear operator. These orthogonal decompositions reduce the operators A and A^*, leading to the injective operators $\mathcal{A} := \mathrm{A}\,|_{\overline{R(\mathrm{A}^*)}}$ and $\mathcal{A}^* := \mathrm{A}^*\,|_{\overline{R(\mathrm{A})}}$, i.e.

$$\begin{aligned} \mathcal{A} &: D(\mathcal{A}) \subset \overline{R(\mathrm{A}^*)} \to \overline{R(\mathrm{A})}, & D(\mathcal{A}) &= D(\mathcal{A}) \cap \overline{R(\mathrm{A}^*)}, \\ \mathcal{A}^* &: D(\mathcal{A}^*) \subset \overline{R(\mathrm{A})} \to \overline{R(\mathrm{A}^*)}, & D(\mathcal{A}^*) &= D(\mathcal{A}^*) \cap \overline{R(\mathrm{A})}. \end{aligned}$$

Note that

$$\overline{R(\mathrm{A}^*)} = N(\mathrm{A})^{\perp_{\mathsf{H}_0}}, \quad \overline{R(\mathrm{A})} = N(\mathrm{A}^*)^{\perp_{\mathsf{H}_1}}$$

and that $\mathcal{A}$ and $\mathcal{A}^*$ are indeed adjoint to each other, i.e., $(\mathcal{A}, \mathcal{A}^*)$ is a dual pair as well. Then the inverse operators

$$\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A}), \quad (\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$$

are well defined and bijective, but possibly unbounded. Furthermore, by (3) we have the refined Helmholtz type decompositions

$$D(\mathrm{A}) = N(\mathrm{A}) \oplus_{\mathsf{H}_0} D(\mathcal{A}), \quad D(\mathrm{A}^*) = N(\mathrm{A}^*) \oplus_{\mathsf{H}_1} D(\mathcal{A}^*) \tag{4}$$

and thus we obtain for the ranges

$$R(\mathrm{A}) = R(\mathcal{A}), \quad R(\mathrm{A}^*) = R(\mathcal{A}^*).$$

2.1.2. *Basic Results.* The following result is a well known and direct consequence of the closed graph theorem and the closed range theorem.

Lemma 2.1 (fa-toolbox lemma 1)**.** *The following assertions are equivalent:*

- **(i)** $\exists\, c_{\mathrm{A}} \in (0, \infty) \quad \forall\, x \in D(\mathcal{A}) \qquad |x|_{\mathsf{H}_0} \leq c_{\mathrm{A}} |\,\mathrm{A}\, x|_{\mathsf{H}_1}$
- **(i*)** $\exists\, c_{\mathrm{A}^*} \in (0, \infty) \quad \forall\, y \in D(\mathcal{A}^*) \qquad |y|_{\mathsf{H}_1} \leq c_{\mathrm{A}^*} |\,\mathrm{A}^*\, y|_{\mathsf{H}_0}$
- **(ii)** $R(\mathrm{A}) = R(\mathcal{A})$ *is closed in* H_1.
- **(ii*)** $R(\mathrm{A}^*) = R(\mathcal{A}^*)$ *is closed in* H_0.
- **(iii)** $\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A})$ *is bounded by* c_{A}.
- **(iii*)** $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$ *is bounded by* c_{A^*}.
- **(iv)** $\mathcal{A} : D(\mathcal{A}) \subset R(\mathrm{A}^*) \to R(\mathrm{A})$ *is bijective with continuous inverse.*
- **(iv*)** $\mathcal{A}^* : D(\mathcal{A}^*) \subset R(\mathrm{A}) \to R(\mathrm{A}^*)$ *is bijective with continuous inverse.*
- **(v)** $\mathcal{A} : D(\mathcal{A}) \to R(\mathrm{A})$ *is a topological isomorphism.*
- **(v*)** $\mathcal{A}^* : D(\mathcal{A}^*) \to R(\mathrm{A}^*)$ *is a topological isomorphism.*

The latter inequalities will be called Friedrichs/Poincaré type estimates. Note that in (iv) and (iv*) we consider $\mathcal{A}$ and $\mathcal{A}^*$ as unbounded linear operators, whereas in (v) and (v*) we consider $\mathcal{A}$ and $\mathcal{A}^*$ as bounded linear operators.

Lemma 2.2 (fa-toolbox lemma 2)**.** *The following assertions are equivalent:*

- **(i)** $D(\mathcal{A}) \hookrightarrow \mathsf{H}_0$ *is compact.*
- **(i*)** $D(\mathcal{A}^*) \hookrightarrow \mathsf{H}_1$ *is compact.*
- **(ii)** $\mathcal{A}^{-1} : R(\mathrm{A}) \to R(\mathrm{A}^*)$ *is compact.*
- **(ii*)** $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to R(\mathrm{A})$ *is compact.*

Moreover: Each of these assumptions imply the assertions of Lemma 2.1 (and of Lemma 2.2).

Remark 2.3 (sufficient assumptions for the fa-toolbox)**.**

- **(i)** *If $R(\mathrm{A})$ is closed, then the assertions of Lemma 2.1 hold.*
- **(ii)** *If $D(\mathcal{A}) \hookrightarrow \mathsf{H}_0$ is compact, then the assertions of Lemma 2.1 and Lemma 2.2 hold. In particular, the Friedrichs/Poincaré type estimates hold, all ranges are closed and the inverse operators are compact.*

2.1.3. *Constants, Spectra, and Eigenvalues.* Let us introduce the "best" constants c_{A}, c_{A^*} by utilising the Rayleigh quotients

$$\frac{1}{c_{\mathrm{A}}} := \inf_{0 \neq x \in D(\mathcal{A})} \frac{|\mathrm{A}\,x|_{\mathsf{H}_1}}{|x|_{\mathsf{H}_0}}, \qquad \frac{1}{c_{\mathrm{A}^*}} := \inf_{0 \neq y \in D(\mathcal{A}^*)} \frac{|\mathrm{A}^*\,y|_{\mathsf{H}_0}}{|y|_{\mathsf{H}_1}}. \tag{5}$$

Then $0 < c_{\mathrm{A}}, c_{\mathrm{A}^*} \leq \infty$ and we refer to c_{A} and c_{A^*} as Friedrichs/Poincaré type constants. From now on, we assume that we always deal with these best constants.

Lemma 2.4 (constant lemma)**.** *The Friedrichs/Poincaré type constants coincide, i.e., $c_{\mathrm{A}} = c_{\mathrm{A}^*}$.*

In the case that $R(\mathrm{A})$ is closed, we shall denote

$$\lambda_{\mathrm{A}} := \frac{1}{c_{\mathrm{A}}} = \frac{1}{c_{\mathrm{A}^*}} > 0.$$

Let us emphasise that

$$\mathrm{A}^*\,\mathrm{A}, \quad \mathrm{A}\,\mathrm{A}^*, \quad \begin{bmatrix} \mathrm{A}^*\,\mathrm{A} & 0 \\ 0 & \mathrm{A}\,\mathrm{A}^* \end{bmatrix}, \quad \begin{bmatrix} \mathrm{A}\,\mathrm{A}^* & 0 \\ 0 & \mathrm{A}^*\,\mathrm{A} \end{bmatrix}, \quad \begin{bmatrix} 0 & \mathrm{A}^* \\ \mathrm{A} & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \mathrm{A} \\ \mathrm{A}^* & 0 \end{bmatrix} \tag{6}$$

are self-adjoint, see Appendix 6, and have essentially - except of 0 and taking square roots - the same spectra contained in $\mathbb{R}$. Moreover, the first four operators are non-negative. The same holds true for the reduced operators $\mathcal{A}$ and $\mathcal{A}^*$. We will give more details in the next lemma.

Lemma 2.5 (constant and eigenvalue lemma)**.** *Let $D(\mathcal{A}) \hookrightarrow \mathsf{H}_0$ be compact. Then the operators in* (6) *have pure and discrete point spectra with no accumulation point in $\mathbb{R}$. Moreover:*

- **(i)** *λ_{A} is the smallest positive eigenvalue of $\begin{bmatrix} 0 & \mathrm{A}^* \\ \mathrm{A} & 0 \end{bmatrix}$ and of $\begin{bmatrix} 0 & \mathrm{A} \\ \mathrm{A}^* & 0 \end{bmatrix}$.*
- **(ii)** *λ_{A}^2 is the smallest positive eigenvalue of $\mathrm{A}^*\,\mathrm{A}$ and of $\mathrm{A}\,\mathrm{A}^*$.*
- **(iii)** *λ_{A}^2 is the smallest positive eigenvalue of $\begin{bmatrix} \mathrm{A}^*\,\mathrm{A} & 0 \\ 0 & \mathrm{A}\,\mathrm{A}^* \end{bmatrix}$ and of $\begin{bmatrix} \mathrm{A}\,\mathrm{A}^* & 0 \\ 0 & \mathrm{A}^*\,\mathrm{A} \end{bmatrix}$.*
- **(iv)** $\sigma(\mathrm{A}^*\,\mathrm{A}) \setminus \{0\} = \sigma(\mathrm{A}\,\mathrm{A}^*) \setminus \{0\} = \sigma\Big(\begin{bmatrix} \mathrm{A}^*\,\mathrm{A} & 0 \\ 0 & \mathrm{A}\,\mathrm{A}^* \end{bmatrix}\Big) \setminus \{0\} = \sigma\Big(\begin{bmatrix} \mathrm{A}\,\mathrm{A}^* & 0 \\ 0 & \mathrm{A}^*\,\mathrm{A} \end{bmatrix}\Big) \setminus \{0\} > 0$
- **(v)** $\sigma\Big(\begin{bmatrix} 0 & \mathrm{A}^* \\ \mathrm{A} & 0 \end{bmatrix}\Big) \setminus \{0\} = \sigma\Big(\begin{bmatrix} 0 & \mathrm{A} \\ \mathrm{A}^* & 0 \end{bmatrix}\Big) \setminus \{0\} = \pm\sqrt{\sigma(\mathrm{A}^*\,\mathrm{A}) \setminus \{0\}}$
- **(vi)** $\sigma(\mathrm{A}^*\,\mathrm{A}) \setminus \{0\} = \sigma(\mathcal{A}^*\,\mathcal{A})$ *and corresponding results hold for all other spectra in* **(iv)**.
- **(vii)** $\sigma\Big(\begin{bmatrix} 0 & \mathrm{A}^* \\ \mathrm{A} & 0 \end{bmatrix}\Big) \setminus \{0\} = \sigma\Big(\begin{bmatrix} 0 & \mathcal{A}^* \\ \mathcal{A} & 0 \end{bmatrix}\Big)$ *and corresponding results hold for all other spectra in* **(v)**.
- **(viii)** $|\mathcal{A}^{-1}|_{R(\mathrm{A}),R(\mathrm{A}^*)} = |(\mathcal{A}^*)^{-1}|_{R(\mathrm{A}^*),R(\mathrm{A})} = c_{\mathrm{A}}$
- **(viii*)** $|\mathcal{A}^{-1}|_{R(\mathrm{A}),D(\mathrm{A})} = |(\mathcal{A}^*)^{-1}|_{R(\mathrm{A}^*),D(\mathrm{A}^*)} = (c_{\mathrm{A}}^2 + 1)^{1/2}$
- **(ix)** $|(\mathcal{A}^*\,\mathcal{A})^{-1}|_{R(\mathrm{A}^*),R(\mathrm{A}^*)} = |(\mathcal{A}\,\mathcal{A}^*)^{-1}|_{R(\mathrm{A}),R(\mathrm{A})} = c_{\mathrm{A}}^2$
- **(x)** $N(\mathrm{A}) = N(\mathrm{A}^*\,\mathrm{A}) = N(\mathrm{A}\,\mathrm{A}^*\,\mathrm{A}) = \dots$ *and* $N(\mathrm{A}^*) = N(\mathrm{A}\,\mathrm{A}^*) = N(\mathrm{A}^*\,\mathrm{A}\,\mathrm{A}^*) = \dots.$
- **(xi)** $R(\mathrm{A}) = R(\mathrm{A}\,\mathrm{A}^*) = R(\mathrm{A}\,\mathrm{A}^*\,\mathrm{A}) = \dots$ *and* $R(\mathrm{A}^*) = R(\mathrm{A}^*\,\mathrm{A}) = R(\mathrm{A}^*\,\mathrm{A}\,\mathrm{A}^*) = \dots$ *and the same holds for the operators $\mathcal{A}$ and $\mathcal{A}^*$.*

For a proof see Appendix 6.

Remark 2.6 (variational formulations)**.** *By Lemma 2.5 the infima in* (5) *are minima, provided that $D(\mathcal{A}) \hookrightarrow \mathsf{H}_0$ is compact. In particular, the respective minimisers x_{A} and y_{A} are the eigenvectors to the eigenvalue λ_{A}^2, i.e,*

$$\inf_{0 \neq x \in D(\mathcal{A})} \frac{|\mathrm{A}\,x|_{\mathsf{H}_1}}{|x|_{\mathsf{H}_0}} = |\mathrm{A}\,x_{\mathrm{A}}|_{\mathsf{H}_1} = \lambda_{\mathrm{A}} = |\mathrm{A}^*\,y_{\mathrm{A}}|_{\mathsf{H}_0} = \inf_{0 \neq y \in D(\mathcal{A}^*)} \frac{|\mathrm{A}^*\,y|_{\mathsf{H}_0}}{|y|_{\mathsf{H}_1}},$$

where we assume without loss of generality $|x_{\mathrm{A}}|_{\mathsf{H}_0} = |y_{\mathrm{A}}|_{\mathsf{H}_1} = 1$. *Moreover,*

$$(\mathrm{A}^* \mathrm{A} - \lambda_{\mathrm{A}}^2) x_{\mathrm{A}} = 0, \qquad x_{\mathrm{A}} \in D(\mathrm{A}^* \mathrm{A}) \cap R(\mathrm{A}^*) = D(\mathcal{A}^* \mathcal{A}) \subset D(\mathcal{A}),$$
$$(\mathrm{A}\,\mathrm{A}^* - \lambda_{\mathrm{A}}^2) y_{\mathrm{A}} = 0, \qquad y_{\mathrm{A}} \in D(\mathrm{A}\,\mathrm{A}^*) \cap R(\mathrm{A}) = D(\mathcal{A}\mathcal{A}^*) \subset D(\mathcal{A}^*),$$

and the eigenvectors satisfy the variational formulations

$$\forall\, \phi \in D(\mathrm{A}) \qquad \langle \mathrm{A}\, x_{\mathrm{A}}, \mathrm{A}\, \phi \rangle_{\mathsf{H}_1} = \lambda_{\mathrm{A}}^2 \langle x_{\mathrm{A}}, \phi \rangle_{\mathsf{H}_0},$$
$$\forall\, \psi \in D(\mathrm{A}^*) \qquad \langle \mathrm{A}^* y_{\mathrm{A}}, \mathrm{A}^* \psi \rangle_{\mathsf{H}_0} = \lambda_{\mathrm{A}}^2 \langle y_{\mathrm{A}}, \psi \rangle_{\mathsf{H}_1}.$$

2.1.4. *Complex Structure Results.* Now, let

$$\mathrm{A}_0 : D(\mathrm{A}_0) \subset \mathsf{H}_0 \to \mathsf{H}_1, \quad \mathrm{A}_1 : D(\mathrm{A}_1) \subset \mathsf{H}_1 \to \mathsf{H}_2$$

be two densely defined and closed linear operators on three Hilbert spaces H_0, H_1, and H_2 with adjoints

$$\mathrm{A}_0^* : D(\mathrm{A}_0^*) \subset \mathsf{H}_1 \to \mathsf{H}_0, \quad \mathrm{A}_1^* : D(\mathrm{A}_1^*) \subset \mathsf{H}_2 \to \mathsf{H}_1$$

as well as reduced operators $\mathcal{A}_0$, $\mathcal{A}_0^*$, and $\mathcal{A}_1$, $\mathcal{A}_1^*$. Furthermore, we assume the complex property (also called sequence property) of A_0 and A_1, that is $\mathrm{A}_1 \mathrm{A}_0 = 0$, i.e.,

$$R(\mathrm{A}_0) \subset N(\mathrm{A}_1), \tag{7}$$

which is equivalent to $\mathrm{A}_0^* \mathrm{A}_1^* = 0$, i.e., $R(\mathrm{A}_1^*) \subset N(\mathrm{A}_0^*)$. Recall that

$$R(\mathrm{A}_0) = R(\mathcal{A}_0), \quad R(\mathrm{A}_0^*) = R(\mathcal{A}_0^*), \quad R(\mathrm{A}_1) = R(\mathcal{A}_1), \quad R(\mathrm{A}_1^*) = R(\mathcal{A}_1^*).$$

From the Helmholtz type decompositions (3) for $\mathrm{A} = \mathrm{A}_0$ and $\mathrm{A} = \mathrm{A}_1$ we get in particular

$$\mathsf{H}_1 = \overline{R(\mathrm{A}_0)} \oplus_{\mathsf{H}_1} N(\mathrm{A}_0^*), \quad \mathsf{H}_1 = \overline{R(\mathrm{A}_1^*)} \oplus_{\mathsf{H}_1} N(\mathrm{A}_1). \tag{8}$$

Introducing the cohomology group

$$N_{0,1} := N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*),$$

we obtain the refined Helmholtz type decompositions

$$\begin{aligned} N(\mathrm{A}_1) &= \overline{R(\mathrm{A}_0)} \oplus_{\mathsf{H}_1} N_{0,1}, & N(\mathrm{A}_0^*) &= \overline{R(\mathrm{A}_1^*)} \oplus_{\mathsf{H}_1} N_{0,1}, \\ D(\mathrm{A}_1) &= \overline{R(\mathrm{A}_0)} \oplus_{\mathsf{H}_1} \big(D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)\big), & D(\mathrm{A}_0^*) &= \overline{R(\mathrm{A}_1^*)} \oplus_{\mathsf{H}_1} \big(D(\mathrm{A}_0^*) \cap N(\mathrm{A}_1)\big), \end{aligned} \tag{9}$$

and therefore

$$\mathsf{H}_1 = \overline{R(\mathrm{A}_0)} \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} \overline{R(\mathrm{A}_1^*)}. \tag{10}$$

Let us remark that the first line of (9) can also be written as

$$\overline{R(\mathrm{A}_0)} = N(\mathrm{A}_1) \cap N_{0,1}^{\perp_{\mathsf{H}_1}}, \quad \overline{R(\mathrm{A}_1^*)} = N(\mathrm{A}_0^*) \cap N_{0,1}^{\perp_{\mathsf{H}_1}}.$$

Note that (10) can be further refined and specialised, e.g., to

$$\begin{aligned} D(\mathrm{A}_1) &= \overline{R(\mathrm{A}_0)} \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} D(\mathcal{A}_1), \\ D(\mathrm{A}_0^*) &= D(\mathcal{A}_0^*) \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} \overline{R(\mathrm{A}_1^*)}, \\ D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) &= D(\mathcal{A}_0^*) \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} D(\mathcal{A}_1). \end{aligned} \tag{11}$$

We observe

$$D(\mathcal{A}_1) = D(\mathrm{A}_1) \cap \overline{R(\mathrm{A}_1^*)} \subset D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*) \subset D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*),$$
$$D(\mathcal{A}_0^*) = D(\mathrm{A}_0^*) \cap \overline{R(\mathrm{A}_0)} \subset D(\mathrm{A}_0^*) \cap N(\mathrm{A}_1) \subset D(\mathrm{A}_0^*) \cap D(\mathrm{A}_1),$$

and using the refined Helmholtz type decompositions (10) and (11) as well as the results of Lemma 2.2 we immediately see:

Lemma 2.7 (fa-toolbox lemma 3)**.** *The following assertions are equivalent:*

- **(i)** $D(\mathcal{A}_0) \hookrightarrow \mathsf{H}_0$, $D(\mathcal{A}_1) \hookrightarrow \mathsf{H}_1$, *and* $N_{0,1} \hookrightarrow \mathsf{H}_1$ *are compact.*
- **(ii)** $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathsf{H}_1$ *is compact.*

In this case, the cohomology group $N_{0,1}$ *has finite dimension.*

We summarise:

Theorem 2.8 (fa-toolbox theorem)**.** *Let the ranges $R(\mathrm{A}_0)$ and $R(\mathrm{A}_1)$ be closed. Then all ranges $R(\mathrm{A}_0)$, $R(\mathrm{A}_0^*)$, and $R(\mathrm{A}_1)$, $R(\mathrm{A}_1^*)$ are closed, and the corresponding Friedrichs/Poincaré type estimates hold, i.e. there exists positive constants $c_{\mathrm{A}_0}, c_{\mathrm{A}_1}$ such that*

$$\begin{aligned}
&\forall\, z \in D(\mathcal{A}_0) = D(\mathrm{A}_0) \cap R(\mathrm{A}_0^*) && |z|_{\mathsf{H}_0} \leq c_{\mathrm{A}_0} |\,\mathrm{A}_0\, z|_{\mathsf{H}_1},\\
&\forall\, x \in D(\mathcal{A}_0^*) = D(\mathrm{A}_0^*) \cap R(\mathrm{A}_0) = D(\mathrm{A}_0^*) \cap N(\mathrm{A}_1) \cap N_{0,1}^{\perp_{\mathsf{H}_1}} && |x|_{\mathsf{H}_1} \leq c_{\mathrm{A}_0} |\,\mathrm{A}_0^*\, x|_{\mathsf{H}_0},\\
&\forall\, x \in D(\mathcal{A}_1) = D(\mathrm{A}_1) \cap R(\mathrm{A}_1^*) = D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*) \cap N_{0,1}^{\perp_{\mathsf{H}_1}} && |x|_{\mathsf{H}_1} \leq c_{\mathrm{A}_1} |\,\mathrm{A}_1\, x|_{\mathsf{H}_2},\\
&\forall\, y \in D(\mathcal{A}_1^*) = D(\mathrm{A}_1^*) \cap R(\mathrm{A}_1) && |y|_{\mathsf{H}_2} \leq c_{\mathrm{A}_1} |\,\mathrm{A}_1^*\, y|_{\mathsf{H}_1},
\end{aligned}$$

and

$$\forall\, x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \cap N_{0,1}^{\perp_{\mathsf{H}_1}} \qquad |x|_{\mathsf{H}_1}^2 \leq c_{\mathrm{A}_1}^2 |\,\mathrm{A}_1\, x|_{\mathsf{H}_2}^2 + c_{\mathrm{A}_0}^2 |\,\mathrm{A}_0^*\, x|_{\mathsf{H}_0}^2.$$

Moreover, all refined Helmholtz type decompositions (9)-(11) hold with closed ranges, in particular, e.g.,

$$\mathsf{H}_1 = R(\mathrm{A}_0) \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} R(\mathrm{A}_1^*).$$

Remark 2.9. *Let us define $c_{\mathrm{A}_0,\mathrm{A}_1} > 0$ by*

$$\frac{1}{c_{\mathrm{A}_0,\mathrm{A}_1}^2} := \inf \frac{|\,\mathrm{A}_1\, x|_{\mathsf{H}_2}^2 + |\,\mathrm{A}_0^*\, x|_{\mathsf{H}_0}^2}{|x|_{\mathsf{H}_0}^2},$$

where the infimum is taken over all $0 \neq x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \cap N_{0,1}^{\perp_{\mathsf{H}_1}}$. Assuming - as mentioned above - that we always take the best constants, we obtain by Theorem 2.8

$$c_{\mathrm{A}_0,\mathrm{A}_1} = \max\{c_{\mathrm{A}_0}, c_{\mathrm{A}_1}\}.$$

This can be seen as follows: Theorem 2.8 shows $c_{\mathrm{A}_0,\mathrm{A}_1} \leq \max\{c_{\mathrm{A}_0}, c_{\mathrm{A}_1}\}$. On the other hand, for

$$x \in D(\mathcal{A}_1) = D(\mathrm{A}_1) \cap R(\mathrm{A}_1^*) = D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*) \cap N_{0,1}^{\perp_{\mathsf{H}_1}}$$

we have $|x|_{\mathsf{H}_1} \leq c_{\mathrm{A}_0,\mathrm{A}_1} |\,\mathrm{A}_1\, x|_{\mathsf{H}_2}$ and hence $c_{\mathrm{A}_1} \leq c_{\mathrm{A}_0,\mathrm{A}_1}$. Analogously we get $c_{\mathrm{A}_0} \leq c_{\mathrm{A}_0,\mathrm{A}_1}$.

Remark 2.10. *If $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathsf{H}_1$ is compact, then $D(\mathcal{A}_0) \hookrightarrow \mathsf{H}_0$, $D(\mathcal{A}_1) \hookrightarrow \mathsf{H}_1$, and $D(\mathcal{A}_0^*) \hookrightarrow \mathsf{H}_1$, $D(\mathcal{A}_1^*) \hookrightarrow \mathsf{H}_2$ are compact, as well as $\dim N_{0,1} < \infty$. Hence all ranges are closed, see Remark 2.3, and all assertions of Theorem 2.8 hold.*

In other words, the primal and dual complex, i.e.,

$$\begin{aligned}
&D(\mathrm{A}_0) \xrightarrow{\ \mathrm{A}_0\ } D(\mathrm{A}_1) \xrightarrow{\ \mathrm{A}_1\ } \mathsf{H}_2,\\
&\mathsf{H}_0 \xleftarrow{\ \mathrm{A}_0^*\ } D(\mathrm{A}_0^*) \xleftarrow{\ \mathrm{A}_1^*\ } D(\mathrm{A}_1^*),
\end{aligned} \tag{12}$$

is a Hilbert complex of closed and densely defined linear operators. The additional assumption that the ranges $R(\mathrm{A}_0)$ and $R(\mathrm{A}_1)$ are closed (and so also the ranges $R(\mathrm{A}_0^*)$ and $R(\mathrm{A}_1^*)$) is equivalent to the closedness of the Hilbert complex. Moreover, the complex is exact if and only if $N_{0,1} = \{0\}$. The complex is called compact, if

$$D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathsf{H}_1 \tag{13}$$

is compact. Remark 2.10 shows that (13) is the crucial assumption for the complex (12).

Finally, we present some results for the (unbounded linear) operator

$$\mathrm{A}_0\, \mathrm{A}_0^* + \mathrm{A}_1^*\, \mathrm{A}_1 : D(\mathrm{A}_0\, \mathrm{A}_0^* + \mathrm{A}_1^*\, \mathrm{A}_1) \subset \mathsf{H}_1 \to \mathsf{H}_1$$

with $D(\mathrm{A}_0\, \mathrm{A}_0^* + \mathrm{A}_1^*\, \mathrm{A}_1) := \big\{x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \,:\, \mathrm{A}_1\, x \in D(\mathrm{A}_1^*) \,\wedge\, \mathrm{A}_0^*\, x \in D(\mathrm{A}_0)\big\}$.

Lemma 2.11 (constant and eigenvalue lemma)**.** *Let $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathsf{H}_1$ be compact. Then:*

- **(i)** $\mathrm{A}_0^*\, \mathrm{A}_0$, $\mathrm{A}_0\, \mathrm{A}_0^*$, $\mathrm{A}_1^*\, \mathrm{A}_1$, $\mathrm{A}_1\, \mathrm{A}_1^*$, *and* $\mathrm{A}_0\, \mathrm{A}_0^* + \mathrm{A}_1^*\, \mathrm{A}_1$ *are self-adjoint and have pure and discrete point spectra with no accumulation point in $\mathbb{R}$.*
- **(ii)** *The results of Lemma 2.5 hold for A_0 and A_1, in particular $\sigma(\mathrm{A}_0^*\, \mathrm{A}_0) \setminus \{0\} = \sigma(\mathrm{A}_0\, \mathrm{A}_0^*) \setminus \{0\}$ and $\sigma(\mathrm{A}_1^*\, \mathrm{A}_1) \setminus \{0\} = \sigma(\mathrm{A}_1\, \mathrm{A}_1^*) \setminus \{0\}$ as well as $N(\mathrm{A}_0\, \mathrm{A}_0^*) = N(\mathrm{A}_0^*)$ and $N(\mathrm{A}_1^*\, \mathrm{A}_1) = N(\mathrm{A}_1)$.*
- **(iii)** $N(\mathrm{A}_0\, \mathrm{A}_0^* + \mathrm{A}_1^*\, \mathrm{A}_1) = N_{0,1}$ *and* $R(\mathrm{A}_0\, \mathrm{A}_0^* + \mathrm{A}_1^*\, \mathrm{A}_1) = N_{0,1}^{\perp_{\mathsf{H}_1}}$*, in particular the range is closed.*
- **(iv)** $\mathrm{A}_0\, \mathrm{A}_0^* + \mathrm{A}_1^*\, \mathrm{A}_1 : D(\mathrm{A}_0\, \mathrm{A}_0^* + \mathrm{A}_1^*\, \mathrm{A}_1) \cap N_{0,1}^{\perp_{\mathsf{H}_1}} \subset N_{0,1}^{\perp_{\mathsf{H}_1}} \to N_{0,1}^{\perp_{\mathsf{H}_1}}$ *is bijective with compact inverse.*
- **(iv*)** $\mathrm{A}_0\, \mathrm{A}_0^* + \mathrm{A}_1^*\, \mathrm{A}_1 : D(\mathrm{A}_0\, \mathrm{A}_0^* + \mathrm{A}_1^*\, \mathrm{A}_1) \cap N_{0,1}^{\perp_{\mathsf{H}_1}} \to N_{0,1}^{\perp_{\mathsf{H}_1}}$ *is a topological isomorphism.*

Moreover, the spectrum of $\mathrm{A}_0\, \mathrm{A}_0^* + \mathrm{A}_1^*\, \mathrm{A}_1$ is given by the spectra of $\mathrm{A}_0\, \mathrm{A}_0^*$ and $\mathrm{A}_1^*\, \mathrm{A}_1$, i.e.,

(v) $\sigma(\mathrm{A}_0\,\mathrm{A}_0^* + \mathrm{A}_1^*\,\mathrm{A}_1)\setminus\{0\} = \big(\sigma(\mathrm{A}_0^*\,\mathrm{A}_0)\setminus\{0\}\big)\cup\big(\sigma(\mathrm{A}_1^*\,\mathrm{A}_1)\setminus\{0\}\big)$.

(v*) *In particular, the smallest positive eigenvalue of* $\mathrm{A}_0\,\mathrm{A}_0^* + \mathrm{A}_1^*\,\mathrm{A}_1$ *is given by* $\min\{\lambda_{\mathrm{A}_0}^2, \lambda_{\mathrm{A}_1}^2\}$.

For a proof see Appendix 6.

Remark 2.12 (Helmholtz decomposition). $\mathrm{A}_0\,\mathrm{A}_0^* + \mathrm{A}_1^*\,\mathrm{A}_1$ *provides the Helmholtz decomposition from Theorem 2.8. To see this, let us denote the orthonormal projector onto the cohomology group* $N_{0,1}$ *by* $\pi_{N_{0,1}} : \mathsf{H}_1 \to N_{0,1}$. *Then, for* $x \in \mathsf{H}_1$ *we have* $(1-\pi_{N_{0,1}})x \in N_{0,1}^{\perp_{\mathsf{H}_1}}$ *and*

$$\begin{aligned} x &= \pi_{N_{0,1}}x + (1-\pi_{N_{0,1}})x \\ &= \pi_{N_{0,1}}x + (\mathrm{A}_0\,\mathrm{A}_0^* + \mathrm{A}_1^*\,\mathrm{A}_1)(\mathrm{A}_0\,\mathrm{A}_0^* + \mathrm{A}_1^*\,\mathrm{A}_1)^{-1}(1-\pi_{N_{0,1}})x \in N_{0,1} \oplus_{\mathsf{H}_1} R(\mathrm{A}_0) \oplus_{\mathsf{H}_1} R(\mathrm{A}_1^*). \end{aligned}$$

2.2. **Laplace and Maxwell Constants in 3D.** Now, we specialise to linear acoustics and electromagnetics in 3D, i.e., to the classical operators of the 3D-de Rham complex, cf. (12),

$$\begin{array}{ccccccc} \mathsf{H}_{\Gamma_\tau}(\mathrm{grad},\Omega) & \xrightarrow{\mathrm{A}_0=\mathrm{grad}_{\Gamma_\tau}} & \mathsf{H}_{\Gamma_\tau}(\mathrm{rot},\Omega) & \xrightarrow{\mathrm{A}_1=\mathrm{rot}_{\Gamma_\tau}} & \mathsf{H}_{\Gamma_\tau}(\mathrm{div},\Omega) & \xrightarrow{\mathrm{A}_2=\mathrm{div}_{\Gamma_\tau}} & \mathsf{L}^2(\Omega), \\ \mathsf{L}^2(\Omega) & \xleftarrow{\mathrm{A}_0^*=-\mathrm{div}_{\Gamma_\nu}} & \mathsf{H}_{\Gamma_\nu}(\mathrm{div},\Omega) & \xleftarrow{\mathrm{A}_1^*=\mathrm{rot}_{\Gamma_\nu}} & \mathsf{H}_{\Gamma_\nu}(\mathrm{rot},\Omega) & \xleftarrow{\mathrm{A}_2^*=-\mathrm{grad}_{\Gamma_\nu}} & \mathsf{H}_{\Gamma_\nu}(\mathrm{grad},\Omega), \end{array} \tag{14}$$

and apply the fa-toolbox to these operators.

More precisely, let $\Omega \subset \mathbb{R}^3$ be a bounded weak Lipschitz domain, see [13, Definition 2.3] for details, with boundary $\Gamma := \partial\,\Omega$, which is divided into two relatively open weak Lipschitz subsets Γ_τ and $\Gamma_\nu := \Gamma\setminus\overline{\Gamma_\tau}$ (its complement), see [13, Definition 2.5] for details. We shall call (Ω,Γ_τ) a bounded weak Lipschitz pair. Moreover, if (Ω,Γ_τ) is a bounded weak Lipschitz pair, so is (Ω,Γ_ν). Note that strong Lipschitz (graph of Lipschitz functions) implies weak Lipschitz (Lipschitz manifolds) for the boundary as well as for the interface. We introduce the usual Lebesgue and Sobolev spaces by $\mathsf{L}^2(\Omega)$ and $\mathsf{H}^k(\Omega)$, $k \in \mathbb{N}_0$. For $k = 1$ we also write

$$\mathsf{H}^1(\Omega) = \mathsf{H}(\mathrm{grad},\Omega) := \big\{u \in \mathsf{L}^2(\Omega) : \mathrm{grad}\,u \in \mathsf{L}^2(\Omega)\big\}.$$

Weak boundary conditions (in the strong sense) are defined by closure of respective test functions, i.e.,

$$\mathsf{H}^1_{\Gamma_\tau}(\Omega) = \mathsf{H}_{\Gamma_\tau}(\mathrm{grad},\Omega) := \overline{\mathsf{C}^\infty_{\Gamma_\tau}(\Omega)}^{\mathsf{H}(\mathrm{grad},\Omega)}, \tag{15}$$

where

$$\mathsf{C}^\infty_{\Gamma_\tau}(\Omega) := \big\{u|_\Omega : u \in \mathsf{C}^\infty(\mathbb{R}^3),\ \mathrm{supp}\,u \text{ compact in } \mathbb{R}^3,\ \mathrm{dist}(\mathrm{supp}\,u,\Gamma_\tau) > 0\big\}.$$

Analogously we define (using test vector fields)

$$\mathsf{H}(\mathrm{rot},\Omega), \quad \mathsf{H}_{\Gamma_\tau}(\mathrm{rot},\Omega), \quad \mathsf{H}(\mathrm{div},\Omega), \quad \mathsf{H}_{\Gamma_\tau}(\mathrm{div},\Omega).$$

All latter definitions extend to $\Omega \subset \mathbb{R}^N$, $N \geq 1$, in an obvious way, see [14, 15] for details. Throughout this paper and until otherwise stated, we shall assume the latter minimal regularity on Ω and Γ_τ.

Assumption 2.13. (Ω,Γ_τ) *is a bounded weak Lipschitz pair.*

As closures of the respective classical operators of vector analysis defined on test functions/vector fields from $\mathsf{C}^\infty_{\Gamma_\tau}(\Omega)$, we consider the densely defined and closed linear operators

$$\begin{aligned} \mathrm{A}_0 &:= \mathrm{grad}_{\Gamma_\tau} : D(\mathrm{grad}_{\Gamma_\tau}) \subset \mathsf{L}^2(\Omega) \longrightarrow \mathsf{L}^2(\Omega); && u \mapsto \mathrm{grad}\,u, \\ \mathrm{A}_1 &:= \mathrm{rot}_{\Gamma_\tau} : D(\mathrm{rot}_{\Gamma_\tau}) \subset \mathsf{L}^2(\Omega) \longrightarrow \mathsf{L}^2(\Omega); && E \mapsto \mathrm{rot}\,E, \\ \mathrm{A}_2 &:= \mathrm{div}_{\Gamma_\tau} : D(\mathrm{div}_{\Gamma_\tau}) \subset \mathsf{L}^2(\Omega) \longrightarrow \mathsf{L}^2(\Omega); && H \mapsto \mathrm{div}\,H, \end{aligned}$$

together with their adjoints, see [13, Theorem 4.5, Section 5.2] and [14, 15, Theorem 4.7, Section 5.2],

$$\begin{aligned} \mathrm{A}_0^* &= \mathrm{grad}_{\Gamma_\tau}^* = -\,\mathrm{div}_{\Gamma_\nu} : D(\mathrm{div}_{\Gamma_\nu}) \subset \mathsf{L}^2(\Omega) \longrightarrow \mathsf{L}^2(\Omega); && H \mapsto -\,\mathrm{div}\,H, \\ \mathrm{A}_1^* &= \mathrm{rot}_{\Gamma_\tau}^* = \mathrm{rot}_{\Gamma_\nu} : D(\mathrm{rot}_{\Gamma_\nu}) \subset \mathsf{L}^2(\Omega) \longrightarrow \mathsf{L}^2(\Omega); && E \mapsto \mathrm{rot}\,E, \\ \mathrm{A}_2^* &= \mathrm{div}_{\Gamma_\tau}^* = -\,\mathrm{grad}_{\Gamma_\nu} : D(\mathrm{grad}_{\Gamma_\nu}) \subset \mathsf{L}^2(\Omega) \longrightarrow \mathsf{L}^2(\Omega); && u \mapsto -\,\mathrm{grad}\,u. \end{aligned}$$

Note that

$$D(\mathrm{grad}_{\Gamma_\tau}) = \mathsf{H}_{\Gamma_\tau}(\mathrm{grad},\Omega), \quad D(\mathrm{rot}_{\Gamma_\tau}) = \mathsf{H}_{\Gamma_\tau}(\mathrm{rot},\Omega), \quad D(\mathrm{div}_{\Gamma_\tau}) = \mathsf{H}_{\Gamma_\tau}(\mathrm{div},\Omega)$$

and that (14) is indeed a Hilbert complex.

Recently, in [13, 14, 15], Weck's selection theorem, also known as the Maxwell compactness property, has been shown to hold for such bounded weak Lipschitz domains and mixed boundary conditions.

Theorem 2.14 (Weck's selection theorem)**.** *The embedding*

$$\mathsf{H}_{\Gamma_\tau}(\mathrm{rot},\Omega)\cap\mathsf{H}_{\Gamma_\nu}(\mathrm{div},\Omega)\hookrightarrow\mathsf{L}^2(\Omega)$$

is compact.

For a proof see [13, 14, 15]. A short historical overview of Weck's selection theorem is given in the introduction of [13], see also the original paper [58] and [49, 57, 26, 59, 34, 36, 50].

Now, Theorem 2.14 implies that the crucial assumption (13) holds for the operators A_n of the de Rham complex (14), cf. the general complex (12). More precisely, by Theorem 2.14

$$\textbf{(a)}\qquad D(\mathrm{A}_1)\cap D(\mathrm{A}_0^*)=\mathsf{H}_{\Gamma_\tau}(\mathrm{rot},\Omega)\cap\mathsf{H}_{\Gamma_\nu}(\mathrm{div},\Omega)\hookrightarrow\mathsf{L}^2(\Omega)=\mathsf{H}_1,$$
$$\textbf{(b)}\qquad D(\mathrm{A}_2)\cap D(\mathrm{A}_1^*)=\mathsf{H}_{\Gamma_\tau}(\mathrm{div},\Omega)\cap\mathsf{H}_{\Gamma_\nu}(\mathrm{rot},\Omega)\hookrightarrow\mathsf{L}^2(\Omega)=\mathsf{H}_2$$

are compact and, hence, (14) is a compact Hilbert complex. Thus, by Theorem 2.8 and Remark 2.10, all ranges are closed, all corresponding Friedrichs/Poincaré type estimates hold, and all refined Helmholtz type decompositions (9)-(11) hold with closed ranges. In particular, denoting the corresponding constants by

$$\begin{aligned}\frac{1}{\lambda_{0,\Gamma_\tau}}&:=c_{0,\Gamma_\tau}:=c_{\mathrm{grad}_{\Gamma_\tau}}:=c_{\mathrm{A}_0}=c_{\mathrm{A}_0^*}=c_{\mathrm{div}_{\Gamma_\nu}}=c_{2,\Gamma_\nu}=\frac{1}{\lambda_{2,\Gamma_\nu}},\\ \frac{1}{\lambda_{1,\Gamma_\tau}}&:=c_{1,\Gamma_\tau}:=c_{\mathrm{rot}_{\Gamma_\tau}}:=c_{\mathrm{A}_1}=c_{\mathrm{A}_1^*}=c_{\mathrm{rot}_{\Gamma_\nu}}=c_{1,\Gamma_\nu}=\frac{1}{\lambda_{1,\Gamma_\nu}},\\ \frac{1}{\lambda_{2,\Gamma_\tau}}&:=c_{2,\Gamma_\tau}:=c_{\mathrm{div}_{\Gamma_\tau}}:=c_{\mathrm{A}_2}=c_{\mathrm{A}_2^*}=c_{\mathrm{grad}_{\Gamma_\nu}}=c_{0,\Gamma_\nu}=\frac{1}{\lambda_{0,\Gamma_\nu}},\end{aligned}\tag{16}$$

and introducing the (finite-dimensional) cohomology groups

$$\begin{aligned}\mathcal{H}_1&:=N_{0,1}:=N(\mathrm{A}_1)\cap N(\mathrm{A}_0^*)=N(\mathrm{rot}_{\Gamma_\tau})\cap N(\mathrm{div}_{\Gamma_\nu}),\\ \mathcal{H}_2&:=N_{1,2}:=N(\mathrm{A}_2)\cap N(\mathrm{A}_1^*)=N(\mathrm{div}_{\Gamma_\tau})\cap N(\mathrm{rot}_{\Gamma_\nu}),\end{aligned}$$

the so-called Dirichlet/Neumann fields, we have by Theorem 2.8 and Remark 2.10 the following inequalities:

Theorem 2.15 (Friedrichs/Poincaré type estimates)**.** *It holds*

$$\begin{aligned}&\forall\,u\in D(\mathcal{A}_0)=D(\mathrm{grad}_{\Gamma_\tau})\cap R(\mathrm{div}_{\Gamma_\nu}) && |u|_{\mathsf{L}^2(\Omega)}\le c_{0,\Gamma_\tau}|\,\mathrm{grad}\,u|_{\mathsf{L}^2(\Omega)},\\ &\forall\,E\in D(\mathcal{A}_0^*)=D(\mathrm{div}_{\Gamma_\nu})\cap R(\mathrm{grad}_{\Gamma_\tau}) && |E|_{\mathsf{L}^2(\Omega)}\le c_{0,\Gamma_\tau}|\,\mathrm{div}\,E|_{\mathsf{L}^2(\Omega)},\\ &\forall\,E\in D(\mathcal{A}_1)=D(\mathrm{rot}_{\Gamma_\tau})\cap R(\mathrm{rot}_{\Gamma_\nu}) && |E|_{\mathsf{L}^2(\Omega)}\le c_{1,\Gamma_\tau}|\,\mathrm{rot}\,E|_{\mathsf{L}^2(\Omega)},\\ &\forall\,H\in D(\mathcal{A}_1^*)=D(\mathrm{rot}_{\Gamma_\nu})\cap R(\mathrm{rot}_{\Gamma_\tau}) && |H|_{\mathsf{L}^2(\Omega)}\le c_{1,\Gamma_\tau}|\,\mathrm{rot}\,H|_{\mathsf{L}^2(\Omega)},\end{aligned}$$

and for all $E\in D(\mathrm{A}_1)\cap D(\mathrm{A}_0^*)\cap N_{0,1}^{\perp_{\mathsf{H}_1}}=D(\mathrm{rot}_{\Gamma_\tau})\cap D(\mathrm{div}_{\Gamma_\nu})\cap\mathcal{H}_1^{\perp_{\mathsf{L}^2(\Omega)}}$

$$|E|^2_{\mathsf{L}^2(\Omega)}\le c^2_{1,\Gamma_\tau}|\,\mathrm{rot}\,E|^2_{\mathsf{L}^2(\Omega)}+c^2_{0,\Gamma_\tau}|\,\mathrm{div}\,E|^2_{\mathsf{L}^2(\Omega)},$$

where

$$\begin{aligned}R(\mathrm{grad}_{\Gamma_\tau})&=N(\mathrm{rot}_{\Gamma_\tau})\cap\mathcal{H}_1^{\perp_{\mathsf{L}^2(\Omega)}}, && R(\mathrm{rot}_{\Gamma_\nu})=N(\mathrm{div}_{\Gamma_\nu})\cap\mathcal{H}_1^{\perp_{\mathsf{L}^2(\Omega)}},\\ R(\mathrm{grad}_{\Gamma_\nu})&=N(\mathrm{rot}_{\Gamma_\nu})\cap\mathcal{H}_2^{\perp_{\mathsf{L}^2(\Omega)}}, && R(\mathrm{rot}_{\Gamma_\tau})=N(\mathrm{div}_{\Gamma_\tau})\cap\mathcal{H}_2^{\perp_{\mathsf{L}^2(\Omega)}}.\end{aligned}$$

Let $c_{0,1,\Gamma_\tau}:=c_{\mathrm{grad}_{\Gamma_\tau},\mathrm{rot}_{\Gamma_\tau}}>0$ be defined by

$$\frac{1}{c^2_{0,1,\Gamma_\tau}}:=\inf\frac{|\,\mathrm{rot}\,E|^2_{\mathsf{L}^2(\Omega)}+|\,\mathrm{div}\,E|^2_{\mathsf{L}^2(\Omega)}}{|E|^2_{\mathsf{L}^2(\Omega)}},$$

where the infimum taken over all $0\neq E\in D(\mathrm{rot}_{\Gamma_\tau})\cap D(\mathrm{div}_{\Gamma_\nu})\cap\mathcal{H}_1^{\perp_{\mathsf{L}^2(\Omega)}}$.

Remark 2.16. *By Remark 2.9 it holds* $c_{0,1,\Gamma_\tau}=\max\{c_{0,\Gamma_\tau},c_{1,\Gamma_\tau}\}$.

Note that by the symmetry of the de Rham complex the corresponding two estimates for $\mathcal{A}_2$ and $\mathcal{A}_2^*$, i.e.,

$$\begin{aligned}&\forall\,H\in D(\mathcal{A}_2)=D(\mathrm{div}_{\Gamma_\tau})\cap R(\mathrm{grad}_{\Gamma_\nu}) && |H|_{\mathsf{L}^2(\Omega)}\le c_{2,\Gamma_\tau}|\,\mathrm{div}\,H|_{\mathsf{L}^2(\Omega)},\\ &\forall\,u\in D(\mathcal{A}_2^*)=D(\mathrm{grad}_{\Gamma_\nu})\cap R(\mathrm{div}_{\Gamma_\tau}) && |u|_{\mathsf{L}^2(\Omega)}\le c_{2,\Gamma_\tau}|\,\mathrm{grad}\,u|_{\mathsf{L}^2(\Omega)},\end{aligned}$$

are redundant, as these are already included in the two estimates for $\mathcal{A}_0$ and $\mathcal{A}_0^*$ just by interchanging the boundary conditions on Γ_τ and Γ_ν. In other words, $c_{2,\Gamma_\tau} = c_{0,\Gamma_\nu}$. Furthermore,

$$N(\operatorname{grad}_{\Gamma_\tau}) = \begin{cases} \{0\} & \text{if } \Gamma_\tau \neq \emptyset, \\ \mathbb{R} & \text{if } \Gamma_\tau = \emptyset, \end{cases}$$

$$R(\operatorname{div}_{\Gamma_\nu}) = N(\operatorname{grad}_{\Gamma_\tau})^{\perp_{\mathsf{L}^2(\Omega)}} = \mathsf{L}^2_{\Gamma_\nu}(\Omega) := \begin{cases} \mathsf{L}^2(\Omega) & \text{if } \Gamma_\nu \neq \Gamma, \\ \mathsf{L}^2(\Omega) \cap \mathbb{R}^{\perp_{\mathsf{L}^2(\Omega)}} & \text{if } \Gamma_\nu = \Gamma, \end{cases}$$

where

$$\mathsf{L}^2(\Omega) \cap \mathbb{R}^{\perp_{\mathsf{L}^2(\Omega)}} = \{u \in \mathsf{L}^2(\Omega) : \langle u, 1\rangle_{\mathsf{L}^2(\Omega)} = 0\} = \{u \in \mathsf{L}^2(\Omega) : \int_\Omega u = 0\}.$$

Combinations of the latter operators give the well known operators from acoustics, Maxwell equations, Laplace equations, and the double rotation equations, i.e.,

$$\mathrm{M}_0 := \begin{bmatrix} 0 & \mathrm{A}_0^* \\ \mathrm{A}_0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \operatorname{grad}_{\Gamma_\tau}^* \\ \operatorname{grad}_{\Gamma_\tau} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\operatorname{div}_{\Gamma_\nu} \\ \operatorname{grad}_{\Gamma_\tau} & 0 \end{bmatrix},$$
$$\mathrm{M}_1 := \begin{bmatrix} 0 & \mathrm{A}_1^* \\ \mathrm{A}_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \operatorname{rot}_{\Gamma_\tau}^* \\ \operatorname{rot}_{\Gamma_\tau} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \operatorname{rot}_{\Gamma_\nu} \\ \operatorname{rot}_{\Gamma_\tau} & 0 \end{bmatrix},$$
$$\mathrm{M}_2 := \begin{bmatrix} 0 & \mathrm{A}_2^* \\ \mathrm{A}_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \operatorname{div}_{\Gamma_\tau}^* \\ \operatorname{div}_{\Gamma_\tau} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\operatorname{grad}_{\Gamma_\nu} \\ \operatorname{div}_{\Gamma_\tau} & 0 \end{bmatrix},$$

and

$$\mathrm{A}_0^*\,\mathrm{A}_0 = \operatorname{grad}_{\Gamma_\tau}^* \operatorname{grad}_{\Gamma_\tau} = -\operatorname{div}_{\Gamma_\nu} \operatorname{grad}_{\Gamma_\tau} =: -\Delta_{\Gamma_\tau}, \quad \mathrm{A}_0\,\mathrm{A}_0^* = \operatorname{grad}_{\Gamma_\tau} \operatorname{grad}_{\Gamma_\tau}^* = -\operatorname{grad}_{\Gamma_\tau} \operatorname{div}_{\Gamma_\nu} = -\Diamond_{\Gamma_\nu},$$
$$\mathrm{A}_1^*\,\mathrm{A}_1 = \operatorname{rot}_{\Gamma_\tau}^* \operatorname{rot}_{\Gamma_\tau} = \operatorname{rot}_{\Gamma_\nu} \operatorname{rot}_{\Gamma_\tau} =: \Box_{\Gamma_\tau}, \quad \mathrm{A}_1\,\mathrm{A}_1^* = \operatorname{rot}_{\Gamma_\tau} \operatorname{rot}_{\Gamma_\tau}^* = \operatorname{rot}_{\Gamma_\tau} \operatorname{rot}_{\Gamma_\nu} = \Box_{\Gamma_\nu},$$
$$\mathrm{A}_2^*\,\mathrm{A}_2 = \operatorname{div}_{\Gamma_\tau}^* \operatorname{div}_{\Gamma_\tau} = -\operatorname{grad}_{\Gamma_\nu} \operatorname{div}_{\Gamma_\tau} =: -\Diamond_{\Gamma_\tau}, \quad \mathrm{A}_2\,\mathrm{A}_2^* = \operatorname{div}_{\Gamma_\tau} \operatorname{div}_{\Gamma_\tau}^* = -\operatorname{div}_{\Gamma_\tau} \operatorname{grad}_{\Gamma_\nu} = -\Delta_{\Gamma_\nu}.$$

Again, M_2 and the operators involving A_2, A_2^* are redundant by interchanging the boundary conditions in M_0 and A_0, A_0^*. Hence, we may focus on c_{0,Γ_τ} and c_{1,Γ_τ}. Section 2.1.3 shows the following:

Theorem 2.17 (Friedrichs/Poincaré type constants)**.** *The Friedrichs/Poincaré type constants can be computed by the four Rayleigh quotations*

$$\frac{1}{c_{0,\Gamma_\tau}} = \lambda_{0,\Gamma_\tau} = \inf_{0 \neq u \in D(\operatorname{grad}_{\Gamma_\tau}) \cap \mathsf{L}^2_{\Gamma_\nu}(\Omega)} \frac{|\operatorname{grad} u|_{\mathsf{L}^2(\Omega)}}{|u|_{\mathsf{L}^2(\Omega)}} = \inf_{0 \neq E \in D(\operatorname{div}_{\Gamma_\nu}) \cap R(\operatorname{grad}_{\Gamma_\tau})} \frac{|\operatorname{div} E|_{\mathsf{L}^2(\Omega)}}{|E|_{\mathsf{L}^2(\Omega)}},$$
$$\frac{1}{c_{1,\Gamma_\tau}} = \lambda_{1,\Gamma_\tau} = \inf_{0 \neq E \in D(\operatorname{rot}_{\Gamma_\tau}) \cap R(\operatorname{rot}_{\Gamma_\nu})} \frac{|\operatorname{rot} E|_{\mathsf{L}^2(\Omega)}}{|E|_{\mathsf{L}^2(\Omega)}} = \inf_{0 \neq H \in D(\operatorname{rot}_{\Gamma_\nu}) \cap R(\operatorname{rot}_{\Gamma_\tau})} \frac{|\operatorname{rot} H|_{\mathsf{L}^2(\Omega)}}{|H|_{\mathsf{L}^2(\Omega)}}.$$

Moreover, λ_{0,Γ_τ} *is the smallest positive eigenvalue of*

$$\begin{bmatrix} 0 & \mathrm{A}_0^* \\ \mathrm{A}_0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\operatorname{div}_{\Gamma_\nu} \\ \operatorname{grad}_{\Gamma_\tau} & 0 \end{bmatrix}$$

and $\lambda_{0,\Gamma_\tau}^2$ *is the smallest (positive) eigenvalue of*

$$\mathrm{A}_0^*\,\mathrm{A}_0 = -\operatorname{div}_{\Gamma_\nu} \operatorname{grad}_{\Gamma_\tau} = -\Delta_{\Gamma_\tau} \quad \text{and} \quad \mathrm{A}_0\,\mathrm{A}_0^* = -\operatorname{grad}_{\Gamma_\tau} \operatorname{div}_{\Gamma_\nu} = -\Diamond_{\Gamma_\nu}.$$

λ_{1,Γ_τ} *is the smallest positive eigenvalue of*

$$\begin{bmatrix} 0 & \mathrm{A}_1^* \\ \mathrm{A}_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \operatorname{rot}_{\Gamma_\nu} \\ \operatorname{rot}_{\Gamma_\tau} & 0 \end{bmatrix}$$

and $\lambda_{1,\Gamma_\tau}^2$ *is the smallest (positive) eigenvalue of*

$$\mathrm{A}_1^*\,\mathrm{A}_1 = \operatorname{rot}_{\Gamma_\nu} \operatorname{rot}_{\Gamma_\tau} = \Box_{\Gamma_\tau} \quad \text{and} \quad \mathrm{A}_1\,\mathrm{A}_1^* = \operatorname{rot}_{\Gamma_\tau} \operatorname{rot}_{\Gamma_\nu} = \Box_{\Gamma_\nu}.$$

Remark 2.18 (variational formulations)**.** *All infima in Theorem 2.17 are minima and the respective minimisers* u_{0,Γ_τ}, E_{0,Γ_ν}, *and* E_{1,Γ_τ}, H_{1,Γ_ν} *are the eigenfunctions to the eigenvalues* $\lambda_{0,\Gamma_\tau}^2$ *and* $\lambda_{1,\Gamma_\tau}^2$, *i.e,*

$$\lambda_{0,\Gamma_\tau} = \frac{|\operatorname{grad} u_{0,\Gamma_\tau}|_{\mathsf{L}^2(\Omega)}}{|u_{0,\Gamma_\tau}|_{\mathsf{L}^2(\Omega)}} = \frac{|\operatorname{div} E_{0,\Gamma_\nu}|_{\mathsf{L}^2(\Omega)}}{|E_{0,\Gamma_\nu}|_{\mathsf{L}^2(\Omega)}}, \qquad \lambda_{1,\Gamma_\tau} = \frac{|\operatorname{rot} E_{1,\Gamma_\tau}|_{\mathsf{L}^2(\Omega)}}{|E_{1,\Gamma_\tau}|_{\mathsf{L}^2(\Omega)}} = \frac{|\operatorname{rot} H_{1,\Gamma_\nu}|_{\mathsf{L}^2(\Omega)}}{|H_{1,\Gamma_\nu}|_{\mathsf{L}^2(\Omega)}},$$

$$(-\Delta_{\Gamma_\tau} - \lambda_{0,\Gamma_\tau}^2) u_{0,\Gamma_\tau} = 0, \qquad u_{0,\Gamma_\tau} \in D(\Delta_{\Gamma_\tau}) \cap \mathsf{L}^2_{\Gamma_\nu}(\Omega) \subset D(\operatorname{grad}_{\Gamma_\tau}) \cap \mathsf{L}^2_{\Gamma_\nu}(\Omega),$$
$$(-\Diamond_{\Gamma_\nu} - \lambda_{0,\Gamma_\tau}^2) E_{0,\Gamma_\nu} = 0, \qquad E_{0,\Gamma_\nu} \in D(\Diamond_{\Gamma_\nu}) \cap R(\operatorname{grad}_{\Gamma_\tau}) \subset D(\operatorname{div}_{\Gamma_\nu}) \cap R(\operatorname{grad}_{\Gamma_\tau}),$$

$$(\square_{\Gamma_\tau} - \lambda_{1,\Gamma_\tau}^2)E_{1,\Gamma_\tau} = 0, \qquad E_{1,\Gamma_\tau} \in D(\square_{\Gamma_\tau}) \cap R(\mathrm{rot}_{\Gamma_\nu}) \subset D(\mathrm{rot}_{\Gamma_\tau}) \cap R(\mathrm{rot}_{\Gamma_\nu}),$$
$$(\square_{\Gamma_\nu} - \lambda_{1,\Gamma_\tau}^2)H_{1,\Gamma_\nu} = 0, \qquad H_{1,\Gamma_\nu} \in D(\square_{\Gamma_\nu}) \cap R(\mathrm{rot}_{\Gamma_\tau}) \subset D(\mathrm{rot}_{\Gamma_\nu}) \cap R(\mathrm{rot}_{\Gamma_\tau}).$$

Moreover, the eigenvectors satisfy the variational formulations

$$\begin{aligned}
&\forall\, \psi \in D(\mathrm{grad}_{\Gamma_\tau}) && \langle \mathrm{grad}\, u_{0,\Gamma_\tau}, \mathrm{grad}\, \psi\rangle_{\mathsf{L}^2(\Omega)} = \lambda_{0,\Gamma_\tau}^2 \langle u_{0,\Gamma_\tau}, \psi\rangle_{\mathsf{L}^2(\Omega)},\\
&\forall\, \Psi \in D(\mathrm{div}_{\Gamma_\nu}) && \langle \mathrm{div}\, E_{0,\Gamma_\nu}, \mathrm{div}\, \Psi\rangle_{\mathsf{L}^2(\Omega)} = \lambda_{0,\Gamma_\tau}^2 \langle E_{0,\Gamma_\nu}, \Psi\rangle_{\mathsf{L}^2(\Omega)},\\
&\forall\, \Phi \in D(\mathrm{rot}_{\Gamma_\tau}) && \langle \mathrm{rot}\, E_{1,\Gamma_\tau}, \mathrm{rot}\, \Phi\rangle_{\mathsf{L}^2(\Omega)} = \lambda_{1,\Gamma_\tau}^2 \langle E_{1,\Gamma_\tau}, \Phi\rangle_{\mathsf{L}^2(\Omega)},\\
&\forall\, \Theta \in D(\mathrm{rot}_{\Gamma_\nu}) && \langle \mathrm{rot}\, H_{1,\Gamma_\nu}, \mathrm{rot}\, \Theta\rangle_{\mathsf{L}^2(\Omega)} = \lambda_{1,\Gamma_\tau}^2 \langle H_{1,\Gamma_\nu}, \Theta\rangle_{\mathsf{L}^2(\Omega)}.
\end{aligned}$$

Remark 2.19. *We emphasise that Lemma 2.11 provides results for the vector Laplacian*

$$\mathrm{A}_0\,\mathrm{A}_0^* + \mathrm{A}_1^*\,\mathrm{A}_1 = -\lozenge_{\Gamma_\nu} + \square_{\Gamma_\tau} = -\,\mathrm{grad}_{\Gamma_\tau}\,\mathrm{div}_{\Gamma_\nu} + \mathrm{rot}_{\Gamma_\nu}\,\mathrm{rot}_{\Gamma_\tau},$$

which has been recently discussed in, e.g., [25].

2.2.1. *Know Results for the Constants in 3D.* Let us summarise and cite some recent results from [37, 38, 39, 41] about the Friedrichs/Poincaré type constants, i.e., about the Friedrichs/Poincaré constants λ_{0,Γ_τ} and the Maxwell constants λ_{1,Γ_τ}.

Theorem 2.20 (Friedrichs/Poincaré/Maxwell constants in 3D)**.** *For $c_{\ell,\Gamma_\tau} = 1/\lambda_{\ell,\Gamma_\tau}$ the following holds:*

(i) *The Friedrichs/Poincaré constants depend monotonically on the boundary conditions, i.e.,*

$$\emptyset \neq \widetilde{\Gamma}_\tau \subset \Gamma_\tau \quad \Rightarrow \quad c_{0,\Gamma_\tau} \leq c_{0,\widetilde{\Gamma}_\tau}.$$

(ii) *The Friedrichs constant is always smaller than the Poincaré constant, i.e.,*

$$c_{0,\Gamma} \leq c_{0,\emptyset},$$

where $c_{0,\Gamma}$ is the classical Friedrichs constant and $c_{0,\emptyset}$ is the classical Poincaré constant. Moreover, $\lambda_{0,\Gamma}$ is usually called the first Dirichlet/Laplace eigenvalue and $\lambda_{0,\emptyset}$ is usually called the second Neumann/Laplace eigenvalue.

(iii) $c_{0,\Gamma} \leq \mathrm{diam}(\Omega)/\pi$

(iv) $c_{0,\Gamma_\tau} = c_{2,\Gamma_\nu}$

(v) $c_{1,\Gamma_\tau} = c_{1,\Gamma_\nu}$

(vi) $c_{0,\Gamma} \leq c_{0,\Gamma_\tau} \leq c_{0,1,\Gamma_\tau} = \max\{c_{0,\Gamma_\tau}, c_{1,\Gamma_\tau}\}$

(vii) *If Ω is convex, then $c_{0,\Gamma} \leq c_{0,\emptyset} \leq \mathrm{diam}(\Omega)/\pi$.*

(viii) *If Ω is convex, then $c_{1,\Gamma} = c_{1,\emptyset} \leq c_{0,\emptyset} \leq \mathrm{diam}(\Omega)/\pi$.*

(ix) *If Ω is convex, then $c_{0,\Gamma} \leq c_{0,1,\Gamma} = \max\{c_{0,\Gamma}, c_{1,\Gamma}\} \leq c_{0,\emptyset} \leq \mathrm{diam}(\Omega)/\pi$.*

(ix') *If Ω is convex, then $c_{0,\Gamma} \leq c_{0,1,\emptyset} = \max\{c_{0,\emptyset}, c_{1,\emptyset}\} = c_{0,\emptyset} \leq \mathrm{diam}(\Omega)/\pi$.*

Remark 2.21. *To the best of our knowledge, it is an open question whether or not*

$$c_{0,\Gamma_\tau} \leq c_{1,\Gamma_\tau} \qquad \text{or at least} \qquad c_{0,\Gamma} \leq c_{1,\Gamma}$$

holds in general.

2.3. **Other Complexes and Constants.** So far, we have discussed the de Rham complex (14) in 3D. While in higher dimensions $N \geq 4$ the situation is very similar to the 3D case (but the adjoint of $\mathrm{rot}_{\Gamma_\tau}$ is no longer a rotation itself), the situations in 1D and 2D are much simpler. Moreover, similar to the 3D-de Rham complex (14), other important complexes of shape (12) fit nicely into our general fa-toolbox and, therefore, are handleable with our theory, see also [42, 40] for details.

2.3.1. *1D-de Rham Complex, Laplace and Maxwell Constants in 1D.* In 1D the domain Ω is an interval and we have just one operator $\mathrm{A}_0 = \mathrm{grad}_{\Gamma_\tau} = (\,\cdot\,)'_{\Gamma_\tau}$ with adjoint $\mathrm{A}_0^* = -\,\mathrm{div}_{\Gamma_\nu} = -(\,\cdot\,)'_{\Gamma_\nu}$, i.e., the complex (12), compare to (14), reads

$$\mathsf{H}^1_{\Gamma_\tau}(\Omega) = \mathsf{H}_{\Gamma_\tau}(\mathrm{grad}, \Omega) \xrightarrow{\mathrm{A}_0 = \mathrm{grad}_{\Gamma_\tau} = (\,\cdot\,)'_{\Gamma_\tau}} \mathsf{L}^2(\Omega),$$
$$\mathsf{L}^2(\Omega) \xleftarrow{\mathrm{A}_0^* = -\,\mathrm{div}_{\Gamma_\nu} = -(\,\cdot\,)'_{\Gamma_\nu}} \mathsf{H}_{\Gamma_\nu}(\mathrm{div}, \Omega) = \mathsf{H}^1_{\Gamma_\nu}(\Omega).$$

Hence, just the Laplacians $\Delta_{\Gamma_\tau} = \operatorname{div}_{\Gamma_\nu} \operatorname{grad}_{\Gamma_\tau} = (\,\cdot\,)''_{\Gamma_\tau}$ and $\Diamond_{\Gamma_\nu} = \operatorname{grad}_{\Gamma_\tau} \operatorname{div}_{\Gamma_\nu} = (\,\cdot\,)''_{\Gamma_\nu}$ exist and there are no Maxwell operators. The crucial compact embedding (13) is simply Rellich's selection theorem, compare to Theorem 2.14. Moreover, here in the 1D case we have

$$\begin{aligned}\lambda_{0,\Gamma_\tau} &= \inf_{0\neq u\in \mathsf{H}^1_{\Gamma_\tau}(\Omega)\cap \mathsf{L}^2_{\Gamma_\nu}(\Omega)} \frac{|\operatorname{grad} u|_{\mathsf{L}^2(\Omega)}}{|u|_{\mathsf{L}^2(\Omega)}} = \inf_{0\neq u\in \mathsf{H}^1_{\Gamma_\tau}(\Omega)\cap \mathsf{L}^2_{\Gamma_\nu}(\Omega)} \frac{|u'|_{\mathsf{L}^2(\Omega)}}{|u|_{\mathsf{L}^2(\Omega)}}\\ &= \underbrace{\inf_{0\neq E\in \mathsf{H}_{\Gamma_\nu}(\operatorname{div},\Omega)\cap R(\operatorname{grad}_{\Gamma_\tau})} \frac{|\operatorname{div} E|_{\mathsf{L}^2(\Omega)}}{|E|_{\mathsf{L}^2(\Omega)}}}_{=\lambda_{2,\Gamma_\nu}} = \inf_{0\neq E\in \mathsf{H}^1_{\Gamma_\nu}(\Omega)\cap \mathsf{L}^2_{\Gamma_\tau}(\Omega)} \frac{|E'|_{\mathsf{L}^2(\Omega)}}{|E|_{\mathsf{L}^2(\Omega)}} = \lambda_{0,\Gamma_\nu},\end{aligned}$$

i.e., it is sufficient to compute the eigenvalues λ_{0,Γ_τ}, and we can also give a meaning to λ_{2,Γ_ν}. Thus

$$\lambda_{0,\Gamma_\tau} = \lambda_{0,\Gamma_\nu} = \lambda_{2,\Gamma_\nu} = \frac{1}{c_{2,\Gamma_\nu}} = \frac{1}{c_{0,\Gamma_\nu}} = \frac{1}{c_{0,\Gamma_\tau}}.$$

Note that

$$\lambda_{0,\Gamma_\tau} = \frac{|\operatorname{grad} u_{0,\Gamma_\tau}|_{\mathsf{L}^2(\Omega)}}{|u_{0,\Gamma_\tau}|_{\mathsf{L}^2(\Omega)}} = \frac{|u'_{0,\Gamma_\tau}|_{\mathsf{L}^2(\Omega)}}{|u_{0,\Gamma_\tau}|_{\mathsf{L}^2(\Omega)}} = \frac{|\operatorname{div} E_{0,\Gamma_\nu}|_{\mathsf{L}^2(\Omega)}}{|E_{0,\Gamma_\nu}|_{\mathsf{L}^2(\Omega)}} = \frac{|E'_{0,\Gamma_\nu}|_{\mathsf{L}^2(\Omega)}}{|E_{0,\Gamma_\nu}|_{\mathsf{L}^2(\Omega)}} = \lambda_{0,\Gamma_\nu}.$$

Theorem 2.20 turns to:

Corollary 2.22 (Friedrichs/Poincaré/Maxwell constants in 1D). *For $c_{\ell,\Gamma_\tau} = 1/\lambda_{\ell,\Gamma_\tau}$ the following holds:*

- **(i)** $\emptyset \neq \widetilde{\Gamma}_\tau \subset \Gamma_\tau \;\Rightarrow\; c_{0,\Gamma_\tau} \leq c_{0,\widetilde{\Gamma}_\tau}$
- **(ii)** $c_{0,\Gamma} = c_{0,\emptyset} \leq \operatorname{diam}(\Omega)/\pi$
- **(iii)** $c_{0,\Gamma} \leq c_{0,\Gamma_\tau} = c_{0,\Gamma_\nu}$
- **(iv)** *There is no c_{1,Γ_τ}, but $c_{2,\Gamma_\nu} = c_{0,\Gamma_\nu} = c_{0,\Gamma_\tau}$.*

2.3.2. *2D-de Rham Complex, Laplace and Maxwell Constants in 2D.* In 2D there are just the two operators $\mathrm{A}_0 = \operatorname{grad}_{\Gamma_\tau}$ and $\mathrm{A}_1 = \operatorname{rot}_{\Gamma_\tau} = \operatorname{div}_{\Gamma_\tau} R$ with adjoints $\mathrm{A}_0^* = -\operatorname{div}_{\Gamma_\nu}$ and $\mathrm{A}_1^* = \vec{\operatorname{rot}}_{\Gamma_\nu} = R \operatorname{grad}_{\Gamma_\nu}$, where

$$\operatorname{rot} E = \operatorname{div} RE = \partial_1 E_2 - \partial_2 E_1, \qquad \vec{\operatorname{rot}} u = R \operatorname{grad} u = \begin{bmatrix} \partial_2 u \\ -\partial_1 u \end{bmatrix}, \qquad R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and the complex (12), compare to (14), reads

$$\mathsf{H}^1_{\Gamma_\tau}(\Omega) = \mathsf{H}_{\Gamma_\tau}(\operatorname{grad},\Omega) \xrightarrow{\mathrm{A}_0=\operatorname{grad}_{\Gamma_\tau}} \mathsf{H}_{\Gamma_\tau}(\operatorname{rot},\Omega) = \mathsf{H}_{\Gamma_\tau}(\operatorname{div},\Omega)R \xrightarrow{\mathrm{A}_1=\operatorname{rot}_{\Gamma_\tau}} \mathsf{L}^2(\Omega),$$

$$\mathsf{L}^2(\Omega) \xleftarrow{\mathrm{A}_0^*=-\operatorname{div}_{\Gamma_\nu}} \mathsf{H}_{\Gamma_\nu}(\operatorname{div},\Omega) \xleftarrow{\mathrm{A}_1^*=\vec{\operatorname{rot}}_{\Gamma_\nu}} \mathsf{H}_{\Gamma_\nu}(\vec{\operatorname{rot}},\Omega) = \mathsf{H}^1_{\Gamma_\nu}(\Omega).$$

Hence, we have the Laplacian $\Delta_{\Gamma_\tau} = \operatorname{div}_{\Gamma_\nu} \operatorname{grad}_{\Gamma_\tau}$ and $\Diamond_{\Gamma_\nu} = \operatorname{grad}_{\Gamma_\tau} \operatorname{div}_{\Gamma_\nu}$, as well as the second order Maxwell operators (related to the 3D notations)

$$\begin{aligned}\Box_{\Gamma_\tau} &= \vec{\operatorname{rot}}_{\Gamma_\nu} \operatorname{rot}_{\Gamma_\tau} = R \operatorname{grad}_{\Gamma_\nu} \operatorname{div}_{\Gamma_\tau} R = R \Diamond_{\Gamma_\tau} R,\\ \Box_{\Gamma_\nu} &= \operatorname{rot}_{\Gamma_\tau} \vec{\operatorname{rot}}_{\Gamma_\nu} = \operatorname{div}_{\Gamma_\tau} RR \operatorname{grad}_{\Gamma_\nu} = -\operatorname{div}_{\Gamma_\tau} \operatorname{grad}_{\Gamma_\nu} = -\Delta_{\Gamma_\nu}.\end{aligned}$$

By Lemma 2.7 the crucial compact embedding (13) is just Rellich's selection theorem, compare to Theorem 2.14. Moreover, here in the 2D case we have

$$\begin{aligned}\lambda_{0,\Gamma_\tau} &= \inf_{0\neq u\in \mathsf{H}^1_{\Gamma_\tau}(\Omega)\cap \mathsf{L}^2_{\Gamma_\nu}(\Omega)} \frac{|\operatorname{grad} u|_{\mathsf{L}^2(\Omega)}}{|u|_{\mathsf{L}^2(\Omega)}} = \inf_{0\neq u\in \mathsf{H}^1_{\Gamma_\tau}(\Omega)\cap \mathsf{L}^2_{\Gamma_\nu}(\Omega)} \frac{|\vec{\operatorname{rot}} u|_{\mathsf{L}^2(\Omega)}}{|u|_{\mathsf{L}^2(\Omega)}}\\ &= \underbrace{\inf_{0\neq E\in \mathsf{H}_{\Gamma_\nu}(\operatorname{div},\Omega)\cap R(\operatorname{grad}_{\Gamma_\tau})} \frac{|\operatorname{div} E|_{\mathsf{L}^2(\Omega)}}{|E|_{\mathsf{L}^2(\Omega)}}}_{=\lambda_{2,\Gamma_\nu}} \overset{E\rightsquigarrow RE}{=} \inf_{0\neq E\in \mathsf{H}_{\Gamma_\nu}(\operatorname{rot},\Omega)\cap R(\vec{\operatorname{rot}}_{\Gamma_\tau})} \frac{|\operatorname{rot} E|_{\mathsf{L}^2(\Omega)}}{|E|_{\mathsf{L}^2(\Omega)}} = \lambda_{1,\Gamma_\nu},\end{aligned}$$

i.e., it is sufficient to compute the eigenvalues λ_{0,Γ_τ}, and we can also give a meaning to λ_{2,Γ_ν}. Thus

$$\lambda_{0,\Gamma_\tau} = \lambda_{1,\Gamma_\nu} = \lambda_{2,\Gamma_\nu} = \frac{1}{c_{2,\Gamma_\nu}} = \frac{1}{c_{1,\Gamma_\nu}} = \frac{1}{c_{0,\Gamma_\tau}}.$$

Note that

$$\lambda_{0,\Gamma_\tau} = \frac{|\operatorname{grad} u_{0,\Gamma_\tau}|_{\mathsf{L}^2(\Omega)}}{|u_{0,\Gamma_\tau}|_{\mathsf{L}^2(\Omega)}} = \frac{|\vec{\operatorname{rot}} u_{0,\Gamma_\tau}|_{\mathsf{L}^2(\Omega)}}{|u_{0,\Gamma_\tau}|_{\mathsf{L}^2(\Omega)}} = \frac{|\operatorname{div} E_{0,\Gamma_\nu}|_{\mathsf{L}^2(\Omega)}}{|E_{0,\Gamma_\nu}|_{\mathsf{L}^2(\Omega)}} = \frac{|\operatorname{rot} E_{1,\Gamma_\nu}|_{\mathsf{L}^2(\Omega)}}{|E_{1,\Gamma_\nu}|_{\mathsf{L}^2(\Omega)}} = \lambda_{1,\Gamma_\nu},$$

i.e., in our 3D-notation $H_{1,\Gamma_\tau} = u_{0,\Gamma_\tau}$ and $E_{0,\Gamma_\nu} = RE_{1,\Gamma_\nu}$. Theorem 2.20 turns to:

Corollary 2.23 (Friedrichs/Poincaré/Maxwell constants in 2D)**.** *For $c_{\ell,\Gamma_\tau}=1/\lambda_{\ell,\Gamma_\tau}$ the following holds:*

(i) *The Friedrichs/Poincaré constants depend monotonically on the boundary conditions, i.e.,*

$$\emptyset\neq\widetilde{\Gamma}_\tau\subset\Gamma_\tau\quad\Rightarrow\quad c_{0,\Gamma_\tau}\leq c_{0,\widetilde{\Gamma}_\tau}.$$

(ii) *The Friedrichs constant is always smaller than the Poincaré constant, i.e., $c_{0,\Gamma}\leq c_{0,\emptyset}$.*

(iii) $c_{0,\Gamma}\leq\operatorname{diam}(\Omega)/\pi$

(iv) $c_{0,\Gamma}\leq c_{0,\Gamma_\tau}=c_{1,\Gamma_\nu}=c_{2,\Gamma_\nu}\leq c_{0,1,\Gamma_\tau}=\max\{c_{0,\Gamma_\tau},c_{1,\Gamma_\tau}\}=\max\{c_{0,\Gamma_\tau},c_{0,\Gamma_\nu}\}$

(v) *If Ω is convex, then $c_{0,\Gamma}\leq c_{0,\emptyset}\leq\operatorname{diam}(\Omega)/\pi$.*

2.3.3. *ND-de Rham Complex, Laplace and Maxwell Constants in ND.* In ND it is helpful to use differential forms instead of functions and vector fields. The de Rham complex of (12) in ND, compare to (14), consists of N differential operators $\mathrm{A}_q:=\mathrm{A}_{q,\Gamma_\tau}:=\mathrm{d}_{q,\Gamma_\tau}$, $q=0,\dots,N-1$, with adjoints $\mathrm{A}_q^*=\mathrm{A}_{q,\Gamma_\tau}^*=-\,\delta_{q+1,\Gamma_\nu}$ acting on alternating q resp. $(q+1)$-forms, i.e.,

$$\cdots\xrightarrow{\ \cdots\ }\mathsf{H}_{\Gamma_\tau}(\mathrm{d}_q,\Omega)\xrightarrow{\mathrm{A}_q=\mathrm{d}_{q,\Gamma_\tau}}\mathsf{H}_{\Gamma_\tau}(\mathrm{d}_{q+1},\Omega)\xrightarrow{\mathrm{A}_{q+1}=\mathrm{d}_{q+1,\Gamma_\tau}}\cdots,$$
$$\cdots\xleftarrow{\mathrm{A}_{q-1}^*=-\,\delta_{q,\Gamma_\nu}}\mathsf{H}_{\Gamma_\nu}(\delta_q,\Omega)\xleftarrow{\mathrm{A}_q^*=-\,\delta_{q+1,\Gamma_\nu}}\mathsf{H}_{\Gamma_\nu}(\delta_{q+1},\Omega)\xleftarrow{\ \cdots\ }\cdots,$$

see, e.g., [32, 33, 6, 5, 8, 2, 23, 24] for details about the complex and numerical applications. Hence, the second order "Laplace" and "Maxwell" operators are simply

$$\mathrm{A}_q^*\,\mathrm{A}_q=-\,\delta_{q+1,\Gamma_\nu}\,\mathrm{d}_{q,\Gamma_\tau},\qquad\mathrm{A}_q\,\mathrm{A}_q^*=-\,\mathrm{d}_{q,\Gamma_\tau}\,\delta_{q+1,\Gamma_\nu},$$

and for the constants and eigenvalues $c_{q,\Gamma_\tau}=1/\lambda_{q,\Gamma_\tau}$ we have

$$\begin{aligned}&\forall\,E\in D(\mathcal{A}_q)=D(\mathrm{d}_{q,\Gamma_\tau})\cap R(\delta_{q+1,\Gamma_\nu})&&|E|_{\mathsf{L}^{2,q}(\Omega)}\leq c_{q,\Gamma_\tau}|\,\mathrm{d}_q\,E|_{\mathsf{L}^{2,q+1}(\Omega)},\\&\forall\,H\in D(\mathcal{A}_q^*)=D(\delta_{q+1,\Gamma_\nu})\cap R(\mathrm{d}_{q,\Gamma_\tau})&&|H|_{\mathsf{L}^{2,q+1}(\Omega)}\leq c_{q,\Gamma_\tau}|\,\delta_{q+1}\,H|_{\mathsf{L}^{2,q}(\Omega)}.\end{aligned}$$

The crucial compact embeddings (13) are given by the following theorem from [15, Theorem 4.9] or [14, Theorem 4.8].

Theorem 2.24 (Weck's selection theorem)**.** *The embeddings*

$$D(\mathrm{A}_q)\cap D(\mathrm{A}_{q-1}^*)=\mathsf{H}_{\Gamma_\tau}(\mathrm{d}_q,\Omega)\cap\mathsf{H}_{\Gamma_\nu}(\delta_q,\Omega)\hookrightarrow\mathsf{L}^{2,q}(\Omega)$$

are compact.

The general theory, the definition $\delta_{q+1,\Gamma_\nu}=\pm\star_{N-q}\mathrm{d}_{N-q-1,\Gamma_\nu}\star_{q+1}$, where $\star_q$ is the Hodge star-operator, and the substitution $E=\star_{q+1}H$ show again a symmetry for the eigenvalues, i.e.,

$$\begin{aligned}\lambda_{q,\Gamma_\tau}&=\inf_{0\neq E\in D(\mathcal{A}_{q,\Gamma_\tau})=D(\mathrm{d}_{q,\Gamma_\tau})\cap R(\delta_{q+1,\Gamma_\nu})}\frac{|\,\mathrm{d}_q\,E|_{\mathsf{L}^{2,q+1}(\Omega)}}{|E|_{\mathsf{L}^{2,q}(\Omega)}}\\&=\inf_{0\neq H\in D(\mathcal{A}_{q,\Gamma_\tau}^*)=D(\delta_{q+1,\Gamma_\nu})\cap R(\mathrm{d}_{q,\Gamma_\tau})}\frac{|\,\delta_{q+1}\,H|_{\mathsf{L}^{2,q}(\Omega)}}{|H|_{\mathsf{L}^{2,q+1}(\Omega)}}\\&=\inf_{0\neq E\in D(\mathcal{A}_{N-q-1,\Gamma_\nu})=D(\mathrm{d}_{N-q-1,\Gamma_\nu})\cap R(\delta_{N-q,\Gamma_\tau})}\frac{|\,\mathrm{d}_{N-q-1}\,E|_{\mathsf{L}^{2,N-q}(\Omega)}}{|E|_{\mathsf{L}^{2,N-q-1}(\Omega)}}=\lambda_{N-q-1,\Gamma_\nu}.\end{aligned}$$

Therefore, we obtain the relations

$$\frac{1}{c_{q,\Gamma_\tau}}=\lambda_{q,\Gamma_\tau}=\lambda_{N-q-1,\Gamma_\nu}=\frac{1}{c_{N-q-1,\Gamma_\nu}},$$

which also confirm (for $N=1,2$) the results of Sections 2.3.1 and 2.3.2. Using the notations from the 3D case we define

$$\frac{1}{c_{q-1,q,\Gamma_\tau}^2}:=\lambda_{q-1,q,\Gamma_\tau}^2:=\inf\frac{|\,\mathrm{d}_q\,E|_{\mathsf{L}^{2,q+1}(\Omega)}^2+|\,\delta_q\,E|_{\mathsf{L}^{2,q-1}(\Omega)}^2}{|E|_{\mathsf{L}^{2,q}(\Omega)}^2},$$

where the infimum is taken over all $0\neq E\in D(\mathrm{A}_{q,\Gamma_\tau})\cap D(\mathrm{A}_{q-1,\Gamma_\tau}^*)=D(\mathrm{d}_{q,\Gamma_\tau})\cap D(\delta_{q,\Gamma_\nu})$ being perpendicular to the respective generalised Dirichlet-Neumann forms $N(\mathrm{A}_{q,\Gamma_\tau})\cap N(\mathrm{A}_{q-1,\Gamma_\tau}^*)$. Theorem 2.20 turns to:

Theorem 2.25 (Friedrichs/Poincaré/Maxwell constants in ND)**.** *For $c_{q,\Gamma_\tau}=1/\lambda_{q,\Gamma_\tau}$ the following holds:*

(i) *The Friedrichs/Poincaré constants depend monotonically on the boundary conditions, i.e.,*

$$\emptyset\neq\widetilde{\Gamma}_\tau\subset\Gamma_\tau\quad\Rightarrow\quad c_{0,\Gamma_\tau}\leq c_{0,\widetilde{\Gamma}_\tau}.$$

(ii) *The Friedrichs constant is always smaller than the Poincaré constant, i.e.,* $c_{0,\Gamma} \leq c_{0,\emptyset}$.
(iii) $c_{0,\Gamma} \leq \operatorname{diam}(\Omega)/\pi$
(iv) $c_{q,\Gamma_\tau} = c_{N-q-1,\Gamma_\nu}$
(v) $c_{q-1,q,\Gamma_\tau} = \max\{c_{q-1,\Gamma_\tau}, c_{q,\Gamma_\tau}\}$
(vi) *If* Ω *is topologically trivial, then* $c_{0,\Gamma} \leq c_{q-1,q,\Gamma_\tau}$.
(vii) *If* Ω *is convex, then* $c_{0,\Gamma} \leq c_{0,\emptyset} \leq \operatorname{diam}(\Omega)/\pi$.
(viii) *If* Ω *is convex, then* $c_{q,\Gamma}$, $c_{q,\emptyset} \leq c_{0,\emptyset} \leq \operatorname{diam}(\Omega)/\pi$.
(ix) *If* Ω *is convex, then* $c_{0,\Gamma} \leq c_{q-1,q,\Gamma} = \max\{c_{q-1,\Gamma}, c_{q,\Gamma}\} \leq c_{0,\emptyset} \leq \operatorname{diam}(\Omega)/\pi$.
(ix') *If* Ω *is convex, then* $c_{0,\Gamma} \leq c_{q-1,q,\emptyset} = \max\{c_{q-1,\emptyset}, c_{q,\emptyset}\} \leq c_{0,\emptyset} \leq \operatorname{diam}(\Omega)/\pi$.

For proofs and details see [41]. To show (vi), for which an argument is missing in [41], let I be a multi-index of order q and let $u \in \mathsf{H}^1_\Gamma(\Omega) = \mathsf{H}_\Gamma(\operatorname{grad}, \Omega)$. Then

$$E := u \,\mathrm{d}\, x^I \in \mathsf{H}^{1,q}_\Gamma(\Omega) \subset \mathsf{H}_{\Gamma_\tau}(\mathrm{d}_q, \Omega) \cap \mathsf{H}_{\Gamma_\nu}(\delta_q, \Omega)$$

and we have by approximation and the triviality of Dirichlet-Neumann forms

$$|u|_{\mathsf{L}^2(\Omega)} = |E|_{\mathsf{L}^{2,q}(\Omega)} \leq c_{q-1,q,\Gamma_\tau} \big(|\,\mathrm{d}_q E|^2_{\mathsf{L}^{2,q+1}(\Omega)} + |\,\delta_q E|^2_{\mathsf{L}^{2,q-1}(\Omega)}\big)^{1/2}$$
$$= c_{q-1,q,\Gamma_\tau} \Big(\sum_{\ell=1}^N |\,\partial_\ell E|^2_{\mathsf{L}^{2,q}(\Omega)}\Big)^{1/2} = c_{q-1,q,\Gamma_\tau} |\operatorname{grad} u|_{\mathsf{L}^2(\Omega)},$$

showing $c_{0,\Gamma} \leq c_{q-1,q,\Gamma_\tau}$.

2.3.4. *3D-Elasticity Complex.* The complex (involving vector as well as symmetric tensor fields)

$$\mathsf{H}_{\Gamma_\tau}(\operatorname{Grad}, \Omega) \xrightarrow{\mathrm{A}_0 = \operatorname{sym}\operatorname{Grad}_{\Gamma_\tau}} \mathsf{H}_{\Gamma_\tau}(\operatorname{Rot}\operatorname{Rot}^\top, \mathbb{S}, \Omega) \xrightarrow{\mathrm{A}_1 = \operatorname{Rot}\operatorname{Rot}^\top_{\Gamma_\tau}} \mathsf{H}_{\Gamma_\tau}(\operatorname{Div}, \mathbb{S}, \Omega) \xrightarrow{\mathrm{A}_2 = \operatorname{Div}_{\Gamma_\tau}} \mathsf{L}^2(\Omega),$$
$$\mathsf{L}^2(\Omega) \xleftarrow{\mathrm{A}_0^* = -\operatorname{Div}_{\Gamma_\nu}} \mathsf{H}_{\Gamma_\nu}(\operatorname{Div}, \mathbb{S}, \Omega) \xleftarrow{\mathrm{A}_1^* = \operatorname{Rot}\operatorname{Rot}^\top_{\Gamma_\nu}} \mathsf{H}_{\Gamma_\nu}(\operatorname{Rot}\operatorname{Rot}^\top, \mathbb{S}, \Omega) \xleftarrow{\mathrm{A}_2^* = \operatorname{sym}\operatorname{Grad}_{\Gamma_\nu}} \mathsf{H}_{\Gamma_\nu}(\operatorname{Grad}, \Omega),$$

is related to elasticity, see, e.g., [8, 9, 10, 7, 3, 53, 4, 16, 46] for details about the complex and numerical applications. Note that, indeed, by Korn's inequality the regularity

$$D(\mathrm{A}_0) = D(\operatorname{sym}\operatorname{Grad}_{\Gamma_\tau}) = \mathsf{H}_{\Gamma_\tau}(\operatorname{sym}\operatorname{Grad}, \Omega) = \mathsf{H}_{\Gamma_\tau}(\operatorname{Grad}, \Omega) = \mathsf{H}^1_{\Gamma_\tau}(\Omega)$$

holds. The "second order Laplace and Maxwell" operators are given by

$$\mathrm{A}_0^* \mathrm{A}_0 = -\operatorname{Div}_{\Gamma_\nu} \operatorname{sym}\operatorname{Grad}_{\Gamma_\tau}, \qquad \mathrm{A}_0 \mathrm{A}_0^* = -\operatorname{sym}\operatorname{Grad}_{\Gamma_\tau} \operatorname{Div}_{\Gamma_\nu},$$
$$\mathrm{A}_1^* \mathrm{A}_1 = \operatorname{Rot}\operatorname{Rot}^\top_{\Gamma_\nu} \operatorname{Rot}\operatorname{Rot}^\top_{\Gamma_\tau}, \qquad \mathrm{A}_1 \mathrm{A}_1^* = \operatorname{Rot}\operatorname{Rot}^\top_{\Gamma_\tau} \operatorname{Rot}\operatorname{Rot}^\top_{\Gamma_\nu},$$

and for the constants and eigenvalues $c^{\mathsf{ela}}_{\ell,\Gamma_\tau} = 1/\lambda^{\mathsf{ela}}_{\ell,\Gamma_\tau}$ we have

$$\forall\, v \in D(\mathcal{A}_0) = D(\operatorname{sym}\operatorname{Grad}_{\Gamma_\tau}) \cap R(\operatorname{Div}_{\Gamma_\nu}) \qquad |v|_{\mathsf{L}^2(\Omega)} \leq c^{\mathsf{ela}}_{0,\Gamma_\tau} |\operatorname{sym}\operatorname{Grad} v|_{\mathsf{L}^2(\mathbb{S},\Omega)},$$
$$\forall\, S \in D(\mathcal{A}_0^*) = D(\operatorname{Div}_{\Gamma_\nu}) \cap R(\operatorname{sym}\operatorname{Grad}_{\Gamma_\tau}) \qquad |S|_{\mathsf{L}^2(\mathbb{S},\Omega)} \leq c^{\mathsf{ela}}_{0,\Gamma_\tau} |\operatorname{Div} S|_{\mathsf{L}^2(\Omega)},$$
$$\forall\, S \in D(\mathcal{A}_1) = D(\operatorname{Rot}\operatorname{Rot}^\top_{\Gamma_\tau}) \cap R(\operatorname{Rot}\operatorname{Rot}^\top_{\Gamma_\nu}) \qquad |S|_{\mathsf{L}^2(\mathbb{S},\Omega)} \leq c^{\mathsf{ela}}_{1,\Gamma_\tau} |\operatorname{Rot}\operatorname{Rot}^\top S|_{\mathsf{L}^2(\mathbb{S},\Omega)},$$
$$\forall\, T \in D(\mathcal{A}_1^*) = D(\operatorname{Rot}\operatorname{Rot}^\top_{\Gamma_\nu}) \cap R(\operatorname{Rot}\operatorname{Rot}^\top_{\Gamma_\tau}) \qquad |T|_{\mathsf{L}^2(\mathbb{S},\Omega)} \leq c^{\mathsf{ela}}_{1,\Gamma_\tau} |\operatorname{Rot}\operatorname{Rot}^\top T|_{\mathsf{L}^2(\mathbb{S},\Omega)},$$
$$\forall\, T \in D(\mathcal{A}_2) = D(\operatorname{Div}_{\Gamma_\tau}) \cap R(\operatorname{sym}\operatorname{Grad}_{\Gamma_\nu}) \qquad |T|_{\mathsf{L}^2(\mathbb{S},\Omega)} \leq c^{\mathsf{ela}}_{2,\Gamma_\tau} |\operatorname{Div} T|_{\mathsf{L}^2(\Omega)},$$
$$\forall\, v \in D(\mathcal{A}_2^*) = D(\operatorname{sym}\operatorname{Grad}_{\Gamma_\nu}) \cap R(\operatorname{Div}_{\Gamma_\tau}) \qquad |v|_{\mathsf{L}^2(\Omega)} \leq c^{\mathsf{ela}}_{2,\Gamma_\tau} |\operatorname{sym}\operatorname{Grad} v|_{\mathsf{L}^2(\mathbb{S},\Omega)}.$$

As in the 3D Maxwell case the last two inequalities are already given by the first two. Note that

$$N(\operatorname{sym}\operatorname{Grad}_{\Gamma_\tau}) = \begin{cases} \{0\} & \text{if } \Gamma_\tau \neq \emptyset, \\ \mathsf{RM} & \text{if } \Gamma_\tau = \emptyset, \end{cases}$$
$$R(\operatorname{Div}_{\Gamma_\nu}) = N(\operatorname{sym}\operatorname{Grad}_{\Gamma_\tau})^{\perp_{\mathsf{L}^2(\Omega)}} = \begin{cases} \mathsf{L}^2(\Omega) & \text{if } \Gamma_\nu \neq \Gamma, \\ \mathsf{L}^2(\Omega) \cap \mathsf{RM}^{\perp_{\mathsf{L}^2(\Omega)}} & \text{if } \Gamma_\nu = \Gamma, \end{cases}$$

where RM denotes the space of global rigid motions. The crucial compact embeddings (13) have recently been proved in [44].

Theorem 2.26 (selection theorems for elasticity)**.** *The embedding*

$$D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) = \mathsf{H}_{\Gamma_\tau}(\operatorname{Rot}\operatorname{Rot}^\top, \mathbb{S}, \Omega) \cap \mathsf{H}_{\Gamma_\nu}(\operatorname{Div}, \mathbb{S}, \Omega) \hookrightarrow \mathsf{L}^2(\mathbb{S}, \Omega)$$

is compact.

Note that by the latter theorem the embedding

$$D(\mathrm{A}_2)\cap D(\mathrm{A}_1^*)=\mathsf{H}_{\Gamma_\tau}(\operatorname{Div},\mathbb{S},\Omega)\cap\mathsf{H}_{\Gamma_\nu}(\operatorname{Rot}\operatorname{Rot}^\top,\mathbb{S},\Omega)\hookrightarrow\mathsf{L}^2(\mathbb{S},\Omega)$$

is compact as well by interchanging Γ_τ and Γ_ν.

Similar to the 3D Maxwell case we get the following theorem, cf. Theorem 2.20.

Theorem 2.27 (Friedrichs/Poincaré type constants for elasticity). *For $c^{\mathsf{ela}}_{\ell,\Gamma_\tau}=1/\lambda^{\mathsf{ela}}_{\ell,\Gamma_\tau}$ the following holds:*

(i) *The Friedrichs/Poincaré type constants depend monotonically on the boundary conditions, i.e.,*

$$\emptyset\neq\widetilde{\Gamma}_\tau\subset\Gamma_\tau\quad\Rightarrow\quad c^{\mathsf{ela}}_{0,\Gamma_\tau}\leq c^{\mathsf{ela}}_{0,\widetilde{\Gamma}_\tau}.$$

(ii) $c^{\mathsf{ela}}_{0,\Gamma_\tau}=c^{\mathsf{ela}}_{2,\Gamma_\nu}$ *and* $c^{\mathsf{ela}}_{1,\Gamma_\tau}=c^{\mathsf{ela}}_{1,\Gamma_\nu}$.

Remark 2.28 (Friedrichs/Poincaré type constants for elasticity). *The Friedrichs/Poincaré type constants of the elasticity complex $c^{\mathsf{ela}}_{0,\Gamma_\tau}=c^{\mathsf{ela}}_{2,\Gamma_\nu}$ are related to the classical Friedrichs/Poincaré constants $c_{0,\Gamma_\tau}=c_{2,\Gamma_\nu}$ by Korn's inequality, i.e.,*

$$\forall\,v\in D(\mathcal{A}_0)=\underbrace{D(\operatorname{sym}\operatorname{Grad}_{\Gamma_\tau})}_{=\mathsf{H}^1_{\Gamma_\tau}(\Omega)}\cap R(\operatorname{Div}_{\Gamma_\nu})\qquad|\operatorname{Grad}v|_{\mathsf{L}^2(\Omega)}\leq c_{\mathsf{k},\Gamma_\tau}|\operatorname{sym}\operatorname{Grad}v|_{\mathsf{L}^2(\mathbb{S},\Omega)}.$$

More precisely,

$$c^{\mathsf{ela}}_{2,\Gamma_\nu}=c^{\mathsf{ela}}_{0,\Gamma_\tau}\leq c_{\mathsf{k},\Gamma_\tau}c_{0,\Gamma_\tau}=c_{\mathsf{k},\Gamma_\tau}c_{2,\Gamma_\nu}$$

holds, as for all $v\in D(\mathcal{A}_0)$

$$|v|_{\mathsf{L}^2(\Omega)}\leq c_{0,\Gamma_\tau}|\operatorname{Grad}v|_{\mathsf{L}^2(\Omega)}\leq c_{\mathsf{k},\Gamma_\tau}c_{0,\Gamma_\tau}|\operatorname{sym}\operatorname{Grad}v|_{\mathsf{L}^2(\mathbb{S},\Omega)}.$$

In particular, for $\Gamma_\tau=\Gamma$ we know $c_{\mathsf{k},\Gamma}\leq\sqrt{2}$, see [12, 11]*, which shows by Theorem 2.20*

$$c^{\mathsf{ela}}_{2,\emptyset}=c^{\mathsf{ela}}_{0,\Gamma}\leq\sqrt{2}c_{0,\Gamma}\leq\sqrt{2}\min\{c_{0,\Gamma_\tau},c_{0,\emptyset},\frac{\operatorname{diam}(\Omega)}{\pi}\}\leq\frac{\sqrt{2}}{\pi}\operatorname{diam}(\Omega).$$

2.3.5. *3D-Biharmonic Complex (div Div-complex).* The complex (involving scalar as well as symmetric and deviatoric tensor fields)

$$\mathsf{H}^2_{\Gamma_\tau}(\Omega)=\mathsf{H}_{\Gamma_\tau}(\operatorname{Grad}\operatorname{grad},\Omega)\xrightarrow{\mathrm{A}_0=\operatorname{Grad}\operatorname{grad}_{\Gamma_\tau}}\mathsf{H}_{\Gamma_\tau}(\operatorname{Rot},\mathbb{S},\Omega)\xrightarrow{\mathrm{A}_1=\operatorname{Rot}_{\Gamma_\tau}}\mathsf{H}_{\Gamma_\tau}(\operatorname{Div},\mathbb{T},\Omega)\xrightarrow{\mathrm{A}_2=\operatorname{Div}_{\Gamma_\tau}}\mathsf{L}^2(\Omega),$$

$$\mathsf{L}^2(\Omega)\xleftarrow{\mathrm{A}_0^*=\operatorname{div}\operatorname{Div}_{\Gamma_\nu}}\mathsf{H}_{\Gamma_\nu}(\operatorname{div}\operatorname{Div},\mathbb{S},\Omega)\xleftarrow{\mathrm{A}_1^*=\operatorname{sym}\operatorname{Rot}_{\Gamma_\nu}}\mathsf{H}_{\Gamma_\nu}(\operatorname{sym}\operatorname{Rot},\mathbb{T},\Omega)\xleftarrow{\mathrm{A}_2^*=-\operatorname{dev}\operatorname{Grad}_{\Gamma_\nu}}\mathsf{H}_{\Gamma_\nu}(\operatorname{Grad},\Omega),$$

arises in general relativity and for the biharmonic equation, see, e.g., [43] for details and, e.g., [60, 35, 51, 47, 48] for numerical applications. Note that, indeed, similar to using Korn's inequality in the latter section, the regularity

$$D(\mathrm{A}_2^*)=D(\operatorname{dev}\operatorname{Grad}_{\Gamma_\nu})=\mathsf{H}_{\Gamma_\nu}(\operatorname{dev}\operatorname{Grad},\Omega)=\mathsf{H}_{\Gamma_\nu}(\operatorname{Grad},\Omega)=\mathsf{H}^1_{\Gamma_\nu}(\Omega)$$

holds, cf. [43, Lemma 3.2]. The "second order Laplace and Maxwell" operators are given by

$$\begin{aligned}
\mathrm{A}_0^*\,\mathrm{A}_0&=\operatorname{div}\operatorname{Div}_{\Gamma_\nu}\operatorname{Grad}\operatorname{grad}_{\Gamma_\tau}, & \mathrm{A}_0\,\mathrm{A}_0^*&=\operatorname{Grad}\operatorname{grad}_{\Gamma_\tau}\operatorname{div}\operatorname{Div}_{\Gamma_\nu},\\
\mathrm{A}_1^*\,\mathrm{A}_1&=\operatorname{sym}\operatorname{Rot}_{\Gamma_\nu}\operatorname{Rot}_{\Gamma_\tau}, & \mathrm{A}_1\,\mathrm{A}_1^*&=\operatorname{Rot}_{\Gamma_\tau}\operatorname{sym}\operatorname{Rot}_{\Gamma_\nu},\\
\mathrm{A}_2^*\,\mathrm{A}_2&=-\operatorname{dev}\operatorname{Grad}_{\Gamma_\nu}\operatorname{Div}_{\Gamma_\tau}, & \mathrm{A}_2\,\mathrm{A}_2^*&=-\operatorname{Div}_{\Gamma_\tau}\operatorname{dev}\operatorname{Grad}_{\Gamma_\nu},
\end{aligned}$$

and for the constants and eigenvalues $c^{\mathsf{bih}}_{\ell,\Gamma_\tau}=1/\lambda^{\mathsf{bih}}_{\ell,\Gamma_\tau}$ we have

$$\begin{aligned}
&\forall\,u\in D(\mathcal{A}_0)=D(\operatorname{Grad}\operatorname{grad}_{\Gamma_\tau})\cap R(\operatorname{div}\operatorname{Div}_{\Gamma_\nu}) && |u|_{\mathsf{L}^2(\Omega)}\leq c^{\mathsf{bih}}_{0,\Gamma_\tau}|\operatorname{Grad}\operatorname{grad}u|_{\mathsf{L}^2(\mathbb{S},\Omega)},\\
&\forall\,S\in D(\mathcal{A}_0^*)=D(\operatorname{div}\operatorname{Div}_{\Gamma_\nu})\cap R(\operatorname{Grad}\operatorname{grad}_{\Gamma_\tau}) && |S|_{\mathsf{L}^2(\mathbb{S},\Omega)}\leq c^{\mathsf{bih}}_{0,\Gamma_\tau}|\operatorname{div}\operatorname{Div}S|_{\mathsf{L}^2(\Omega)},\\
&\forall\,S\in D(\mathcal{A}_1)=D(\operatorname{Rot}_{\Gamma_\tau})\cap R(\operatorname{sym}\operatorname{Rot}_{\Gamma_\nu}) && |S|_{\mathsf{L}^2(\mathbb{S},\Omega)}\leq c^{\mathsf{bih}}_{1,\Gamma_\tau}|\operatorname{Rot}S|_{\mathsf{L}^2(\mathbb{T},\Omega)},\\
&\forall\,T\in D(\mathcal{A}_1^*)=D(\operatorname{sym}\operatorname{Rot}_{\Gamma_\nu})\cap R(\operatorname{Rot}_{\Gamma_\tau}) && |T|_{\mathsf{L}^2(\mathbb{T},\Omega)}\leq c^{\mathsf{bih}}_{1,\Gamma_\tau}|\operatorname{sym}\operatorname{Rot}T|_{\mathsf{L}^2(\mathbb{S},\Omega)},\\
&\forall\,T\in D(\mathcal{A}_2)=D(\operatorname{Div}_{\Gamma_\tau})\cap R(\operatorname{dev}\operatorname{Grad}_{\Gamma_\nu}) && |T|_{\mathsf{L}^2(\mathbb{T},\Omega)}\leq c^{\mathsf{bih}}_{2,\Gamma_\tau}|\operatorname{Div}T|_{\mathsf{L}^2(\Omega)},\\
&\forall\,v\in D(\mathcal{A}_2^*)=D(\operatorname{dev}\operatorname{Grad}_{\Gamma_\nu})\cap R(\operatorname{Div}_{\Gamma_\tau}) && |v|_{\mathsf{L}^2(\Omega)}\leq c^{\mathsf{bih}}_{2,\Gamma_\tau}|\operatorname{dev}\operatorname{Grad}v|_{\mathsf{L}^2(\mathbb{T},\Omega)}.
\end{aligned}$$

We emphasise that this complex is the first non-symmetric one and we get additional results for the operators involving A_2. Note that

$$N(\operatorname{Grad}\operatorname{grad}_{\Gamma_\tau}) = \begin{cases} \{0\} & \text{if } \Gamma_\tau \neq \emptyset, \\ \mathsf{P}^1 & \text{if } \Gamma_\tau = \emptyset, \end{cases}$$

$$R(\operatorname{div}\operatorname{Div}_{\Gamma_\nu}) = N(\operatorname{Grad}\operatorname{grad}_{\Gamma_\tau})^{\perp_{\mathsf{L}^2(\Omega)}} = \begin{cases} \mathsf{L}^2(\Omega) & \text{if } \Gamma_\nu \neq \Gamma, \\ \mathsf{L}^2(\Omega) \cap (\mathsf{P}^1)^{\perp_{\mathsf{L}^2(\Omega)}} & \text{if } \Gamma_\nu = \Gamma, \end{cases}$$

$$N(\operatorname{dev}\operatorname{Grad}_{\Gamma_\nu}) = \begin{cases} \{0\} & \text{if } \Gamma_\nu \neq \emptyset, \\ \mathsf{RT} & \text{if } \Gamma_\nu = \emptyset, \end{cases}$$

$$R(\operatorname{Div}_{\Gamma_\tau}) = N(\operatorname{dev}\operatorname{Grad}_{\Gamma_\nu})^{\perp_{\mathsf{L}^2(\Omega)}} = \begin{cases} \mathsf{L}^2(\Omega) & \text{if } \Gamma_\tau \neq \Gamma, \\ \mathsf{L}^2(\Omega) \cap \mathsf{RT}^{\perp_{\mathsf{L}^2(\Omega)}} & \text{if } \Gamma_\tau = \Gamma, \end{cases}$$

where P^1 denotes the polynomials of order less then 1 and RT the space of global Raviart-Thomas vector fields. The crucial compact embeddings (13) have recently been proved in [43, Lemma 3.22].

Theorem 2.29 (selection theorems for the biharmonic complex)**.** *The embeddings*

$$D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) = \mathsf{H}_{\Gamma_\tau}(\operatorname{Rot}, \mathbb{S}, \Omega) \cap \mathsf{H}_{\Gamma_\nu}(\operatorname{div}\operatorname{Div}, \mathbb{S}, \Omega) \hookrightarrow \mathsf{L}^2(\mathbb{S}, \Omega),$$
$$D(\mathrm{A}_2) \cap D(\mathrm{A}_1^*) = \mathsf{H}_{\Gamma_\tau}(\operatorname{Div}, \mathbb{T}, \Omega) \cap \mathsf{H}_{\Gamma_\nu}(\operatorname{sym}\operatorname{Rot}, \mathbb{T}, \Omega) \hookrightarrow \mathsf{L}^2(\mathbb{T}, \Omega)$$

are compact.

Similar to the 3D Maxwell case and the 3D elasticity case we get the following result, cf. Theorem 2.20, Theorem 2.27, and Remark 2.28.

Remark 2.30 (Friedrichs/Poincaré type constants for the biharmonic complex)**.** *For $c_{\ell,\Gamma_\tau}^{\mathsf{bit}} = 1/\lambda_{\ell,\Gamma_\tau}^{\mathsf{bit}}$ the following holds:*

- **(i)** *The Friedrichs/Poincaré type constants depend monotonically on the boundary conditions, i.e.,*
$$\emptyset \neq \widetilde{\Gamma}_\tau \subset \Gamma_\tau \quad \Rightarrow \quad c_{0,\Gamma_\tau}^{\mathsf{bih}} \leq c_{0,\widetilde{\Gamma}_\tau}^{\mathsf{bih}}.$$
- **(ii)** *Due to the lack a symmetry in the biharmonic complex there are no further formulas relating $c_{0,\Gamma_\tau}^{\mathsf{bih}}$ to $c_{2,\Gamma_\nu}^{\mathsf{bih}}$ or $c_{1,\Gamma_\tau}^{\mathsf{bih}}$ to $c_{1,\Gamma_\nu}^{\mathsf{bih}}$.*
- **(iii)** *As pointed out in Remark 2.28 for the elasticity complex, there is a similar relation between the Friedrichs/Poincaré type constants of the biharmonic complex $c_{0,\Gamma_\tau}^{\mathsf{bih}}$ and $c_{2,\Gamma_\tau}^{\mathsf{bih}}$ and the classical Friedrichs/Poincaré constants $c_{0,\Gamma_\tau} = c_{2,\Gamma_\nu}$ by the classical Friedrichs/Poincaré estimate and a Korn like inequality, i.e.,*
$$\forall\, v \in D(\mathcal{A}_2^*) = \underbrace{D(\operatorname{dev}\operatorname{Grad}_{\Gamma_\nu})}_{=\mathsf{H}_{\Gamma_\nu}^1(\Omega)} \cap R(\operatorname{Div}_{\Gamma_\tau}) \qquad |\operatorname{Grad} v|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{dev},\Gamma_\nu} |\operatorname{dev}\operatorname{Grad} v|_{\mathsf{L}^2(\mathbb{T},\Omega)},$$
cf. [43, Lemma 3.2]*. More precisely, $c_{0,\Gamma_\tau}^{\mathsf{bih}} \leq c_{0,\Gamma_\tau}^2$ and $c_{2,\Gamma_\tau}^{\mathsf{bih}} \leq c_{\mathsf{dev},\Gamma_\nu} c_{0,\Gamma_\nu}$ hold, as*
$$\forall\, v \in D(\mathcal{A}_2^*) \qquad |v|_{\mathsf{L}^2(\Omega)} \leq c_{0,\Gamma_\nu} |\operatorname{Grad} v|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{dev},\Gamma_\nu} c_{0,\Gamma_\nu} |\operatorname{dev}\operatorname{Grad} v|_{\mathsf{L}^2(\mathbb{T},\Omega)}.$$

3. Analytical Examples

In the sequel we will compute all Friedrichs/Poincaré and Maxwell eigenvalues for the unit cube in 1D, 2D, and 3D with mixed boundary conditions on canonical boundary parts. We emphasise that the completeness of the respective eigensystems can be shown as in [29].

3.1. **1D.** Let $\Omega := I := (0,1)$, $\Gamma = \{0,1\}$, and $\Gamma_\tau \in P\big(\{0,1\}\big) = \big\{\emptyset, \{0\}, \{1\}, \Gamma\big\}$, and recall Section 2.3.1. From Appendix 7.1 we see

$$c_{0,\Gamma} = c_{0,\emptyset} = \frac{1}{\pi}, \qquad c_{0,\{0\}} = c_{0,\{1\}} = \frac{2}{\pi}. \tag{17}$$

Note that from $c_{0,\Gamma_\tau} = c_{0,\Gamma_\nu}$, see Corollary 2.22, we already know $c_{0,\Gamma} = c_{0,\emptyset}$ and $c_{0,\{0\}} = c_{0,\{1\}}$.

Remark 3.1. *Corollary 2.22 may be verified by this example.*

- **(i)** $\emptyset \neq \{0\}, \{1\} \subset \Gamma \quad \Rightarrow \quad c_{0,\Gamma} = \dfrac{1}{\pi} \leq \dfrac{2}{\pi} = c_{0,\{0\}} = c_{0,\{1\}}$
- **(ii)** $c_{0,\Gamma} = c_{0,\emptyset} = \dfrac{1}{\pi} = \dfrac{\operatorname{diam}(\Omega)}{\pi}$

3.2. **2D.** Let $\Omega := I^2$, $I := (0,1)$, $\Gamma = \overline{\Gamma_b \cup \Gamma_t \cup \Gamma_l \cup \Gamma_r}$, where Γ_b, Γ_t, Γ_l, Γ_r are the open bottom, top, left, and right boundary parts of Γ, respectively, and $\Gamma_\tau \in P\big(\{\Gamma_b, \Gamma_t, \Gamma_l, \Gamma_r\}\big)$, and recall Section 2.3.2. We shall use canonical index notations such as

$$\Gamma_{b,l} := \operatorname{int}(\overline{\Gamma_b \cup \Gamma_l}), \quad \Gamma_{b,l,t} := \operatorname{int}(\overline{\Gamma_b \cup \Gamma_l \cup \Gamma_t}).$$

From Appendix 7.2 we see

$$\begin{aligned} c_{0,\emptyset} &= \frac{1}{\pi}, & c_{0,\Gamma_{b,l}} = c_{0,\Gamma_{b,r}} = c_{0,\Gamma_{t,l}} = c_{0,\Gamma_{t,r}} &= \frac{\sqrt{2}}{\pi}, \\ c_{0,\Gamma_b} = c_{0,\Gamma_t} = c_{0,\Gamma_l} = c_{0,\Gamma_r} &= \frac{2}{\pi}, & c_{0,\Gamma_{b,l,r}} = c_{0,\Gamma_{t,l,r}} = c_{0,\Gamma_{b,t,l}} = c_{0,\Gamma_{b,t,r}} &= \frac{2}{\sqrt{5}\pi}, \\ c_{0,\Gamma_{b,t}} = c_{0,\Gamma_{l,r}} &= \frac{1}{\pi}, & c_{0,\Gamma} &= \frac{1}{\sqrt{2}\pi}. \end{aligned} \tag{18}$$

Remark 3.2. *Corollary 2.23 may be verified by this example.*

- **(i)** $\emptyset \neq \Gamma_b \subset \Gamma_{b,l} \subset \Gamma_{b,l,r} \subset \Gamma \quad \Rightarrow \quad c_{0,\Gamma} = \frac{1}{\sqrt{2}\pi} \leq c_{0,\Gamma_{b,l,r}} = \frac{2}{\sqrt{5}\pi} \leq c_{0,\Gamma_{b,l}} = \frac{2}{\sqrt{2}\pi} \leq c_{0,\Gamma_b} = \frac{2}{\pi}$
- **(i')** $\emptyset \neq \Gamma_l \subset \Gamma_{l,r} \subset \Gamma_{b,l,r} \subset \Gamma \quad \Rightarrow \quad c_{0,\Gamma} = \frac{1}{\sqrt{2}\pi} \leq c_{0,\Gamma_{b,l,r}} = \frac{2}{\sqrt{5}\pi} \leq c_{0,\Gamma_{l,r}} = \frac{1}{\pi} \leq c_{0,\Gamma_l} = \frac{2}{\pi}$
- **(ii)** $c_{0,\Gamma} = \frac{1}{\sqrt{2}\pi} \leq \frac{1}{\pi} = c_{0,\emptyset}$
- **(iii)** $c_{0,\Gamma} = \frac{1}{\sqrt{2}\pi} \leq \frac{\sqrt{2}}{\pi} = \frac{\operatorname{diam}(\Omega)}{\pi}$
- **(iv)** Ω *is convex and* $c_{0,\Gamma} = \frac{1}{\sqrt{2}\pi} \leq \frac{1}{\pi} = c_{0,\emptyset} \leq \frac{\sqrt{2}}{\pi} = \frac{\operatorname{diam}(\Omega)}{\pi}$.

3.3. **3D.** Let $\Omega := \widehat{\Omega} \times I = I^3$, $\widehat{\Omega} := I^2$, $I := (0,1)$, $\Gamma = \overline{\Gamma_b \cup \Gamma_t \cup \Gamma_l \cup \Gamma_r \cup \Gamma_f \cup \Gamma_{bk}}$, where Γ_b, Γ_t, Γ_l, Γ_r, Γ_f, Γ_{bk} are the open bottom, top, left, right, front, and back boundary parts of Γ, respectively, and $\Gamma_\tau \in P\big(\{\Gamma_b, \Gamma_t, \Gamma_l, \Gamma_r, \Gamma_f, \Gamma_{bk}\}\big)$, and recall Section 2.2 as well as Theorem 2.20. Again, we use canonical index notations such as

$$\Gamma_{b,r} := \operatorname{int}(\overline{\Gamma_b \cup \Gamma_l}), \quad \Gamma_{b,r,bk,f} := \operatorname{int}(\overline{\Gamma_b \cup \Gamma_r \cup \Gamma_{bk} \cup \Gamma_f}).$$

From Appendix 7.3 we see for c_{0,Γ_τ}

$$\begin{aligned} c_{0,\emptyset} &= \frac{1}{\pi}, \\ c_{0,\Gamma_b} = c_{0,\Gamma_t} = c_{0,\Gamma_l} = c_{0,\Gamma_r} = c_{0,\Gamma_f} = c_{0,\Gamma_{bk}} &= \frac{2}{\pi}, \\ c_{0,\Gamma_{b,t}} = c_{0,\Gamma_{l,r}} = c_{0,\Gamma_{f,bk}} &= \frac{1}{\pi}, \\ c_{0,\Gamma_{b,l}} = c_{0,\Gamma_{b,r}} = c_{0,\Gamma_{b,f}} = c_{0,\Gamma_{b,bk}} & \\ = c_{0,\Gamma_{t,l}} = c_{0,\Gamma_{t,r}} = c_{0,\Gamma_{t,f}} = c_{0,\Gamma_{t,bk}} = c_{0,\Gamma_{f,l}} = c_{0,\Gamma_{f,r}} = c_{0,\Gamma_{bk,l}} = c_{0,\Gamma_{bk,r}} &= \frac{\sqrt{2}}{\pi}, \\ c_{0,\Gamma_{b,t,l}} = c_{0,\Gamma_{b,t,r}} = c_{0,\Gamma_{b,t,f}} = c_{0,\Gamma_{b,t,bk}} = c_{0,\Gamma_{l,r,b}} & \\ = c_{0,\Gamma_{l,r,t}} = c_{0,\Gamma_{l,r,f}} = c_{0,\Gamma_{l,r,bk}} = c_{0,\Gamma_{f,bk,l}} = c_{0,\Gamma_{f,bk,r}} = c_{0,\Gamma_{f,bk,b}} = c_{0,\Gamma_{f,bk,t}} &= \frac{2}{\sqrt{5}\pi}, \\ c_{0,\Gamma_{b,bk,l}} = c_{0,\Gamma_{b,l,f}} = c_{0,\Gamma_{b,f,r}} = c_{0,\Gamma_{b,r,bk}} = c_{0,\Gamma_{t,bk,l}} = c_{0,\Gamma_{t,l,f}} = c_{0,\Gamma_{t,f,r}} = c_{0,\Gamma_{t,r,bk}} &= \frac{2}{\sqrt{3}\pi}, \\ c_{0,\Gamma_{b,t,l,r}} = c_{0,\Gamma_{b,t,f,bk}} = c_{0,\Gamma_{l,r,f,bk}} &= \frac{1}{\sqrt{2}\pi}, \\ c_{0,\Gamma_{b,t,l,bk}} = c_{0,\Gamma_{b,t,f,l}} = c_{0,\Gamma_{b,t,r,f}} = c_{0,\Gamma_{b,t,r,bk}} = c_{0,\Gamma_{l,r,f,t}} = c_{0,\Gamma_{l,r,f,b}} & \\ = c_{0,\Gamma_{l,r,t,bk}} = c_{0,\Gamma_{l,r,b,bk}} = c_{0,\Gamma_{f,bk,b,l}} = c_{0,\Gamma_{f,bk,t,l}} = c_{0,\Gamma_{f,bk,b,r}} = c_{0,\Gamma_{f,bk,r,r}} &= \frac{2}{\sqrt{6}\pi}, \\ c_{0,\Gamma_{b,t,l,r,bk}} = c_{0,\Gamma_{b,t,l,r,f}} = c_{0,\Gamma_{b,t,l,f,bk}} = c_{0,\Gamma_{b,t,r,f,bk}} = c_{0,\Gamma_{b,l,r,f,bk}} = c_{0,\Gamma_{t,l,r,f,bk}} &= \frac{2}{3\pi}, \\ c_{0,\Gamma} &= \frac{1}{\sqrt{3}\pi}, \end{aligned} \tag{19}$$

and for c_{1,Γ_τ}

$$
\begin{aligned}
c_{1,\emptyset} = c_{1,\Gamma} &= \frac{1}{\sqrt{2}\pi},\\
c_{1,\Gamma_b} = c_{1,\Gamma_t} = c_{1,\Gamma_l} = c_{1,\Gamma_r} = c_{1,\Gamma_f} = c_{1,\Gamma_{bk}} &= \frac{2}{\sqrt{5}\pi},\\
c_{1,\Gamma_{l,r}} = c_{1,\Gamma_{b,t}} = c_{1,\Gamma_{f,bk}} &= \frac{1}{\pi},\\
c_{1,\Gamma_{b,l}} = c_{1,\Gamma_{b,r}} = c_{1,\Gamma_{b,f}} = c_{1,\Gamma_{b,bk}} &\\
= c_{1,\Gamma_{t,l}} = c_{1,\Gamma_{t,r}} = c_{1,\Gamma_{t,f}} = c_{1,\Gamma_{t,bk}} = c_{1,\Gamma_{f,l}} = c_{1,\Gamma_{l,bk}} = c_{1,\Gamma_{bk,r}} = c_{1,\Gamma_{f,r}} &= \frac{\sqrt{2}}{\pi},\\
c_{1,\Gamma_{b,l,t}} = c_{1,\Gamma_{b,r,t}} = c_{1,\Gamma_{b,f,t}} = c_{1,\Gamma_{b,bk,t}} = c_{1,\Gamma_{r,l,t}} &\\
= c_{1,\Gamma_{r,l,b}} = c_{1,\Gamma_{r,l,f}} = c_{1,\Gamma_{r,l,bk}} = c_{1,\Gamma_{f,bk,l}} = c_{1,\Gamma_{f,bk,r}} = c_{1,\Gamma_{f,bk,t}} = c_{1,\Gamma_{f,bk,b}} &= \frac{2}{\pi},\\
c_{1,\Gamma_{b,l,bk}} = c_{1,\Gamma_{b,r,bk}} = c_{1,\Gamma_{b,l,f}} = c_{1,\Gamma_{b,r,f}} = c_{1,\Gamma_{t,l,bk}} = c_{1,\Gamma_{t,r,bk}} = c_{1,\Gamma_{t,l,f}} = c_{1,\Gamma_{t,r,f}} &= \frac{2}{\sqrt{3}\pi},
\end{aligned}
\tag{20}
$$

and all the other remaining cases follow by $c_{1,\Gamma_\nu} = c_{1,\Gamma_\tau}$ as well as symmetry.

Remark 3.3. *Theorem 2.20 may be verified by these examples. E.g.:*

(i) $\emptyset \neq \Gamma_b \subset \Gamma_{b,l} \subset \Gamma_{b,l,r} \subset \Gamma_{b,t,l,r} \subset \Gamma_{b,t,l,r,f} \subset \Gamma \quad \Rightarrow$

$$c_{0,\Gamma} = \frac{1}{\sqrt{3}\pi} \leq c_{0,\Gamma_{b,t,l,r,f}} = \frac{2}{3\pi} \leq c_{0,\Gamma_{b,t,l,r}} = \frac{1}{\sqrt{2}\pi} \leq c_{0,\Gamma_{b,l,r}} = \frac{2}{\sqrt{5}\pi} \leq c_{0,\Gamma_{b,l}} = \frac{2}{\sqrt{2}\pi} \leq c_{0,\Gamma_b} = \frac{2}{\pi}$$

(i') $\emptyset \neq \Gamma_b \subset \Gamma_{b,l} \subset \Gamma_{b,l,r} \subset \Gamma_{b,f,l,r} \subset \Gamma_{b,f,l,r,t} \subset \Gamma \quad \Rightarrow$

$$c_{0,\Gamma} = \frac{1}{\sqrt{3}\pi} \leq c_{0,\Gamma_{b,f,l,r,t}} = \frac{2}{3\pi} \leq c_{0,\Gamma_{b,f,l,r}} = \frac{2}{\sqrt{6}\pi} \leq c_{0,\Gamma_{b,l,r}} = \frac{2}{\sqrt{5}\pi} \leq c_{0,\Gamma_{b,l}} = \frac{2}{\sqrt{2}\pi} \leq c_{0,\Gamma_b} = \frac{2}{\pi}$$

(i") $\emptyset \neq \Gamma_l \subset \Gamma_{l,r} \subset \Gamma_{b,l,r} \subset \Gamma_{b,f,l,r} \subset \Gamma_{b,f,l,r,t} \subset \Gamma \quad \Rightarrow$

$$c_{0,\Gamma} = \frac{1}{\sqrt{3}\pi} \leq c_{0,\Gamma_{b,f,l,r,t}} = \frac{2}{3\pi} \leq c_{0,\Gamma_{b,f,l,r}} = \frac{2}{\sqrt{6}\pi} \leq c_{0,\Gamma_{b,l,r}} = \frac{2}{\sqrt{5}\pi} \leq c_{0,\Gamma_{l,r}} = \frac{1}{\pi} \leq c_{0,\Gamma_l} = \frac{2}{\pi}$$

(i"') $\emptyset \neq \Gamma_l \subset \Gamma_{b,l} \subset \Gamma_{b,f,l} \subset \Gamma_{b,f,l,r} \subset \Gamma_{b,f,l,r,t} \subset \Gamma \quad \Rightarrow$

$$c_{0,\Gamma} = \frac{1}{\sqrt{3}\pi} \leq c_{0,\Gamma_{b,f,l,r,t}} = \frac{2}{3\pi} \leq c_{0,\Gamma_{b,f,l,r}} = \frac{2}{\sqrt{6}\pi} \leq c_{0,\Gamma_{b,f,l}} = \frac{2}{\sqrt{3}\pi} \leq c_{0,\Gamma_{b,l}} = \frac{2}{\sqrt{2}\pi} \leq c_{0,\Gamma_l} = \frac{2}{\pi}$$

(ii) $c_{0,\Gamma} = \frac{1}{\sqrt{3}\pi} \leq \frac{1}{\pi} = c_{0,\emptyset}$

(iii) $c_{0,\Gamma} = \frac{1}{\sqrt{3}\pi} \leq \frac{\sqrt{3}}{\pi} = \frac{\operatorname{diam}(\Omega)}{\pi}$

(iv) Ω *is convex and* $c_{0,\Gamma} = \frac{1}{\sqrt{3}\pi} \leq \frac{1}{\pi} = c_{0,\emptyset} \leq \frac{\sqrt{3}}{\pi} = \frac{\operatorname{diam}(\Omega)}{\pi}$.

(v) Ω *is convex and* $c_{1,\Gamma} = c_{1,\emptyset} = \frac{1}{\sqrt{2}\pi} \leq \frac{1}{\pi} = c_{0,\emptyset} \leq \frac{\sqrt{3}}{\pi} = \frac{\operatorname{diam}(\Omega)}{\pi}$.

(vi) Ω *is convex and*

$$c_{0,\Gamma} = \frac{1}{\sqrt{3}\pi} \leq \frac{1}{\sqrt{2}\pi} = c_{0,1,\Gamma} = \max\{c_{0,\Gamma}, c_{1,\Gamma}\} = \max\{\frac{1}{\sqrt{3}\pi}, \frac{1}{\sqrt{2}\pi}\} \leq \frac{1}{\pi} = c_{0,\emptyset} \leq \frac{\sqrt{3}}{\pi} = \frac{\operatorname{diam}(\Omega)}{\pi},$$

$$c_{0,\Gamma} = \frac{1}{\sqrt{3}\pi} \leq \frac{1}{\pi} = c_{0,1,\emptyset} = \max\{c_{0,\emptyset}, c_{1,\emptyset}\} = \max\{\frac{1}{\pi}, \frac{1}{\sqrt{2}\pi}\} = c_{0,\emptyset} \leq \frac{\sqrt{3}}{\pi} = \frac{\operatorname{diam}(\Omega)}{\pi}.$$

Remark 3.4. *In general, the Maxwell constants do not have monotonicity properties, which can also be verified by the latter examples. In fact, in our examples, the Maxwell constants are monotone increasing up to a certain situation in the 'middle', where the tangential and the normal boundary condition are equally strong, and from there on the Maxwell constants are monotone decreasing. E.g.:*

- $\emptyset \neq \Gamma_b \subset \Gamma_{b,l} \subset \Gamma_{b,l,r} \subset \Gamma_{b,t,l,r} \subset \Gamma_{b,t,l,r,f} \subset \Gamma$, *but*

$$c_{1,\Gamma} = \frac{1}{\sqrt{2}\pi} \overset{\text{ok}}{\leq} c_{1,\Gamma_{b,t,l,r,f}} = c_{1,\Gamma_{bk}} = \frac{2}{\sqrt{5}\pi} \overset{\text{ok}}{\leq} c_{1,\Gamma_{b,t,l,r}} = c_{1,\Gamma_{f,bk}} = \frac{1}{\pi}$$

$$\overset{\text{ok}}{\leq} c_{1,\Gamma_{b,l,r}} = \frac{2}{\pi} \overset{\text{not ok}}{\underset{>}{\not\leq}} c_{1,\Gamma_{b,l}} = \frac{2}{\sqrt{2}\pi} \overset{\text{not ok}}{\underset{>}{\not\leq}} c_{1,\Gamma_{b}} = \frac{2}{\sqrt{5}\pi}.$$

- $\emptyset \neq \Gamma_b \subset \Gamma_{b,l} \subset \Gamma_{b,l,r} \subset \Gamma_{b,f,l,r} \subset \Gamma_{b,f,l,r,t} \subset \Gamma$, *but*

$$c_{1,\Gamma} = \frac{1}{\sqrt{2}\pi} \overset{\text{ok}}{\leq} c_{1,\Gamma_{b,t,l,r,f}} = c_{1,\Gamma_{bk}} = \frac{2}{\sqrt{5}\pi} \overset{\text{ok}}{\leq} c_{1,\Gamma_{b,f,l,r}} = c_{1,\Gamma_{t,bk}} = \frac{2}{\sqrt{2}\pi}$$
$$\overset{\text{ok}}{\leq} c_{1,\Gamma_{b,l,r}} = \frac{2}{\pi} \overset{\text{not ok}}{\underset{>}{\not\leq}} c_{1,\Gamma_{b,l}} = \frac{2}{\sqrt{2}\pi} \overset{\text{not ok}}{\underset{>}{\not\leq}} c_{1,\Gamma_{b}} = \frac{2}{\sqrt{5}\pi}.$$

- $\emptyset \neq \Gamma_l \subset \Gamma_{l,r} \subset \Gamma_{b,l,r} \subset \Gamma_{b,f,l,r} \subset \Gamma_{b,f,l,r,t} \subset \Gamma$, *but*

$$c_{1,\Gamma} = \frac{1}{\sqrt{2}\pi} \overset{\text{ok}}{\leq} c_{1,\Gamma_{b,t,l,r,f}} = c_{1,\Gamma_{bk}} = \frac{2}{\sqrt{5}\pi} \overset{\text{ok}}{\leq} c_{1,\Gamma_{b,f,l,r}} = c_{1,\Gamma_{t,bk}} = \frac{2}{\sqrt{2}\pi}$$
$$\overset{\text{ok}}{\leq} c_{1,\Gamma_{b,l,r}} = \frac{2}{\pi} \overset{\text{not ok}}{\underset{>}{\not\leq}} c_{1,\Gamma_{l,r}} = \frac{1}{\pi} \overset{\text{not ok}}{\underset{>}{\not\leq}} c_{1,\Gamma_{l}} = \frac{2}{\sqrt{5}\pi}.$$

- $\emptyset \neq \Gamma_l \subset \Gamma_{b,l} \subset \Gamma_{b,f,l} \subset \Gamma_{b,f,l,r} \subset \Gamma_{b,f,l,r,t} \subset \Gamma$, *but*

$$c_{1,\Gamma} = \frac{1}{\sqrt{2}\pi} \overset{\text{ok}}{\leq} c_{1,\Gamma_{b,t,l,r,f}} = c_{1,\Gamma_{bk}} = \frac{2}{\sqrt{5}\pi} \overset{\text{ok}}{\leq} c_{1,\Gamma_{b,f,l,r}} = c_{1,\Gamma_{t,bk}} = \frac{2}{\sqrt{2}\pi}$$
$$\overset{\text{not ok}}{\underset{>}{\not\leq}} c_{1,\Gamma_{b,l,f}} = \frac{2}{\sqrt{3}\pi} \overset{\text{ok}}{\leq} c_{1,\Gamma_{l,b}} = \frac{2}{\sqrt{2}\pi} \overset{\text{not ok}}{\underset{>}{\not\leq}} c_{1,\Gamma_{l}} = \frac{2}{\sqrt{5}\pi}.$$

4. Numerical Examples

The finite element method (FEM) is applied for evaluation of the Rayleigh quotients on finite dimensional subspaces. Constants are therefore approximated and convergence to their exact values is expected for higher dimensions. Assuming that Ω is discretised by a triangular (2D) or a tetrahedral (3D) mesh $\mathcal{T}$, we use only the lowest order finite elements available:

- Linear Lagrange (P1) nodal elements θ_i^{P1} for approximations of $\mathsf{H}_{\Gamma_\tau}(\mathrm{grad},\Omega)$ spaces,
- Linear Nédélec (N) edge elements Θ_i^{N} for approximations of $\mathsf{H}_{\Gamma_\tau}(\mathrm{rot},\Omega)$ spaces,
- Linear Raviart-Thomas (RT) face elements Θ_i^{RT} for approximations of $\mathsf{H}_{\Gamma_\tau}(\mathrm{div},\Omega)$ spaces.

We assemble the mass matrices M^{P1}, M^{N}, M^{RT} and the stiffness matrices $\mathrm{K}^{\mathsf{P1}}, \mathrm{K}^{\mathsf{N}}, \mathrm{K}^{\mathsf{RT}}$ defined by

$$\begin{aligned} \mathrm{M}_{ij}^{\mathsf{P1}} &= \langle \theta_i^{\mathsf{P1}}, \theta_j^{\mathsf{P1}} \rangle_{\mathsf{L}^2(\Omega)}, & \mathrm{K}_{ij}^{\mathsf{P1}} &= \langle \mathrm{grad}\, \theta_i^{\mathsf{P1}}, \mathrm{grad}\, \theta_j^{\mathsf{P1}} \rangle_{\mathsf{L}^2(\Omega)}, \\ \mathrm{M}_{ij}^{\mathsf{N}} &= \langle \Theta_i^{\mathsf{N}}, \Theta_j^{\mathsf{N}} \rangle_{\mathsf{L}^2(\Omega)}, & \mathrm{K}_{ij}^{\mathsf{N}} &= \langle \mathrm{rot}\, \Theta_i^{\mathsf{N}}, \mathrm{rot}\, \Theta_j^{\mathsf{N}} \rangle_{\mathsf{L}^2(\Omega)}, \\ \mathrm{M}_{ij}^{\mathsf{RT}} &= \langle \Theta_i^{\mathsf{RT}}, \Theta_j^{\mathsf{RT}} \rangle_{\mathsf{L}^2(\Omega)}, & \mathrm{K}_{ij}^{\mathsf{RT}} &= \langle \mathrm{div}\, \Theta_i^{\mathsf{RT}}, \mathrm{div}\, \Theta_j^{\mathsf{RT}} \rangle_{\mathsf{L}^2(\Omega)}, \end{aligned}$$

where the indices i, j are the global numbers of the corresponding degrees of freedom, i.e., they are related to mesh nodes (for P1 elements) or mesh edges or faces (for N and RT elements). By using the affine mappings (for P1 elements) or Piola mappings (for N and RT elements) from reference elements we can assemble the local matrices. Detailed implementation of finite element assemblies is explained in [1, 52]. Squares of terms from Theorem 2.17 are easy to evaluate as quadratic forms with mass and stiffness matrices:

$$\begin{aligned} |u|_{\mathsf{L}^2(\Omega)}^2 &= \mathrm{M}^{\mathsf{P1}}\, u^{\mathsf{P1}} \cdot u^{\mathsf{P1}}, & |\,\mathrm{grad}\, u|_{\mathsf{L}^2(\Omega)}^2 &= \mathrm{K}^{\mathsf{P1}}\, u^{\mathsf{P1}} \cdot u^{\mathsf{P1}}, \\ |E|_{\mathsf{L}^2(\Omega)}^2 &= \mathrm{M}^{\mathsf{N}}\, E^{\mathsf{N}} \cdot E^{\mathsf{N}}, & |\,\mathrm{rot}\, E|_{\mathsf{L}^2(\Omega)}^2 &= \mathrm{K}^{\mathsf{N}}\, E^{\mathsf{N}} \cdot E^{\mathsf{N}}, \\ |H|_{\mathsf{L}^2(\Omega)}^2 &= \mathrm{M}^{\mathsf{RT}}\, H^{\mathsf{RT}} \cdot H^{\mathsf{RT}}, & |\,\mathrm{div}\, H|_{\mathsf{L}^2(\Omega)}^2 &= \mathrm{K}^{\mathsf{RT}}\, H^{\mathsf{RT}} \cdot H^{\mathsf{RT}}, \end{aligned}$$

where u^{P1}, E^{N}, and H^{RT} represent (column) vectors of coefficients with respect to their finite element bases of P1, N, and RT, respectively.

4.1. Friedrichs/Poincaré and Divergence Constants.

The classical Friedrichs constant $c_{0,\Gamma}$ is approximated as

$$\frac{1}{c_{0,\Gamma,\mathsf{P1}}^2} = \lambda_{0,\Gamma,\mathsf{P1}}^2 = \min_{\substack{0 \neq u^{\mathsf{P1}},\\ u_\Gamma^{\mathsf{P1}} = 0}} \frac{\mathrm{K}^{\mathsf{P1}}\, u^{\mathsf{P1}} \cdot u^{\mathsf{P1}}}{\mathrm{M}^{\mathsf{P1}}\, u^{\mathsf{P1}} \cdot u^{\mathsf{P1}}},$$

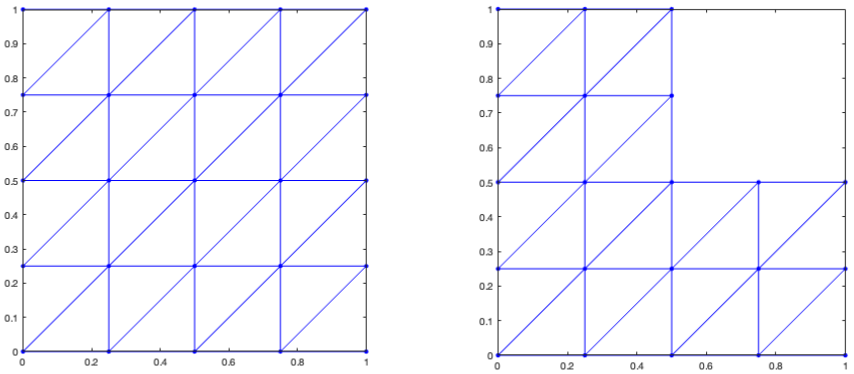

FIGURE 1. Coarse (level 1) computational meshes for the unit square and the L-shape domain.

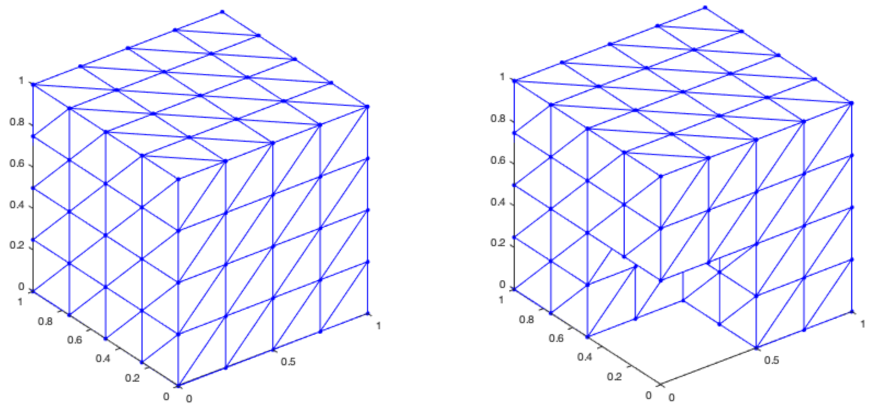

FIGURE 2. Coarse (level 1) computational meshes for the unit cube and the Fichera corner domain.

where u_{Γ}^{P1} denotes a subvector of u^{P1} in indices corresponding to boundary nodes. $\lambda^2_{0,\Gamma,\mathsf{P1}}$ is the minimal (positive) eigenvalue of the generalized eigenvalue problem

$$\mathrm{K}^{\mathsf{P1}}\, u^{\mathsf{P1}} = \lambda^2\, \mathrm{M}^{\mathsf{P1}}\, u^{\mathsf{P1}}, \qquad u_{\Gamma}^{\mathsf{P1}} = 0,$$

and may also be found by computing the minimal (positive) eigenvalue of

$$\mathrm{K}^{\mathsf{P1}}_{\mathsf{int}}\, u^{\mathsf{P1}}_{\mathsf{int}} = \lambda^2\, \mathrm{M}^{\mathsf{P1}}_{\mathsf{int}}\, u^{\mathsf{P1}}_{\mathsf{int}},$$

where $\mathrm{K}^{\mathsf{P1}}_{\mathsf{int}}$, $\mathrm{M}^{\mathsf{P1}}_{\mathsf{int}}$, and $u^{\mathsf{P1}}_{\mathsf{int}}$ are restrictions of the matrices K^{P1}, M^{P1}, and the vector u^{P1}, respectively, to indices corresponding to internal mesh nodes only. Note that $\mathrm{K}^{\mathsf{P1}}_{\mathsf{int}}$ is regular.

The classical Poincaré constant $c_{0,\emptyset}$ is approximated as

$$\frac{1}{c^2_{0,\emptyset,\mathsf{P1}}} = \lambda^2_{0,\emptyset,\mathsf{P1}} = \min_{\substack{0\neq u^{\mathsf{P1}},\\ u^{\mathsf{P1}}\cdot 1^{\mathsf{P1}}=0}} \frac{\mathrm{K}^{\mathsf{P1}}\, u^{\mathsf{P1}}\cdot u^{\mathsf{P1}}}{\mathrm{M}^{\mathsf{P1}}\, u^{\mathsf{P1}}\cdot u^{\mathsf{P1}}},$$

where the constraint $u^{\mathsf{P1}} \cdot 1^{\mathsf{P1}} = 0$ means that the vector u^{P1} has to be perpendicular to the constant vector of ones. $\lambda^2_{0,\emptyset,\mathsf{P1}}$ is the minimal positive eigenvalue of the generalized eigenvalue problem

$$\mathrm{K}^{\mathsf{P1}}\, u^{\mathsf{P1}} = \lambda^2\, \mathrm{M}^{\mathsf{P1}}\, u^{\mathsf{P1}}.$$

The minimal eigenvalue here is $\lambda^2 = 0$ and the corresponding eigenvector is the constant vector of ones. Analogously, the Friedrichs/Poincaré (Laplace) constants for mixed boundary conditions c_{0,Γ_τ} is approximated by using the same techniques and finite elements $\mathsf{P1}$. More precisely, for $\Gamma_\tau \neq \emptyset$ we have

$$\frac{1}{c^2_{0,\Gamma_\tau,\mathsf{P1}}} = \lambda^2_{0,\Gamma_\tau,\mathsf{P1}} = \min_{\substack{0\neq u^{\mathsf{P1}},\\ u^{\mathsf{P1}}_{\Gamma_\tau}=0}} \frac{\mathrm{K}^{\mathsf{P1}}\, u^{\mathsf{P1}}\cdot u^{\mathsf{P1}}}{\mathrm{M}^{\mathsf{P1}}\, u^{\mathsf{P1}}\cdot u^{\mathsf{P1}}}, \tag{21}$$

where $u^{\mathsf{P1}}_{\Gamma_\tau}$ denotes a subvector of u^{P1} in indices corresponding to boundary nodes of Γ_τ. $\lambda^2_{0,\Gamma_\tau,\mathsf{P1}}$ is the minimal (positive) eigenvalue of the generalized eigenvalue problem

$$\mathrm{K}^{\mathsf{P1}}\ u^{\mathsf{P1}} = \lambda^2\ \mathrm{M}^{\mathsf{P1}}\ u^{\mathsf{P1}}, \qquad u^{\mathsf{P1}}_{\Gamma_\tau} = 0,$$

and may be computed again by solving a restricted problem (to internal nodes and some boundary nodes) with a regular stiffness matrix $\mathrm{K}^{\mathsf{P1}}_{\mathsf{int},\Gamma_\tau}$, i.e.,

$$\mathrm{K}^{\mathsf{P1}}_{\mathsf{int},\Gamma_\tau}\ u^{\mathsf{P1}}_{\mathsf{int},\Gamma_\tau} = \lambda^2\ \mathrm{M}^{\mathsf{P1}}_{\mathsf{int},\Gamma_\tau}\ u^{\mathsf{P1}}_{\mathsf{int},\Gamma_\tau}.$$

As in any dimension the Friedrichs/Poincaré constants can be computed either as a gradient or as a divergence constant, see Theorem 2.17, we can approximate

$$c_{0,\Gamma_\tau} = c_{2,\Gamma_\nu}$$

either by (21) or by

$$\frac{1}{c^2_{2,\Gamma_\nu,\mathsf{RT}}} = \lambda^2_{2,\Gamma_\nu,\mathsf{RT}} = \min_{\substack{0\neq H^{\mathsf{RT}},\\ H^{\mathsf{RT}}_{\Gamma_\nu}=0,\\ H^{\mathsf{RT}}\perp N(\mathrm{K}^{\mathsf{RT}})}} \frac{\mathrm{K}^{\mathsf{RT}}\, H^{\mathsf{RT}}\cdot H^{\mathsf{RT}}}{\mathrm{M}^{\mathsf{RT}}\, H^{\mathsf{RT}}\cdot H^{\mathsf{RT}}},$$

where $H^{\mathsf{RT}}_{\Gamma_\nu}$ denotes a subvector of H^{RT} in indices corresponding to boundary faces of Γ_ν (boundary edges in 2D). $\lambda^2_{2,\Gamma_\nu,\mathsf{RT}}$ is the minimal positive eigenvalue of the generalized eigenvalue problem

$$\mathrm{K}^{\mathsf{RT}}\ H^{\mathsf{RT}} = \lambda^2\ \mathrm{M}^{\mathsf{RT}}\ H^{\mathsf{RT}}, \qquad H^{\mathsf{RT}}_{\Gamma_\nu} = 0,$$

respectively,

$$\mathrm{K}^{\mathsf{RT}}_{\mathsf{int},\Gamma_\nu}\ H^{\mathsf{RT}}_{\mathsf{int},\Gamma_\nu} = \lambda^2\ \mathrm{M}^{\mathsf{RT}}_{\mathsf{int},\Gamma_\nu}\ H^{\mathsf{RT}}_{\mathsf{int},\Gamma_\nu},$$

where $\mathrm{K}^{\mathsf{RT}}_{\mathsf{int},\Gamma_\nu}$, $\mathrm{M}^{\mathsf{RT}}_{\mathsf{int},\Gamma_\nu}$, and $H^{\mathsf{RT}}_{\mathsf{int},\Gamma_\nu}$ are restrictions of the matrices K^{RT}, M^{RT}, and the vector H^{RT} to indices corresponding to 'free' mesh faces (edges in 2D) only. Note that there are a lot of first zero eigenvalues $\lambda^2 = 0$ as neither K^{RT} nor $\mathrm{K}^{\mathsf{RT}}_{\mathsf{int},\Gamma_\nu}$ are regular due to the existence of large kernels $N(\mathrm{K}^{\mathsf{RT}})$ and $N(\mathrm{K}^{\mathsf{RT}}_{\mathsf{int},\Gamma_\nu})$ since all rotations belong to the kernel of the divergence.

4.2. **Maxwell Constants.** While the computation of the Friedrichs/Poincaré constants $c_{0,\Gamma_\tau} = c_{2,\Gamma_\nu}$ is more or less independent of the dimension, the computation of the Maxwell constants is different in 2D and 3D, or generally, in ND. By Remark 3.2 (v) we have in 2D

$$c_{1,\Gamma_\nu} = c_{0,\Gamma_\tau},$$

and thus the Maxwell constants can simply be computed by the corresponding Friedrichs/Poincaré (Laplace) constants. In particular, for the tangential (electric) and normal (magnetic) Maxwell constants it holds

$$c_{1,\Gamma} = c_{0,\emptyset}, \qquad c_{1,\emptyset} = c_{0,\Gamma}.$$

By Remark 3.3 (vii) we have in 3D

$$c_{1,\Gamma_\tau} = c_{1,\Gamma_\nu},$$

and thus the Maxwell constants have to be computed separately. In particular, for the tangential (electric) and normal (magnetic) Maxwell constants it holds

$$c_{1,\Gamma} = c_{1,\emptyset}.$$

The Maxwell constants are approximated as

$$\frac{1}{c^2_{1,\Gamma_\tau,\mathsf{N}}} = \lambda^2_{1,\Gamma_\tau,\mathsf{N}} = \min_{\substack{0\neq E^{\mathsf{N}},\\ E^{\mathsf{N}}_{\Gamma_\tau}=0,\\ E^{\mathsf{N}}\perp N(\mathrm{K}^{\mathsf{N}})}} \frac{\mathrm{K}^{\mathsf{N}}\, E^{\mathsf{N}}\cdot E^{\mathsf{N}}}{\mathrm{M}^{\mathsf{N}}\, E^{\mathsf{N}}\cdot E^{\mathsf{N}}},$$

where $E^{\mathsf{N}}_{\Gamma_\tau}$ denotes a subvector of E^{N} in indices corresponding to boundary edges of Γ_τ. $\lambda^2_{1,\Gamma_\tau,\mathsf{N}}$ is the minimal positive eigenvalue of the generalized eigenvalue problem

$$\mathrm{K}^{\mathsf{N}}\ E^{\mathsf{N}} = \lambda^2\ \mathrm{M}^{\mathsf{N}}\ E^{\mathsf{N}}, \qquad E^{\mathsf{N}}_{\Gamma_\tau} = 0,$$

respectively,

$$\mathrm{K}^{\mathsf{N}}_{\mathsf{int},\Gamma_\tau}\ E^{\mathsf{N}}_{\mathsf{int},\Gamma_\tau} = \lambda^2\ \mathrm{M}^{\mathsf{N}}_{\mathsf{int},\Gamma_\tau}\ E^{\mathsf{N}}_{\mathsf{int},\Gamma_\tau},$$

where $\mathrm{K}^{\mathsf{N}}_{\mathsf{int},\Gamma_\tau}$, $\mathrm{M}^{\mathsf{N}}_{\mathsf{int},\Gamma_\tau}$, and $E^{\mathsf{N}}_{\mathsf{int},\Gamma_\tau}$ are restrictions of the matrices K^{N}, M^{N}, and the vector E^{N} to indices corresponding to 'free' mesh edges only. Note that similar to the computation of the divergence constants

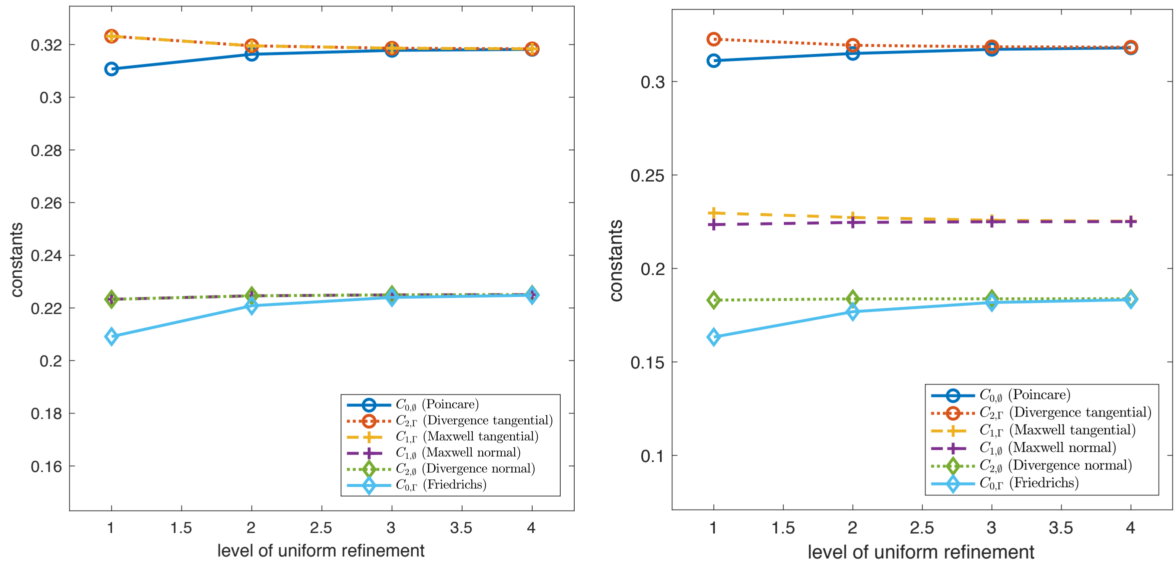

FIGURE 3. Constants computed for the unit square (left) and unit cube (right) with full boundary conditions. Theoretically, it holds in 2D (left)

$$c_{0,\Gamma} = c_{2,\emptyset} = c_{1,\emptyset} = \frac{1}{\sqrt{2}\pi} \approx 0.225 < c_{0,\emptyset} = c_{2,\Gamma} = c_{1,\Gamma} = \frac{1}{\pi} \approx 0.318$$

and in 3D (right)

$$c_{0,\Gamma} = c_{2,\emptyset} = \frac{1}{\sqrt{3}\pi} \approx 0.184 < c_{1,\Gamma} = c_{1,\emptyset} = \frac{1}{\sqrt{2}\pi} \approx 0.225 < c_{0,\emptyset} = c_{2,\Gamma} = \frac{1}{\pi} \approx 0.318.$$

there are a lot of first zero eigenvalues $\lambda^2 = 0$ as neither K^{N} nor $\mathrm{K}^{\mathsf{N}}_{\mathsf{int},\Gamma_\tau}$ are regular due to the existence of large kernels $N(\mathrm{K}^{\mathsf{N}})$ and $N(\mathrm{K}^{\mathsf{N}}_{\mathsf{int},\Gamma_\tau})$ since now all gradients belong to the kernel of the rotation.

We emphasise that the Maxwell constants are also approximated by

$$\frac{1}{c^2_{1,\Gamma_\nu,\mathsf{N}}} = \lambda^2_{1,\Gamma_\nu,\mathsf{N}} = \min_{\substack{0 \neq E^{\mathsf{N}}, \\ E^{\mathsf{N}}_{\Gamma_\nu} = 0, \\ E^{\mathsf{N}} \perp N(\mathrm{K}^{\mathsf{N}})}} \frac{\mathrm{K}^{\mathsf{N}} E^{\mathsf{N}} \cdot E^{\mathsf{N}}}{\mathrm{M}^{\mathsf{N}} E^{\mathsf{N}} \cdot E^{\mathsf{N}}}.$$

4.3. **2D Computations.** We demonstrate two benchmarks with the unit square and the L-shape domain, the first with known and the second with unknown values of the constants. Their coarse (level 1) meshes are displayed in Figure 1. For the unit square we have by Remark 3.2 exact values

$$c_{0,\Gamma} = c_{1,\emptyset} = \frac{1}{\sqrt{2}\pi} \approx 0.22507907, \qquad c_{0,\emptyset} = c_{1,\Gamma} = \frac{1}{\pi} \approx 0.31830988,$$

and our approximative values converge to them, see Table 1. This extends results of [54, Table 1]. For one case of mixed boundary conditions (with missing boundary part Γ_b) we have by Remark 3.2 and (18) exact values

$$c_{0,\Gamma_{t,l,r}} = c_{1,\Gamma_b} = \frac{2}{\sqrt{5}\pi} \approx 0.28470501, \qquad c_{0,\Gamma_b} = c_{1,\Gamma_{t,l,r}} = \frac{2}{\pi} \approx 0.63661977,$$

and our approximative values converge again to them, see Table 2. Approximative values for the L-shape domain are provided in Tables 3 and 4. We notice a quadratic convergence of all constants with respect to the mesh size h. It means that an absolute error of any considered constant approximation is reduced by a factor of 4 after each uniform mesh refinement. A finest (level 7) mesh for the square domain used in our computations consists of 131.072 triangles with 66.049 nodes and 197.120 edges.

4.4. **3D Computations.** We present two benchmarks with the unit cube and the Fichera corner domain, the first with known and the second with unknown values of the constants. Their coarse (level 1) meshes

level	$c_{0,\emptyset,\mathsf{P1}}$	$c_{2,\Gamma,\mathsf{RT}}$	$c_{1,\Gamma,\mathsf{N}}$	$c_{1,\emptyset,\mathsf{N}}$	$c_{2,\emptyset,\mathsf{RT}}$	$c_{0,\Gamma,\mathsf{P1}}$
1	0.31072999	0.32316745	0.32316745	0.22328039	0.22328039	0.20912552
2	0.31631302	0.31953907	0.31953907	0.22460517	0.22460517	0.22083319
3	0.31780225	0.31861815	0.31861815	0.22495907	0.22495907	0.22400032
4	0.31818232	0.31838701	0.31838701	0.22504898	0.22504898	0.22480828
5	0.31827795	0.31832917	0.31832917	0.22507155	0.22507155	0.22501131
6	0.31830190	0.31831471	0.31831471	0.22507720	0.22507720	0.22506213
7	0.31830789	0.31831109	0.31831109	0.22507861	0.22507861	0.22507484
∞	0.31830988	0.31830988	0.31830988	0.22507907	0.22507907	0.22507907

TABLE 1. Constants computed for the unit square and full boundary conditions.

level	$c_{0,\Gamma_b,\mathsf{P1}}$	$c_{2,\Gamma_{t,l,r},\mathsf{RT}}$	$c_{1,\Gamma_{t,l,r},\mathsf{N}}$	$c_{1,\Gamma_b,\mathsf{N}}$	$c_{2,\Gamma_b,\mathsf{RT}}$	$c_{0,\Gamma_{t,l,r},\mathsf{P1}}$
1	0.63267458	0.63798842	0.63798842	0.28486798	0.28486798	0.27318834
2	0.63560893	0.63696095	0.63696095	0.28473767	0.28473767	0.28172459
3	0.63636500	0.63670501	0.63670501	0.28471277	0.28471277	0.28395286
4	0.63655592	0.63664108	0.63664108	0.28470693	0.28470693	0.28451652
5	0.63660380	0.63662510	0.63662510	0.28470549	0.28470549	0.28465786
6	0.63661578	0.63662110	0.63662110	0.28470514	0.28470514	0.28469323
7	0.63661877	0.63662011	0.63662011	0.28470505	0.28470505	0.28470207
∞	0.63661977	0.63661977	0.63661977	0.28470501	0.28470501	0.28470501

TABLE 2. Constants computed for the unit square and mixed boundary conditions.

level	$c_{0,\emptyset,\mathsf{P1}}$	$c_{2,\Gamma,\mathsf{RT}}$	$c_{1,\Gamma,\mathsf{N}}$	$c_{1,\emptyset,\mathsf{N}}$	$c_{2,\emptyset,\mathsf{RT}}$	$c_{0,\Gamma,\mathsf{P1}}$
1	0.39156654	0.43611331	0.43611331	0.16795692	0.16795692	0.13325394
2	0.40370423	0.42045050	0.42045050	0.16377267	0.16377267	0.15232573
3	0.40850306	0.41492017	0.41492017	0.16214127	0.16214127	0.15838355
4	0.41038725	0.41287500	0.41287500	0.16148392	0.16148392	0.16020361
5	0.41112643	0.41209870	0.41209870	0.16121879	0.16121879	0.16076463
6	0.41141712	0.41179918	0.41179918	0.16111230	0.16111230	0.16094566
7	0.41153175	0.41168242	0.41168242	0.16106970	0.16106970	0.16100698

TABLE 3. Constants computed for the L-shape domain and full boundary conditions.

level	$c_{0,\Gamma_b,\mathsf{P1}}$	$c_{2,\Gamma_{t,l,r},\mathsf{RT}}$	$c_{1,\Gamma_{t,l,r},\mathsf{N}}$	$c_{1,\Gamma_b,\mathsf{N}}$	$c_{2,\Gamma_b,\mathsf{RT}}$	$c_{0,\Gamma_{t,l,r},\mathsf{P1}}$
1	0.55287499	0.58356116	0.58356116	0.24038804	0.24038804	0.21444362
2	0.56332946	0.57483917	0.57483917	0.23765352	0.23765352	0.22916286
3	0.56716139	0.57152377	0.57152377	0.23648111	0.23648111	0.23363643
4	0.56857589	0.57025101	0.57025101	0.23597974	0.23597974	0.23498908
5	0.56910703	0.56975715	0.56975715	0.23577100	0.23577100	0.23541318
6	0.56930976	0.56956402	0.56956402	0.23568569	0.23568569	0.23555262
7	0.56938813	0.56948808	0.56948808	0.23565122	0.23565122	0.23560066

TABLE 4. Constants computed for the L-shape domain and mixed boundary conditions.

are displayed in Figure 2. For the unit cube we have by Remark 3.3 exact values

$$c_{0,\Gamma} = \frac{1}{\sqrt{3}\pi} \approx 0.18377629, \quad c_{1,\Gamma} = c_{1,\emptyset} = \frac{1}{\sqrt{2}\pi} \approx 0.22507907, \quad c_{0,\emptyset} = \frac{1}{\pi} \approx 0.31830988,$$

and our approximative values converge to them, see Table 5. For one case of mixed boundary conditions (with missing boundary part Γ_b) we have by Remark 3.3 and (19), (20) exact values

$$c_{0,\Gamma_{t,l,r,f,bk}} = \frac{2}{3\pi} \approx 0.21220659, \quad c_{1,\Gamma_{t,l,r,f,bk}} = c_{1,\Gamma_b} = \frac{2}{\sqrt{5}\pi} \approx 0.28470501, \quad c_{0,\Gamma_b} = \frac{2}{\pi} \approx 0.63661977,$$

and our approximative values converge again to them, see Table 6. Approximative values for the Fichera corner domain are provided in Tables 7 and 8. We notice a slightly lower than quadratic convergence of all constants with respect to the mesh size h. A finest (level 4) mesh for the cube domain used in our computations consists of 196.608 tetrahedra with 399.360 faces, 238.688 edges and 35.937 nodes.

level	$c_{0,\emptyset,\mathsf{P1}}$	$c_{2,\Gamma,\mathsf{RT}}$	$c_{1,\Gamma,\mathsf{N}}$	$c_{1,\emptyset,\mathsf{N}}$	$c_{2,\emptyset,\mathsf{RT}}$	$c_{0,\Gamma,\mathsf{P1}}$
1	0.31114284	0.32265677	0.22964649	0.22346361	0.18305860	0.16330104
2	0.31500347	0.31939334	0.22727295	0.22461307	0.18369611	0.17685247
3	0.31720303	0.31857551	0.22577016	0.22497862	0.18375776	0.18178558
4	0.31799426	0.31837527	0.22526682	0.22505528	0.18377095	0.18324991
∞	0.31830988	0.31830988	0.22507907	0.22507907	0.18377629	0.18377629

TABLE 5. Constants computed for the unit cube and full boundary conditions.

level	$c_{0,\Gamma_b,\mathsf{P1}}$	$c_{2,\Gamma_{t,l,r,f,bk},\mathsf{RT}}$	$c_{1,\Gamma_{t,l,r,f,bk},\mathsf{N}}$	$c_{1,\Gamma_b,\mathsf{N}}$	$c_{2,\Gamma_b,\mathsf{RT}}$	$c_{0,\Gamma_{t,l,r,f,bk},\mathsf{P1}}$
1	0.63279353	0.63799454	0.28810408	0.28568645	0.21207495	0.19466267
2	0.63563506	0.63694323	0.28621730	0.28506833	0.21221199	0.20590030
3	0.63636820	0.63669754	0.28518535	0.28483451	0.21220585	0.21033840
4	0.63655623	0.63663874	0.28483637	0.28474355	0.21220553	0.21170560
∞	0.63661977	0.63661977	0.28470501	0.28470501	0.21220659	0.21220659

TABLE 6. Constants computed for the unit cube and mixed boundary conditions.

level	$c_{0,\emptyset,\mathsf{P1}}$	$c_{2,\Gamma,\mathsf{RT}}$	$c_{1,\Gamma,\mathsf{N}}$	$c_{1,\emptyset,\mathsf{N}}$	$c_{2,\emptyset,\mathsf{RT}}$	$c_{0,\Gamma,\mathsf{P1}}$
1	0.34328060	0.37118723	0.30375245	0.26905796	0.15938388	0.12490491
2	0.35193318	0.36341148	0.28961049	0.27500043	0.15638922	0.14329827
3	0.35628919	0.36072519	0.28329415	0.27728510	0.15500593	0.15042074
4	0.35808207	0.35976508	0.28054899	0.27811443	0.15444291	0.15286199

TABLE 7. Constants computed for the Fichera corner domain and full boundary conditions.

level	$c_{0,\Gamma_b,\mathsf{P1}}$	$c_{2,\Gamma_{t,l,r,f,bk},\mathsf{RT}}$	$c_{1,\Gamma_{t,l,r,f,bk},\mathsf{N}}$	$c_{1,\Gamma_b,\mathsf{N}}$	$c_{2,\Gamma_b,\mathsf{RT}}$	$c_{0,\Gamma_{t,l,r,f,bk},\mathsf{P1}}$
1	0.71426099	0.73298472	0.40148747	0.36201635	0.16864924	0.13435096
2	0.72081163	0.72802794	0.38625667	0.36900931	0.16395187	0.15055805
3	0.72341261	0.72621410	0.37907733	0.37196215	0.16217563	0.15739881
4	0.72444626	0.72553497	0.37599825	0.37313348	0.16150269	0.15983418

TABLE 8. Constants computed for the Fichera corner domain and mixed boundary conditions.

4.5. **Testing of the Monotonicity Properties.** We perform some monotonicity tests on the constants depending on the respective boundary conditions, i.e., we display the mapping

$$\Gamma_\nu \longmapsto (c_{0,\Gamma_\tau}, c_{0,\Gamma_\nu}, c_{1,\Gamma_\tau}, c_{1,\Gamma_\nu}, c_{2,\Gamma_\tau}, c_{2,\Gamma_\nu})$$

for a monotone increasing sequence of Γ_ν. Figures 4 and 5 depict examples of such sequences in 2D/3D. The boundary part Γ_ν is represented discretely as a set of Neumann faces in 3D or a set of Neumann edges in 2D. Boundary faces or edges are checked for their connectivity and a breadth-first search (BFS) algorithm is applied to order them in a sequence. All constants are then evaluated for every element of the sequence and the results are displayed in Figures 6 and 7.

4.6. **Computational Details and MATLAB Code.** A generalized eigenvalue system

$$Kv = \lambda^2 Mv \tag{22}$$

with a positive semidefinite and symmetric matrix $K \in \mathbb{R}^{n\times n}$ and a positive definite and symmetric matrix $M \in \mathbb{R}^{n\times n}$ is solved for a smallest positive eigenvalue $\lambda^2 > 0$. We apply two computational techniques.

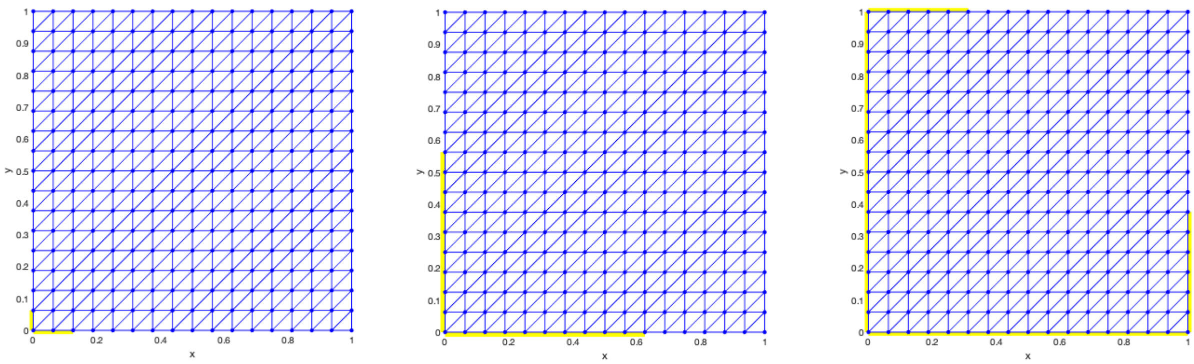

FIGURE 4. A sequence of monotone increasing Neumann boundary parts Γ_ν (yellow) for the unit square domain with 3, 19, and 45 Neumann edges. A full boundary Γ consists of 64 edges.

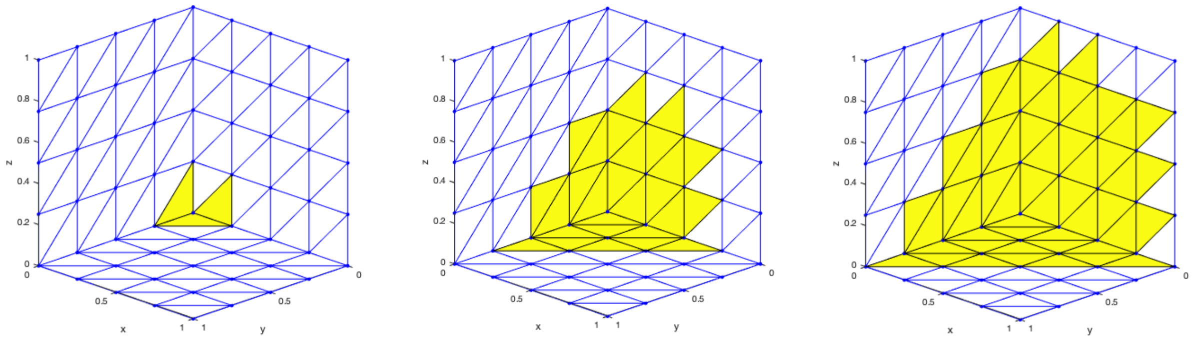

FIGURE 5. A sequence of monotone increasing Neumann boundary parts Γ_ν (yellow) for the unit cube with 3, 27, and 51 Neumann faces. A full boundary Γ consists of 192 faces.

4.6.1. *A projection to the range of K.* We apply the QR-decomposition of K in the form

$$KE = \tilde{Q}\tilde{R},$$

where $E \in \mathbb{R}^{n \times n}$ is a permutation matrix, $\tilde{Q} \in \mathbb{R}^{n \times n}$ is an orthogonal matrix and $\tilde{R} \in \mathbb{R}^{n \times n}$ is an upper triangular matrix with diagonal entries ordered in decreasing order as

$$|\tilde{R}_{1,1}| \geq \cdots \geq |\tilde{R}_{r,r}| \geq \cdots \geq |\tilde{R}_{n,n}|.$$

The number $r \leq n$ of nonzero entries of the sequence above then determines the range K and all rows of $\tilde{R}$ with indices larger than r are zero rows, cf. Figure 8. Therefore, we can also decompose

$$KE = QR \tag{23}$$

where $Q \in \mathbb{R}^{n \times r}$ is a restriction of $\tilde{Q}$ to its first r columns and $R \in \mathbb{R}^{r \times n}$ a restriction of $\tilde{R}$ to its first r rows. Then, a mapping $v = Qz$ for $z \in \mathbb{R}^{r \times 1}$ projects a vector z to the range of K and we can transform the generalized eigenvalue system (22) to $KQz = \lambda^2 MQz$ or equivalently (since $Q^{-1} = Q^\top$) to a standard eigenvalue problem

$$Q^\top M^{-1} KQz = \lambda^2 z. \tag{24}$$

A matrix M^{-1} is full and expensive to compute, its memory storage is large and the multiplication with $Q^\top M^{-1}$ is costly. In view of (23), the symmetry of K and the orthogonality of E, it holds

$$KQ = (QRE^\top)Q = (QRE^\top)^\top Q = ER^\top Q^\top Q = ER^\top$$

and it slightly reduces assembly times in our practical computations. Therefore, this projection technique is applied to coarser meshes only with several thousands degrees of freedom.

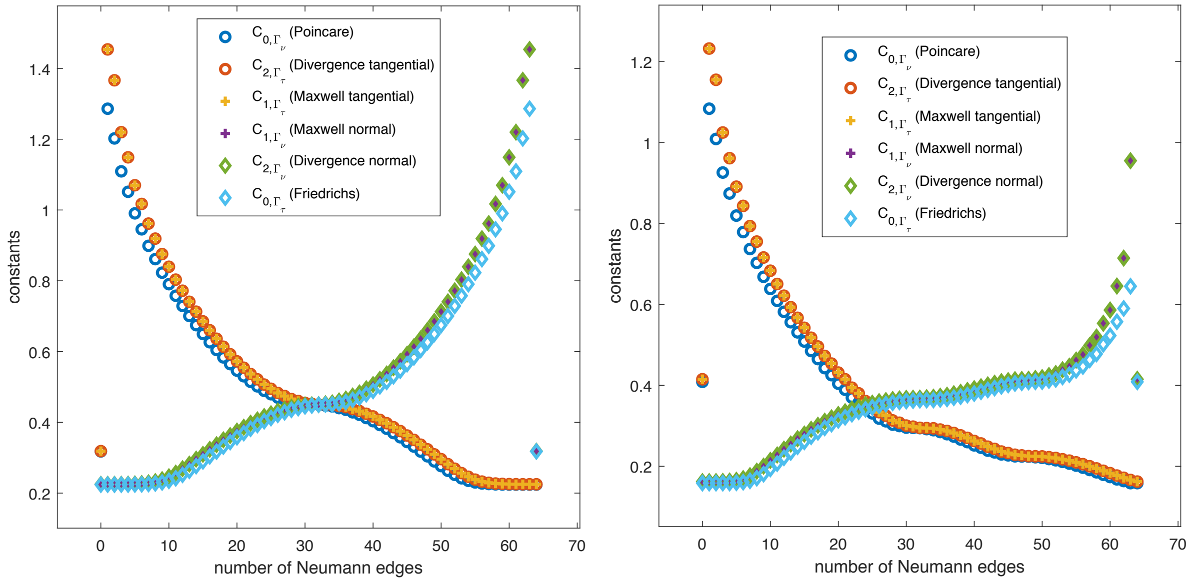

FIGURE 6. Constants computed for the unit square (left) and the L-shape domain (right) - monotonicity test for a monotone increasing sequence of Γ_ν.

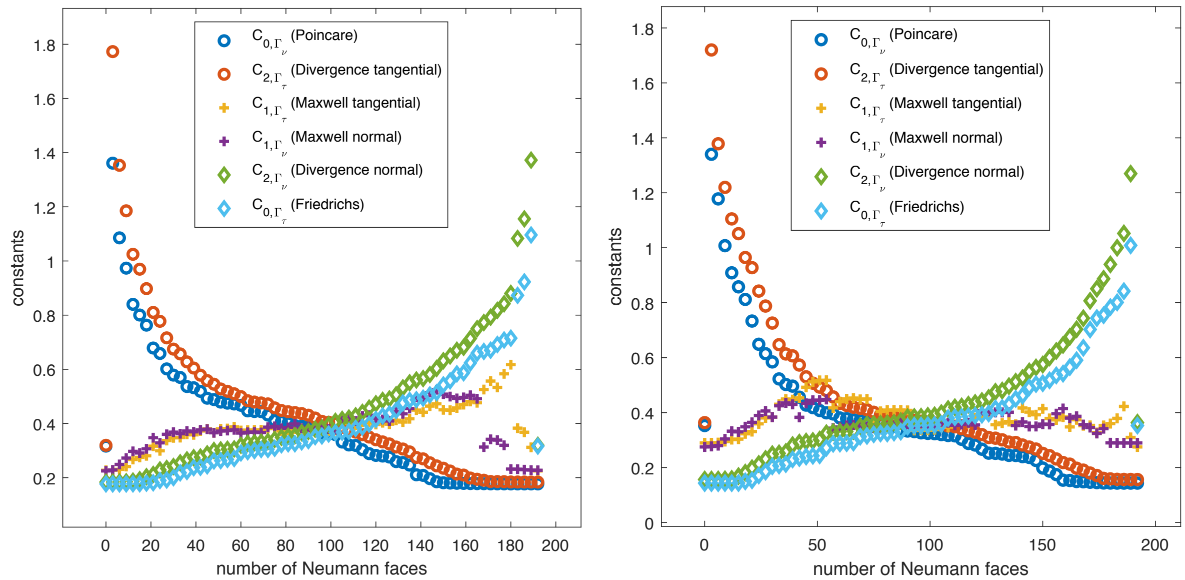

FIGURE 7. Constants computed for the unit cube (left) and the Fichera corner domain (right) - monotonicity test for a monotone increasing sequence of Γ_ν.

4.6.2. *A nested iteration technique.* An eigenvalue evaluated on a coarser mesh (by the technique above) is used as initial guess on a finer (uniformly refined) mesh, where an inbuilt MATLAB function *eigs* is applied for the search of the closest eigenvalue. It was noticed that for some cases of mixed boundary conditions in 3D the nested iterations technique did not converge (a sequence of corresponding Laplace constant in the monotonicity test did not form a monotone sequence). Without additional preconditioning (multigrid, domain decompositions) of eigenvalue solvers we are able to evaluate smallest positive eigenvalues for finer meshes with several hundred thousands degrees of freedom.

Alternatively, for computations of Laplace eigenvalues, the dimension of the nullspace is 0 or 1 and it is faster to compute the eigenvalues directly by the MATLAB function `eigs`. It is more computationally demanding to evaluate divergence and Maxwell constants than Laplace constants, since the numbers of faces (in 3D) and edges are significantly higher than the number of nodes. A MATLAB code used for

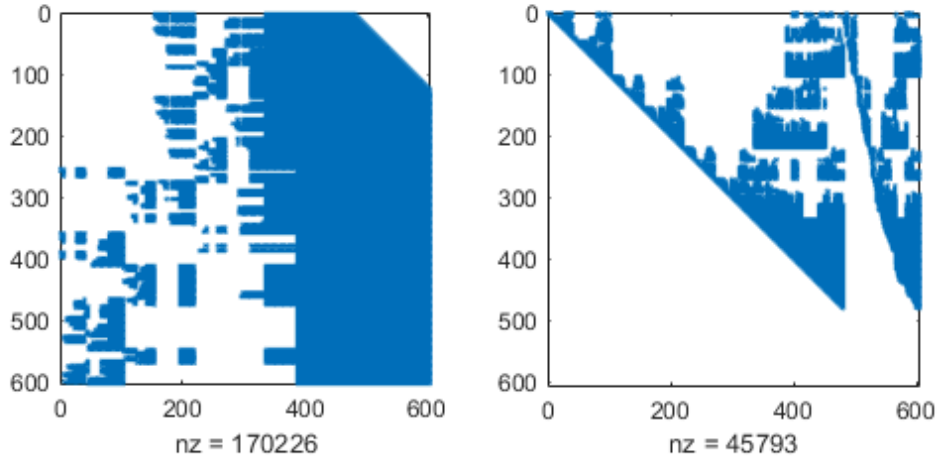

FIGURE 8. An example: A positive semidefinite matrix $K \in \mathbb{R}^{604\times 604}$ is decomposed as a product of $\tilde{Q} \in \mathbb{R}^{604\times 604}$ (left) and $\tilde{R} \in \mathbb{R}^{604\times 604}$ (right). Here, the dimension of the range of K is 480 and for the restricted matrices we have $Q \in \mathbb{R}^{604\times 480}$ and $R \in \mathbb{R}^{480\times 604}$.

numerical evaluations is based on finite element matrix assemblies from [1, 52] and also utilizes a 3D cube mesh and mesh visualizations from [55]. The code is freely available for download and testing at:

`https://www.mathworks.com/matlabcentral/fileexchange/23991`

It can be easily modified to other domains and boundary conditions. The starting scripts for testing are `start_2D` and `start_3D`. To a given mesh, it automatically determines its boundary. In 2D, the code can also visualize eigenfunctions, see Figure 9 for the case of the L-shape domain.

5. Discussion of the Numerical Results and Conclusions

Our numerical results, especially in 3D, did verify all the theoretical assertions of Theorem 2.20, see also Remark 3.3 and Remark 3.2, in particular,

- the monotone dependence of the Friedrichs/Poincaré and divergence constants on the boundary conditions, i.e., the monotonicity of the mapping
$$\Gamma_\nu \mapsto c_{0,\Gamma_\tau} = c_{2,\Gamma_\nu},$$
- the 'independence' of the Maxwell constants on the boundary conditions, i.e.,
$$\forall\, \Gamma_\tau \subset \Gamma \qquad c_{1,\Gamma_\tau} = c_{1,\Gamma_\nu},$$
- as well as the boundedness of the full tangential and normal Maxwell constants by the Poincaré constant for convex Ω, i.e.,
$$c_{1,\Gamma} = c_{1,\emptyset} \leq c_{0,\emptyset} = c_{2,\Gamma}.$$

While the first two assertions hold for general bounded Lipschitz domains and Lipschitz interfaces, the third assertion is analytically proved only for convex domains and the full boundary conditions. In our numerical experiments, the unit cube served as a prototype for a convex domain, and we picked the Fichera corner domain as a typical example of a non-convex domain, see Figure 2 for both initial meshes.

To our surprise, even for mixed boundary conditions and for non-convex geometries, the *extended inequalities*

$$c_{0,\Gamma} \leq \min\{c_{0,\Gamma_\tau}, c_{0,\Gamma_\nu}\} \leq c_{1,\Gamma_\tau} = c_{1,\Gamma_\nu} \leq \max\{c_{0,\Gamma_\tau}, c_{0,\Gamma_\nu}\} \leq \sup_{\Gamma_\tau \neq \emptyset} c_{0,\Gamma_\tau} = \sup_{\Gamma_\nu \neq \Gamma} c_{2,\Gamma_\nu} \tag{25}$$

seem to hold for our examples, see Figure 7. In these special cases the Maxwell constants are always in between the Friedrichs/Poincaré (Laplace) constants. We emphasise that our examples possess (piecewise) vanishing curvature. It remains an open question if (25) is true - at least partially - in general or, e.g., for polyhedra. Moreover, if Γ_τ approaches $\emptyset$, the Friedrichs/Poincaré constants c_{0,Γ_τ} seem to be bounded, i.e., the suprema in (25) appear to be bounded, although a kernel of dimension 1 (constants) is approximated.

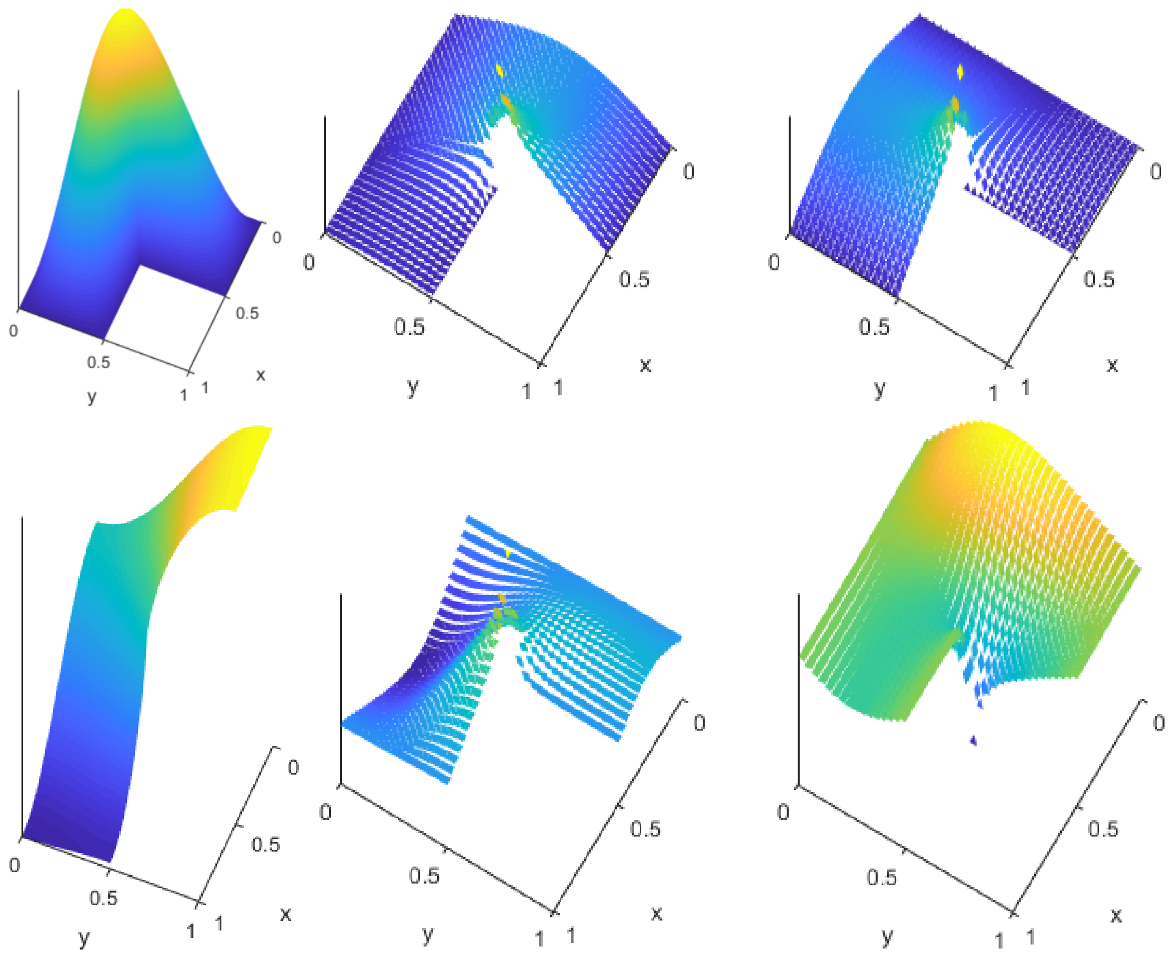

FIGURE 9. Eigenfunctions - Friedrichs (top left), Poincaré (bottom left), tangential Maxwell (top middle and right), normal Maxwell (bottom middle and right) - for the L-shape domain with full boundary conditions.

5.1. **Hints for the Extended Inequalities.** We note the well known integration by parts formula

$$|\operatorname{grad} E|^2_{\mathsf{L}^2(\Omega)} = |\operatorname{rot} E|^2_{\mathsf{L}^2(\Omega)} + |\operatorname{div} E|^2_{\mathsf{L}^2(\Omega)}, \tag{26}$$

being valid for all vector fields $E \in \mathsf{H}_\Gamma(\operatorname{grad}, \Omega)$, the closure of Ω-compactly supported test fields, see (15). Using a more sophisticated integration by parts formula from [12, Corollary 6], which has been proved already in, e.g., [28, Theorem 2.3] for the case of full boundary conditions, we see that (26) remains true for *polyhedral* domains Ω and for vector fields

$$E \in \mathsf{H}_{\Gamma_\tau,\Gamma_\nu}(\operatorname{grad}, \Omega) := \overline{\mathsf{C}^\infty_{\Gamma_\tau,\Gamma_\nu}(\overline{\Omega})}^{\mathsf{H}(\operatorname{grad},\Omega)} \subset \mathsf{H}(\operatorname{grad}, \Omega) \cap \mathsf{H}_{\Gamma_\tau}(\operatorname{rot}, \Omega) \cap \mathsf{H}_{\Gamma_\nu}(\operatorname{div}, \Omega), \tag{27}$$

where

$$\mathsf{C}^\infty_{\Gamma_\tau,\Gamma_\nu}(\overline{\Omega}) := \big\{\Phi|_\Omega : \Phi \in \mathsf{C}^\infty(\mathbb{R}^3), \operatorname{supp}\Phi \text{compact in } \mathbb{R}^3, \, n \times \Phi|_{\Gamma_\tau} = 0, \, n \cdot \Phi|_{\Gamma_\nu} = 0\big\}.$$

Note that these results at least go back to the book of Grisvard [31, Theorem 3.1.1.2], see also the book of Leis [36, p. 156-157].

A first hint for a possible explanation of (25) is then the following observation: Let E_{1,Γ_τ} be the minimiser from Remark 2.18. Then

$$E_{1,\Gamma_\tau} \in D(\operatorname{rot}_{\Gamma_\tau}) \cap R(\operatorname{rot}_{\Gamma_\nu}) \subset \mathsf{H}_{\Gamma_\tau}(\operatorname{rot}, \Omega) \cap \mathsf{H}_{\Gamma_\nu}(\operatorname{div}, \Omega), \qquad \operatorname{div} E_{1,\Gamma_\tau} = 0.$$

Hence, if Ω is a *polyhedron* and if E_{1,Γ_τ} is regular[1] enough, i.e., $E_{1,\Gamma_\tau} \in \mathsf{H}_{\Gamma_\tau,\Gamma_\nu}(\operatorname{grad},\Omega)$, then by (26) and (27)

$$\lambda_{1,\Gamma_\tau} = \frac{|\operatorname{rot} E_{1,\Gamma_\tau}|_{\mathsf{L}^2(\Omega)}}{|E_{1,\Gamma_\tau}|_{\mathsf{L}^2(\Omega)}} = \frac{|\operatorname{grad} E_{1,\Gamma_\tau}|_{\mathsf{L}^2(\Omega)}}{|E_{1,\Gamma_\tau}|_{\mathsf{L}^2(\Omega)}}.$$

Moreover, if E_{1,Γ_τ} admits the additional regularity $E_{1,\Gamma_\tau} \in \mathsf{H}_{\Gamma_\tau}(\operatorname{grad},\Omega)$, then

$$\lambda_{1,\Gamma_\tau} \geq \inf_{0 \neq E \in \mathsf{H}_{\Gamma_\tau}(\operatorname{grad},\Omega)} \frac{|\operatorname{grad} E|_{\mathsf{L}^2(\Omega)}}{|E|_{\mathsf{L}^2(\Omega)}} = \lambda_{0,\Gamma_\tau}.$$

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6. Appendix: Some Proofs

Proof of (6). To show that, e.g., $\mathrm{A}^* \mathrm{A}$ is self-adjoint, we first observe that $\mathrm{A}^* \mathrm{A}$ is symmetric. Hence, so is $\mathrm{A}^* \mathrm{A} + 1$. By Riesz' representation theorem, for any $f \in \mathsf{H}_0$ there exists a unique $x \in D(\mathrm{A})$ such that

$$\forall\, \varphi \in D(\mathrm{A}) \qquad \langle \mathrm{A}\, x, \mathrm{A}\, \varphi \rangle_{\mathsf{H}_1} + \langle x, \varphi \rangle_{\mathsf{H}_0} = \langle f, \varphi \rangle_{\mathsf{H}_0}.$$

Thus, $\mathrm{A}\,x \in D(\mathrm{A}^*)$ and $\mathrm{A}^*\mathrm{A}\,x = f - x$, i.e., $x \in D(\mathrm{A}^*\mathrm{A})$ and $(\mathrm{A}^*\mathrm{A}+1)x = f$. In other words, $\mathrm{A}^*\mathrm{A}+1$ is onto. Therefore, $\mathrm{A}^*\mathrm{A}+1$ is self-adjoint and so is $\mathrm{A}^*\mathrm{A}$. Note that we did not need the additional assumption that $R(\mathrm{A})$ is closed or that A resp. A^* is onto.

We also present an alternative proof of the self-adjointness of $\mathrm{A}^*\mathrm{A}$ in the case that $R(\mathrm{A})$ is closed. For this, let $y \in \mathsf{H}_0$ such that there exists $z \in \mathsf{H}_0$ with

$$\forall\, x \in D(\mathrm{A}^*\mathrm{A}) \qquad \langle \mathrm{A}^*\mathrm{A}\,x, y\rangle_{\mathsf{H}_0} = \langle x, z\rangle_{\mathsf{H}_0}. \tag{28}$$

Picking $x \in N(\mathrm{A})$ shows that $z \perp_{\mathsf{H}_0} N(\mathrm{A})$, i.e., we have $z \in R(\mathrm{A}^*)$ by (3). For $\varphi \in D(\mathrm{A}^*)$ we note $\mathrm{A}^*\varphi \in R(\mathrm{A}^*) = R(\mathcal{A}^*)$. Thus there is

$$\psi_\varphi := (\mathcal{A}^*)^{-1}\,\mathrm{A}^*\varphi \in D(\mathcal{A}^*) \subset R(\mathrm{A}) = R(\mathcal{A}) \quad \text{with} \quad \mathrm{A}^*\psi_\varphi = \mathrm{A}^*\varphi.$$

Moreover, there exists $x_\varphi := \mathcal{A}^{-1}\psi_\varphi \in D(\mathcal{A})$ with $\mathrm{A}\,x_\varphi = \psi_\varphi$ and thus $x_\varphi \in D(\mathcal{A}^*\mathcal{A})$. By (28) we see

$$\begin{aligned}\langle \mathrm{A}^*\varphi, y\rangle_{\mathsf{H}_0} &= \langle \mathrm{A}^*\mathrm{A}\,x_\varphi, y\rangle_{\mathsf{H}_0} = \langle x_\varphi, z\rangle_{\mathsf{H}_0} = \big\langle x_\varphi, \mathrm{A}^*(\mathcal{A}^*)^{-1}z\big\rangle_{\mathsf{H}_0} = \big\langle \mathrm{A}\,x_\varphi, (\mathcal{A}^*)^{-1}z\big\rangle_{\mathsf{H}_1}\\ &= \big\langle \varphi, (\mathcal{A}^*)^{-1}z\big\rangle_{\mathsf{H}_1} + \underbrace{\big\langle \psi_\varphi - \varphi, (\mathcal{A}^*)^{-1}z\big\rangle_{\mathsf{H}_1}}_{=0},\end{aligned}$$

as $\psi_\varphi - \varphi \in N(\mathrm{A}^*) \perp_{\mathsf{H}_1} R(\mathrm{A}) \supset D(\mathcal{A}^*) \ni (\mathcal{A}^*)^{-1}z$. Therefore, $y \in D(\mathrm{A})$ and $\mathrm{A}\,y = (\mathcal{A}^*)^{-1}z \in D(\mathrm{A}^*)$, showing $y \in D(\mathrm{A}^*\mathrm{A})$ and $\mathrm{A}^*\mathrm{A}\,y = z$. This proves $(\mathrm{A}^*\mathrm{A})^* = \mathrm{A}^*\mathrm{A}$. □

Proof of Lemma 2.5. We show a few selected assertions of Lemma 2.5.

• For an eigenvalue $\lambda > 0$ and an eigenvector (x, y) of $\begin{bmatrix} 0 & \mathrm{A}^* \\ \mathrm{A} & 0 \end{bmatrix}$ it holds $\mathrm{A}^* y = \lambda x$ and $\mathrm{A}\,x = \lambda y$. Note that $x = 0$ implies $y = 0$. Thus $0 \neq x \in D(\mathrm{A}^*\mathrm{A})$ and $\mathrm{A}^*\mathrm{A}\,x = \lambda\,\mathrm{A}^* y = \lambda^2 x$, i.e., x is an eigenvector and λ^2 is an eigenvalue of $\mathrm{A}^*\mathrm{A}$.

• If $\lambda^2 > 0$ is an eigenvalue and x is an eigenvector of $\mathrm{A}^*\mathrm{A}$, then $y_\pm := \pm\lambda^{-1}\,\mathrm{A}\,x \in D(\mathrm{A}^*)$ and $\mathrm{A}^* y_\pm = \pm\lambda^{-1}\,\mathrm{A}^*\mathrm{A}\,x = \pm\lambda x$, i.e., $(x, y_\pm)$ is an eigenvector and $\pm\lambda$ is an eigenvalue of $\begin{bmatrix} 0 & \mathrm{A}^* \\ \mathrm{A} & 0 \end{bmatrix}$. Note that $y_\pm \neq 0$ as $y_\pm = 0$ implies $x = 0$.

• If $\lambda^2 > 0$ is an eigenvalue and x is an eigenvector of $\mathrm{A}^*\mathrm{A}$, then $y := \mathrm{A}\,x \in D(\mathrm{A}^*)$ and we have $\mathrm{A}^* y = \mathrm{A}^*\mathrm{A}\,x = \lambda^2 x \in D(\mathrm{A})$. Hence $y \in D(\mathrm{A}\,\mathrm{A}^*)$ and $\mathrm{A}\,\mathrm{A}^* y = \lambda^2\,\mathrm{A}\,x = \lambda^2 y$, i.e., λ^2 is an eigenvalue and y is an eigenvector of $\mathrm{A}\,\mathrm{A}^*$. Note that $y \neq 0$ as $y = 0$ implies $x = 0$.

• To show that indeed, e.g., λ_{A}^2 is the smallest positive eigenvalue of $\mathrm{A}^*\mathrm{A}$, let us consider a sequence $(\tilde{x}_n)$ in $D(\mathcal{A}) \setminus \{0\}$ with

$$\frac{|\,\mathrm{A}\,\tilde{x}_n|_{\mathsf{H}_1}}{|\tilde{x}_n|_{\mathsf{H}_0}} \to \inf_{0 \neq x \in D(\mathcal{A})} \frac{|\,\mathrm{A}\,x|_{\mathsf{H}_1}}{|x|_{\mathsf{H}_0}} = \lambda_{\mathrm{A}} > 0.$$

Then $(x_n) := (\tilde{x}_n / |\tilde{x}_n|_{\mathsf{H}_0}) \subset D(\mathcal{A})$ with $|x_n|_{\mathsf{H}_0} = 1$ and

$$\lambda_{\mathrm{A}} \leq |\,\mathrm{A}\,x_n|_{\mathsf{H}_1} \to \lambda_{\mathrm{A}}.$$

Hence (x_n) is bounded in $D(\mathcal{A})$, yielding a subsequence - again denoted by (x_n) - as well as $x_{\mathrm{A}} \in \mathsf{H}_0$ and $y_{\mathrm{A}} \in \mathsf{H}_1$ with $x_n \rightharpoonup x_{\mathrm{A}}$ in H_0, $\mathrm{A}\,x_n \rightharpoonup y_{\mathrm{A}}$ in H_1, and $x_n \to x_{\mathrm{A}}$ in H_0. Then $x_{\mathrm{A}} \in D(\mathrm{A})$ and $\mathrm{A}\,x_{\mathrm{A}} = y_{\mathrm{A}}$ as for all $\psi \in D(\mathrm{A}^*)$

$$\langle y_{\mathrm{A}}, \psi\rangle_{\mathsf{H}_1} \leftarrow \langle \mathrm{A}\,x_n, \psi\rangle_{\mathsf{H}_1} = \langle x_n, \mathrm{A}^*\psi\rangle_{\mathsf{H}_0} \to \langle x_{\mathrm{A}}, \mathrm{A}^*\psi\rangle_{\mathsf{H}_0}.$$

Note that $x_{\mathrm{A}} \in R(\mathrm{A}^*)$ as $x_n \in R(\mathrm{A}^*) = N(\mathrm{A})^{\perp_{\mathsf{H}_0}}$, especially, $x_{\mathrm{A}} \in D(\mathcal{A})$. Moreover, $|x_{\mathrm{A}}|_{\mathsf{H}_0} = 1$.

By elementary calculations[2] we obtain for all $\varphi, \phi \in D(\mathcal{A})$

$$\big|\langle \mathrm{A}\,\varphi, \mathrm{A}\,\phi\rangle_{\mathsf{H}_1} - \lambda_{\mathrm{A}}^2\langle \varphi, \phi\rangle_{\mathsf{H}_0}\big|^2 \leq 2\big(|\,\mathrm{A}\,\varphi|_{\mathsf{H}_1}^2 - \lambda_{\mathrm{A}}^2|\varphi|_{\mathsf{H}_0}^2\big)\big(|\,\mathrm{A}\,\phi|_{\mathsf{H}_1}^2 - \lambda_{\mathrm{A}}^2|\phi|_{\mathsf{H}_0}^2\big).$$

In particular, for $\varphi := x_n$ we get for all $\phi \in D(\mathcal{A})$

$$\big|\langle \mathrm{A}\,x_n, \mathrm{A}\,\phi\rangle_{\mathsf{H}_1} - \lambda_{\mathrm{A}}^2\langle x_n, \phi\rangle_{\mathsf{H}_0}\big|^2 \leq 2\big(|\,\mathrm{A}\,x_n|_{\mathsf{H}_1}^2 - \lambda_{\mathrm{A}}^2\big)\big(|\,\mathrm{A}\,\phi|_{\mathsf{H}_1}^2 - \lambda_{\mathrm{A}}^2|\phi|_{\mathsf{H}_0}^2\big) \to 0 \tag{29}$$

[2]For all $\varphi, \phi \in D(\mathcal{A})$ and for all $\varepsilon \in \mathbb{R}$ it holds $\lambda_{\mathrm{A}}|\varphi + \varepsilon\phi|_{\mathsf{H}_0} \leq |\,\mathrm{A}(\varphi + \varepsilon\phi)|_{\mathsf{H}_1}$, i.e.,

$$0 \leq \underbrace{\big(|\,\mathrm{A}\,\varphi|_{\mathsf{H}_1}^2 - \lambda_{\mathrm{A}}^2|\varphi|_{\mathsf{H}_0}^2\big)}_{=:\alpha\geq 0} + 2\varepsilon\Re\underbrace{\big(\langle \mathrm{A}\,\varphi, \mathrm{A}\,\phi\rangle_{\mathsf{H}_1} - \lambda_{\mathrm{A}}^2\langle \varphi, \phi\rangle_{\mathsf{H}_0}\big)}_{=:\delta} + \varepsilon^2\underbrace{\big(|\,\mathrm{A}\,\phi|_{\mathsf{H}_1}^2 - \lambda_{\mathrm{A}}^2|\phi|_{\mathsf{H}_0}^2\big)}_{=:\gamma\geq 0}.$$

Let $\beta := \Re\delta$ and $0 \leq f(\varepsilon) := \alpha + 2\beta\varepsilon + \gamma\varepsilon^2$. If $\gamma = 0$ then $\beta = 0$. For $\gamma > 0$ the minimum of f is attained at $\varepsilon = -\beta/\gamma$ and thus $0 \leq f(-\beta/\gamma) = \alpha - 2\beta^2/\gamma + \beta^2/\gamma = \alpha - \beta^2/\gamma$ yielding $\beta^2 \leq \alpha\gamma$. Replacing ε by $-i\varepsilon$ shows the same inequality $\beta^2 \leq \alpha\gamma$ for $\beta := \Im\delta$. Hence $|\delta|^2 \leq 2\alpha\gamma$.

and thus

$$\begin{aligned}&\big|\langle \mathrm{A}\,x_{\mathrm{A}}, \mathrm{A}\,\phi\rangle_{\mathsf{H}_1} - \lambda_{\mathrm{A}}^2\langle x_{\mathrm{A}}, \phi\rangle_{\mathsf{H}_0}\big|\\ &\leq \big|\langle \mathrm{A}(x_{\mathrm{A}} - x_n), \mathrm{A}\,\phi\rangle_{\mathsf{H}_1}\big| + \lambda_{\mathrm{A}}^2\big|\langle x_{\mathrm{A}} - x_n, \phi\rangle_{\mathsf{H}_0}\big| + \big|\langle \mathrm{A}\,x_n, \mathrm{A}\,\phi\rangle_{\mathsf{H}_1} - \lambda_{\mathrm{A}}^2\langle x_n, \phi\rangle_{\mathsf{H}_0}\big| \to 0.\end{aligned}$$

Hence, for all $\phi \in D(\mathcal{A})$

$$\langle \mathrm{A}\,x_{\mathrm{A}}, \mathrm{A}\,\phi\rangle_{\mathsf{H}_1} = \lambda_{\mathrm{A}}^2\langle x_{\mathrm{A}}, \phi\rangle_{\mathsf{H}_0}. \tag{30}$$

For $\phi \in D(\mathrm{A}) = N(\mathrm{A}) \oplus_{\mathsf{H}_0} D(\mathcal{A})$, see the Helmholtz type decomposition (4), we decompose

$$\phi = \phi_N + \phi_{\mathcal{A}} \in N(\mathrm{A}) \oplus_{\mathsf{H}_0} D(\mathcal{A})$$

and compute by using (30), $\mathrm{A}\,\phi = \mathrm{A}\,\phi_{\mathcal{A}}$, and $x_{\mathrm{A}} \in R(\mathrm{A}^*)\perp_{\mathsf{H}_0} N(\mathrm{A})$

$$\langle \mathrm{A}\,x_{\mathrm{A}}, \mathrm{A}\,\phi\rangle_{\mathsf{H}_1} = \langle \mathrm{A}\,x_{\mathrm{A}}, \mathrm{A}\,\phi_{\mathcal{A}}\rangle_{\mathsf{H}_1} = \lambda_{\mathrm{A}}^2\langle x_{\mathrm{A}}, \phi_{\mathcal{A}}\rangle_{\mathsf{H}_0} = \lambda_{\mathrm{A}}^2\langle x_{\mathrm{A}}, \phi\rangle_{\mathsf{H}_0}.$$

Therefore, (30) holds for all $\phi \in D(\mathrm{A})$, i.e.,

$$\forall\,\phi \in D(\mathrm{A}) \qquad \langle \mathrm{A}\,x_{\mathrm{A}}, \mathrm{A}\,\phi\rangle_{\mathsf{H}_1} = \lambda_{\mathrm{A}}^2\langle x_{\mathrm{A}}, \phi\rangle_{\mathsf{H}_0}. \tag{31}$$

This implies $\mathrm{A}\,x_{\mathrm{A}} \in D(\mathrm{A}^*)$, i.e., $x_{\mathrm{A}} \in D(\mathrm{A}^*\,\mathrm{A})$ and $\mathrm{A}^*\,\mathrm{A}\,x_{\mathrm{A}} = \lambda_{\mathrm{A}}^2 x_{\mathrm{A}}$. We even have $x_{\mathrm{A}} \in D(\mathcal{A}^*\,\mathcal{A})$. Thus, λ_{A}^2 is an eigenvalue and x_{A} is an eigenvector of $\mathrm{A}^*\,\mathrm{A}$. Note that (30) or (31) implies (for $\phi = x_{\mathrm{A}}$) $|\,\mathrm{A}\,x_{\mathrm{A}}|_{\mathsf{H}_1}^2 = \lambda_{\mathrm{A}}^2|x_{\mathrm{A}}|_{\mathsf{H}_0}^2$, i.e., $|\,\mathrm{A}\,x_{\mathrm{A}}|_{\mathsf{H}_1} = \lambda_{\mathrm{A}}$.

Finally, we show that (x_n) even converges strongly in $D(\mathcal{A})$, i.e., $(\mathrm{A}\,x_n)$ converges strongly in H_1 respectively in $R(\mathrm{A})$. For this, we get for all $\phi \in D(\mathcal{A})$ by (29) and (30)

$$\begin{aligned}&\big|\langle \mathrm{A}(x_n - x_{\mathrm{A}}), \mathrm{A}\,\phi\rangle_{\mathsf{H}_1} - \lambda_{\mathrm{A}}^2\langle x_n - x_{\mathrm{A}}, \phi\rangle_{\mathsf{H}_0}\big|^2\\ &= \big|\langle \mathrm{A}\,x_n, \mathrm{A}\,\phi\rangle_{\mathsf{H}_1} - \lambda_{\mathrm{A}}^2\langle x_n, \phi\rangle_{\mathsf{H}_0}\big|^2 \leq 2\big(|\,\mathrm{A}\,x_n|_{\mathsf{H}_1}^2 - \lambda_{\mathrm{A}}^2\big)\big(|\,\mathrm{A}\,\phi|_{\mathsf{H}_1}^2 - \lambda_{\mathrm{A}}^2|\phi|_{\mathsf{H}_0}^2\big).\end{aligned}$$

In particular, for $\phi = x_n - x_{\mathrm{A}}$ we see

$$\big||\,\mathrm{A}(x_n - x_{\mathrm{A}})|_{\mathsf{H}_1}^2 - \lambda_{\mathrm{A}}^2|x_n - x_{\mathrm{A}}|_{\mathsf{H}_0}^2\big|^2 \leq c\big(|\,\mathrm{A}\,x_n|_{\mathsf{H}_1}^2 - \lambda_{\mathrm{A}}^2\big) \to 0,$$

and hence

$$|\,\mathrm{A}(x_n - x_{\mathrm{A}})|_{\mathsf{H}_1}^2 \leq \lambda_{\mathrm{A}}^2|x_n - x_{\mathrm{A}}|_{\mathsf{H}_0}^2 + \big||\,\mathrm{A}(x_n - x_{\mathrm{A}})|_{\mathsf{H}_1}^2 - \lambda_{\mathrm{A}}^2|x_n - x_{\mathrm{A}}|_{\mathsf{H}_0}^2\big| \to 0.$$

• For $0 \neq \lambda \in \sigma(\mathrm{A}^*\,\mathrm{A})$ we have $\mathrm{A}^*\,\mathrm{A}\,x = \lambda x$ for some $0 \neq x \in D(\mathrm{A}^*\,\mathrm{A}) = D(\mathcal{A}^*\,\mathrm{A})$. Hence $x \in R(\mathrm{A}^*)$ and thus $x \in D(\mathcal{A})$, showing $0 \neq x \in D(\mathcal{A}^*\,\mathcal{A})$. So $\lambda \in \sigma(\mathcal{A}^*\,\mathcal{A})$.

• For $0 \neq \lambda \in \sigma\Big(\begin{bmatrix} 0 & \mathrm{A}^* \\ \mathrm{A} & 0 \end{bmatrix}\Big)\setminus\{0\}$ we have $\mathrm{A}^*\,y = \lambda x$ and $\mathrm{A}\,x = \lambda y$ for some $0 \neq (x, y) \in D(\mathrm{A}) \times D(\mathrm{A}^*)$. Hence $(x, y) \in R(\mathrm{A}^*) \times R(\mathrm{A})$ and thus $(x, y) \in D(\mathcal{A}) \times D(\mathcal{A}^*)$, showing $0 \neq (x, y) \in D(\mathcal{A}) \times D(\mathcal{A}^*)$. So $\lambda \in \sigma\Big(\begin{bmatrix} 0 & \mathcal{A}^* \\ \mathcal{A} & 0 \end{bmatrix}\Big)$.

• It holds

$$|\,\mathcal{A}^{-1}\,|_{R(\mathrm{A}),R(\mathrm{A}^*)} = \sup_{0\neq y\in D(\mathcal{A}^{-1})} \frac{|\,\mathcal{A}^{-1}\,y|_{\mathsf{H}_0}}{|y|_{\mathsf{H}_1}} = \sup_{0\neq x\in D(\mathcal{A})} \frac{|x|_{\mathsf{H}_0}}{|\,\mathrm{A}\,x|_{\mathsf{H}_1}} = \Big(\inf_{0\neq x\in D(\mathcal{A})} \frac{|\,\mathrm{A}\,x|_{\mathsf{H}_1}}{|x|_{\mathsf{H}_0}}\Big)^{-1} = c_{\mathrm{A}}.$$

• Let $x_{\mathrm{A}^*\,\mathrm{A}}$ with $|x_{\mathrm{A}^*\,\mathrm{A}}|_{\mathsf{H}_0} = 1$ be an eigenvector of $\mathrm{A}^*\,\mathrm{A}$ to the eigenvalue λ_{A}^2. Then $x_{\mathrm{A}^*\,\mathrm{A}} \in R(\mathrm{A}^*)$ and $\lambda_{\mathrm{A}}^2(\mathcal{A}^*\,\mathcal{A})^{-1}x_{\mathrm{A}^*\,\mathrm{A}} = x_{\mathrm{A}^*\,\mathrm{A}}$. Thus

$$\begin{aligned}\big|(\mathcal{A}^*\,\mathcal{A})^{-1}\big|_{R(\mathrm{A}^*),R(\mathrm{A}^*)} &\leq |\,\mathcal{A}^{-1}\,|_{R(\mathrm{A}),R(\mathrm{A}^*)}\big|(\mathcal{A}^*)^{-1}\big|_{R(\mathrm{A}^*),R(\mathrm{A})} = c_{\mathrm{A}}^2\\ &= \sup_{0\neq x\in D((\mathcal{A}^*\,\mathcal{A})^{-1})} \frac{\big|(\mathcal{A}^*\,\mathcal{A})^{-1}x\big|_{\mathsf{H}_0}}{|x|_{\mathsf{H}_0}} \geq \big|(\mathcal{A}^*\,\mathcal{A})^{-1}x_{\mathrm{A}^*\,\mathrm{A}}\big|_{\mathsf{H}_0} = \frac{1}{\lambda_{\mathrm{A}}^2} = c_{\mathrm{A}}^2.\end{aligned}$$

• For $x \in N(\mathrm{A}^*\,\mathrm{A})$ we have $0 = \langle \mathrm{A}^*\,\mathrm{A}\,x, x\rangle_{\mathsf{H}_0} = |\,\mathrm{A}\,x|_{\mathsf{H}_1}^2$, i.e., $x \in N(\mathrm{A})$. Analogously, we see $N(\mathrm{A}\,\mathrm{A}^*) = N(\mathrm{A}^*)$. For $x \in N(\mathrm{A}\,\mathrm{A}^*\,\mathrm{A})$ we have $\mathrm{A}\,x \in N(\mathrm{A}\,\mathrm{A}^*) = N(\mathrm{A}^*)$, i.e., $x \in N(\mathrm{A}^*\,\mathrm{A}) = N(\mathrm{A})$. The latter arguments can be repeated for any higher power.

• For $y \in R(\mathrm{A})$ we see $x := \mathcal{A}^{-1}\,y \in D(\mathcal{A}) \subset R(\mathrm{A}^*)$ and $z := (\mathcal{A}^*)^{-1}x \in D(\mathcal{A}^*) \subset R(\mathrm{A})$. Thus $z \in D(\mathcal{A}\,\mathcal{A}^*)$ and $\mathrm{A}^*\,z = x$ and $\mathrm{A}\,x = y$ as well as $\mathrm{A}\,\mathrm{A}^*\,z = \mathrm{A}\,x = y \in R(\mathcal{A}\,\mathcal{A}^*) = R(\mathrm{A}\,\mathrm{A}^*)$. The latter arguments can be repeated for any higher power, completing the proof. □

Proof of Lemma 2.11. **(i)** By (6) we just have to show that $\mathrm{A}_0\,\mathrm{A}_0^* + \mathrm{A}_1^*\,\mathrm{A}_1$ is self-adjoint. For this, let $y \in \mathsf{H}_1$ such that there exists $z \in \mathsf{H}_1$ with

$$\forall\, x \in D(\mathrm{A}_0\,\mathrm{A}_0^* + \mathrm{A}_1^*\,\mathrm{A}_1) \qquad \big\langle (\mathrm{A}_0\,\mathrm{A}_0^* + \mathrm{A}_1^*\,\mathrm{A}_1)x, y\big\rangle_{\mathsf{H}_1} = \langle x, z\rangle_{\mathsf{H}_1}. \tag{32}$$

Picking $x \in N_{0,1}$ shows that $z \perp_{\mathsf{H}_1} N_{0,1}$ and hence, according to Theorem 2.8, y and z can be orthogonally decomposed into

$$y = y_{R(\mathrm{A}_0)} + y_{R(\mathrm{A}_1^*)} + y_{N_{0,1}} \in R(\mathrm{A}_0) \oplus_{\mathsf{H}_1} R(\mathrm{A}_1^*) \oplus_{\mathsf{H}_1} N_{0,1},$$
$$z = z_{R(\mathrm{A}_0)} + z_{R(\mathrm{A}_1^*)} \in R(\mathrm{A}_0) \oplus_{\mathsf{H}_1} R(\mathrm{A}_1^*).$$

(32) implies for all $x \in D(\mathrm{A}_0\,\mathrm{A}_0^* + \mathrm{A}_1^*\,\mathrm{A}_1)$

$$\begin{aligned}\langle \mathrm{A}_0\,\mathrm{A}_0^*\,x, y_{R(\mathrm{A}_0)}\rangle_{\mathsf{H}_1} + \langle \mathrm{A}_1^*\,\mathrm{A}_1\,x, y_{R(\mathrm{A}_1^*)}\rangle_{\mathsf{H}_1} &= \big\langle (\mathrm{A}_0\,\mathrm{A}_0^* + \mathrm{A}_1^*\,\mathrm{A}_1)x, y\big\rangle_{\mathsf{H}_1}\\ &= \langle x, z\rangle_{\mathsf{H}_1} = \langle x, z_{R(\mathrm{A}_0)}\rangle_{\mathsf{H}_1} + \langle x, z_{R(\mathrm{A}_1^*)}\rangle_{\mathsf{H}_1}.\end{aligned} \tag{33}$$

For $x \in D(\mathcal{A}_1^*\,\mathcal{A}_1) \subset R(\mathrm{A}_1^*) \subset N(\mathrm{A}_0^*)$ we see by (33) that $\langle \mathrm{A}_1^*\,\mathrm{A}_1\,x, y_{R(\mathrm{A}_1^*)}\rangle_{\mathsf{H}_1} = \langle x, z_{R(\mathrm{A}_1^*)}\rangle_{\mathsf{H}_1}$ holds, yielding by (6), i.e., $\mathcal{A}_1^*\,\mathcal{A}_1$ is self-adjoint, that $y_{R(\mathrm{A}_1^*)} \in D(\mathcal{A}_1^*\,\mathcal{A}_1) \subset N(\mathrm{A}_0^*)$ with $\mathrm{A}_1^*\,\mathrm{A}_1\,y_{R(\mathrm{A}_1^*)} = z_{R(\mathrm{A}_1^*)}$. Analogously we see by using $x \in D(\mathcal{A}_0\,\mathcal{A}_0^*)$ that $y_{R(\mathrm{A}_0)} \in D(\mathcal{A}_0\,\mathcal{A}_0^*) \subset N(\mathrm{A}_1)$ with $\mathrm{A}_0\,\mathrm{A}_0^*\,y_{R(\mathrm{A}_0)} = z_{R(\mathrm{A}_0)}$. Thus $y \in D(\mathrm{A}_0\,\mathrm{A}_0^* + \mathrm{A}_1^*\,\mathrm{A}_1)$ with $(\mathrm{A}_0\,\mathrm{A}_0^* + \mathrm{A}_1^*\,\mathrm{A}_1)y = z_{R(\mathrm{A}_0)} + z_{R(\mathrm{A}_1^*)} = z$, i.e., we have shown $(\mathrm{A}_0\,\mathrm{A}_0^* + \mathrm{A}_1^*\,\mathrm{A}_1)^* = \mathrm{A}_0\,\mathrm{A}_0^* + \mathrm{A}_1^*\,\mathrm{A}_1$.

(v) Let $0 \neq \lambda \in \sigma(\mathrm{A}_0\,\mathrm{A}_0^* + \mathrm{A}_1^*\,\mathrm{A}_1)$ and let $0 \neq x \in D(\mathrm{A}_0\,\mathrm{A}_0^* + \mathrm{A}_1^*\,\mathrm{A}_1)$ be an eigenvector to the eigenvalue λ. Then $y := \mathrm{A}_1^*\,\mathrm{A}_1\,x = \lambda x - \mathrm{A}_0\,\mathrm{A}_0^*\,x \in D(\mathrm{A}_1^*\,\mathrm{A}_1)$ and

$$\mathrm{A}_1^*\,\mathrm{A}_1\,y = \lambda\,\mathrm{A}_1^*\,\mathrm{A}_1\,x = \lambda y.$$

Thus, as long as $y \neq 0$, λ is an eigenvalue of $\mathrm{A}_1^*\,\mathrm{A}_1$ with eigenvector y. On the other hand, if $y = 0$, then $z := \mathrm{A}_0\,\mathrm{A}_0^*\,x = \lambda x \in D(\mathrm{A}_0\,\mathrm{A}_0^*) \setminus \{0\}$ and $\mathrm{A}_0\,\mathrm{A}_0^*\,z = \lambda\,\mathrm{A}_0\,\mathrm{A}_0^*\,x = \lambda z$. Hence λ is an eigenvalue of $\mathrm{A}_0\,\mathrm{A}_0^*$ with eigenvector z. This shows

$$\sigma(\mathrm{A}_0\,\mathrm{A}_0^* + \mathrm{A}_1^*\,\mathrm{A}_1) \setminus \{0\} \subset \big(\sigma(\mathrm{A}_0\,\mathrm{A}_0^*) \setminus \{0\}\big) \cup \big(\sigma(\mathrm{A}_1^*\,\mathrm{A}_1) \setminus \{0\}\big).$$

For the other inclusion, let, e.g., $0 \neq \lambda \in \sigma(\mathrm{A}_1^*\,\mathrm{A}_1)$ and let $0 \neq x \in D(\mathrm{A}_1^*\,\mathrm{A}_1)$ be an eigenvector to the eigenvalue λ. Then $x \in R(\mathrm{A}_1^*) \subset N(\mathrm{A}_0^*)$ and thus $(\mathrm{A}_0\,\mathrm{A}_0^* + \mathrm{A}_1^*\,\mathrm{A}_1)x = \mathrm{A}_1^*\,\mathrm{A}_1\,x = \lambda x$, i.e., λ is an eigenvalue of $\mathrm{A}_0\,\mathrm{A}_0^* + \mathrm{A}_1^*\,\mathrm{A}_1$ with eigenvector x. Thus

$$\sigma(\mathrm{A}_1^*\,\mathrm{A}_1) \setminus \{0\} \subset \sigma(\mathrm{A}_0\,\mathrm{A}_0^* + \mathrm{A}_1^*\,\mathrm{A}_1) \setminus \{0\},$$

and analogously we show $\sigma(\mathrm{A}_0\,\mathrm{A}_0^*) \setminus \{0\} \subset \sigma(\mathrm{A}_0\,\mathrm{A}_0^* + \mathrm{A}_1^*\,\mathrm{A}_1) \setminus \{0\}$.

(iii) Let $x \in N(\mathrm{A}_0\,\mathrm{A}_0^* + \mathrm{A}_1^*\,\mathrm{A}_1)$. Then

$$0 = \big\langle (\mathrm{A}_0\,\mathrm{A}_0^* + \mathrm{A}_1^*\,\mathrm{A}_1)x, x\big\rangle_{\mathsf{H}_1} = |\,\mathrm{A}_0^*\,x|_{\mathsf{H}_0}^2 + |\,\mathrm{A}_1\,x|_{\mathsf{H}_2}^2,$$

showing $x \in N_{0,1}$. As $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathsf{H}_1$ is compact, so is $D(\mathrm{A}_0\,\mathrm{A}_0^* + \mathrm{A}_1^*\,\mathrm{A}_1) \hookrightarrow \mathsf{H}_1$, showing that the range $R(\mathrm{A}_0\,\mathrm{A}_0^* + \mathrm{A}_1^*\,\mathrm{A}_1)$ is closed by Remark 2.3 (ii). Thus

$$R(\mathrm{A}_0\,\mathrm{A}_0^* + \mathrm{A}_1^*\,\mathrm{A}_1) = N(\mathrm{A}_0\,\mathrm{A}_0^* + \mathrm{A}_1^*\,\mathrm{A}_1)^{\perp_{\mathsf{H}_1}} = N_{0,1}^{\perp_{\mathsf{H}_1}},$$

finishing the proof. □

7. Appendix: Analytical Calculations

We compute the exact eigenvalues and eigenfunctions of Section 3 in detail.

7.1. **1D.**

Recall the situation and notations from Section 2.3.1 and Section 3.1. In particular,

$$\frac{1}{c_{0,\Gamma_\tau}} = \lambda_{0,\Gamma_\tau} = \lambda_{0,\Gamma_\nu} = \frac{1}{c_{0,\Gamma_\nu}}.$$

Let $u = u_{0,\Gamma_\tau}$ be the first eigenfunction for the eigenvalue λ^2 with $\lambda = \lambda_{0,\Gamma_\tau} > 0$ of $-\Delta_{\Gamma_\tau}$. Hence, we have $E_{0,\Gamma_\nu} = \operatorname{grad} u_{0,\Gamma_\tau}$ and

$$u \in D(\Delta_{\Gamma_\tau}) \cap \mathsf{L}^2_{\Gamma_\nu}(\Omega) \subset \mathsf{H}^1_{\Gamma_\tau}(\Omega) \cap \mathsf{L}^2_{\Gamma_\nu}(\Omega), \qquad \operatorname{grad} u = u' \in D(\operatorname{div}_{\Gamma_\nu}) = \mathsf{H}^1_{\Gamma_\nu}(\Omega),$$

as well as

$$(-\Delta - \lambda^2)u = -u'' - \lambda^2 u = 0.$$

Then

$$u = \alpha \sin(\lambda x) + \beta \cos(\lambda x), \qquad u'(x) = \alpha\lambda \cos(\lambda x) - \beta\lambda \sin(\lambda x).$$

For the different boundary conditions we get:

• $\Gamma_\tau = \emptyset$ and $\Gamma_\nu = \Gamma$, i.e., $u'(0) = u'(1) = 0$: $\alpha = 0$, $\lambda = n\pi$, $n \in \mathbb{N}_0$, i.e.,

$$\lambda_{0,\emptyset} = \pi, \qquad u_{0,\emptyset}(x) = \beta \cos(\pi x).$$

Note that in this case the first eigenvalue is $\lambda = 0$.

• $\Gamma_\tau = \{0\}$ and $\Gamma_\nu = \{1\}$, i.e., $u(0) = u'(1) = 0$: $\beta = 0$, $\lambda = (n - 1/2)\pi$, $n \in \mathbb{N}$, i.e.,

$$\lambda_{0,\{0\}} = \frac{\pi}{2}, \qquad u_{0,\{0\}}(x) = \alpha \sin(\frac{\pi}{2} x).$$

• $\Gamma_\tau = \{1\}$ and $\Gamma_\nu = \{0\}$, i.e., $u'(0) = u(1) = 0$: $\alpha = 0$, $\lambda = (n - 1/2)\pi$, $n \in \mathbb{N}$, i.e.,

$$\lambda_{0,\{1\}} = \frac{\pi}{2}, \qquad u_{0,\{1\}}(x) = \beta \cos(\frac{\pi}{2} x).$$

• $\Gamma_\tau = \Gamma$ and $\Gamma_\nu = \emptyset$, i.e., $u(0) = u(1) = 0$: $\beta = 0$, $\lambda = n\pi$, $n \in \mathbb{N}$, i.e.,

$$\lambda_{0,\Gamma} = \pi, \qquad u_{0,\Gamma}(x) = \alpha \sin(\pi x).$$

Note that from $\lambda_{0,\Gamma_\tau} = \lambda_{0,\Gamma_\nu}$ we already know $\lambda_{0,\Gamma} = \lambda_{0,\emptyset}$ and $\lambda_{0,\{0\}} = \lambda_{0,\{1\}}$, i.e.,

$$\lambda_{0,\Gamma} = \lambda_{0,\emptyset} = \pi, \qquad \lambda_{0,\{0\}} = \lambda_{0,\{1\}} = \frac{\pi}{2}.$$

7.2. **2D.** Recall the situation and notations from Section 2.3.2 and Section 3.2. In particular,

$$\frac{1}{c_{0,\Gamma_\tau}} = \lambda_{0,\Gamma_\tau} = \lambda_{1,\Gamma_\nu} = \frac{1}{c_{1,\Gamma_\nu}}.$$

Let $u = u_{0,\Gamma_\tau}$ be the first eigenfunction for the eigenvalue λ^2 with $\lambda = \lambda_{0,\Gamma_\tau} > 0$ of $-\Delta_{\Gamma_\tau}$. Hence, we have $E_{0,\Gamma_\nu} = \operatorname{grad} u_{0,\Gamma_\tau}$ and[3]

$$u \in D(\Delta_{\Gamma_\tau}) \cap \mathsf{L}^2_{\Gamma_\nu}(\Omega) \subset \mathsf{H}^1_{\Gamma_\tau}(\Omega) \cap \mathsf{L}^2_{\Gamma_\nu}(\Omega), \qquad \operatorname{grad} u \in D(\operatorname{div}_{\Gamma_\nu}) = \mathsf{H}_{\Gamma_\nu}(\operatorname{div}, \Omega),$$

as well as

$$(-\Delta - \lambda^2)u = 0.$$

Separation of variables shows with $u(x) = u_1(x_1)u_2(x_2)$ and $\operatorname{grad} u(x) = \begin{bmatrix} u_1'(x_1)u_2(x_2) \\ u_1(x_1)u_2'(x_2) \end{bmatrix}$

$$0 = (-\Delta - \lambda^2)u(x) = -u_1''(x_1)u_2(x_2) - u_1(x_1)u_2''(x_2) - \lambda^2 u_1(x_1)u_2(x_2).$$

For fixed x_2 with $u_2(x_2) \neq 0$ we get

$$-u_1''(x_1) - \mu_1^2 u_1(x_1) = 0, \qquad \mu_1^2 = \frac{u_2''(x_2)}{u_2(x_2)} + \lambda^2,$$

i.e.,

$$-u_1''(x_1) - \mu_1^2 u_1(x_1) = 0, \quad -u_2''(x_2) - \mu_2^2 u_2(x_2) = 0, \quad \lambda^2 = \mu_1^2 + \mu_2^2.$$

The Dirichlet boundary conditions, i.e.,

$$u = 0 \quad \text{on} \quad \Gamma_\tau,$$

reduce to Dirichlet boundary conditions for u_1 and u_2, respectively, and the Neumann boundary conditions, i.e.,

$$n \cdot \operatorname{grad} u = 0 \quad \text{on} \quad \Gamma_\nu,$$

reduce to Dirichlet boundary conditions for u_1' and u_2', respectively. More precisely, we have:

• Γ_l, $n = -e^1$, $x_1 = 0$:

$$0 = u|_{\Gamma_l} = u_1 u_2|_{\Gamma_l} \qquad \Rightarrow \qquad u_1(0) = 0,$$
$$0 = n \cdot \operatorname{grad} u|_{\Gamma_l} = -u_1' u_2|_{\Gamma_l} \qquad \Rightarrow \qquad u_1'(0) = 0.$$

• Γ_r, $n = e^1$, $x_1 = 1$:

$$0 = u|_{\Gamma_r} = u_1 u_2|_{\Gamma_r} \qquad \Rightarrow \qquad u_1(1) = 0,$$
$$0 = n \cdot \operatorname{grad} u|_{\Gamma_r} = u_1' u_2|_{\Gamma_r} \qquad \Rightarrow \qquad u_1'(1) = 0.$$

[3]Note that

$$E_{1,\Gamma_\tau} \in D(\square_{\Gamma_\tau}) \cap R(\vec{\operatorname{rot}}_{\Gamma_\nu}) \subset \mathsf{H}_{\Gamma_\tau}(\operatorname{rot}, \Omega) \cap R(\vec{\operatorname{rot}}_{\Gamma_\nu}),$$
$$H_{1,\Gamma_\nu} = \operatorname{rot} E_{1,\Gamma_\tau} \in D(\vec{\operatorname{rot}}_{\Gamma_\nu}) \cap R(\operatorname{rot}_{\Gamma_\tau}) = \mathsf{H}_{\Gamma_\nu}(\vec{\operatorname{rot}}, \Omega) \cap \mathsf{L}^2_{\Gamma_\tau}(\Omega) = \mathsf{H}^1_{\Gamma_\nu}(\Omega) \cap \mathsf{L}^2_{\Gamma_\tau}(\Omega).$$

• Γ_b, $n = -e^2$, $x_2 = 0$:

$$0 = u|_{\Gamma_b} = u_1 u_2|_{\Gamma_b} \qquad \Rightarrow \qquad u_2(0) = 0,$$
$$0 = n \cdot \operatorname{grad} u|_{\Gamma_b} = -u_1 u_2'|_{\Gamma_b} \qquad \Rightarrow \qquad u_2'(0) = 0.$$

• Γ_t, $n = e^2$, $x_2 = 1$:

$$0 = u|_{\Gamma_t} = u_1 u_2|_{\Gamma_t} \qquad \Rightarrow \qquad u_2(1) = 0,$$
$$0 = n \cdot \operatorname{grad} u|_{\Gamma_t} = u_1 u_2'|_{\Gamma_t} \qquad \Rightarrow \qquad u_2'(1) = 0.$$

The 1D case shows for the different boundary conditions the following:

• $\Gamma_\tau = \emptyset$ and $\Gamma_\nu = \Gamma$, i.e., $u_1'(0) = u_1'(1) = u_2'(0) = u_2'(1) = 0$: $\mu_1 = n\pi$, $\mu_2 = m\pi$, i.e., $\lambda = \sqrt{n^2 + m^2}\pi$, $n, m \in \mathbb{N}_0$, and

$$\lambda_{0,\emptyset} = \pi, \qquad u_{0,\emptyset}(x) = \alpha \cos(\pi x_1) + \beta \cos(\pi x_2).$$

Note that in this case the first eigenvalue is $\lambda = 0$.

• $\Gamma_\tau = \Gamma_b$ and $\Gamma_\nu = \Gamma_{t,l,r}$, i.e., $u_1'(0) = u_1'(1) = u_2(0) = u_2'(1) = 0$: $\mu_1 = n\pi$, $\mu_2 = (m - 1/2)\pi$, i.e., $\lambda = \sqrt{n^2 + (m - 1/2)^2}\pi$, $n \in \mathbb{N}_0$, $m \in \mathbb{N}$, and

$$\lambda_{0,\Gamma_b} = \frac{1}{2}\pi, \qquad u_{0,\Gamma_b}(x) = \alpha \sin(\frac{\pi}{2} x_2).$$

• $\Gamma_\tau = \Gamma_{b,t}$ and $\Gamma_\nu = \Gamma_{l,r}$, i.e., $u_1'(0) = u_1'(1) = u_2(0) = u_2(1) = 0$: $\mu_1 = n\pi$, $\mu_2 = m\pi$, i.e., $\lambda = \sqrt{n^2 + m^2}\pi$, $n \in \mathbb{N}_0$, $m \in \mathbb{N}$, and

$$\lambda_{0,\Gamma_{b,t}} = \pi, \qquad u_{0,\Gamma_{b,t}}(x) = \alpha \sin(\pi x_2).$$

• $\Gamma_\tau = \Gamma_{b,l}$ and $\Gamma_\nu = \Gamma_{t,r}$, i.e., $u_1(0) = u_1'(1) = u_2(0) = u_2'(1) = 0$: $\mu_1 = (n - 1/2)\pi$, $\mu_2 = (m - 1/2)\pi$, i.e., $\lambda = \sqrt{(n - 1/2)^2 + (m - 1/2)^2}\pi$, $n, m \in \mathbb{N}$, and

$$\lambda_{0,\Gamma_{b,l}} = \frac{\sqrt{2}}{2}\pi, \qquad u_{0,\Gamma_{b,l}}(x) = \alpha \sin(\frac{\pi}{2} x_1) \sin(\frac{\pi}{2} x_2).$$

• $\Gamma_\tau = \Gamma_{b,l,r}$ and $\Gamma_\nu = \Gamma_t$, i.e., $u_1(0) = u_1(1) = u_2(0) = u_2'(1) = 0$: $\mu_1 = n\pi$, $\mu_2 = (m - 1/2)\pi$, i.e., $\lambda = \sqrt{n^2 + (m - 1/2)^2}\pi$, $n, m \in \mathbb{N}$, and

$$\lambda_{0,\Gamma_{b,l,r}} = \frac{\sqrt{5}}{2}\pi, \qquad u_{0,\Gamma_{b,l,r}}(x) = \alpha \sin(\pi x_1) \sin(\frac{\pi}{2} x_2).$$

• $\Gamma_\tau = \Gamma$ and $\Gamma_\nu = \emptyset$, i.e., $u_1(0) = u_1(1) = u_2(0) = u_2(1) = 0$: $\mu_1 = n\pi$, $\mu_2 = m\pi$, i.e., $\lambda = \sqrt{n^2 + m^2}\pi$, $n, m \in \mathbb{N}$, and

$$\lambda_{0,\Gamma} = \sqrt{2}\pi, \qquad u_{0,\Gamma}(x) = \alpha \sin(\pi x_1) \sin(\pi x_2).$$

All other cases follow by symmetry, i.e.,

$$\lambda_{0,\emptyset} = \pi, \qquad \lambda_{0,\Gamma_{b,l}} = \lambda_{0,\Gamma_{b,r}} = \lambda_{0,\Gamma_{t,l}} = \lambda_{0,\Gamma_{t,r}} = \frac{\sqrt{2}}{2}\pi,$$
$$\lambda_{0,\Gamma_b} = \lambda_{0,\Gamma_t} = \lambda_{0,\Gamma_l} = \lambda_{0,\Gamma_r} = \frac{1}{2}\pi, \qquad \lambda_{0,\Gamma_{b,l,r}} = \lambda_{0,\Gamma_{t,l,r}} = \lambda_{0,\Gamma_{b,t,l}} = \lambda_{0,\Gamma_{b,t,r}} = \frac{\sqrt{5}}{2}\pi,$$
$$\lambda_{0,\Gamma_{b,t}} = \lambda_{0,\Gamma_{l,r}} = \pi, \qquad \lambda_{0,\Gamma} = \sqrt{2}\pi.$$

7.3. **3D.** Recall the situation and notations from Section 2.2, Theorem 2.20, and Section 3.3. In particular,

$$\frac{1}{c_{0,\Gamma_\tau}} = \lambda_{0,\Gamma_\tau} = \lambda_{2,\Gamma_\nu} = \frac{1}{c_{2,\Gamma_\nu}}, \qquad \frac{1}{c_{1,\Gamma_\tau}} = \lambda_{1,\Gamma_\tau} = \lambda_{1,\Gamma_\nu} = \frac{1}{c_{1,\Gamma_\nu}}.$$

Let $u = u_{0,\Gamma_\tau}$ be the first eigenfunction for the eigenvalue λ^2 with $\lambda = \lambda_{0,\Gamma_\tau} > 0$ of $-\Delta_{\Gamma_\tau}$. Analogously, let $E = E_{1,\Gamma_\tau}$ be the first eigenfunction for the eigenvalue $\widetilde{\lambda}^2$ with $\widetilde{\lambda} = \lambda_{1,\Gamma_\tau} > 0$ of $\Box_{\Gamma_\tau}$. Hence,

$$u \in D(\Delta_{\Gamma_\tau}) \cap \mathsf{L}^2_{\Gamma_\nu}(\Omega) \subset \mathsf{H}^1_{\Gamma_\tau}(\Omega) \cap \mathsf{L}^2_{\Gamma_\nu}(\Omega), \qquad E \in D(\Box_{\Gamma_\tau}) \cap R(\operatorname{rot}_{\Gamma_\nu}) \subset \mathsf{H}_{\Gamma_\tau}(\operatorname{rot}, \Omega) \cap R(\operatorname{rot}_{\Gamma_\nu}),$$
$$\operatorname{grad} u \in D(\operatorname{div}_{\Gamma_\nu}) = \mathsf{H}_{\Gamma_\nu}(\operatorname{div}, \Omega), \qquad \operatorname{rot} E \in D(\operatorname{rot}_{\Gamma_\nu}) \cap R(\operatorname{rot}_{\Gamma_\tau}) = \mathsf{H}_{\Gamma_\nu}(\operatorname{rot}, \Omega) \cap R(\operatorname{rot}_{\Gamma_\tau}),$$

and we have by $-\Delta = \operatorname{rot}\operatorname{rot} - \operatorname{grad}\operatorname{div} = \Box - \operatorname{grad}\operatorname{div}$

$$(-\Delta - \lambda^2)u = 0, \qquad (-\Delta - \widetilde{\lambda}^2)E = (\Box - \widetilde{\lambda}^2)E = 0,$$

as div $E = 0$. Let us first discuss u. Separation of variables shows with

$$u(x) = \widehat{u}(\widehat{x})u_3(x_3) = u_1(x_1)u_2(x_2)u_3(x_3), \quad \widehat{u}(\widehat{x}) = u_1(x_1)u_2(x_2), \qquad x = \begin{bmatrix} \widehat{x} \\ x_3 \end{bmatrix}, \quad \widehat{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and

$$\operatorname{grad} u(x) = \begin{bmatrix} u_3(x_3) \operatorname{grad} \widehat{u}(\widehat{x}) \\ u_3'(x_3)\widehat{u}(\widehat{x}) \end{bmatrix} = \begin{bmatrix} u_1'(x_1)u_2(x_2)u_3(x_3) \\ u_1(x_1)u_2'(x_2)u_3(x_3) \\ u_1(x_1)u_2(x_2)u_3'(x_3) \end{bmatrix}, \quad \operatorname{grad} \widehat{u}(\widehat{x}) = \begin{bmatrix} u_1'(x_1)u_2(x_2) \\ u_1(x_1)u_2'(x_2) \end{bmatrix}$$

that

$$0 = (-\Delta - \lambda^2)u(x) = -\Delta\widehat{u}(\widehat{x})u_3(x_3) - \widehat{u}(\widehat{x})u_3''(x_3) - \lambda^2\widehat{u}(\widehat{x})u_3(x_3).$$

For fixed x_3 with $u_3(x_3) \neq 0$ we get

$$-\Delta\widehat{u}(\widehat{x}) - \widehat{\mu}^2\widehat{u}(\widehat{x}) = 0, \qquad \widehat{\mu}^2 = \frac{u_3''(x_3)}{u_3(x_3)} + \lambda^2,$$

i.e.,

$$-\Delta\widehat{u}(\widehat{x}) - \widehat{\mu}^2\widehat{u}(\widehat{x}) = 0, \quad -u_3''(x_3) - \mu_3^2 u_3(x_3) = 0, \quad \lambda^2 = \widehat{\mu}^2 + \mu_3^2.$$

From the 2D case we already know $\widehat{\mu}^2 = \mu_1^2 + \mu_2^2$ and the splitting of $\widehat{u}$, i.e.,

$$\lambda^2 = \mu_1^2 + \mu_2^2 + \mu_3^2$$

and

$$-u_1''(x_1) - \mu_1^2 u_1(x_1) = 0, \quad -u_2''(x_2) - \mu_2^2 u_2(x_2) = 0, \quad -u_3''(x_3) - \mu_3^2 u_3(x_3) = 0.$$

The Dirichlet boundary conditions, i.e.,

$$u = 0 \quad \text{on} \quad \Gamma_\tau,$$

reduce to Dirichlet boundary conditions for u_1, u_2, and u_3, respectively, and the Neumann boundary conditions, i.e.,

$$n \cdot \operatorname{grad} u = 0 \quad \text{on} \quad \Gamma_\nu,$$

reduce to Neumann boundary conditions for $\widehat{u}$ and u_3 and hence to Dirichlet boundary conditions for u_1', u_2', and u_3', respectively.

- Γ_{bk}, $n = -e^1$, $x_1 = 0$:

$$\begin{aligned} 0 = u|_{\Gamma_{bk}} &= u_1u_2u_3|_{\Gamma_{bk}} &&\Rightarrow && u_1(0) = 0, \\ 0 = n \cdot \operatorname{grad} u|_{\Gamma_{bk}} &= -u_1'u_2u_3|_{\Gamma_{bk}} &&\Rightarrow && u_1'(0) = 0. \end{aligned}$$

- Γ_f, $n = e^1$, $x_1 = 1$:

$$\begin{aligned} 0 = u|_{\Gamma_f} &= u_1u_2u_3|_{\Gamma_f} &&\Rightarrow && u_1(1) = 0, \\ 0 = n \cdot \operatorname{grad} u|_{\Gamma_f} &= u_1'u_2u_3|_{\Gamma_f} &&\Rightarrow && u_1'(1) = 0. \end{aligned}$$

- Γ_l, $n = -e^2$, $x_2 = 0$:

$$\begin{aligned} 0 = u|_{\Gamma_l} &= u_1u_2u_3|_{\Gamma_l} &&\Rightarrow && u_2(0) = 0, \\ 0 = n \cdot \operatorname{grad} u|_{\Gamma_l} &= -u_1u_2'u_3|_{\Gamma_l} &&\Rightarrow && u_2'(0) = 0. \end{aligned}$$

- Γ_r, $n = e^2$, $x_2 = 1$:

$$\begin{aligned} 0 = u|_{\Gamma_r} &= u_1u_2u_3|_{\Gamma_r} &&\Rightarrow && u_2(1) = 0, \\ 0 = n \cdot \operatorname{grad} u|_{\Gamma_r} &= u_1u_2'u_3|_{\Gamma_r} &&\Rightarrow && u_2'(1) = 0. \end{aligned}$$

- Γ_b, $n = -e^3$, $x_3 = 0$:

$$\begin{aligned} 0 = u|_{\Gamma_b} &= u_1u_2u_3|_{\Gamma_b} &&\Rightarrow && u_3(0) = 0, \\ 0 = n \cdot \operatorname{grad} u|_{\Gamma_b} &= -u_1u_2u_3'|_{\Gamma_b} &&\Rightarrow && u_3'(0) = 0. \end{aligned}$$

- Γ_t, $n = e^3$, $x_3 = 1$:

$$\begin{aligned} 0 = u|_{\Gamma_t} &= u_1u_2u_3|_{\Gamma_t} &&\Rightarrow && u_3(1) = 0, \\ 0 = n \cdot \operatorname{grad} u|_{\Gamma_t} &= u_1u_2u_3'|_{\Gamma_t} &&\Rightarrow && u_3'(1) = 0. \end{aligned}$$

The 1D case shows for the different boundary conditions the following:

• $\Gamma_\tau = \emptyset$ and $\Gamma_\nu = \Gamma$, i.e., $u_1'(0) = u_1'(1) = u_2'(0) = u_2'(1) = u_3'(0) = u_3'(1) = 0$: $\mu_1 = n\pi$, $\mu_2 = m\pi$, $\mu_3 = k\pi$, i.e., $\lambda = \sqrt{n^2+m^2+k^2}\pi$, $n,m,k \in \mathbb{N}_0$, and

$$\lambda_{0,\emptyset} = \pi, \qquad u_{0,\emptyset}(x) = \alpha\cos(\pi x_1) + \beta\cos(\pi x_2) + \gamma\cos(\pi x_3).$$

Note that in this case the first eigenvalue is $\lambda = 0$.

• $\Gamma_\tau = \Gamma_b$ and $\Gamma_\nu = \Gamma_{t,l,r,f,bk}$, i.e., $u_1'(0) = u_1'(1) = u_2'(0) = u_2'(1) = u_3(0) = u_3'(1) = 0$: $\mu_1 = n\pi$, $\mu_2 = m\pi$, $\mu_3 = (k-1/2)\pi$, i.e., $\lambda = \sqrt{n^2+m^2+(k-1/2)^2}\pi$, $n,m \in \mathbb{N}_0$, $k \in \mathbb{N}$, and

$$\lambda_{0,\Gamma_b} = \frac{1}{2}\pi, \qquad u_{0,\Gamma_b}(x) = \alpha\sin(\frac{\pi}{2}x_3).$$

• $\Gamma_\tau = \Gamma_{b,t}$ and $\Gamma_\nu = \Gamma_{l,r,f,bk}$, i.e., $u_1'(0) = u_1'(1) = u_2'(0) = u_2'(1) = u_3(0) = u_3(1) = 0$: $\mu_1 = n\pi$, $\mu_2 = m\pi$, $\mu_3 = k\pi$, i.e., $\lambda = \sqrt{n^2+m^2+k^2}\pi$, $n,m \in \mathbb{N}_0$, $k \in \mathbb{N}$, and

$$\lambda_{0,\Gamma_{b,t}} = \pi, \qquad u_{0,\Gamma_{b,t}}(x) = \alpha\sin(\pi x_3).$$

• $\Gamma_\tau = \Gamma_{b,l}$ and $\Gamma_\nu = \Gamma_{t,r,f,bk}$, i.e., $u_1'(0) = u_1'(1) = u_2(0) = u_2'(1) = u_3(0) = u_3'(1) = 0$: $\mu_1 = n\pi$, $\mu_2 = (m-1/2)\pi$, $\mu_3 = (k-1/2)\pi$, i.e., $\lambda = \sqrt{n^2+(m-1/2)^2+(k-1/2)^2}\pi$, $n \in \mathbb{N}_0$, $m,k \in \mathbb{N}$, and

$$\lambda_{0,\Gamma_{b,l}} = \frac{\sqrt{2}}{2}\pi, \qquad u_{0,\Gamma_{b,l}}(x) = \alpha\sin(\frac{\pi}{2}x_2)\sin(\frac{\pi}{2}x_3).$$

• $\Gamma_\tau = \Gamma_{b,t,l}$ and $\Gamma_\nu = \Gamma_{r,f,bk}$, i.e., $u_1'(0) = u_1'(1) = u_2(0) = u_2'(1) = u_3(0) = u_3(1) = 0$: $\mu_1 = n\pi$, $\mu_2 = (m-1/2)\pi$, $\mu_3 = k\pi$, i.e., $\lambda = \sqrt{n^2+(m-1/2)^2+k^2}\pi$, $n \in \mathbb{N}_0$, $m,k \in \mathbb{N}$, and

$$\lambda_{0,\Gamma_{b,t,l}} = \frac{\sqrt{5}}{2}\pi, \qquad u_{0,\Gamma_{b,t,l}}(x) = \alpha\sin(\frac{\pi}{2}x_2)\sin(\pi x_3).$$

• $\Gamma_\tau = \Gamma_{b,l,bk}$ and $\Gamma_\nu = \Gamma_{r,f,t}$, i.e., $u_1(0) = u_1'(1) = u_2(0) = u_2'(1) = u_3(0) = u_3'(1) = 0$: $\mu_1 = (n-1/2)\pi$, $\mu_2 = (m-1/2)\pi$, $\mu_3 = (k-1/2)\pi$, i.e., $\lambda = \sqrt{(n-1/2)^2+(m-1/2)^2+(k-1/2)^2}\pi$, $n,m,k \in \mathbb{N}$, and

$$\lambda_{0,\Gamma_{b,l,bk}} = \frac{\sqrt{3}}{2}\pi, \qquad u_{0,\Gamma_{b,l,bk}}(x) = \alpha\sin(\frac{\pi}{2}x_1)\sin(\frac{\pi}{2}x_2)\sin(\frac{\pi}{2}x_3).$$

• $\Gamma_\tau = \Gamma_{b,t,l,r}$ and $\Gamma_\nu = \Gamma_{f,bk}$, i.e., $u_1'(0) = u_1'(1) = u_2(0) = u_2(1) = u_3(0) = u_3(1) = 0$: $\mu_1 = n\pi$, $\mu_2 = m\pi$, $\mu_3 = k\pi$, i.e., $\lambda = \sqrt{n^2+m^2+k^2}\pi$, $n \in \mathbb{N}_0$, $m,k \in \mathbb{N}$, and

$$\lambda_{0,\Gamma_{b,t,l,r}} = \sqrt{2}\pi, \qquad u_{0,\Gamma_{b,t,l,r}}(x) = \alpha\sin(\pi x_2)\sin(\pi x_3).$$

• $\Gamma_\tau = \Gamma_{b,t,l,bk}$ and $\Gamma_\nu = \Gamma_{f,r}$, i.e., $u_1(0) = u_1'(1) = u_2(0) = u_2'(1) = u_3(0) = u_3(1) = 0$: $\mu_1 = (n-1/2)\pi$, $\mu_2 = (m-1/2)\pi$, $\mu_3 = k\pi$, i.e., $\lambda = \sqrt{(n-1/2)^2+(m-1/2)^2+k^2}\pi$, $n,m,k \in \mathbb{N}$, and

$$\lambda_{0,\Gamma_{b,t,l,bk}} = \frac{\sqrt{6}}{2}\pi, \qquad u_{0,\Gamma_{b,t,l,bk}}(x) = \alpha\sin(\frac{\pi}{2}x_1)\sin(\frac{\pi}{2}x_2)\sin(\pi x_3).$$

• $\Gamma_\tau = \Gamma_{b,t,l,r,bk}$ and $\Gamma_\nu = \Gamma_f$, i.e., $u_1(0) = u_1'(1) = u_2(0) = u_2(1) = u_3(0) = u_3(1) = 0$: $\mu_1 = (n-1/2)\pi$, $\mu_2 = m\pi$, $\mu_3 = k\pi$, i.e., $\lambda = \sqrt{(n-1/2)^2+m^2+k^2}\pi$, $n,m,k \in \mathbb{N}$, and

$$\lambda_{0,\Gamma_{b,t,l,r,bk}} = \frac{3}{2}\pi, \qquad u_{0,\Gamma_{b,t,l,r,bk}}(x) = \alpha\sin(\frac{\pi}{2}x_1)\sin(\pi x_2)\sin(\pi x_3).$$

• $\Gamma_\tau = \Gamma$ and $\Gamma_\nu = \emptyset$, i.e., $u_1(0) = u_1(1) = u_2(0) = u_2(1) = u_3(0) = u_3(1) = 0$: $\mu_1 = n\pi$, $\mu_2 = m\pi$, $\mu_3 = k\pi$, i.e., $\lambda = \sqrt{n^2+m^2+k^2}\pi$, $n,m,k \in \mathbb{N}$, and

$$\lambda_{0,\Gamma} = \sqrt{3}\pi, \qquad u_{0,\Gamma}(x) = \alpha\sin(\pi x_1)\sin(\pi x_2)\sin(\pi x_3).$$

All other cases follow by symmetry, i.e.,

$$\lambda_{0,\emptyset} = \pi,$$

$$\lambda_{0,\Gamma_b} = \lambda_{0,\Gamma_t} = \lambda_{0,\Gamma_l} = \lambda_{0,\Gamma_r} = \lambda_{0,\Gamma_f} = \lambda_{0,\Gamma_{bk}} = \frac{1}{2}\pi,$$

$$\lambda_{0,\Gamma_{b,t}} = \lambda_{0,\Gamma_{l,r}} = \lambda_{0,\Gamma_{f,bk}} = \pi,$$

$$\begin{aligned}\lambda_{0,\Gamma_{b,l}} &= \lambda_{0,\Gamma_{b,r}} = \lambda_{0,\Gamma_{b,f}} = \lambda_{0,\Gamma_{b,bk}} \\ &= \lambda_{0,\Gamma_{t,l}} = \lambda_{0,\Gamma_{t,r}} = \lambda_{0,\Gamma_{t,f}} = \lambda_{0,\Gamma_{t,bk}} = \lambda_{0,\Gamma_{f,l}} = \lambda_{0,\Gamma_{f,r}} = \lambda_{0,\Gamma_{bk,l}} = \lambda_{0,\Gamma_{bk,r}} = \frac{\sqrt{2}}{2}\pi,\end{aligned}$$

$$\lambda_{0,\Gamma_{b,t,l}} = \lambda_{0,\Gamma_{b,t,r}} = \lambda_{0,\Gamma_{b,t,f}} = \lambda_{0,\Gamma_{b,t,bk}} = \lambda_{0,\Gamma_{l,r,b}}$$

$$= \lambda_{0,\Gamma_{l,r,t}} = \lambda_{0,\Gamma_{l,r,f}} = \lambda_{0,\Gamma_{l,r,bk}} = \lambda_{0,\Gamma_{f,bk,l}} = \lambda_{0,\Gamma_{f,bk,r}} = \lambda_{0,\Gamma_{f,bk,b}} = \lambda_{0,\Gamma_{f,bk,t}} = \frac{\sqrt{5}}{2}\pi,$$

$$\lambda_{0,\Gamma_{b,bk,l}} = \lambda_{0,\Gamma_{b,l,f}} = \lambda_{0,\Gamma_{b,f,r}} = \lambda_{0,\Gamma_{b,r,bk}} = \lambda_{0,\Gamma_{t,bk,l}} = \lambda_{0,\Gamma_{t,l,f}} = \lambda_{0,\Gamma_{t,f,r}} = \lambda_{0,\Gamma_{t,r,bk}} = \frac{\sqrt{3}}{2}\pi,$$

$$\lambda_{0,\Gamma_{b,t,l,r}} = \lambda_{0,\Gamma_{b,t,f,bk}} = \lambda_{0,\Gamma_{l,r,f,bk}} = \sqrt{2}\pi,$$

$$\lambda_{0,\Gamma_{b,t,l,bk}} = \lambda_{0,\Gamma_{b,t,f,l}} = \lambda_{0,\Gamma_{b,t,r,f}} = \lambda_{0,\Gamma_{b,t,r,bk}} = \lambda_{0,\Gamma_{l,r,f,t}} = \lambda_{0,\Gamma_{l,r,f,b}}$$

$$= \lambda_{0,\Gamma_{l,r,t,bk}} = \lambda_{0,\Gamma_{l,r,b,bk}} = \lambda_{0,\Gamma_{f,bk,b,l}} = \lambda_{0,\Gamma_{f,bk,t,l}} = \lambda_{0,\Gamma_{f,bk,b,r}} = \lambda_{0,\Gamma_{f,bk,r,r}} = \frac{\sqrt{6}}{2}\pi,$$

$$\lambda_{0,\Gamma_{b,t,l,r,bk}} = \lambda_{0,\Gamma_{b,t,l,r,f}} = \lambda_{0,\Gamma_{b,t,l,f,bk}} = \lambda_{0,\Gamma_{b,t,r,f,bk}} = \lambda_{0,\Gamma_{b,l,r,f,bk}} = \lambda_{0,\Gamma_{t,l,r,f,bk}} = \frac{3}{2}\pi,$$

$$\lambda_{0,\Gamma} = \sqrt{3}\pi.$$

Now, we take care of E. As $\operatorname{div} E = 0$ and $(-\Delta - \widetilde{\lambda}^2)E = 0$, a simple ansatz is given by, e.g.,

$$E := \operatorname{rot} U = \begin{bmatrix} \partial_2 u \\ -\partial_1 u \\ 0 \end{bmatrix}, \quad U(x) := u(x)\, e^3 = u(x) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

where u is a solution of $(-\Delta - \widetilde{\lambda}^2)u = 0$, i.e., $(-\Delta - \widetilde{\lambda}^2)U = 0$. Then $\operatorname{div} E = 0$ and

$$-\Delta E = \operatorname{rot}\operatorname{rot} E = \operatorname{rot}\operatorname{rot}\operatorname{rot} U = -\operatorname{rot}\Delta U = \widetilde{\lambda}^2 \operatorname{rot} U = \widetilde{\lambda}^2 E.$$

As u solves $(-\Delta - \widetilde{\lambda}^2)u = 0$ we have again by separation of variables

$$u(x) = u_1(x_1)u_2(x_2)u_3(x_3), \quad \operatorname{grad} u(x) = \begin{bmatrix} u_1'(x_1)u_2(x_2)u_3(x_3) \\ u_1(x_1)u_2'(x_2)u_3(x_3) \\ u_1(x_1)u_2(x_2)u_3'(x_3) \end{bmatrix}$$

as well as

$$\widetilde{\lambda}^2 = \mu_1^2 + \mu_2^2 + \mu_3^2$$

and

$$-u_1''(x_1) - \mu_1^2 u_1(x_1) = 0, \quad -u_2''(x_2) - \mu_2^2 u_2(x_2) = 0, \quad -u_3''(x_3) - \mu_3^2 u_3(x_3) = 0.$$

Moreover, by the complex property $R(\operatorname{rot}_{\Gamma_\nu}) \subset N(\operatorname{div}_{\Gamma_\nu})$, E must satisfy

$$E \in D(\operatorname{rot}_{\Gamma_\tau}) \cap R(\operatorname{rot}_{\Gamma_\nu}) \subset D(\operatorname{rot}_{\Gamma_\tau}) \cap N(\operatorname{div}_{\Gamma_\nu}),$$

$$\operatorname{rot} E \in D(\operatorname{rot}_{\Gamma_\nu}) \cap R(\operatorname{rot}_{\Gamma_\tau}) \subset D(\operatorname{rot}_{\Gamma_\nu}) \cap N(\operatorname{div}_{\Gamma_\tau}),$$

i.e., in classical terms

$$n \times E|_{\Gamma_\tau} = 0, \quad n \cdot E|_{\Gamma_\nu} = 0, \quad n \times \operatorname{rot} E|_{\Gamma_\nu} = 0, \quad n \cdot \operatorname{rot} E|_{\Gamma_\tau} = 0.$$

As the fourth boundary condition is implied by the first one and the second boundary condition is implied by the third one ($n \times \operatorname{rot} E|_{\Gamma_\nu} = 0 \Rightarrow 0 = n \cdot \operatorname{rot}\operatorname{rot} E|_{\Gamma_\nu} = \widetilde{\lambda}^2 n \cdot E|_{\Gamma_\nu}$), the third and fourth ones are (almost) redundant, and we are (almost) left with the simple boundary conditions

$$n \times E|_{\Gamma_\tau} = 0, \quad n \cdot E|_{\Gamma_\nu} = 0,$$

except for some special cases, where also the third one

$$n \times \operatorname{rot} E|_{\Gamma_\nu} = 0$$

is needed. For the computations of the boundary conditions we note[4]

$$\operatorname{rot} E = \begin{bmatrix} -\partial_3 E_2 \\ \partial_3 E_1 \\ \partial_1 E_2 - \partial_2 E_1 \end{bmatrix} = \begin{bmatrix} \partial_1 \partial_3 u \\ \partial_2 \partial_3 u \\ -\partial_1^2 u - \partial_2^2 u \end{bmatrix},$$

and thus

$$E = \begin{bmatrix} E_1 \\ E_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \partial_2 u \\ -\partial_1 u \\ 0 \end{bmatrix}, \qquad e^1 \times E = \begin{bmatrix} 0 \\ 0 \\ E_2 \end{bmatrix}, \qquad e^2 \times E = \begin{bmatrix} 0 \\ 0 \\ -E_1 \end{bmatrix}, \qquad e^3 \times E = \begin{bmatrix} -E_2 \\ E_1 \\ 0 \end{bmatrix},$$

[4]Alternatively, $\operatorname{rot} E = \operatorname{rot}\operatorname{rot} U = -\Delta U + \operatorname{grad}\operatorname{div} U = -\Delta u\, e^3 + \operatorname{grad} \partial_3 u = \begin{bmatrix} \partial_1 \partial_3 u \\ \partial_2 \partial_3 u \\ -\partial_1^2 u - \partial_2^2 u \end{bmatrix} = \begin{bmatrix} -\partial_3 E_2 \\ \partial_3 E_1 \\ \partial_1 E_2 - \partial_2 E_1 \end{bmatrix}$.

$$e^3 \times \operatorname{rot} E = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} -\partial_3 E_2 \\ \partial_3 E_1 \\ \partial_1 E_2 - \partial_2 E_1 \end{bmatrix} = -\begin{bmatrix} \partial_3 E_1 \\ \partial_3 E_2 \\ 0 \end{bmatrix} = \begin{bmatrix} -\partial_2 \partial_3 u \\ \partial_1 \partial_3 u \\ 0 \end{bmatrix}.$$

As an alternative we can also set boundary conditions for U directly. Since

$$E = \operatorname{rot} U \in R(\operatorname{rot}_{\Gamma_\nu}),$$

we get $n \times U|_{\Gamma_\nu} = u\, n \times e^3|_{\Gamma_\nu} = 0$.

- Γ_{bk}, $n = -e^1$, $x_1 = 0$:

$$0 = n \times E|_{\Gamma_{bk}} = -E_2 e^3|_{\Gamma_{bk}} = \partial_1 u\, e^3|_{\Gamma_{bk}} = u_1' u_2 u_3\, e^3|_{\Gamma_{bk}} \quad \Rightarrow \quad u_1'(0) = 0,$$
$$0 = n \cdot E|_{\Gamma_{bk}} = -E_1|_{\Gamma_{bk}} = -\partial_2 u|_{\Gamma_{bk}} = -u_1 u_2' u_3|_{\Gamma_{bk}} \quad \Rightarrow \quad u_1(0) = 0.$$

Alternatively,

$$0 = u\, n \times e^3|_{\Gamma_{bk}} = u\, e^2|_{\Gamma_{bk}} = u_1 u_2 u_3\, e^2|_{\Gamma_{bk}} \quad \Rightarrow \quad u_1(0) = 0.$$

- Γ_f, $n = e^1$, $x_1 = 1$:

$$0 = n \times E|_{\Gamma_f} = E_2 e^3|_{\Gamma_f} = -\partial_1 u\, e^3|_{\Gamma_f} = -u_1' u_2 u_3\, e^3|_{\Gamma_f} \quad \Rightarrow \quad u_1'(1) = 0,$$
$$0 = n \cdot E|_{\Gamma_f} = E_1|_{\Gamma_f} = \partial_2 u|_{\Gamma_f} = u_1 u_2' u_3|_{\Gamma_f} \quad \Rightarrow \quad u_1(1) = 0.$$

Alternatively,

$$0 = u\, n \times e^3|_{\Gamma_f} = -u\, e^2|_{\Gamma_f} = -u_1 u_2 u_3\, e^2|_{\Gamma_f} \quad \Rightarrow \quad u_1(1) = 0.$$

- Γ_l, $n = -e^2$, $x_2 = 0$:

$$0 = n \times E|_{\Gamma_l} = E_1 e^3|_{\Gamma_l} = \partial_2 u\, e^3|_{\Gamma_l} = u_1 u_2' u_3 u\, e^3|_{\Gamma_l} \quad \Rightarrow \quad u_2'(0) = 0,$$
$$0 = n \cdot E|_{\Gamma_l} = -E_2|_{\Gamma_l} = \partial_1 u|_{\Gamma_l} = u_1' u_2 u_3|_{\Gamma_l} \quad \Rightarrow \quad u_2(0) = 0.$$

Alternatively,

$$0 = u\, n \times e^3|_{\Gamma_l} = -u\, e^1|_{\Gamma_l} = -u_1 u_2 u_3\, e^1|_{\Gamma_l} \quad \Rightarrow \quad u_2(0) = 0.$$

- Γ_r, $n = e^2$, $x_2 = 1$:

$$0 = n \times E|_{\Gamma_r} = -E_1 e^3|_{\Gamma_r} = -\partial_2 u\, e^3|_{\Gamma_r} = -u_1 u_2' u_3\, e^3|_{\Gamma_r} \quad \Rightarrow \quad u_2'(1) = 0,$$
$$0 = n \cdot E|_{\Gamma_r} = E_2|_{\Gamma_r} = -\partial_1 u|_{\Gamma_r} = -u_1' u_2 u_3|_{\Gamma_r} \quad \Rightarrow \quad u_2(1) = 0.$$

Alternatively,

$$0 = u\, n \times e^3|_{\Gamma_r} = u\, e^1|_{\Gamma_r} = u_1 u_2 u_3\, e^1|_{\Gamma_r} \quad \Rightarrow \quad u_2(1) = 0.$$

- Γ_b, $n = -e^3$, $x_3 = 0$:

$$0 = n \times E|_{\Gamma_b} = \begin{bmatrix} E_2 \\ -E_1 \\ 0 \end{bmatrix}|_{\Gamma_b} = -\begin{bmatrix} \partial_1 u \\ \partial_2 u \\ 0 \end{bmatrix}|_{\Gamma_b} = -\begin{bmatrix} u_1' u_2 u_3 \\ u_1 u_2' u_3 \\ 0 \end{bmatrix}|_{\Gamma_b} \quad \Rightarrow \quad u_3(0) = 0,$$
$$0 = n \cdot E|_{\Gamma_b} = 0 \quad \text{(no condition)},$$
$$0 = n \times \operatorname{rot} E|_{\Gamma_b} = \begin{bmatrix} \partial_3 E_1 \\ \partial_3 E_2 \\ 0 \end{bmatrix}|_{\Gamma_b} = \begin{bmatrix} \partial_2 \partial_3 u \\ -\partial_1 \partial_3 u \\ 0 \end{bmatrix}|_{\Gamma_b} = \begin{bmatrix} u_1 u_2' u_3' \\ -u_1' u_2 u_3' \\ 0 \end{bmatrix}|_{\Gamma_b} \quad \Rightarrow \quad u_3'(0) = 0.$$

Alternatively,

$$0 = u\, n \times e^3|_{\Gamma_b} = 0 \quad \text{(no condition)}.$$

- Γ_t, $n = e^3$, $x_3 = 1$:

$$0 = n \times E|_{\Gamma_t} = \begin{bmatrix} -E_2 \\ E_1 \\ 0 \end{bmatrix}|_{\Gamma_t} = \begin{bmatrix} \partial_1 u \\ \partial_2 u \\ 0 \end{bmatrix}|_{\Gamma_t} = \begin{bmatrix} u_1' u_2 u_3 \\ u_1 u_2' u_3 \\ 0 \end{bmatrix}|_{\Gamma_t} \quad \Rightarrow \quad u_3(1) = 0,$$
$$0 = n \cdot E|_{\Gamma_t} = 0 \quad \text{(no condition)},$$

$$0 = n \times \operatorname{rot} E|_{\Gamma_t} = -\begin{bmatrix} \partial_3 E_1 \\ \partial_3 E_2 \\ 0 \end{bmatrix} |_{\Gamma_t} = \begin{bmatrix} -\partial_2 \partial_3 u \\ \partial_1 \partial_3 u \\ 0 \end{bmatrix} |_{\Gamma_t} = \begin{bmatrix} -u_1 u_2' u_3' \\ u_1' u_2 u_3' \\ 0 \end{bmatrix} |_{\Gamma_t} \qquad \Rightarrow \qquad u_3'(1) = 0.$$

Alternatively,

$$0 = u\, n \times e^3|_{\Gamma_t} = 0 \quad \text{(no condition)}.$$

By construction, i.e.,

$$E = \begin{bmatrix} \partial_2 u \\ -\partial_1 u \\ 0 \end{bmatrix},$$

u can be constant in one variable x_n and simultaneously in two variables x_1, x_3 and x_2, x_3, respectively, but not simultaneously in the two variables x_1, x_2 since this implies $E = 0$. The 1D case shows for the different boundary conditions the following:

- $\Gamma_\tau = \emptyset$ and $\Gamma_\nu = \Gamma$, i.e., $u_1(0) = u_1(1) = u_2(0) = u_2(1) = u_3'(0) = u_3'(1) = 0$: $\mu_1 = n\pi$, $\mu_2 = m\pi$, $\mu_3 = k\pi$, i.e., $\widetilde{\lambda} = \sqrt{n^2 + m^2 + k^2}\pi$, $n, m \in \mathbb{N}$, $k \in \mathbb{N}_0$, and

$$\lambda_{1,\emptyset} = \sqrt{2}\pi, \qquad u_{1,\emptyset}(x) = \alpha \sin(\pi x_1) \sin(\pi x_2), \qquad E_{1,\emptyset}(x) = \alpha\pi \begin{bmatrix} \sin(\pi x_1)\cos(\pi x_2) \\ -\cos(\pi x_1)\sin(\pi x_2) \\ 0 \end{bmatrix}.$$

Note that we already know from the theory that

$$\lambda_{1,\Gamma} = \lambda_{1,\emptyset} = \sqrt{2}\pi.$$

In the particular computation we get $u_1'(0) = u_1'(1) = u_2'(0) = u_2'(1) = u_3(0) = u_3(1) = 0$ for $\Gamma_\tau = \Gamma$ and $\Gamma_\nu = \emptyset$: $\mu_1 = n\pi$, $\mu_2 = m\pi$, $\mu_3 = k\pi$, i.e., $\widetilde{\lambda} = \sqrt{n^2 + m^2 + k^2}\pi$, $n, k \in \mathbb{N}$, $m \in \mathbb{N}_0$, or $m, k \in \mathbb{N}$, $n \in \mathbb{N}_0$. We emphasise that here the actual case $n = m = 0$ is not allowed as this would imply $E = 0$, see our discussion above. The eigenvectors are

$$u_{1,\Gamma}(x) = \alpha \cos(\pi x_1) \sin(\pi x_3) + \beta \cos(\pi x_2) \sin(\pi x_3),$$
$$E_{1,\Gamma}(x) = \alpha\pi \sin(\pi x_1) \sin(\pi x_3) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \beta\pi \sin(\pi x_2) \sin(\pi x_3) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

- $\Gamma_\tau = \Gamma_b$ and $\Gamma_\nu = \Gamma_{t,l,r,f,bk}$, i.e., $u_1(0) = u_1(1) = u_2(0) = u_2(1) = u_3(0) = u_3'(1) = 0$: $\mu_1 = n\pi$, $\mu_2 = m\pi$, $\mu_3 = (k - 1/2)\pi$, i.e., $\widetilde{\lambda} = \sqrt{n^2 + m^2 + (k-1/2)^2}\pi$, $n, m, k \in \mathbb{N}$, and the minimum and eigenvectors are

$$\widetilde{\lambda} = \frac{3}{2}\pi, \qquad u(x) = \alpha \sin(\pi x_1) \sin(\pi x_2) \sin(\frac{\pi}{2} x_3), \qquad E(x) = \alpha\pi \sin(\frac{\pi}{2} x_3) \begin{bmatrix} \sin(\pi x_1)\cos(\pi x_2) \\ -\cos(\pi x_1)\sin(\pi x_2) \\ 0 \end{bmatrix}.$$

If $\Gamma_\tau = \Gamma_l$ and $\Gamma_\nu = \Gamma_{t,b,r,f,bk}$, i.e., $u_1(0) = u_1(1) = u_2'(0) = u_2(1) = u_3'(0) = u_3'(1) = 0$: $\mu_1 = n\pi$, $\mu_2 = (m - 1/2)\pi$, $\mu_3 = k\pi$, i.e., $\widetilde{\lambda} = \sqrt{n^2 + (m-1/2)^2 + k^2}\pi$, $n, m \in \mathbb{N}$, $k \in \mathbb{N}_0$, and the minimum and eigenvectors are

$$\lambda_{1,\Gamma_l} = \frac{\sqrt{5}}{2}\pi, \qquad u_{1,\Gamma_l}(x) = \alpha \sin(\pi x_1) \cos(\frac{\pi}{2} x_2), \qquad E_{1,\Gamma_l}(x) = \alpha\frac{\pi}{2} \begin{bmatrix} -\sin(\pi x_1)\sin(\frac{\pi}{2} x_2) \\ 2\cos(\pi x_1)\cos(\frac{\pi}{2} x_2) \\ 0 \end{bmatrix}.$$

This shows that by replacing the ansatz for E by, e.g.,

$$E := \operatorname{rot} U = \begin{bmatrix} 0 \\ \partial_3 u \\ -\partial_2 u \end{bmatrix}, \quad U(x) := u(x)\, e^1 = u(x) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

we get also the smaller eigenvalue $\widetilde{\lambda} = (\sqrt{5}/2)\pi$ in the case $\Gamma_\tau = \Gamma_b$. Hence, by symmetry

$$\lambda_{1,\Gamma_b} = \lambda_{1,\Gamma_t} = \lambda_{1,\Gamma_l} = \lambda_{1,\Gamma_r} = \lambda_{1,\Gamma_f} = \lambda_{1,\Gamma_{bk}} = \frac{\sqrt{5}}{2}\pi.$$

• $\Gamma_\tau = \Gamma_{b,t}$ and $\Gamma_\nu = \Gamma_{l,r,f,bk}$, i.e., $u_1(0) = u_1(1) = u_2(0) = u_2(1) = u_3(0) = u_3(1) = 0$: $\mu_1 = n\pi$, $\mu_2 = m\pi$, $\mu_3 = k\pi$, i.e., $\widetilde{\lambda} = \sqrt{n^2 + m^2 + k^2}\pi$, $n, m, k \in \mathbb{N}$, and the minimum and eigenvectors are

$$\widetilde{\lambda} = \sqrt{3}\pi, \qquad u(x) = \alpha \sin(\pi x_1)\sin(\pi x_2)\sin(\pi x_3), \qquad E(x) = \alpha\pi \sin(\pi x_3)\begin{bmatrix} \sin(\pi x_1)\cos(\pi x_2) \\ -\cos(\pi x_1)\sin(\pi x_2) \\ 0 \end{bmatrix}.$$

If $\Gamma_\tau = \Gamma_{l,r}$ and $\Gamma_\nu = \Gamma_{b,t,f,bk}$, i.e., $u_1(0) = u_1(1) = u_2'(0) = u_2'(1) = u_3'(0) = u_3'(1) = 0$: $\mu_1 = n\pi$, $\mu_2 = m\pi$, $\mu_3 = k\pi$, i.e., $\widetilde{\lambda} = \sqrt{n^2 + m^2 + k^2}\pi$, $n \in \mathbb{N}$, $m, k \in \mathbb{N}_0$, and the minimum and eigenvectors are

$$\lambda_{1,\Gamma_{l,r}} = \pi, \qquad u_{1,\Gamma_{l,r}}(x) = \alpha \sin(\pi x_1), \qquad E_{1,\Gamma_{l,r}}(x) = -\alpha\pi \cos(\pi x_1)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Hence, again by changing the ansatz, we get also the smaller eigenvalue $\widetilde{\lambda} = \pi$ in the case $\Gamma_\tau = \Gamma_{b,t}$. Thus, by symmetry

$$\lambda_{1,\Gamma_{l,r}} = \lambda_{1,\Gamma_{b,t}} = \lambda_{1,\Gamma_{f,bk}} = \pi.$$

• $\Gamma_\tau = \Gamma_{b,l}$ and $\Gamma_\nu = \Gamma_{t,r,f,bk}$, i.e., $u_1(0) = u_1(1) = u_2'(0) = u_2(1) = u_3(0) = u_3'(1) = 0$: $\mu_1 = n\pi$, $\mu_2 = (m - 1/2)\pi$, $\mu_3 = (k - 1/2)\pi$, i.e., $\widetilde{\lambda} = \sqrt{n^2 + (m - 1/2)^2 + (k - 1/2)^2}\pi$ with $n, m, k \in \mathbb{N}$, and the minimum and eigenvectors are

$$\widetilde{\lambda} = \frac{\sqrt{6}}{2}\pi, \quad u(x) = \alpha \sin(\pi x_1)\cos(\frac{\pi}{2}x_2)\sin(\frac{\pi}{2}x_3), \quad E(x) = -\alpha\frac{\pi}{2}\sin(\frac{\pi}{2}x_3)\begin{bmatrix} \sin(\pi x_1)\sin(\frac{\pi}{2}x_2) \\ 2\cos(\pi x_1)\cos(\frac{\pi}{2}x_2) \\ 0 \end{bmatrix}.$$

If $\Gamma_\tau = \Gamma_{f,l}$ and $\Gamma_\nu = \Gamma_{b,t,r,bk}$, i.e., $u_1(0) = u_1'(1) = u_2'(0) = u_2(1) = u_3'(0) = u_3'(1) = 0$: $\mu_1 = (n - 1/2)\pi$, $\mu_2 = (m - 1/2)\pi$, $\mu_3 = k\pi$, i.e., $\widetilde{\lambda} = \sqrt{(n - 1/2)^2 + (m - 1/2)^2 + k^2}\pi$, $n, m \in \mathbb{N}$, $k \in \mathbb{N}_0$, and the minimum and eigenvectors are

$$\lambda_{1,\Gamma_{f,l}} = \frac{\sqrt{2}}{2}\pi, \qquad u_{1,\Gamma_{f,l}}(x) = \alpha \sin(\frac{\pi}{2}x_1)\cos(\frac{\pi}{2}x_2), \qquad E_{1,\Gamma_{f,l}}(x) = -\alpha\frac{\pi}{2}\begin{bmatrix} \sin(\frac{\pi}{2}x_1)\sin(\frac{\pi}{2}x_2) \\ \cos(\frac{\pi}{2}x_1)\cos(\frac{\pi}{2}x_2) \\ 0 \end{bmatrix}.$$

Hence, again by changing the ansatz, we obtain also the smaller eigenvalue $\widetilde{\lambda} = (\sqrt{2}/2)\pi$ in the case $\Gamma_\tau = \Gamma_{b,l}$. Thus, by symmetry

$$\begin{aligned} &\lambda_{1,\Gamma_{b,l}} = \lambda_{1,\Gamma_{b,r}} = \lambda_{1,\Gamma_{b,f}} = \lambda_{1,\Gamma_{b,bk}} \\ &= \lambda_{1,\Gamma_{t,l}} = \lambda_{1,\Gamma_{t,r}} = \lambda_{1,\Gamma_{t,f}} = \lambda_{1,\Gamma_{t,bk}} = \lambda_{1,\Gamma_{f,l}} = \lambda_{1,\Gamma_{l,bk}} = \lambda_{1,\Gamma_{bk,r}} = \lambda_{1,\Gamma_{f,r}} = \frac{\sqrt{2}}{2}\pi. \end{aligned}$$

• $\Gamma_\tau = \Gamma_{b,l,t}$ and $\Gamma_\nu = \Gamma_{r,f,bk}$, i.e., $u_1(0) = u_1(1) = u_2'(0) = u_2(1) = u_3(0) = u_3(1) = 0$: $\mu_1 = n\pi$, $\mu_2 = (m - 1/2)\pi$, $\mu_3 = k\pi$, i.e., $\widetilde{\lambda} = \sqrt{n^2 + (m - 1/2)^2 + k^2}\pi$, $n, m, k \in \mathbb{N}$, and the minimum and eigenvectors are

$$\widetilde{\lambda} = \frac{3}{2}\pi, \quad u(x) = \alpha \sin(\pi x_1)\cos(\frac{\pi}{2}x_2)\sin(\pi x_3), \quad E(x) = -\alpha\frac{\pi}{2}\sin(\pi x_3)\begin{bmatrix} \sin(\pi x_1)\sin(\frac{\pi}{2}x_2) \\ 2\cos(\pi x_1)\cos(\frac{\pi}{2}x_2) \\ 0 \end{bmatrix}.$$

If $\Gamma_\tau = \Gamma_{f,l,bk}$ and $\Gamma_\nu = \Gamma_{t,r,b}$, i.e., $u_1'(0) = u_1'(1) = u_2'(0) = u_2(1) = u_3'(0) = u_3'(1) = 0$: $\mu_1 = n\pi$, $\mu_2 = (m - 1/2)\pi$, $\mu_3 = k\pi$, i.e., $\widetilde{\lambda} = \sqrt{n^2 + (m - 1/2)^2 + k^2}\pi$, $n, k \in \mathbb{N}_0$, $m \in \mathbb{N}$, and the minimum and eigenvectors are

$$\lambda_{1,\Gamma_{f,l,bk}} = \frac{1}{2}\pi, \qquad u_{1,\Gamma_{f,l,bk}}(x) = \alpha \cos(\frac{\pi}{2}x_2), \qquad E_{1,\Gamma_{f,l,bk}}(x) = -\alpha\frac{\pi}{2}\sin(\frac{\pi}{2}x_2)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Hence, again by changing the ansatz, we obtain also the smaller eigenvalue $\widetilde{\lambda} = (1/2)\pi$ in the case $\Gamma_\tau = \Gamma_{b,l,t}$. Thus, by symmetry

$$\begin{aligned} &\lambda_{1,\Gamma_{b,l,t}} = \lambda_{1,\Gamma_{b,r,t}} = \lambda_{1,\Gamma_{b,f,t}} = \lambda_{1,\Gamma_{b,bk,t}} = \lambda_{1,\Gamma_{r,l,t}} = \lambda_{1,\Gamma_{r,l,b}} \\ &= \lambda_{1,\Gamma_{r,l,f}} = \lambda_{1,\Gamma_{r,l,bk}} = \lambda_{1,\Gamma_{f,bk,l}} = \lambda_{1,\Gamma_{f,bk,r}} = \lambda_{1,\Gamma_{f,bk,t}} = \lambda_{1,\Gamma_{f,bk,b}} = \frac{1}{2}\pi. \end{aligned}$$

• $\Gamma_\tau = \Gamma_{b,l,bk}$ and $\Gamma_\nu = \Gamma_{t,r,f}$, i.e., $u_1'(0) = u_1(1) = u_2'(0) = u_2(1) = u_3(0) = u_3'(1) = 0$: $\mu_1 = (n-1/2)\pi$, $\mu_2 = (m-1/2)\pi$, $\mu_3 = (k-1/2)\pi$, i.e., $\widetilde{\lambda} = \sqrt{(n-1/2)^2+(m-1/2)^2+(k-1/2)^2}\pi$, $n,m,k \in \mathbb{N}$, and the minimum and eigenvectors are

$$\lambda_{1,\Gamma_{b,l,bk}} = \frac{\sqrt{3}}{2}\pi,$$

$$u_{1,\Gamma_{b,l,bk}}(x) = \alpha\cos(\frac{\pi}{2}x_1)\cos(\frac{\pi}{2}x_2)\sin(\frac{\pi}{2}x_3), \quad E_{1,\Gamma_{b,l,bk}}(x) = \alpha\frac{\pi}{2}\sin(\frac{\pi}{2}x_3)\begin{bmatrix}-\cos(\frac{\pi}{2}x_1)\sin(\frac{\pi}{2}x_2)\\ \sin(\frac{\pi}{2}x_1)\cos(\frac{\pi}{2}x_2)\\ 0\end{bmatrix}.$$

By symmetry

$$\lambda_{1,\Gamma_{b,l,bk}} = \lambda_{1,\Gamma_{b,r,bk}} = \lambda_{1,\Gamma_{b,l,f}} = \lambda_{1,\Gamma_{b,r,f}} = \lambda_{1,\Gamma_{t,l,bk}} = \lambda_{1,\Gamma_{t,r,bk}} = \lambda_{1,\Gamma_{t,l,f}} = \lambda_{1,\Gamma_{t,r,f}} = \frac{\sqrt{3}}{2}\pi.$$

We summarise

$$\begin{aligned}
\lambda_{1,\emptyset} = \lambda_{1,\Gamma} &= \sqrt{2}\pi,\\
\lambda_{1,\Gamma_b} = \lambda_{1,\Gamma_t} = \lambda_{1,\Gamma_l} = \lambda_{1,\Gamma_r} = \lambda_{1,\Gamma_f} = \lambda_{1,\Gamma_{bk}} &= \frac{\sqrt{5}}{2}\pi,\\
\lambda_{1,\Gamma_{l,r}} = \lambda_{1,\Gamma_{b,t}} = \lambda_{1,\Gamma_{f,bk}} &= \pi,\\
\lambda_{1,\Gamma_{b,l}} = \lambda_{1,\Gamma_{b,r}} = \lambda_{1,\Gamma_{b,f}} = \lambda_{1,\Gamma_{b,bk}}&\\
= \lambda_{1,\Gamma_{t,l}} = \lambda_{1,\Gamma_{t,r}} = \lambda_{1,\Gamma_{t,f}} = \lambda_{1,\Gamma_{t,bk}} = \lambda_{1,\Gamma_{f,l}} = \lambda_{1,\Gamma_{l,bk}} = \lambda_{1,\Gamma_{bk,r}} = \lambda_{1,\Gamma_{f,r}} &= \frac{\sqrt{2}}{2}\pi,\\
\lambda_{1,\Gamma_{b,l,t}} = \lambda_{1,\Gamma_{b,r,t}} = \lambda_{1,\Gamma_{b,f,t}} = \lambda_{1,\Gamma_{b,bk,t}} = \lambda_{1,\Gamma_{r,l,t}}&\\
= \lambda_{1,\Gamma_{r,l,b}} = \lambda_{1,\Gamma_{r,l,f}} = \lambda_{1,\Gamma_{r,l,bk}} = \lambda_{1,\Gamma_{f,bk,l}} = \lambda_{1,\Gamma_{f,bk,r}} = \lambda_{1,\Gamma_{f,bk,t}} = \lambda_{1,\Gamma_{f,bk,b}} &= \frac{1}{2}\pi,\\
\lambda_{1,\Gamma_{b,l,bk}} = \lambda_{1,\Gamma_{b,r,bk}} = \lambda_{1,\Gamma_{b,l,f}} = \lambda_{1,\Gamma_{b,r,f}} = \lambda_{1,\Gamma_{t,l,bk}} = \lambda_{1,\Gamma_{t,r,bk}} = \lambda_{1,\Gamma_{t,l,f}} = \lambda_{1,\Gamma_{t,r,f}} &= \frac{\sqrt{3}}{2}\pi,
\end{aligned}$$

and all other cases follow by $\lambda_{1,\Gamma_\nu} = \lambda_{1,\Gamma_\tau}$ as well as symmetry.

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