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Low Frequency Asymptotics and Electro-Magneto-Statics for Time-Harmonic Maxwell's Equations in Exterior Weak Lipschitz Domains with Mixed Boundary Conditions

by

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Abstract. We prove that the time-harmonic solutions to Maxwell's equations in a 3D exterior domain converge to a certain static solution as the frequency tends to zero. We work in weighted Sobolev spaces and construct new compactly supported replacements for Dirichlet-Neumann fields. Moreover, we even show convergence in operator norm.

Contents

1. Introduction

Applying a time-harmonic ansatz (or Fourier-transformation with respect to time) to the classical time-dependent Maxwell equations in some domain $\Omega \subset \mathbb{R}^3$, we are led to consider the *time-harmonic Maxwell system*

$$\operatorname{rot} E + i\omega B = G\,, \qquad -\operatorname{rot} H + i\omega D = -F\,, \qquad \text{in } \Omega \tag{1.1}$$

with frequency $\omega \in \mathbb{C}$. Here, E and H denote the electric and magnetic field, $D = \varepsilon E$ and $B = \mu H$ represent the displacement current and magnetic induction, respectively, and F, G are known source terms. The matrix valued functions ε and μ describe the permittivity and permeability of the medium filling Ω and are assumed to be time-independent. In the following we are specifically interested in the case of an exterior weak Lipschitz domain $\Omega \subset \mathbb{R}^3$ (i.e., a connected open subset with compact complement) with boundary $\Gamma := \partial\Omega$ (Lipschitz submanifold) decomposed into two relatively open subsets Γ_1 and $\Gamma_2 := \Gamma \setminus \overline{\Gamma}_1$ being itself Lipschitz submanifolds of Γ. We impose mixed homogeneous boundary conditions, which in classical terms can be written as

$$n \times E = 0 \text{ on } \Gamma_1, \qquad n \times H = 0 \text{ on } \Gamma_2, \qquad (n : \text{ outward unit normal}), \tag{1.2}$$

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and, in order to separate outgoing from incoming waves, we require the so called Silver-Müller radiation condition

$$\xi \times H + E\,,\ \xi \times E - H = \mathcal{O}(r^{-1}) \qquad \text{for} \qquad r \longrightarrow \infty\,. \tag{1.3}$$

First existence results concerning boundary value problems for the time-harmonic Maxwell system in exterior domains have been given by Müller [16, 15] in domains with smooth boundaries and homogeneous, isotropic media, i.e. $\varepsilon = \mu = \mathbb{1}$. In [10] Leis used the limiting absorption principle to obtain existence and uniqueness for media, which are possibly inhomogeneous and anisotropic within a bounded subset of Ω. Nevertheless, Leis still needed strong assumptions on the boundary regularity. In the bounded domain case, even for general inhomogeneous and anisotropic media (cf. Leis [11]), it is sufficient that Ω allows for a certain selection theorem, later called *Weck's selection theorem* or *Maxwell compactness property*, which holds for a class of boundaries much larger than those accessible by the detour over H^1 (cf. Weck [32], Weber [31], Picard [28], Costabel [2], Witsch [36], and Picard, Weck, and Witsch [30]). The most recent result for a solution theory in the exterior domain case is due to the second author [19, 23] (see also [18]) and in its structure comparable to the results of [30]. While all these results handle the case of full boundary conditions, in [17] the authors treated for the first time mixed boundary conditions. Using the framework of polynomially weighted Sobolev spaces from [30], we have been able to show that the time-harmonic boundary value problem (1.1), (1.2), and (1.3) admits unique solutions. In particular, by means of Eidus limiting absorption principle [3] (see also [4, 5]) for the physically interesting case of real frequencies ω a Fredholm alternative type result holds true. Similar to the bounded domain case, the crucial tool for existence is again a compact embedding result, now being a local version of Weck's selection theorem.

In this paper we investigate the low frequency behaviour of the corresponding time-harmonic solution operator. To this end we first have to provide a solution theory for the static boundary value problem, i.e., $\omega = 0$, which reads

$$\begin{aligned}
\operatorname{rot} E &= G && \text{in } \Omega\,, & \operatorname{rot} H &= F && \text{in } \Omega\,,\\
\operatorname{div} \varepsilon E &= f && \text{in } \Omega\,, & \operatorname{div} \mu H &= g && \text{in } \Omega\,,\\
n \times E &= 0 && \text{on } \Gamma_1\,, & n \times H &= 0 && \text{on } \Gamma_2\,,\\
n \cdot \varepsilon E &= 0 && \text{on } \Gamma_2\,, & n \cdot \mu H &= 0 && \text{on } \Gamma_1\,.
\end{aligned} \tag{1.4}$$

There are two major challenges:

- Problems in exterior domains require to work in weighted Sobolev spaces.
- The systems (1.4) have non trivial kernels, forcing us to work with orthogonality constraints on solutions in weighted Sobolev spaces to achieve uniqueness. This specific difficulty is overcome by a construction of special compactly supported fields and certain functionals, see Theorem 3.11.

In the case of full homogeneous boundary conditions and homogeneous, isotropic media Kress [8] (using integral equation methods) and Picard [26] (using Hilbert space methods) established solution theories in the generalized setting of alternating differential forms on Riemannian manifolds of arbitrary dimensions (see also [29] for nonlinear materials). For the classical threedimensional case of electro-magneto-statics with full homogeneous boundary conditions, we refer to Picard [27] (see also [14]) as well as the references therein. Following the Hilbert space approach, in Section 3 we will present Helmholtz type decompositions in weighted Sobolev spaces which then together with Weck's local selection theorem will provide a powerful setting for solving system (1.4).

In Section 4 we shortly present the time-harmonic solution theory summarizing the results obtained in [17]. This results follow by the same methods as in [19, 23] (see also Picard, Weck, and Witsch [30], Weck and Witsch [33, 34, 35]). For nonreal frequencies the solution is obtained by standard Hilbert space methods as ω belongs to the resolvent set of the *Maxwell operator*

$$\mathcal{M} : \mathbf{R}_{\Gamma_1}(\Omega) \times \mathbf{R}_{\Gamma_2}(\Omega) \subset \mathsf{L}^2_\Lambda(\Omega) \longrightarrow \mathsf{L}^2_\Lambda(\Omega)\,,\ \ (E,H) \longmapsto i\Lambda^{-1}\mathrm{M}\,, \qquad \mathsf{L}^2_\Lambda(\Omega) := \mathsf{L}^2_\varepsilon(\Omega) \times \mathsf{L}^2_\mu(\Omega)\,,$$

where

$$\Lambda := \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix}, \qquad \mathrm{M} := \begin{pmatrix} 0 & -\operatorname{rot} \\ \operatorname{rot} & 0 \end{pmatrix}, \qquad \mathsf{L}^2_\gamma(\Omega) := \big(\mathsf{L}^2(\Omega), \langle\, \gamma\,\cdot\,, \,\cdot\, \rangle_{\mathsf{L}^2_\gamma(\Omega)}\big)\,.$$

The case of real frequencies $\omega \neq 0$ is much more challenging, since here we want to solve in the continuous spectrum of the Maxwell operator. Nevertheless, restricting to data $(F,G) \in \mathsf{L}^2_{>\frac{1}{2}}(\Omega) \times \mathsf{L}^2_{>\frac{1}{2}}(\Omega)$, we are able to obtain radiating solutions $(E,H) \in \mathsf{L}^2_{<-\frac{1}{2}}(\Omega) \times \mathsf{L}^2_{<-\frac{1}{2}}(\Omega)$ by means of Eidus' limiting absorption principle [3, 4], i.e., as limit of solutions corresponding to frequencies $\omega \in \mathbb{C}_+ \setminus \mathbb{R}$. In other words, the resolvent $(\mathcal{M} - \omega)^{-1}$ and hence also $\mathcal{L}_{\Lambda,\omega} = i(\mathcal{M} - \omega)^{-1}\Lambda^{-1}$ may be extended continuously to the real axis (cf. [12]). An a-priori-estimate and the polynomial decay of eigenfunctions needed in the limit process are obtained by transferring well known results for the Helmholtz equation in the whole space using a suitable decomposition of the fields E and H and perturbation arguments. This will be sufficient to show that a generalized Fredholm alternative holds, see Theorem 4.3. We have to admit finite dimensional eigenspaces for certain eigenvalues $\omega \neq 0$, which can not accumulate in $\mathbb{R} \setminus \{0\}$. Next by proving an estimate for the solutions of the homogeneous and isotropic whole space problem together with an perturbation argument, we show that these possible eigenvalues do not accumulate even at $\omega = 0$. Therefore, for small $\omega \neq 0$ the time-harmonic solution operator $\mathcal{L}_{\Lambda,\omega}$ is well defined on $\mathsf{L}^2_{>\frac{1}{2}}(\Omega) \times \mathsf{L}^2_{>\frac{1}{2}}(\Omega)$ and a low frequency analysis is reasonable.

Finally, in Section 5 we investigate the low frequency behavior of time-harmonic solutions, in particular the question under which conditions radiating solutions converge to a static solution of system (1.4). In the case of a bounded domain the low frequency asymptotics is simply given by a Neumann series of the static solution operator $\mathcal{L}_0$, which directly follows by applying $\mathcal{L}_0$ to the time-harmonic system (1.1). More precisely, in the case that Ω is a bounded Lipschitz domain, by Weck's selection theorem the range $\mathcal{R}(\mathcal{M})$ of the Maxwell operator is closed and the reduced Maxwell operator

$$\mathcal{M}_{\mathrm{red}} : \mathcal{D}(\mathcal{M}) \cap \mathcal{R}(\mathcal{M}) \subset \mathcal{R}(\mathcal{M}) \longrightarrow \mathcal{R}(\mathcal{M}) \,, \quad (E,H) \longmapsto (-i\varepsilon^{-1} \operatorname{rot} H, i\mu^{-1} \operatorname{rot} E)$$

has a continuous inverse $\mathcal{L}_0 : \mathcal{R}(\mathcal{M}) \longrightarrow \mathcal{D}(\mathcal{M}) \cap \mathcal{R}(\mathcal{M})$, which interpreted as operator into $\mathcal{R}(\mathcal{M})$ is even compact. Moreover, arbitrary powers $\mathcal{L}_0^j$ of $\mathcal{L}_0$ are well defined. Hence, for small $|\omega| > 0$ the time-harmonic solution operator $\mathcal{L}_\omega : \mathsf{L}^2_\Lambda(\Omega) \longrightarrow \mathcal{D}(\mathcal{M})$ is well defined (Fredholm alternative) and is given by the Neumann series

$$\mathcal{L}_\omega = -\omega^{-1}\pi_{\mathcal{N}(\mathcal{M})} + \sum_{j=0}^{\infty} \omega^j \mathcal{L}_0^{j+1} \pi_{\mathcal{R}(\mathcal{M})} \,. \tag{1.5}$$

Here, $\pi_{\mathcal{N}(\mathcal{M})}$ and $\pi_{\mathcal{R}(\mathcal{M})}$ are the projections onto the kernel and the range of $\mathcal{M}$, respectively.

In the exterior domain case this simple low frequency asymptotics does not hold. It is even not well defined in an obvious way, since now the static solution operator $\mathcal{L}_0$ maps data from a polynomially weighted Sobolev space to solutions belonging to a less weighted Sobolev space (cf. Theorem 3.15 resp. Theorem 3.16). However, using an estimate for the solutions of the homogeneous, isotropic whole space problem together with a perturbation argumentwe can prove the convergence of the time-harmonic solutions $\mathcal{L}_\omega(F,G)$ to a specific static solution $\mathcal{L}_0(F,G)$ on a certain subspace, i.e.,

$$\mathcal{L}_\omega \longrightarrow \mathcal{L}_0 \,.$$

A proper and corrected version of the low frequency asymptotics (1.5) for the case of an exterior domain will be addressed in a forthcoming publication (see [21, 22, 20, 19, 18] for the case of full boundary conditions).

2. Preliminaries

In the following, $\Omega \subset \mathbb{R}^3$ is an exterior weak Lipschitz domain, see [1, Definition 2.3], with boundary $\Gamma := \partial\Omega$ decomposed into two relatively open weak Lipschitz subdomains Γ_1 and $\Gamma_2 := \Gamma \setminus \overline{\Gamma}_1$, see [1, Definition 2.5]. For $x \in \mathbb{R}^3$ with $x \neq 0$ let $r(x) := |\,x\,|$ and $\xi(x) := x/|\,x\,|$ $\big(\,|\cdot|$: Euclidean norm in $\mathbb{R}^3\big)$. Moreover, we fix $\hat{r} > 0$ such that $\mathbb{R}^3 \setminus \Omega \Subset \mathrm{U}_{\hat{r}}$ (compactly included) and define

$$\Omega_\delta := \Omega \cap \mathrm{U}_\delta \,, \qquad \Gamma_\delta := \Gamma_1 \cup \mathrm{S}_\delta \,, \qquad \breve{\mathrm{U}}_\delta := \mathbb{R}^3 \setminus \overline{\mathrm{U}}_\delta \,, \qquad (\,\delta \geq \hat{r}\,) \,,$$

where U_δ and S_δ denote the open ball resp. sphere of radius δ centered at the origin. We also pick some

$$\tilde{\eta} \in \mathsf{C}^\infty(\mathbb{R}) \qquad \text{with} \qquad 0 \leq \tilde{\eta} \leq 1 \,, \quad \operatorname{supp} \tilde{\eta} \subset (1,\infty) \,, \quad \tilde{\eta}|_{[2,\infty)} = 1 \,,$$

and define for $\delta \geq \hat{r}$ functions $\eta_\delta \in \mathsf{C}^\infty(\mathbb{R}^3)$ by

$$\eta_\delta(x) := \tilde{\eta}(r(x)/\delta).$$

These functions satisfy $\operatorname{supp}\eta_\delta \subset \check{\mathrm{U}}_\delta$ as well as $\eta_\delta = 1$ on $\check{\mathrm{U}}_{2\delta}$ and will later be used for particular cut-off procedures. The usual Lebesgue and Sobolev spaces will be denoted by $\mathsf{L}^2(\Omega)$, $\mathbf{H}^m(\Omega)$ and

$$\mathbf{R}(\Omega) := \big\{E \in \mathsf{L}^2(\Omega) \,:\, \operatorname{rot} E \in \mathsf{L}^2(\Omega)\big\}, \qquad \mathbf{D}(\Omega) := \big\{E \in \mathsf{L}^2(\Omega) \,:\, \operatorname{div} E \in \mathsf{L}^2(\Omega)\big\}, \tag{2.1}$$

where we prefer to write rot instead of curl. However, for our purposes this spaces are not rich enough, as even for square-integrable right hand sides the system (1.1), (1.2) does not admit square-integrable solutions (cf. [26], [17]). Hence we have to generalize the solution concept and work in polynomially weighted Sobolev spaces. For $\rho := (1+r^2)^{1/2}$, $m \in \mathbb{N}$, and $t \in \mathbb{R}$ we introduce

$$\mathsf{L}^2_t(\Omega) := \big\{u \in \mathsf{L}^2_{\mathrm{loc}}(\Omega) \,:\, \rho^t u \in \mathsf{L}^2(\Omega)\big\},$$

as well as

$$\mathbf{H}^m_t(\Omega) := \big\{u \in \mathsf{L}^2_t(\Omega) \,:\, \partial^\alpha u \in \mathsf{L}^2_t(\Omega) \text{ for all } |\alpha| \leq m\big\},$$
$$\mathsf{H}^m_t(\Omega) := \big\{u \in \mathsf{L}^2_t(\Omega) \,:\, \partial^\alpha u \in \mathsf{L}^2_{t+|\alpha|}(\Omega) \text{ for all } |\alpha| \leq m\big\},$$

$$\mathbf{R}_t(\Omega) := \big\{E \in \mathsf{L}^2_t(\Omega) \,:\, \operatorname{rot} E \in \mathsf{L}^2_t(\Omega)\big\}, \qquad \mathsf{R}_t(\Omega) := \big\{E \in \mathsf{L}^2_t(\Omega) \,:\, \operatorname{rot} E \in \mathsf{L}^2_{t+1}(\Omega)\big\},$$
$$\mathbf{D}_t(\Omega) := \big\{H \in \mathsf{L}^2_t(\Omega) \,:\, \operatorname{div} H \in \mathsf{L}^2_t(\Omega)\big\}, \qquad \mathsf{D}_t(\Omega) := \big\{H \in \mathsf{L}^2_t(\Omega) \,:\, \operatorname{div} H \in \mathsf{L}^2_{t+1}(\Omega)\big\}.$$

We do not distinguish between vector fields resp. functions and (in accordance with (2.1)) we skip the weight if $t = 0$, i.e.,

$$\mathbf{H}^1(\Omega) = \mathbf{H}^1_0(\Omega), \qquad \mathsf{R}(\Omega) = \mathsf{R}_0(\Omega), \qquad \mathbf{D}(\Omega) = \mathbf{D}_0(\Omega), \qquad \ldots .$$

If $\Gamma_1 \neq \emptyset$, homogeneous scalar, tangential or normal traces are encoded in

$$\mathbf{H}^1_{\Gamma_1}(\Omega) := \overline{\mathsf{C}^\infty_{\Gamma_1}(\Omega)}^{\|\cdot\|_{\mathbf{H}^1(\Omega)}}, \quad \mathbf{R}_{\Gamma_1}(\Omega) := \overline{\mathsf{C}^\infty_{\Gamma_1}(\Omega)}^{\|\cdot\|_{\mathbf{R}(\Omega)}}, \quad \mathbf{D}_{\Gamma_1}(\Omega) := \overline{\mathsf{C}^\infty_{\Gamma_1}(\Omega)}^{\|\cdot\|_{\mathbf{D}(\Omega)}},$$

as well as

$$\mathbf{H}^1_{t,\Gamma_1}(\Omega) := \overline{\mathsf{C}^\infty_{\Gamma_1}(\Omega)}^{\|\cdot\|_{\mathbf{H}^1_t(\Omega)}}, \quad \mathbf{R}_{t,\Gamma_1}(\Omega) := \overline{\mathsf{C}^\infty_{\Gamma_1}(\Omega)}^{\|\cdot\|_{\mathbf{R}_t(\Omega)}}, \quad \mathbf{D}_{t,\Gamma_1}(\Omega) := \overline{\mathsf{C}^\infty_{\Gamma_1}(\Omega)}^{\|\cdot\|_{\mathbf{D}_t(\Omega)}},$$
$$\mathsf{H}^1_{t,\Gamma_1}(\Omega) := \overline{\mathsf{C}^\infty_{\Gamma_1}(\Omega)}^{\|\cdot\|_{\mathsf{H}^1_t(\Omega)}}, \quad \mathsf{R}_{t,\Gamma_1}(\Omega) := \overline{\mathsf{C}^\infty_{\Gamma_1}(\Omega)}^{\|\cdot\|_{\mathsf{R}_t(\Omega)}}, \quad \mathsf{D}_{t,\Gamma_1}(\Omega) := \overline{\mathsf{C}^\infty_{\Gamma_1}(\Omega)}^{\|\cdot\|_{\mathsf{D}_t(\Omega)}}.$$

where the set of test fields (resp. test functions) is given by

$$\mathsf{C}^\infty_{\Gamma_1}(\Omega) := \big\{ \varphi|_\Omega \,:\, \varphi \in \mathsf{C}^\infty(\mathbb{R}^3),\ \operatorname{supp}\varphi \text{ compact in } \mathbb{R}^3,\ \operatorname{dist}(\operatorname{supp}\varphi, \Gamma_1) > 0 \big\}.$$

We emphasize that in the case of a bounded domain, weighted and unweighted spaces coincide. Moreover, by [17, Lemma 2.2], see also [1, Theorem 4.5], it holds

$$\begin{aligned}
\mathbf{H}^1_{t,\Gamma_1}(\Omega) &= \big\{u \in \mathbf{H}^1_t(\Omega) \,:\, \langle u, \operatorname{div}\Phi\rangle_{\mathsf{L}^2(\Omega)} = -\langle \nabla u, \Phi\rangle_{\mathsf{L}^2(\Omega)} \text{ for all } \Phi \in \mathsf{C}^\infty_{\Gamma_2}(\Omega)\big\},\\
\mathbf{R}_{t,\Gamma_1}(\Omega) &= \big\{E \in \mathbf{R}_t(\Omega) \,:\, \langle E, \operatorname{rot}\Phi\rangle_{\mathsf{L}^2(\Omega)} = \langle \operatorname{rot} E, \Phi\rangle_{\mathsf{L}^2(\Omega)} \text{ for all } \Phi \in \mathsf{C}^\infty_{\Gamma_2}(\Omega)\big\},\\
\mathbf{D}_{t,\Gamma_1}(\Omega) &= \big\{H \in \mathbf{D}_t(\Omega) \,:\, \langle H, \nabla\phi\rangle_{\mathsf{L}^2(\Omega)} = -\langle \operatorname{div} H, \phi\rangle_{\mathsf{L}^2(\Omega)} \text{ for all } \phi \in \mathsf{C}^\infty_{\Gamma_2}(\Omega)\big\},
\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}
\mathsf{H}^1_{t,\Gamma_1}(\Omega) &= \big\{u \in \mathsf{H}^1_t(\Omega) \,:\, \langle u, \operatorname{div}\Phi\rangle_{\mathsf{L}^2(\Omega)} = -\langle \nabla u, \Phi\rangle_{\mathsf{L}^2(\Omega)} \text{ for all } \Phi \in \mathsf{C}^\infty_{\Gamma_2}(\Omega)\big\},\\
\mathsf{R}_{t,\Gamma_1}(\Omega) &= \big\{E \in \mathsf{R}_t(\Omega) \,:\, \langle E, \operatorname{rot}\Phi\rangle_{\mathsf{L}^2(\Omega)} = \langle \operatorname{rot} E, \Phi\rangle_{\mathsf{L}^2(\Omega)} \text{ for all } \Phi \in \mathsf{C}^\infty_{\Gamma_2}(\Omega)\big\},\\
\mathsf{D}_{t,\Gamma_1}(\Omega) &= \big\{H \in \mathsf{D}_t(\Omega) \,:\, \langle H, \nabla\phi\rangle_{\mathsf{L}^2(\Omega)} = -\langle \operatorname{div} H, \phi\rangle_{\mathsf{L}^2(\Omega)} \text{ for all } \phi \in \mathsf{C}^\infty_{\Gamma_2}(\Omega)\big\}.
\end{aligned} \tag{2.3}$$

Equipped with their natural inner products, all these spaces are Hilbert spaces. Vanishing rotation resp. divergence will be indicated by an index zero in the lower left corner, e.g.,

$$
{}_0\mathsf{R}_t(\Omega) := \Big\{ E \in \mathsf{R}_t(\Omega) \,:\, \operatorname{rot} E = 0 \Big\}\,, \qquad {}_0\mathsf{D}_t(\Omega) := \Big\{ E \in \mathsf{D}_t(\Omega) \,:\, \operatorname{div} E = 0 \Big\}\,,
$$

$$
{}_0\mathsf{R}_{t,\Gamma_1}(\Omega) := {}_0\mathsf{R}_t(\Omega) \cap \mathsf{R}_{t,\Gamma_1}(\Omega)\,, \qquad {}_0\mathsf{D}_{t,\Gamma_1}(\Omega) := {}_0\mathsf{D}_t(\Omega) \cap \mathsf{D}_{t,\Gamma_1}(\Omega)\,.
$$

For simplification and to shorten notation we write

$$
\mathsf{V}_{<s} := \bigcap_{t<s} \mathsf{V}_t \qquad \text{and} \qquad \mathsf{V}_{>s} := \bigcup_{t>s} \mathsf{V}_t \qquad (\, s \in \mathbb{R}\,)\,,
$$

for $\mathrm{V_t}$ being any of the spaces above and skip the space reference, i.e.,

$$
\mathbf{H}^m_t = \mathbf{H}^m_t(\Omega)\,, \qquad \mathsf{R}_{t,\Gamma_1} = \mathsf{R}_{t,\Gamma_1}(\Omega)\,, \qquad \mathbf{D}_t = \mathbf{D}_t(\Omega)\,, \qquad \mathsf{H}^m_{t,\Gamma_2} = \mathsf{H}^m_{t,\Gamma_2}(\Omega)\,, \qquad \ldots\,,
$$

if $\Omega = \mathbb{R}^3$.

Definition 2.1. *Let $\kappa \geq 0$. We call a transformation γ "κ-decaying", if*

- $\gamma : \Omega \longrightarrow \mathbb{R}^{3\times 3}$ *is an L^∞-matrix field,*
- γ *is symmetric, i.e.,*

$$
\forall\, E, H \in \mathsf{L}^2(\Omega) : \quad \langle\, E\,, \gamma H\,\rangle_{\mathsf{L}^2(\Omega)} = \langle\, \gamma E\,, H\,\rangle_{\mathsf{L}^2(\Omega)}\,,
$$

- γ *is uniformly positive definite, i.e.,*

$$
\exists\, c > 0 \ \ \forall\, E \in \mathsf{L}^2(\Omega) : \quad \langle\, E\,, \gamma E\,\rangle_{\mathsf{L}^2(\Omega)} \geq c \cdot \|\, E\,\|^2_{\mathsf{L}^2(\Omega)}\,,
$$

- γ *is asymptotically a multiple of the identity, i.e.,*

$$
\gamma = \gamma_0 \cdot \mathbb{1} + \hat{\gamma} \ \textit{with}\ \gamma_0 \in \mathbb{R}_+ \ \textit{and}\ \hat{\gamma} = \mathcal{O}\big(r^{-\kappa}\big) \ \textit{as}\ r \longrightarrow \infty\,.
$$

General Assumption 2.2. *From now on and through this paper we assume the following:*

- $\Omega \subset \mathbb{R}^3$ *is an exterior weak Lipschitz domain with boundary Γ and weak Lipschitz interfaces Γ_1 and $\Gamma_2 = \Gamma \setminus \overline{\Gamma}_1$ as introduced in the beginning of this section.*
- $\varepsilon = \varepsilon_0 \cdot \mathbb{1} + \hat{\varepsilon}$ *and* $\mu = \mu_0 \cdot \mathbb{1} + \hat{\mu}$ *are κ-decaying with $\kappa \geq 0$.*

For most of our results we need the slightly stronger assumption on the perturbations $\hat{\varepsilon}$ and $\hat{\mu}$. That is, $\hat{\varepsilon}$ resp. $\hat{\mu}$ have to be differentiable outside of an arbitrarily large ball with decaying derivative. More precisely:

Definition 2.3. *Let $\kappa \geq 0$. We call a transformation γ "$\kappa - \mathsf{C}^1$−decaying", if*

- $\gamma = \gamma_0 \cdot \mathbb{1} + \hat{\gamma}$ *is κ-decaying*
- *and for some $\tilde{r} > \hat{r}$ we have*

$$
\hat{\gamma} \in \mathsf{C}^1(\check{\mathrm{U}}_{\tilde{r}}) \quad \textit{with} \quad \partial_j \hat{\gamma} = \mathcal{O}\big(r^{-1-\kappa}\big) \ \ \textit{as}\ \ r \longrightarrow \infty\,, \qquad (\, j = 1, 2, 3\,).
$$

Note that a κ-decaying (resp. $\kappa - \mathsf{C}^1$− decaying) transformation γ is pointwise invertible for sufficiently large x. In this sense, γ^{-1} is κ-decaying (resp. $\kappa - \mathsf{C}^1$− decaying) as well. Moreover,

$$
\langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2_\gamma(\Omega)} := \langle\, \gamma\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2(\Omega)} \qquad \text{resp.} \qquad \langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2_{t,\gamma}(\Omega)} := \langle\, \gamma\rho^t\,\cdot\,,\, \rho^t\,\cdot\,\rangle_{\mathsf{L}^2(\Omega)}
$$

define inner products on $\mathsf{L}^2(\Omega)$ resp. $\mathsf{L}^2_t(\Omega)$ inducing norms equivalent to the standard ones. Thus

$$
\mathsf{L}^2_\gamma(\Omega) := \big(\, \mathsf{L}^2(\Omega), \langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2_\gamma(\Omega)}\,\big) \qquad \text{and} \qquad \mathsf{L}^2_{t,\gamma}(\Omega) := \big(\, \mathsf{L}^2_t(\Omega), \langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2_{t,\gamma}(\Omega)}\,\big)
$$

are Hilbert spaces and we use $\oplus_\gamma$, $\oplus_{\mathrm{t},\gamma}$ resp. $\perp_\gamma$, $\perp_{\mathrm{t},\gamma}$ to indicate orthogonal sum and orthogonal complement in this spaces. If $\gamma = \mathbb{1}$ we put $\oplus_\gamma =: \oplus$ as well as $\perp_\gamma =: \perp$. Finally we introduce for $s \in \mathbb{R}$ the (weighted) *"Dirichlet-Neumann fields"*

$$
{}_\gamma\mathcal{H}_{s,\Gamma_1,\Gamma_2}(\Omega) := {}_0\mathsf{R}_{s,\Gamma_1}(\Omega) \cap \gamma^{-1}{}_0\mathsf{D}_{s,\Gamma_2}(\Omega)\,, \qquad \mathcal{H}_{s,\Gamma_1,\Gamma_2}(\Omega) := {}_{\mathbb{1}}\mathcal{H}_{s,\Gamma_1,\Gamma_2}(\Omega)\,,
$$

where as before we skip the weight if $s = 0$.

3. The Static Problem $\omega = 0$

We start our considerations with the supposedly simpler case of electro-magneto-statics, which in fact possesses its own difficulties. First, as Ω is an exterior domain we are forced to work in polynomially weighted Sobolev spaces. Second, for $\omega = 0$ the time-harmonic Maxwell system (1.1),(1.2), i.e.,

$$\operatorname{rot} E = G \quad \text{in} \quad \Omega\,, \qquad\qquad n \times E = 0 \quad \text{on} \quad \Gamma_1\,,$$
$$\operatorname{rot} H = F \quad \text{in} \quad \Omega\,, \qquad\qquad n \times H = 0 \quad \text{on} \quad \Gamma_2\,,$$

is no longer coupled and in order to determine E and H we have to add two more equations[i]

$$\operatorname{div} \varepsilon E = f\,, \qquad\qquad \operatorname{div} \mu H = g\,, \qquad\qquad \text{in } \Omega\,,$$

as well as additional boundary conditions

$$n \cdot \varepsilon E = 0 \quad \text{on} \quad \Gamma_2\,, \qquad\qquad n \cdot \mu H = 0 \quad \text{on} \quad \Gamma_1\,.$$

The resulting boundary value problems of electro- resp. magneto-statics (cf. (1.4))

$$\operatorname{rot} E = G\,, \qquad \operatorname{div} \varepsilon E = f\,, \qquad n \times E = 0 \quad \text{on} \quad \Gamma_1\,, \qquad n \cdot \varepsilon E = 0 \quad \text{on} \quad \Gamma_2\,, \tag{3.1}$$

$$\operatorname{rot} H = F\,, \qquad \operatorname{div} \mu H = g\,, \qquad n \times H = 0 \quad \text{on} \quad \Gamma_2\,, \qquad n \cdot \mu H = 0 \quad \text{on} \quad \Gamma_1\,, \tag{3.2}$$

still have non-trivial but finite-dimensional kernels ${}_{\varepsilon}\mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega)$ and ${}_{\mu}\mathcal{H}_{\Gamma_2,\Gamma_1}(\Omega)$, respectively, demanding for finitely many orthogonality constraints to achieve unique solutions. Due to the similarity between (3.1) and (3.2) we concentrate on the electro-static problem (3.1), keeping in mind that interchanging Γ_1 and Γ_2 as well as ε and μ we also solve the magneto-static system.

Let Θ be a domain in $\mathbb{R}^3$. Considering the densely defined and closed linear operators

$$\mathrm{A}_1 := \operatorname{grad}_{\Gamma_1} : \mathcal{D}(\mathrm{A}_1) := \mathbf{H}^1_{\Gamma_1}(\Theta) \subset \mathsf{L}^2(\Theta) \;\longrightarrow\; \mathsf{L}^2_{\varepsilon}(\Theta)\,, \quad w \;\longmapsto\; \nabla w\,,$$
$$\mathrm{A}_2 := \operatorname{rot}_{\Gamma_1} : \mathcal{D}(\mathrm{A}_2) := \mathbf{R}_{\Gamma_1}(\Theta) \subset \mathsf{L}^2_{\varepsilon}(\Theta) \;\longrightarrow\; \mathsf{L}^2(\Theta)\,, \quad u \;\longmapsto\; \operatorname{rot} u\,,$$

the Hilbert space adjoints are (cf. [17, Lemma 2.2], [1, Theorem 4.5])

$$\mathrm{A}_1^* = \operatorname{grad}_{\Gamma_1}^* = -\operatorname{div}_{\Gamma_2} \varepsilon : \mathcal{D}(\mathrm{A}_1^*) = \varepsilon^{-1}\mathbf{D}_{\Gamma_2}(\Theta) \subset \mathsf{L}^2_{\varepsilon}(\Theta) \;\longrightarrow\; \mathsf{L}^2(\Theta)\,, \quad u \;\longmapsto\; -\operatorname{div} \varepsilon u\,,$$
$$\mathrm{A}_2^* = \operatorname{rot}_{\Gamma_1}^* = \varepsilon^{-1} \operatorname{rot}_{\Gamma_2} : \mathcal{D}(\mathrm{A}_2^*) = \mathbf{R}_{\Gamma_2}(\Theta) \subset \mathsf{L}^2(\Theta) \;\longrightarrow\; \mathsf{L}^2_{\varepsilon}(\Theta)\,, \quad u \;\longmapsto\; \varepsilon^{-1} \operatorname{rot} u\,.$$

These operators satisfy

$$\mathcal{R}(\mathrm{A}_1) \subset \mathcal{N}(\mathrm{A}_2)\,, \qquad\qquad \mathcal{R}(\mathrm{A}_2^*) \subset \mathcal{N}(\mathrm{A}_1^*)\,, \tag{3.3}$$

and by the projection theorem the Helmholtz-type decompositions

$$\left\{\begin{aligned}
\mathsf{L}^2(\Theta) &= \overline{\mathcal{R}(\mathrm{A}_1)} \oplus_{\varepsilon} \mathcal{N}(\mathrm{A}_1^*) & \mathsf{L}^2(\Theta) &= \overline{\mathcal{R}(\mathrm{A}_2^*)} \oplus_{\varepsilon} \mathcal{N}(\mathrm{A}_2)\,,\\
&= \overline{\nabla \mathbf{H}^1_{\Gamma_1}(\Theta)} \oplus_{\varepsilon} \varepsilon^{-1}{}_0\mathsf{D}_{\Gamma_2}(\Theta)\,, & &= \varepsilon^{-1}\overline{\operatorname{rot} \mathbf{R}_{\Gamma_2}(\Theta)} \oplus_{\varepsilon} {}_0\mathsf{R}_{\Gamma_1}(\Theta)\,,\\
{}_0\mathsf{R}_{\Gamma_1}(\Theta) &= \mathcal{N}(\mathrm{A}_2) & \varepsilon^{-1}{}_0\mathsf{D}_{\Gamma_2}(\Theta) &= \mathcal{N}(\mathrm{A}_1^*)\\
&= \overline{\mathcal{R}(\mathrm{A}_1)} \oplus_{\varepsilon} \big(\mathcal{N}(\mathrm{A}_1^*) \cap \mathcal{N}(\mathrm{A}_2)\big) & &= \overline{\mathcal{R}(\mathrm{A}_2^*)} \oplus_{\varepsilon} \big(\mathcal{N}(\mathrm{A}_2) \cap \mathcal{N}(\mathrm{A}_1^*)\big)\\
&= \overline{\nabla \mathbf{H}^1_{\Gamma_1}(\Theta)} \oplus_{\varepsilon} {}_{\varepsilon}\mathcal{H}_{\Gamma_1,\Gamma_2}(\Theta)\,, & &= \varepsilon^{-1}\overline{\operatorname{rot} \mathbf{R}_{\Gamma_2}(\Theta)} \oplus_{\varepsilon} {}_{\varepsilon}\mathcal{H}_{\Gamma_1,\Gamma_2}(\Theta)
\end{aligned}\right. \tag{3.4}$$

hold true. As shown in [24], rewriting (3.1) into

$$\mathrm{A}_2 E = G\,, \qquad \mathrm{A}_1^* E = f\,, \qquad E \in \mathcal{D}(\mathrm{A}_2) \cap \mathcal{D}(\mathrm{A}_1^*) \tag{3.5}$$

---

[i] For $\omega \neq 0$ these equations are implicitly given, as by differentiating (1.1) we immediately get

$$i\omega \operatorname{div} \varepsilon E = \operatorname{div}(-\operatorname{rot} H + i\omega\varepsilon E) = -\operatorname{div} F\,, \qquad i\omega \operatorname{div} \mu H = \operatorname{div}(\operatorname{rot} E + i\omega\mu H) = \operatorname{div} G \qquad \text{in } \Omega\,.$$

and using (3.4) together with standard functional analysis tools, we immediately obtain an L^2-solution theory for electro-magneto-statics, provided Θ satisfies *"Weck's selection theorem"*, a compactness result comparable to Rellich's selection theorem well suited for Maxwell's equations.

Definition 3.1. *A domain $\Theta \subset \mathbb{R}^3$ satisfies "Weck's selection theorem" (WST) (or possesses the "Maxwell compactness property") if the embedding*

$$\mathbf{R}_{\Gamma_1}(\Theta) \cap \varepsilon^{-1}\mathbf{D}_{\Gamma_2}(\Theta) \longrightarrow \mathsf{L}^2(\Theta) \text{ is compact}. \tag{3.6}$$

In particular, as shown in [1] (see also [24, Section 5]), it holds:

Lemma 3.2. *Let $\Theta \subset \mathbb{R}^3$ be a bounded weak Lipschitz domain with boundary Γ and weak Lipschitz interfaces Γ_1 and $\Gamma_2 := \Gamma \setminus \overline{\Gamma}_1$. Then Weck's selection theorem holds true and implies the following:*

(*i*) (Maxwell estimate) *There is $c > 0$ such that for all $E \in \mathbf{R}_{\Gamma_1}(\Theta) \cap \varepsilon^{-1}\mathbf{D}_{\Gamma_2}(\Theta) \cap {}_{\varepsilon}\mathcal{H}_{\Gamma_1,\Gamma_2}(\Theta)^{\perp_\varepsilon}$*

$$\| E \|_{\mathsf{L}^2(\Theta)} \leq c \Big(\| \operatorname{rot} E \|_{\mathsf{L}^2(\Theta)} + \| \operatorname{div} \varepsilon E \|_{\mathsf{L}^2(\Theta)} \Big).$$

(*ii*) (Finite dimensional kernel) *The unit ball in ${}_{\varepsilon}\mathcal{H}_{\Gamma_1,\Gamma_2}(\Theta)$ is compact, i.e.,*

$$\dim {}_{\varepsilon}\mathcal{H}_{\Gamma_1,\Gamma_2}(\Theta) < \infty.$$

(*iii*) (Closed ranges) *The ranges of* $\operatorname{grad}_{\Gamma_1}$ *and* $\operatorname{rot}_{\Gamma_1}$ *resp.* $\operatorname{div}_{\Gamma_2} \varepsilon$ *and* $\varepsilon^{-1} \operatorname{rot}_{\Gamma_2}$ *are closed, i.e.,*

$$\overline{\nabla \mathbf{H}^1_{\Gamma_1}(\Theta)} = \nabla \mathbf{H}^1_{\Gamma_1}(\Theta), \qquad \overline{\operatorname{rot} \mathbf{R}_{\Gamma_1}(\Theta)} = \operatorname{rot} \mathbf{R}_{\Gamma_1}(\Theta),$$
$$\overline{\operatorname{div} \mathbf{D}_{\Gamma_2}(\Theta)} = \operatorname{div} \mathbf{D}_{\Gamma_2}(\Theta), \qquad \overline{\operatorname{rot} \mathbf{R}_{\Gamma_2}(\Theta)} = \operatorname{rot} \mathbf{R}_{\Gamma_2}(\Theta),$$

and the following Helmholtz type decompositions are valid

$$\mathsf{L}^2(\Theta) = \nabla \mathbf{H}^1_{\Gamma_1}(\Theta) \oplus_\varepsilon \varepsilon^{-1}{}_0\mathsf{D}_{\Gamma_2}(\Theta), \qquad \mathsf{L}^2(\Theta) = \varepsilon^{-1} \operatorname{rot} \mathbf{R}_{\Gamma_2}(\Theta) \oplus_\varepsilon {}_0\mathsf{R}_{\Gamma_1}(\Theta),$$
$${}_0\mathsf{R}_{\Gamma_1}(\Theta) = \nabla \mathbf{H}^1_{\Gamma_1}(\Theta) \oplus_\varepsilon {}_{\varepsilon}\mathcal{H}_{\Gamma_1,\Gamma_2}(\Theta), \qquad \varepsilon^{-1}{}_0\mathsf{D}_{\Gamma_2}(\Theta) = \varepsilon^{-1} \operatorname{rot} \mathbf{R}_{\Gamma_2}(\Theta) \oplus_\varepsilon {}_{\varepsilon}\mathcal{H}_{\Gamma_1,\Gamma_2}(\Theta).$$

Remark 3.3. *In the latter lemma and the previous arguments (involving Θ) it is sufficient that ε is $\kappa-$decaying with $\kappa \geq 0$.*

While Weck's selection theorem holds true for bounded weak Lipschitz domains, it fails for unbounded such as exterior domains (cf. [1], [7] and also [6] for strong Lipschitz domains). Thus, we cannot retreat on the functional analysis toolbox from [24], especially Lemma 3.2, to solve system (3.1). Instead we will use a slightly weaker version of (3.6) to prove similar results in weighted L^2- spaces. More precisely, as Ω_δ is a bounded weak Lipschitz domain with boundary $\Gamma = \overline{\Gamma}_\delta \cup \overline{\Gamma}_2$, Lemma 3.2 yields, e.g.,

$$\forall\, \delta \geq \hat{r}: \quad \mathbf{R}_{\Gamma_\delta}(\Omega_\delta) \cap \varepsilon^{-1}\mathbf{D}_{\Gamma_2}(\Omega_\delta) \longrightarrow \mathsf{L}^2(\Omega_\delta) \quad \text{is compact}.$$

Hence by [17, Lemma 3.3] it holds:

Theorem 3.4 (Weck's local selection theorem)**.** *The embedding*

$$\mathbf{R}_{\Gamma_1}(\Omega) \cap \varepsilon^{-1}\mathbf{D}_{\Gamma_2}(\Omega) \longrightarrow \mathsf{L}^2_{\mathrm{loc}}(\overline{\Omega})$$

is compact. Equivalently for all $s, t \in \mathbb{R}$ with $t < s$ the embedding

$$\mathbf{R}_{s,\Gamma_1}(\Omega) \cap \varepsilon^{-1}\mathbf{D}_{s,\Gamma_2}(\Omega) \longrightarrow \mathsf{L}^2_t(\Omega)$$

is compact.

As we will show in the following, by Theorem 3.4 we are indeed able to reconstruct the results of Lemma 3.2 in the framework of weighted Sobolev spaces, leading to a solution theory for (3.1) resp. (3.2) in exterior domains.

3.1. **Poincaré and Maxwell Estimates in Exterior Domains.** We start out proving a weighted version of the Poincaré estimate. From [25, Lemma 15], see also [12, Poincare's estimate III], we have

$$\|\phi\|_{\mathsf{L}^2_{-1}(\Omega)} \leq \|r^{-1}\phi\|_{\mathsf{L}^2(\Omega)} \leq 2\,\|\nabla\phi\|_{\mathsf{L}^2(\Omega)} \qquad \forall\,\phi\in\mathsf{C}^\infty_\Gamma(\Omega)\,, \tag{3.7}$$

which by continuity extends to all $u\in\mathsf{H}^1_{-1,\Gamma}(\Omega)$ and can even be generalized to functions in $\mathsf{H}^1_{-1}(\Omega)$.

Lemma 3.5. *The following Poincaré estimate holds:*

$$\exists\,c>0 \quad \forall\,u\in\mathsf{H}^1_{-1}(\Omega)\;: \qquad \|u\|_{\mathsf{L}^2_{-1}(\Omega)} \leq c\,\|\nabla u\|_{\mathsf{L}^2(\Omega)}$$

Proof. For $u\in\mathsf{H}^1_{-1}(\Omega)$ it holds $\eta_{\hat r}u\in\mathsf{H}^1_{-1,\Gamma}(\Omega)$ and from (3.7) we obtain

$$\|u\|_{\mathsf{L}^2_{-1}(\Omega)} \leq 2\,\|\nabla(\eta_{\hat r}u)\|_{\mathsf{L}^2_{-1}(\Omega)} + \|(1-\eta_{\hat r})u\|_{\mathsf{L}^2_{-1}(\Omega)} \leq c\Big(\|\nabla u\|_{\mathsf{L}^2(\Omega)} + \|u\|_{\mathsf{L}^2(\Omega_{2\hat r})}\Big)\,, \tag{3.8}$$

with $c>0$. Assuming the asserted estimate is wrong, there exists a sequence $(u_n)_{n\in\mathbb{N}}\subset\mathsf{H}^1_{-1}(\Omega)$ with

$$\|u_n\|_{\mathsf{L}^2_{-1}(\Omega)} = 1 \qquad \text{and} \qquad \|\nabla u_n\|_{\mathsf{L}^2(\Omega)} < \frac{1}{n} \xrightarrow{n\to\infty} 0\,.$$

Hence, $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathsf{H}^1(\Omega_{2\hat r})$ and by Rellich's selection theorem[ii] we can extract a subsequence $(u_{\pi(n)})_{n\in\mathbb{N}}$ converging in $\mathsf{L}^2(\Omega_{2\hat r})$. By (3.8) the sequence $(u_{\pi(n)})_{n\in\mathbb{N}}$ is even a Cauchy sequence in $\mathsf{H}^1_{-1}(\Omega)$ and therefore converging to some $u\in\mathsf{H}^1_{-1}(\Omega)$ with $\nabla u=0$. Consequently u is constant in Ω and as $u\in\mathsf{L}^2_{-1}(\Omega)$ we have $u=0$, a contradiction. □

Similarly, Weck's local selection theorem yields a weighted version of the Maxwell estimate. Again we start with testfields $\Phi\in\mathsf{C}^\infty_\Gamma(\Omega)$ stating that by (3.7) and $-\Delta\Phi = \operatorname{rot}\operatorname{rot}\Phi - \nabla\operatorname{div}\Phi$ we have

$$\|\Phi\|_{\mathsf{L}^2_{-1}(\Omega)} \leq c\,\|\nabla\Phi\|_{\mathsf{L}^2(\Omega)} \leq c\,\Big(\|\operatorname{rot}\Phi\|_{\mathsf{L}^2(\Omega)} + \|\operatorname{div}\Phi\|_{\mathsf{L}^2(\Omega)}\Big) \qquad \forall\,\Phi\in\mathsf{C}^\infty_\Gamma(\Omega)\,. \tag{3.9}$$

which directly extends to $\Phi\in\mathsf{H}^1_{-1,\Gamma}(\Omega)$ and can also be generalized.

Lemma 3.6. *Let $s\in\mathbb{R}$, $\tilde r>\hat r$, and $\Xi\subset\check{\mathrm{U}}_{\tilde r}\subset\mathbb{R}^3$ be an exterior domain with $\operatorname{dist}(\Xi,\mathrm{S}_{\tilde r})>0$. Furthermore, let ε be $\kappa-\mathsf{C}^1-$decaying with $\kappa>0$ such that $\varepsilon\in\mathsf{C}^1(\check{\mathrm{U}}_{\tilde r})$. Then the conditions $E\in\mathsf{L}^2_s(\check{\mathrm{U}}_{\tilde r})$, $\operatorname{rot}E\in\mathsf{L}^2_{s+1}(\check{\mathrm{U}}_{\tilde r})$, and $\operatorname{div}\varepsilon E\in\mathsf{L}^2_{s+1}(\check{\mathrm{U}}_{\tilde r})$ imply $E\in\mathsf{H}^1_s(\Xi)$ and it holds*

$$\|E\|_{\mathsf{H}^1_s(\Xi)} \leq c\,\Big(\|E\|_{\mathsf{L}^2_s(\check{\mathrm{U}}_{\tilde r})} + \|\operatorname{rot}E\|_{\mathsf{L}^2_{s+1}(\check{\mathrm{U}}_{\tilde r})} + \|\operatorname{div}\varepsilon E\|_{\mathsf{L}^2_{s+1}(\check{\mathrm{U}}_{\tilde r})}\Big)$$

with $c>0$ independent of E.

Proof. This regularity result is a direct consequence of [9, Lemma 4.2] . A detailed proof can be found in [18, Korollar 3.7]. □

Remark 3.7. *By obvious modifications on ε and the assumptions imposed on E, the previous result can also be formulated for the bold Hilbert spaces, e.g., $\mathbf{H}^1_s(\Xi)$. Beyond that it may even be generalized to higher regularity for E, e.g., $E\in\mathsf{H}^m_s(\Xi)$. We also note that the assumptions on ε may be reduced to a $\kappa-$decaying $\varepsilon=\varepsilon_0\cdot\mathbb{1}+\hat\varepsilon$ with $\kappa>0$ and $\hat\varepsilon\in\mathsf{C}^1(\check{\mathrm{U}}_{\tilde r})$ such that $\partial_j\hat\varepsilon=\mathcal{O}(r^{-1})$.*

Lemma 3.8. *Let ε be $\kappa-\mathsf{C}^1-$decaying with order $\kappa>0$. Then there exist $c,\delta>0$ such that*

$$\|E\|_{\mathsf{L}^2_{-1}(\Omega)} \leq c\,\Big(\|\operatorname{rot}E\|_{\mathsf{L}^2(\Omega)} + \|\operatorname{div}\varepsilon E\|_{\mathsf{L}^2(\Omega)} + \|E\|_{\mathsf{L}^2(\Omega_\delta)}\Big)$$

holds for all $E\in\mathsf{R}_{-1}(\Omega)\cap\varepsilon^{-1}\mathsf{D}_{-1}(\Omega)$.

[ii] Note that also Rellich's selection theorem holds in bounded weak Lipschitz domains (cf. [1, Theorem 4.8]).

Proof. By assumption ε is of the form $\varepsilon = \varepsilon_0 \cdot \mathbb{1} + \hat{\varepsilon}$ with $\varepsilon_0 \in \mathbb{R}_+$ and there exists $\tilde{r} > \hat{r}$ such that

$$\hat{\varepsilon} \in \mathsf{C}^1(\check{\mathrm{U}}_{\tilde{r}}) \quad \text{with} \quad \hat{\varepsilon} = \mathcal{O}\big(r^{-\kappa}\big)\,, \ \partial_j\hat{\varepsilon} = \mathcal{O}\big(r^{-1-\kappa}\big) \quad \text{as} \quad r \longrightarrow \infty\,, \qquad (\, j = 1,2,3\,)\,. \tag{3.10}$$

Using the cut-off function from above, we define $\tilde{E} := \eta_{\tilde{r}} E$ and as Lemma 3.6 yields $E \in \mathsf{H}^1_{-1}(\operatorname{supp}\eta_{\tilde{r}})$, we have $\tilde{E} \in \mathsf{H}^1_{-1,\Gamma}(\Omega)$. Hence by (3.9)

$$\begin{aligned}
\|\, \tilde{E} \,\|_{\mathsf{L}^2_{-1}(\Omega)} &\leq c \Big(\|\operatorname{rot} \tilde{E} \,\|_{\mathsf{L}^2(\check{\mathrm{U}}_{\tilde{r}})} + \|\operatorname{div} \varepsilon_0 \tilde{E} \,\|_{\mathsf{L}^2(\operatorname{supp}\eta_{\tilde{r}})} \Big) \\
&\leq c \Big(\|\operatorname{rot} \tilde{E} \,\|_{\mathsf{L}^2(\check{\mathrm{U}}_{\tilde{r}})} + \|\operatorname{div} \varepsilon \tilde{E} \,\|_{\mathsf{L}^2(\operatorname{supp}\eta_{\tilde{r}})} + \|\operatorname{div} \hat{\varepsilon} \tilde{E} \,\|_{\mathsf{L}^2(\operatorname{supp}\eta_{\tilde{r}})} \Big) \\
&\leq c \Big(\|\operatorname{rot} \tilde{E} \,\|_{\mathsf{L}^2(\check{\mathrm{U}}_{\tilde{r}})} + \|\operatorname{div} \varepsilon \tilde{E} \,\|_{\mathsf{L}^2(\check{\mathrm{U}}_{\tilde{r}})} + \sum_{j=1,2,3} \|(\partial_j \hat{\varepsilon})\tilde{E} \,\|_{\mathsf{L}^2(\operatorname{supp}\eta_{\tilde{r}})} + \|\, \hat{\varepsilon} : \nabla \tilde{E} \,\|_{\mathsf{L}^2(\operatorname{supp}\eta_{\tilde{r}})} \Big)\,.
\end{aligned}$$

With (3.10) and the regularity estimate from Lemma 3.6 we obtain

$$\begin{aligned}
\|\, \tilde{E} \,\|_{\mathsf{L}^2_{-1}(\Omega)} &\leq c \Big(\|\operatorname{rot} \tilde{E} \,\|_{\mathsf{L}^2(\check{\mathrm{U}}_{\tilde{r}})} + \|\operatorname{div} \varepsilon \tilde{E} \,\|_{\mathsf{L}^2(\check{\mathrm{U}}_{\tilde{r}})} + \|\, \tilde{E} \,\|_{\mathsf{H}^1_{-1-\kappa}(\operatorname{supp}\eta_{\tilde{r}})} \Big) \\
&\leq c \Big(\|\operatorname{rot} \tilde{E} \,\|_{\mathsf{L}^2(\check{\mathrm{U}}_{\tilde{r}})} + \|\operatorname{div} \varepsilon \tilde{E} \,\|_{\mathsf{L}^2(\check{\mathrm{U}}_{\tilde{r}})} + \|\, \tilde{E} \,\|_{\mathsf{L}^2_{-1-\kappa}(\check{\mathrm{U}}_{\tilde{r}})} \Big) \\
&\leq c \Big(\|\operatorname{rot} \tilde{E} \,\|_{\mathsf{L}^2(\Omega)} + \|\operatorname{div} \varepsilon \tilde{E} \,\|_{\mathsf{L}^2(\Omega)} + \|\, \tilde{E} \,\|_{\mathsf{L}^2_{-1-\kappa}(\Omega)} \Big)\,,
\end{aligned}$$

such that by

$$\operatorname{rot} \tilde{E} = \eta_{\tilde{r}} \operatorname{rot} E + (\nabla \eta_{\tilde{r}}) \times E\,, \qquad\qquad \operatorname{div} \varepsilon \tilde{E} = \eta_{\tilde{r}} \operatorname{div} \varepsilon E + (\nabla \eta_{\tilde{r}}) \cdot \varepsilon E\,,$$

we end up with

$$\begin{aligned}
\|\, E \,\|_{\mathsf{L}^2_{-1}(\Omega)} &\leq c \Big(\|\, \tilde{E} \,\|_{\mathsf{L}^2_{-1}(\Omega)} + \|\, E \,\|_{\mathsf{L}^2(\Omega_{2\tilde{r}})} \Big) \\
&\leq c \Big(\|\operatorname{rot} \tilde{E} \,\|_{\mathsf{L}^2(\Omega)} + \|\operatorname{div} \varepsilon \tilde{E} \,\|_{\mathsf{L}^2(\Omega)} + \|\, \tilde{E} \,\|_{\mathsf{L}^2_{-1-\kappa}(\Omega)} + \|\, E \,\|_{\mathsf{L}^2(\Omega_{2\tilde{r}})} \Big) \\
&\leq c \Big(\|\operatorname{rot} E \,\|_{\mathsf{L}^2(\Omega)} + \|\operatorname{div} \varepsilon E \,\|_{\mathsf{L}^2(\Omega)} + \|\, E \,\|_{\mathsf{L}^2_{-1-\kappa}(\Omega)} \Big)\,.
\end{aligned}$$

Finally, as $\kappa > 0$, the assertion follows by

$$\|\, E \,\|^2_{\mathsf{L}^2_{-1-\kappa}(\Omega)} = \|\, E \,\|^2_{\mathsf{L}^2_{-1-\kappa}(\Omega_\delta)} + \|\, E \,\|^2_{\mathsf{L}^2_{-1-\kappa}(\check{\mathrm{U}}_\delta)} \leq \|\, E \,\|^2_{\mathsf{L}^2(\Omega_\delta)} + (1+\delta^2)^{-\kappa} \cdot \|\, E \,\|^2_{\mathsf{L}^2_{-1}(\Omega)}$$

choosing $\delta > \hat{r}$ big enough. $\square$

Now, analogously to the proof of Lemma 3.5, we use Theorem 3.4 to eliminate the extra term on the right hand side. But unlike there, here the kernels of the involved operators "rot" and "div ε" are not necessarily trivial. Therefore, we aim for a weighted version of the Maxwell estimate excluding the *Dirichlet-Neumann fields*

$$_\varepsilon\mathcal{H}_{-1,\Gamma_1,\Gamma_2}(\Omega) = {}_0\mathsf{R}_{-1,\Gamma_1}(\Omega) \cap \varepsilon^{-1}{}_0\mathsf{D}_{-1,\Gamma_2}(\Omega)\,.$$

Fortunately, the space $_\varepsilon\mathcal{H}_{-1,\Gamma_1,\Gamma_2}(\Omega)$ is only finite dimensional.

Lemma 3.9. *Let ε be $\kappa - \mathsf{C}^1-$ decaying with $\kappa > 0$. Then:*

(i) (Maxwell estimate) *There is $c > 0$ s. t. for all $E \in \mathsf{R}_{-1,\Gamma_1}(\Omega) \cap \varepsilon^{-1}\mathsf{D}_{-1,\Gamma_2}(\Omega) \cap {}_\varepsilon\mathcal{H}_{-1,\Gamma_1,\Gamma_2}(\Omega)^{\perp_{-1,\varepsilon}}$*

$$\|\, E \,\|_{\mathsf{L}^2_{-1}(\Omega)} \leq c \cdot \Big(\|\operatorname{rot} E \,\|_{\mathsf{L}^2(\Omega)} + \|\operatorname{div} \varepsilon E \,\|_{\mathsf{L}^2(\Omega)} \Big)\,.$$

(ii) (Finite dimensional kernel) *The unit ball in $_\varepsilon\mathcal{H}_{-1,\Gamma_1,\Gamma_2}(\Omega)$ is compact, i.e.,*

$$\dim {}_\varepsilon\mathcal{H}_{-1,\Gamma_1,\Gamma_2}(\Omega) < \infty\,.$$

(iii) (Closed ranges) *The ranges of* $\operatorname{grad}_{\Gamma_1}$, $\operatorname{rot}_{\Gamma_1}$ *and* $\operatorname{div}_{\Gamma_2}\varepsilon$ *are **not** closed, but it holds*

(*a*) $\overline{\nabla\mathbf{H}^1_{\Gamma_1}(\Omega)} = \overline{\nabla\mathsf{H}^1_{-1,\Gamma_1}(\Omega)} = \nabla\mathsf{H}^1_{-1,\Gamma_1}(\Omega)$,

(*b*) $\overline{\operatorname{rot}\mathbf{R}_{\Gamma_1}(\Omega)} = \overline{\operatorname{rot}\mathsf{R}_{-1,\Gamma_1}(\Omega)}$
$$= \operatorname{rot}\mathsf{R}_{-1,\Gamma_1}(\Omega) = \operatorname{rot}\Big(\mathsf{R}_{-1,\Gamma_1}(\Omega)\cap\varepsilon^{-1}{}_0\mathsf{D}_{-1,\Gamma_2}(\Omega)\cap{}_\varepsilon\mathcal{H}_{-1,\Gamma_1,\Gamma_2}(\Omega)^{\perp_{-1,\varepsilon}}\Big),$$

(*c*) $\overline{\operatorname{div}\mathbf{D}_{\Gamma_2}(\Omega)} = \overline{\operatorname{div}\mathsf{D}_{-1,\Gamma_2}(\Omega)}$
$$= \operatorname{div}\mathsf{D}_{-1,\Gamma_2}(\Omega) = \operatorname{div}\Big(\mathsf{D}_{-1,\Gamma_2}(\Omega)\cap\varepsilon\,{}_0\mathsf{R}_{-1,\Gamma_1}(\Omega)\cap\varepsilon\,{}_\varepsilon\mathcal{H}_{-1,\Gamma_1,\Gamma_2}(\Omega)^{\perp_{-1,\varepsilon}}\Big).$$

Proof. Statement (*ii*) just follows by Weck's local selection theorem and Lemma 3.8. For (*i*) suppose the estimate is wrong, i.e., there exists $(E_n)_{n\in\mathbb{N}}\subset\mathsf{R}_{-1,\Gamma_1}(\Omega)\cap\varepsilon^{-1}\mathsf{D}_{-1,\Gamma_2}(\Omega)\cap{}_\varepsilon\mathcal{H}_{-1,\Gamma_1,\Gamma_2}(\Omega)^{\perp_{-1,\varepsilon}}$ with

$$\|E_n\|_{\mathsf{L}^2_{-1}(\Omega)} = 1 \qquad\text{and}\qquad \|\operatorname{rot}E_n\|_{\mathsf{L}^2(\Omega)} + \|\operatorname{div}\varepsilon E_n\|_{\mathsf{L}^2(\Omega)} \xrightarrow{n\to\infty} 0. \tag{3.11}$$

Then the sequence $(E_n)_{n\in\mathbb{N}}$ is bounded in $\mathsf{R}_{-1,\Gamma_1}(\Omega)\cap\varepsilon^{-1}\mathsf{D}_{-1,\Gamma_2}(\Omega)$ and Weck's local selection theorem provides a subsequence $(E_{\pi(n)})_{n\in\mathbb{N}}$ converging in $\mathsf{L}^2_{\mathrm{loc}}(\overline{\Omega})$. By Lemma 3.8 the sequence $(E_{\pi(n)})_{n\in\mathbb{N}}$ is an L^2_{-1}−Cauchy-sequence and we obtain

$$E := \lim_{n\to\infty}E_{\pi(n)} \in \mathsf{R}_{-1,\Gamma_1}(\Omega)\cap\varepsilon^{-1}\mathsf{D}_{-1,\Gamma_2}(\Omega) \qquad\text{with}\qquad \operatorname{rot}E = 0 \qquad\text{resp.}\qquad \operatorname{div}\varepsilon E = 0.$$

Additionally, $(E_{\pi(n)})_{n\in\mathbb{N}}\subset(E_n)_{n\in\mathbb{N}}\subset{}_\varepsilon\mathcal{H}_{-1,\Gamma_1,\Gamma_2}(\Omega)^{\perp_{-1,\varepsilon}}$ such that

$$\forall\, H\in{}_\varepsilon\mathcal{H}_{-1,\Gamma_1,\Gamma_2}(\Omega): \qquad \langle E, H\rangle_{\mathsf{L}^2_{-1,\varepsilon}(\Omega)} = \lim_{n\to\infty}\langle E_{\pi(n)}, H\rangle_{\mathsf{L}^2_{-1,\varepsilon}(\Omega)} = 0.$$

hence

$$E\in{}_\varepsilon\mathcal{H}_{-1,\Gamma_1,\Gamma_2}(\Omega)\cap{}_\varepsilon\mathcal{H}_{-1,\Gamma_1,\Gamma_2}(\Omega)^{\perp_{-1,\varepsilon}} = \{0\},$$

a contradiction. Let us finally turn to statement (*iii*). By definition we clearly have

$$\overline{\nabla\mathbf{H}^1_{\Gamma_1}(\Omega)} = \overline{\nabla\mathsf{H}^1_{-1,\Gamma_1}(\Omega)}, \qquad \overline{\operatorname{rot}\mathbf{R}_{\Gamma_1}(\Omega)} = \overline{\operatorname{rot}\mathsf{R}_{-1,\Gamma_1}(\Omega)}, \qquad \overline{\operatorname{div}\mathbf{D}_{\Gamma_2}(\Omega)} = \overline{\operatorname{div}\mathsf{D}_{-1,\Gamma_2}(\Omega)}. \tag{3.12}$$

Now, for $u^\nabla\in\overline{\nabla\mathsf{H}^1_{-1,\Gamma_1}(\Omega)}$ there exists $(u_n)_{n\in\mathbb{N}}\subset\mathsf{H}^1_{-1,\Gamma_1}(\Omega)$ with $\nabla u_n\xrightarrow{n\to\infty}u^\nabla$ in $\mathsf{L}^2(\Omega)$. The Poincaré estimate, Lemma 3.5, shows that $(u_n)_{n\in\mathbb{N}}$ is converging in $\mathsf{L}^2_{-1}(\Omega)$ to some $u\in\mathsf{L}^2_{-1}(\Omega)$ and we have

$$\langle u, \operatorname{div}\Phi\rangle_{\mathsf{L}^2(\Omega)} = \lim_{n\to\infty}\langle u_n, \operatorname{div}\Phi\rangle_{\mathsf{L}^2(\Omega)} = -\lim_{n\to\infty}\langle\nabla u_n, \Phi\rangle_{\mathsf{L}^2(\Omega)} = -\langle u^\nabla, \Phi\rangle_{\mathsf{L}^2(\Omega)} \quad \forall\,\Phi\in\mathsf{C}^\infty_{\Gamma_2}(\Omega).$$

Thus, by (2.3) we have $u\in\mathsf{H}^1_{-1,\Gamma_1}(\Omega)$ and $\nabla u = u^\nabla$, which shows (*a*). For (*b*) let $E^{\mathrm{rot}}\in\overline{\operatorname{rot}\mathsf{R}_{-1,\Gamma_1}(\Omega)}$ and $(E_n)_{n\in\mathbb{N}}\subset\mathbf{R}_{\Gamma_1}(\Omega)$ a sequence with $\operatorname{rot}E_n\xrightarrow{n\to\infty}E^{\mathrm{rot}}$ in $\mathsf{L}^2(\Omega)$. Using the decompositions from (3.4) and statement (*a*), we obtain $E_n = E_n^\nabla + \hat{E}_n\in\nabla\mathsf{H}^1_{-1,\Gamma_1}(\Omega)\oplus_\varepsilon\varepsilon^{-1}\,{}_0\mathsf{D}_{\Gamma_2}(\Omega)$, hence

$$\hat{E}_n = E_n - E_n^\nabla\in\mathbf{R}_{\Gamma_1}(\Omega)\cap\varepsilon^{-1}{}_0\mathsf{D}_{\Gamma_2}(\Omega) \quad\text{with}\quad \operatorname{rot}\hat{E}_n = \operatorname{rot}E_n\xrightarrow{\mathsf{L}^2(\Omega)}E^{\mathrm{rot}}.$$

As $\hat{E}_n\subset\mathsf{L}^2(\Omega)\subset\mathsf{L}^2_{-1}(\Omega)$ and ${}_\varepsilon\mathcal{H}_{-1,\Gamma_1,\Gamma_2}(\Omega)\subset\mathsf{L}^2_{-1}(\Omega)$ is finite-dimensional, we continue splitting

$$\hat{E}_n = E_n^{\mathcal{H}} + \tilde{E}_n\in{}_\varepsilon\mathcal{H}_{-1,\Gamma_1,\Gamma_2}(\Omega)\oplus_{-1,\varepsilon}{}_\varepsilon\mathcal{H}_{-1,\Gamma_1,\Gamma_2}(\Omega)^{\perp_{-1,\varepsilon}},$$

and end up with

$$\tilde{E}_n = \hat{E}_n - E_n^{\mathcal{H}}\in\mathsf{R}_{-1,\Gamma_1}(\Omega)\cap\varepsilon^{-1}{}_0\mathsf{D}_{-1,\Gamma_2}(\Omega)\cap{}_\varepsilon\mathcal{H}_{-1,\Gamma_1,\Gamma_2}(\Omega)^{\perp_{-1,\varepsilon}}, \qquad \operatorname{rot}\tilde{E}_n = \operatorname{rot}\hat{E}_n\xrightarrow{\mathsf{L}^2(\Omega)}E^{\mathrm{rot}}.$$

Now the weighted Maxwell estimate from (*i*) shows that $(\tilde{E}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathsf{L}^2_{-1}(\Omega)$, hence converging to some $\tilde{E}\in\mathsf{L}^2_{-1}(\Omega)$. In addition we have

- $\forall\,\Phi\in\mathsf{C}^\infty_{\Gamma_2}(\Omega)$: $\langle\tilde{E}, \operatorname{rot}\Phi\rangle_{\mathsf{L}^2(\Omega)}\xleftarrow{n\to\infty}\langle\tilde{E}_n, \operatorname{rot}\Phi\rangle_{\mathsf{L}^2(\Omega)} = \langle\operatorname{rot}\tilde{E}_n, \Phi\rangle_{\mathsf{L}^2(\Omega)}\xrightarrow{n\to\infty}\langle E^{\mathrm{rot}}, \Phi\rangle_{\mathsf{L}^2(\Omega)}$,

- $\forall\,\phi \in \mathsf{C}^{\infty}_{\Gamma_1}(\Omega)$: $\quad \langle\, \varepsilon \tilde{E}\,, \nabla\phi\,\rangle_{\mathsf{L}^2(\Omega)} \xleftarrow{n\to\infty} \langle\, \varepsilon \tilde{E}_n\,, \nabla\phi\,\rangle_{\mathsf{L}^2(\Omega)} = -\langle\, \operatorname{div}\varepsilon\tilde{E}_n\,, \phi\,\rangle_{\mathsf{L}^2(\Omega)} = 0\,,$
- $\forall\, H \in {}_{\varepsilon}\mathcal{H}_{-1,\Gamma_1,\Gamma_2}(\Omega)$: $\quad \langle\, \tilde{E}\,, H\,\rangle_{\mathsf{L}^2_{-1,\varepsilon}(\Omega)} \xleftarrow{n\to\infty} \langle\, \tilde{E}_n\,, H\,\rangle_{\mathsf{L}^2_{-1,\varepsilon}(\Omega)} = 0\,,$

such that by (2.3)

$$\tilde{E} \in \mathsf{R}_{-1,\Gamma_1}(\Omega) \cap \varepsilon^{-1}{}_0\mathsf{D}_{-1,\Gamma_2}(\Omega) \cap {}_{\varepsilon}\mathcal{H}_{-1,\Gamma_1,\Gamma_2}(\Omega)^{\perp_{-1,\varepsilon}} \quad \text{with} \quad \operatorname{rot}\tilde{E} = E^{\mathrm{rot}}$$

and (b) is proven. The last assertion (c) follows by similar arguments. □

Remark 3.10. *Under the assumptions of Lemma 3.9 we have in particular*

$$\mathsf{L}^2(\Omega) = \overline{\mathcal{R}(\operatorname{div}_{\Gamma_2}\varepsilon)} \oplus_{\varepsilon} \mathcal{N}(\operatorname{grad}_{\Gamma_1}) = \operatorname{div}\mathsf{D}_{-1,\Gamma_2}(\Omega) \oplus_{\varepsilon} \{0\} = \operatorname{div}\mathsf{D}_{-1,\Gamma_2}(\Omega)$$

and by (3.4) *the following Helmholtz type decompositions hold true:*

$$\mathsf{L}^2(\Omega) = \nabla\mathsf{H}^1_{-1,\Gamma_1}(\Omega) \oplus_{\varepsilon} \varepsilon^{-1}{}_0\mathsf{D}_{\Gamma_2}(\Omega)\,, \qquad \mathsf{L}^2(\Omega) = \varepsilon^{-1}\operatorname{rot}\mathsf{R}_{-1,\Gamma_1}(\Omega) \oplus_{\varepsilon} {}_0\mathsf{R}_{\Gamma_2}(\Omega)\,,$$
$$ {}_0\mathsf{R}_{\Gamma_1}(\Omega) = \nabla\mathsf{H}^1_{-1,\Gamma_1}(\Omega) \oplus_{\varepsilon} {}_{\varepsilon}\mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega)\,, \qquad \varepsilon^{-1}{}_0\mathsf{D}_{\Gamma_1}(\Omega) = \varepsilon^{-1}\operatorname{rot}\mathsf{R}_{-1,\Gamma_1}(\Omega) \oplus_{\varepsilon} {}_{\varepsilon}\mathcal{H}_{\Gamma_2,\Gamma_1}(\Omega)\,.$$

3.2. **Dirichlet-Neumann Fields in Exterior Domains.** As noted before, to solve (3.1) resp. (3.2) with Hilbert space methods we have to deal with ${}_{\varepsilon}\mathcal{H}_{-1,\Gamma_1,\Gamma_2}(\Omega)$ resp. ${}_{\mu}\mathcal{H}_{-1,\Gamma_2,\Gamma_1}(\Omega)$. Therefore, a more thorough investigation of these fields is needed.

From the literature, it is well known, that the existence of Dirichlet-Neumann fields is strongly related to the topological properties of the domain Ω. For example, as shown in [27] (see also [14]) in the limit cases $\Gamma_1 = \Gamma$ resp. $\Gamma_1 = \emptyset$ the dimension of $\mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega) = {}_{\mathbb{1}}\mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega)$ is essentially given by the number of connected components of the boundary Γ resp. the number of handles of Ω. In addition, as in [20, Lemma 3.8] we obtain for γ $\kappa - \mathsf{C}^1-$decaying with $\kappa > 0$

$$(3.13) \qquad {}_{\gamma}\mathcal{H}_{-\frac{3}{2},\Gamma_1,\Gamma_2}(\Omega) = {}_{\gamma}\mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega) = {}_{\gamma}\mathcal{H}_{<\frac{1}{2},\Gamma_1,\Gamma_2}(\Omega)\,,$$

and an easy application of the Helmholtz decompositions (3.4) shows that the dimension of the Dirichlet-Neumann fields ${}_{\gamma}\mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega)$ does not depend on the transformation γ, i.e.,

$$(3.14) \qquad d_{1,2} := \dim\mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega) = \dim{}_{\gamma}\mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega) = \dim{}_{\gamma}\mathcal{H}_{-1,\Gamma_1,\Gamma_2}(\Omega) < \infty\,.$$

As a crucial technical trick we will show that there exists a finite set of compactly supported vector fields $\mathfrak{B}_1(\Omega)$, whose projections form a basis of ${}_{\gamma}\mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega)$. The underlying idea is, that $\Omega_{\hat{r}}$ and Ω have essentially the same topological properties. Hence, choosing a basis of ${}_{\gamma}\mathcal{H}_{\Gamma_{\hat{r}},\Gamma_2}(\Omega_{\hat{r}})$, extending their elements by zero to Ω and projecting them onto ${}_{\gamma}\mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega)$, we obtain a basis of ${}_{\gamma}\mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega)$. Moreover, the extensions by zero define exactly the set $\mathfrak{B}_1(\Omega)$, which will also serve as a set of linear functionals ensuring uniqueness of static solutions.

Theorem 3.11. *There exist a finite set*

$$\mathfrak{B}_1(\Omega) = \{\mathcal{B}_{1,1}, \mathcal{B}_{1,2}, \ldots, \mathcal{B}_{1,d_{1,2}}\} \subset {}_0\mathsf{R}_{\Gamma_1}(\Omega) \qquad \textit{with} \qquad {}_{\gamma}\mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega) \cap \mathfrak{B}_1(\Omega)^{\perp_{\gamma}} = \{0\}\,.$$

In addition, the elements of $\mathfrak{B}_1(\Omega)$ *have compact support and their projections (in* $\mathsf{L}^2_{\gamma}(\Omega)$*) along* $\overline{\nabla\mathbf{H}^1_{\Gamma_1}(\Omega)}$ *form a basis of the Dirichlet-Neumann fields* ${}_{\gamma}\mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega)$.

Proof. The proof is given in the Appendix. □

Note that, as $\mathfrak{B}_1(\Omega)$ contains only compactly supported functions, we obviously have

$$(3.15) \qquad \forall\, s \in \mathbb{R}: \qquad \overline{\operatorname{rot}\mathsf{R}_{s-1,\Gamma_2}(\Omega)}^{\|\cdot\|_{\mathsf{L}^2_s(\Omega)}} \cup \overline{\operatorname{rot}\mathbf{R}_{s,\Gamma_2}(\Omega)}^{\|\cdot\|_{\mathsf{L}^2_s(\Omega)}} \subset \mathfrak{B}_1(\Omega)^{\perp}\,.$$

Therefore, $\mathfrak{B}_1(\Omega)$ allows for an alternative characterization for $\overline{\mathcal{R}(\operatorname{rot}_{\Gamma_2})}$ and, in particular, we may generalize the weighted Maxwell estimate from Lemma 3.9.

Lemma 3.12. *Let* ε *be* $\kappa-\mathsf{C}^1-$*decaying with order* $\kappa > 0$ *and* $\mathfrak{B}_1(\Omega)$ *be the finite set from Theorem 3.11.*

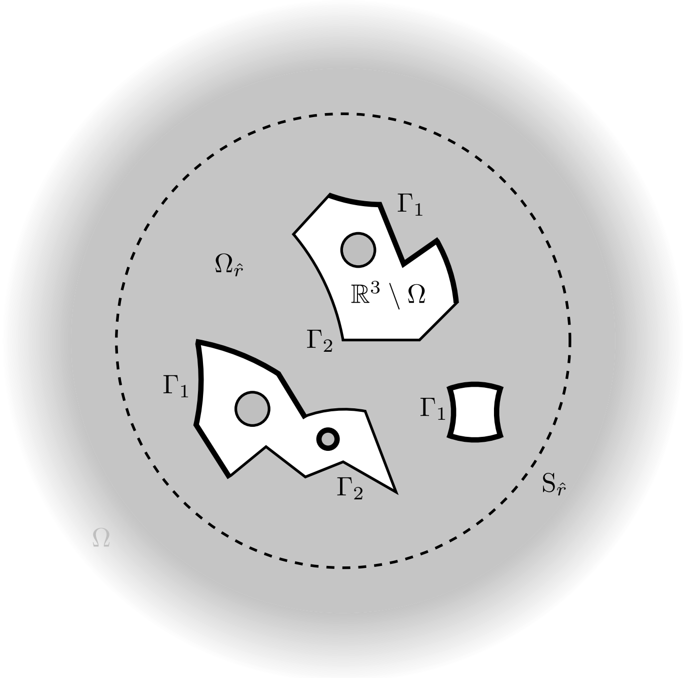


FIGURE 1. $\mathbb{R}^3 \setminus \Omega$ surrounded by the boundary parts Γ_1 (thick black lines) and Γ_2 (thin black lines) as well as the artificial boundary sphere $\mathrm{S}_{\hat{r}}$ (dashed line).

(*i*) *It holds*

$$\varepsilon^{-1}{}_0\mathsf{D}_{\Gamma_2}(\Omega) \cap \mathfrak{B}_1(\Omega)^{\perp_\varepsilon} = \varepsilon^{-1}{}_0\mathsf{D}_{\Gamma_2}(\Omega) \cap {}_\varepsilon\mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega)^{\perp_\varepsilon} = \varepsilon^{-1}\overline{\operatorname{rot} \mathbf{R}_{\Gamma_2}(\Omega)}\,.$$

(*ii*) *There exists $c > 0$ such that for all $E \in \mathsf{R}_{-1,\Gamma_1}(\Omega) \cap \varepsilon^{-1}\mathsf{D}_{-1,\Gamma_2}(\Omega)$ it holds*

$$\| E \|_{\mathsf{L}^2_{-1}(\Omega)} \le c\Big(\| \operatorname{rot} E \|_{\mathsf{L}^2(\Omega)} + \| \operatorname{div} \varepsilon E \|_{\mathsf{L}^2(\Omega)} + \sum_{\ell=1,\dots,d_{1,2}} |\langle E , \mathcal{B}_{1,\ell} \rangle_{\mathsf{L}^2_\varepsilon(\Omega)}| \Big)\,.$$

Proof. By (3.3), (3.4) and (3.15) we clearly have

$$\varepsilon^{-1}{}_0\mathsf{D}_{\Gamma_2}(\Omega) \cap {}_\varepsilon\mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega)^{\perp_\varepsilon} = \varepsilon^{-1}\overline{\operatorname{rot} \mathbf{R}_{\Gamma_2}(\Omega)} \subset \varepsilon^{-1}{}_0\mathsf{D}_{\Gamma_2}(\Omega) \cap \mathfrak{B}_1(\Omega)^{\perp_\varepsilon}\,.$$

Now let $E \in \varepsilon^{-1}{}_0\mathsf{D}_{\Gamma_2}(\Omega) \cap \mathfrak{B}_1(\Omega)^{\perp_\varepsilon}$. Then, by (3.4), we decompose

$$E = \mathcal{E} + \mathcal{H} \in \varepsilon^{-1}\overline{\operatorname{rot} \mathbf{R}_{\Gamma_2}(\Omega)} \oplus_\varepsilon {}_\varepsilon\mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega)\,,$$

hence, by (3.15) and Theorem 3.11 we have $\mathcal{H} = E - \mathcal{E} \in {}_\varepsilon\mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega) \cap \mathfrak{B}_1(\Omega)^{\perp_\varepsilon} = \{0\}$, which proves statement (*i*). In order to show (*ii*) we assume the estimate to be wrong. Then there exists

$$(E_n)_{n\in\mathbb{N}} \subset \mathsf{R}_{-1,\Gamma_1}(\Omega) \cap \varepsilon^{-1}\mathsf{D}_{-1,\Gamma_2}(\Omega) \quad \text{with} \quad \| E_n \|_{\mathsf{L}^2_{-1}(\Omega)} = 1$$

and

$$\operatorname{rot} E_n \xrightarrow{\mathsf{L}^2(\Omega)} 0\,, \qquad \operatorname{div} \varepsilon E_n \xrightarrow{\mathsf{L}^2(\Omega)} 0\,, \qquad \langle E_n , \mathcal{B}_{1,\ell} \rangle_{\mathsf{L}^2_\varepsilon(\Omega)} \xrightarrow{\mathbb{C}} 0 \qquad (\ell = 1,\dots,d_{1,2})$$

for $n \longrightarrow \infty$. Thus $(E_n)_{n\in\mathbb{N}}$ is bounded in $\mathsf{R}_{-1,\Gamma_1}(\Omega) \cap \varepsilon^{-1}\mathsf{D}_{-1,\Gamma_2}(\Omega)$ and by Weck's local selection theorem it has a subsequence $(E_{\pi(n)})_{n\in\mathbb{N}}$ converging in $\mathsf{L}^2_{\mathrm{loc}}(\overline{\Omega})$. By Lemma 3.8, this sequence even converges in $\mathsf{R}_{-1,\Gamma_1}(\Omega)\cap\varepsilon^{-1}\mathsf{D}_{-1,\Gamma_2}(\Omega)$ to some

$$E\in\mathsf{R}_{-1,\Gamma_1}(\Omega)\cap\varepsilon^{-1}\mathsf{D}_{-1,\Gamma_2}(\Omega)\quad\text{with}\quad\operatorname{rot}E=0\quad\text{resp. }\operatorname{div}\varepsilon E=0\,.$$

We obtain $E\in{}_\varepsilon\mathcal{H}_{-1,\Gamma_1,\Gamma_2}(\Omega)\overset{(3.13)}{=}{}_\varepsilon\mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega)$ and additionally

$$\langle\,E\,,\mathcal{B}_{1,\ell}\,\rangle_{\mathsf{L}^2_\varepsilon(\Omega)}=\lim_{n\to\infty}\langle\,E_{\pi(n)}\,,\mathcal{B}_{1,\ell}\,\rangle_{\mathsf{L}^2_\varepsilon(\Omega)}=0\,,\qquad\qquad \ell=1,\ldots,d_{1,2}\,,$$

hence $E\in{}_\varepsilon\mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega)\cap\mathfrak{B}_1(\Omega)^{\perp_\varepsilon}=\{0\}$ by Theorem 3.11, a contradiction. □

Remark 3.13. *Note that in Theorem 3.11 and Lemma 3.12 (i) no assumption on γ resp. ε is required, except of the General Assumption 2.2.*

3.3. **Static Solution Theory.** Let us turn back to the boundary value problem of electro-magneto-statics, using (3.1) as an illustrative example. As indicated by Lemma 3.9 we will solve (3.1) for given data $(G,f)\in\mathsf{L}^2(\Omega)\times\mathsf{L}^2(\Omega)$ by constructing a solution $E\in\mathsf{L}^2_{-1}(\Omega)$. In order to obtain uniqueness, we have to impose some additional conditions, but instead of projecting to Dirichlet-Neumann fields, we use projections to $\mathfrak{B}_1(\Omega)$.

Definition 3.14. *Let $(G,f,\zeta)\in\mathsf{L}^2_{\mathrm{loc}}(\overline{\Omega})\times\mathsf{L}^2_{\mathrm{loc}}(\overline{\Omega})\times\mathbb{C}^{d_{1,2}}$. We call E "(static) solution" of (3.1), if*

$$E\in\mathsf{R}_{-1,\Gamma_1}(\Omega)\cap\varepsilon^{-1}\mathsf{D}_{-1,\Gamma_2}(\Omega)$$

satisfies

$$\operatorname{rot}E=G\,,\qquad\operatorname{div}\varepsilon E=f\,,\qquad\langle\,E\,,\mathcal{B}_{1,\ell}\,\rangle_{\mathsf{L}^2_\varepsilon(\Omega)}=\zeta_\ell\qquad(\,\ell=1,\ldots,d_{1,2}\,)\,,\tag{3.16}$$

where $\{\,\mathcal{B}_{1,1},\mathcal{B}_{1,2},\ldots,\mathcal{B}_{1,d_{1,2}}\,\}$ are the elements in $\mathfrak{B}_1(\Omega)$ from Theorem 3.11.

Let $G\in\mathsf{L}^2(\Omega)$, $f\in\mathsf{L}^2(\Omega)$, $\zeta\in\mathbb{C}^{d_{1,2}}$, and let ε decay with order $\kappa>0$. First of all note that (3.1) admits at most one *static solution*, as for the homogeneous problem $E\in{}_\varepsilon\mathcal{H}_{-1,\Gamma_1,\Gamma_2}(\Omega)\cap\mathfrak{B}_1(\Omega)^{\perp_\varepsilon}$ together with (3.13) and Theorem 3.11 yields $E=0$. Turning to existence, necessary conditions are obviously

$$G\in\operatorname{rot}\mathsf{R}_{-1,\Gamma_1}(\Omega)\qquad\text{and}\qquad f\in\operatorname{div}\mathsf{D}_{-1,\Gamma_2}(\Omega),$$

the latter one being no further restriction as by Lemma 3.9, Remark 3.10 we have $\operatorname{div}\mathsf{D}_{-1,\Gamma_2}(\Omega)=\mathsf{L}^2(\Omega)$. But in fact this conditions are already sufficient since Lemma 3.9 also yields

$$E_1\in\mathsf{R}_{-1,\Gamma_1}(\Omega)\cap\varepsilon^{-1}{}_0\mathsf{D}_{-1,\Gamma_2}(\Omega)\qquad\text{and}\qquad E_2\in\mathsf{D}_{-1,\Gamma_2}(\Omega)\cap\varepsilon\,{}_0\mathsf{R}_{-1,\Gamma_1}(\Omega)$$

with $\operatorname{rot}E_1=G$ and $\operatorname{div}E_2=f$. Thus,

$$\widehat{E}:=E_1+\varepsilon^{-1}E_2\in\mathsf{R}_{-1,\Gamma_1}(\Omega)\cap\varepsilon^{-1}\mathsf{D}_{-1,\Gamma_2}(\Omega)$$

already satisfies

$$\operatorname{rot}\widehat{E}=\operatorname{rot}E_1=G\qquad\text{and}\qquad\operatorname{div}\varepsilon\widehat{E}=\operatorname{div}E_2=f\,.$$

Moreover, assuming we are able to construct $H\in{}_\varepsilon\mathcal{H}_{-1,\Gamma_1,\Gamma_2}(\Omega)={}_\varepsilon\mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega)$ with

$$\langle\,H\,,\mathcal{B}_{1,\ell}\,\rangle_{\mathsf{L}^2_\varepsilon(\Omega)}=\zeta_\ell-\langle\,\widehat{E}\,,\mathcal{B}_{1,\ell}\,\rangle_{\mathsf{L}^2_\varepsilon(\Omega)}:=\tilde{\zeta}_\ell\,,\qquad\qquad\ell=1,\ldots,d_{1,2}\,,\tag{3.17}$$

the sum

$$E:=\widehat{E}+H\in\mathsf{R}_{-1,\Gamma_1}(\Omega)\cap\varepsilon^{-1}\mathsf{D}_{-1,\Gamma_2}(\Omega)$$

solves

$$\operatorname{rot}E=G\,,\qquad\operatorname{div}\varepsilon E=f\,,\qquad\langle\,E\,,\mathcal{B}_{1,\ell}\,\rangle_{\mathsf{L}^2_\varepsilon(\Omega)}=\zeta_\ell\qquad(\,\ell=1,\ldots,d_{1,2}\,)\,,$$

hence E is a static solution of (3.1). It remains to construct H such that (3.17) holds. For that we decompose $\mathcal{B}_\ell$ according to Remark 3.10 in

$$\mathcal{B}_{1,\ell}=\nabla w_\ell+H_\ell\in\nabla\mathsf{H}^1_{-1,\Gamma_1}(\Omega)\oplus_\varepsilon{}_\varepsilon\mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega)\,,\qquad\qquad\ell=1,\ldots,d_{1,2}\,,$$

noting that by Theorem 3.11 $\{H_\ell\}_\ell$ is a basis of ${}_\varepsilon\mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega)$ and w.l.o.g. orthonormal in $\mathsf{L}^2_\varepsilon(\Omega)$. Then

$$H := \sum_{j=1,\dots,d_{1,2}} \tilde{\zeta}_j H_j \in {}_\varepsilon\mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega)$$

indeed satisfies

$$\langle H , \mathcal{B}_{1,\ell} \rangle_{\mathsf{L}^2_\varepsilon(\Omega)} = \underbrace{\langle H , \nabla w_\ell \rangle_{\mathsf{L}^2_\varepsilon(\Omega)}}_{=0} + \sum_{j=1,\dots,d_{1,2}} \tilde{\zeta}_j \langle H_j , H_\ell \rangle_{\mathsf{L}^2_\varepsilon(\Omega)} = \tilde{\zeta}_\ell , \qquad \ell = 1, \dots , d_{1,2} .$$

and we have solved the electro-static problem (3.1).

Theorem 3.15. *Let ε be $\kappa - \mathsf{C}^1-$decaying with $\kappa > 0$. For all $(G, f) \in \mathsf{L}^2(\Omega) \times \mathsf{L}^2(\Omega)$ with*

$$G \in {}_0\mathbb{D}_{\Gamma_1}(\Omega) := {}_0\mathsf{D}_{\Gamma_1}(\Omega) \cap \mathfrak{B}_2(\Omega)^\perp$$

and $\zeta \in \mathbb{C}^{d_{1,2}}$ there exists a unique static solution

$$E \in \mathsf{R}_{-1,\Gamma_1}(\Omega) \cap \varepsilon^{-1}\mathsf{D}_{-1,\Gamma_2}(\Omega)$$

of (3.1)*. In addition, the corresponding solution operator*

$$\begin{array}{rccc} \mathcal{L}_{\varepsilon,0} : & {}_0\mathbb{D}_{\Gamma_1}(\Omega) \times \mathsf{L}^2(\Omega) \times \mathbb{C}^{d_{1,2}} & \longrightarrow & \mathsf{R}_{-1,\Gamma_1}(\Omega) \cap \varepsilon^{-1}\mathsf{D}_{-1,\Gamma_2}(\Omega) \\ & (G, f, \zeta) & \longmapsto & E \end{array}$$

is continuous.

Proof. It remains to show that $\mathcal{L}_{\varepsilon,0}$ is bounded. But this is a direct consequence of Lemma 3.12, (*ii*). □

Swapping Γ_1 and Γ_2 resp. ε and μ we obtain a corresponding result for the magneto-static problem (3.2).

Theorem 3.16. *Let μ be $\kappa - \mathsf{C}^1-$decaying with $\kappa > 0$. For all $(F, g) \in \mathsf{L}^2(\Omega) \times \mathsf{L}^2(\Omega)$ with*

$$F \in {}_0\mathbb{D}_{\Gamma_2}(\Omega) := {}_0\mathsf{D}_{\Gamma_2}(\Omega) \cap \mathfrak{B}_1(\Omega)^\perp$$

and $\theta \in \mathbb{C}^{d_{2,1}}$ there exists a unique static solution

$$H \in \mathsf{R}_{-1,\Gamma_2}(\Omega) \cap \mu^{-1}\mathsf{D}_{-1,\Gamma_1}(\Omega)$$

of (3.2)*. In addition, the corresponding solution operator*

$$\begin{array}{rccc} \mathcal{L}_{\mu,0} : & {}_0\mathbb{D}_{\Gamma_2}(\Omega) \times \mathsf{L}^2(\Omega) \times \mathbb{C}^{d_{2,1}} & \longrightarrow & \mathsf{R}_{-1,\Gamma_2}(\Omega) \cap \mu^{-1}\mathsf{D}_{-1,\Gamma_1}(\Omega) \\ & (F, g, \theta) & \longmapsto & H \end{array}$$

is continuous.

Remark 3.17. *By Theorem 3.15 and Theorem 3.16 for all*

$$\big(F, g, G, f, \zeta, \theta\big) \in {}_0\mathbb{D}_{\Gamma_2}(\Omega) \times \mathsf{L}^2(\Omega) \times {}_0\mathbb{D}_{\Gamma_1}(\Omega) \times \mathsf{L}^2(\Omega) \times \mathbb{C}^{d_{1,2}} \times \mathbb{C}^{d_{2,1}}$$

the electro-magneto static system (3.1)*,* (3.2) *has a unique solution*

$$(E, H) \in \big(\mathsf{R}_{-1,\Gamma_1}(\Omega) \cap \varepsilon^{-1}\mathsf{D}_{-1,\Gamma_2}(\Omega)\big) \times \big(\mathsf{R}_{-1,\Gamma_2}(\Omega) \cap \mu^{-1}\mathsf{D}_{-1,\Gamma_1}(\Omega)\big) .$$

The corresponding solution operator is continuous and will be denoted by $\mathcal{L}_{\Lambda,0}$

4. The Time-Harmonic Problem $\omega \neq 0$

Having established the static solution theory we treat the time-harmonic case. For sake of brevity we just concentrate on the main results and refer to [17] for the details and some additional results. Let

$$\omega \in \mathbb{C}_+ := \{z \in \mathbb{C} \,:\, \operatorname{Im}(z) \geq 0\} \qquad \text{with} \qquad \omega \neq 0\,.$$

We are looking for an electro-magnetic field $(E,H) \in \mathsf{R}_{\mathrm{loc},\Gamma_1}(\Omega) \times \mathsf{R}_{\mathrm{loc},\Gamma_2}(\Omega)$ such that for given data $(F,G) \in \mathsf{L}^2_{\mathrm{loc}}(\overline{\Omega}) \times \mathsf{L}^2_{\mathrm{loc}}(\overline{\Omega})$ it holds

$$(\,\mathrm{M} + i\omega\Lambda\,)(E,H) = (F,G)\,.$$

By (2.2) the *"Maxwell-operator"*

$$\mathcal{M} : \mathbf{R}_{\Gamma_1}(\Omega) \times \mathbf{R}_{\Gamma_2}(\Omega) \subset \mathsf{L}^2_{\varepsilon}(\Omega) \times \mathsf{L}^2_{\mu}(\Omega) \;\longrightarrow\; \mathsf{L}^2_{\varepsilon}(\Omega) \times \mathsf{L}^2_{\mu}(\Omega)\,,\;\; (E,H) \;\longmapsto\; i\Lambda^{-1}\mathrm{M}(E,H)\,,$$

is self-adjoint which in the case of $\omega \in \mathbb{C} \setminus \mathbb{R}$ immediately yields an L^2-solution theory.

Theorem 4.1. *Let $\omega \in \mathbb{C}\setminus\mathbb{R}$. For every $(F,G) \in \mathsf{L}^2(\Omega)\times\mathsf{L}^2(\Omega)$ system* (1.1), (1.2) *has a unique solution*

$$(E,H) \in \mathbf{R}_{\Gamma_1}(\Omega) \times \mathbf{R}_{\Gamma_2}(\Omega)\,.$$

Moreover, the solution operator, which we denote by $\mathcal{L}_{\Lambda,\omega} := i(\,\mathcal{M} - \omega\,)^{-1}\Lambda^{-1}$ is continuous.

The case $\omega \in \mathbb{R} \setminus \{0\}$ is more challenging, since we want to solve in the continuous spectrum of $\mathcal{M}$. Clearly this cannot be done for every $(F,G) \in \mathsf{L}^2(\Omega) \times \mathsf{L}^2(\Omega)$, since otherwise $\omega \notin \sigma(\mathcal{M})$. Thus we have to work in certain subspaces of $\mathsf{L}^2(\Omega) \times \mathsf{L}^2(\Omega)$ and we have to generalize the solution concept.

Definition 4.2. *Let $\omega \in \mathbb{R}\setminus\{0\}$ and $(F,G) \in \mathsf{L}^2_{\mathrm{loc}}(\Omega)\times\mathsf{L}^2_{\mathrm{loc}}(\Omega)$. We call (E,H) "(radiating) solution" of the time-harmonic boundary value problem* (1.1), (1.2), *if*

$$(E,H) \in \mathbf{R}_{<-\frac{1}{2},\Gamma_1}(\Omega) \times \mathbf{R}_{<-\frac{1}{2},\Gamma_2}(\Omega)$$

and satisfies

$$\big(\,\mathrm{M} + i\omega\Lambda\,\big)(E,H) = (F,G)\,, \qquad \big(\,\Lambda_0 + \sqrt{\varepsilon_0\mu_0}\;\Xi\,\big)(E,H) \in \mathsf{L}^2_{>-\frac{1}{2}}(\Omega) \times \mathsf{L}^2_{>-\frac{1}{2}}(\Omega)\,, \tag{4.1}$$

where

$$\Lambda_0 := \begin{pmatrix} \varepsilon_0 & 0 \\ 0 & \mu_0 \end{pmatrix} \qquad \textit{and} \qquad \Xi := \begin{pmatrix} 0 & -\xi\times \\ \xi\times & 0 \end{pmatrix}\,.$$

Conveniently, we can apply the same methods as in [19], see also [30, 33, 34], to obtain a solution theory. In particular, we use the limiting absorption principle introduced by Eidus and approximate solutions to $\omega \in \mathbb{R}\setminus\{0\}$ by solutions corresponding to $\omega \in \mathbb{C}_+\setminus\mathbb{R}$. Again, Weck's local selection theorem is the crucial tool in the limit process. Additionally, the polynomial decay of eigenfunctions as well as an a-priori estimate for solutions corresponding to non-real frequencies are needed and both are obtained by reduction to similar results known for the Helmholtz equation in the whole of $\mathbb{R}^3$. For the details see [17].

Theorem 4.3 (Generalized Fredholm Alternative)**.** *Let $\omega \in \mathbb{R} \setminus \{0\}$ and let ε, μ be κ-decaying with $\kappa > 1$. Moreover, let*

$$\mathcal{N}_{\mathsf{gen}}(\,\mathcal{M} - \omega\,) := \big\{(E,H) \,:\, (E,H) \textit{ is a radiating solution of } (\,\mathrm{M} + i\omega\Lambda\,)(E,H) = 0\,\big\}\,,$$

$$\sigma_{\mathsf{gen}}(\mathcal{M}) := \big\{\,\omega \in \mathbb{C} \setminus \{0\} \,:\, \mathcal{N}_{\mathsf{gen}}(\,\mathcal{M} - \omega\,) \neq \{0\}\,\big\}\,.$$

Then:

(*i*) *For all $t \in \mathbb{R}$*

$$\mathcal{N}_{\mathsf{gen}}(\,\mathcal{M} - \omega\,) \subset \Big(\mathbf{R}_{t,\Gamma_1}(\Omega) \cap \varepsilon^{-1}\operatorname{rot}\mathbf{R}_{t,\Gamma_2}(\Omega)\Big) \times \Big(\mathbf{R}_{t,\Gamma_2}(\Omega) \cap \mu^{-1}\operatorname{rot}\mathbf{R}_{t,\Gamma_1}(\Omega)\Big)\,.$$

(*ii*) $\dim \mathcal{N}_{\mathsf{gen}}(\,\mathcal{M} - \omega\,) < \infty\,.$

(*iii*) $\sigma_{\mathsf{gen}}(\mathcal{M}) \subset \mathbb{R} \setminus \{0\}$ *and* $\sigma_{\mathsf{gen}}(\mathcal{M})$ *has no accumulation point in* $\mathbb{R} \setminus \{0\}\,.$

(*iv*) *For all* $(F,G) \in \mathsf{L}^2_{>\frac{1}{2}}(\Omega) \times \mathsf{L}^2_{>\frac{1}{2}}(\Omega)$ *there exists a radiating solution* (E,H) *of* (1.1), (1.2), *if and only if*

$$\forall\,(e,h) \in \mathcal{N}_{\mathsf{gen}}(\,\mathcal{M}-\omega\,): \qquad \langle\,(F,G)\,,(e,h)\,\rangle_{\mathsf{L}^2(\Omega)} = 0\,.$$

Moreover, the solution (E,H) *can be chosen, such that*

$$\forall\,(e,h) \in \mathcal{N}_{\mathsf{gen}}(\,\mathcal{M}-\omega\,): \qquad \langle\,(E,H)\,,(e,h)\,\rangle_{\mathsf{L}^2_\Lambda(\Omega)} = 0\,.$$

Then (E,H) *is uniquely determined.*

(*v*) *For all* $s,-t > 1/2$ *the solution operator*

$$\mathcal{L}_{\Lambda,\omega} : \Big(\,\mathsf{L}^2_s(\Omega) \times \mathsf{L}^2_s(\Omega)\,\Big) \cap \mathcal{N}_{\mathsf{gen}}(\,\mathcal{M}-\omega\,)^{\perp} \longrightarrow \Big(\,\mathbf{R}_{t,\Gamma_1}(\Omega) \times \mathbf{R}_{t,\Gamma_2}(\Omega)\,\Big) \cap \mathcal{N}_{\mathsf{gen}}(\,\mathcal{M}-\omega\,)^{\perp_\Lambda}$$

defined by (4) *is continuous. Here* $\perp_\Lambda$ *indicates the orthogonal complement in* $\mathsf{L}^2_\Lambda(\Omega)$.

Remark 4.4. *By Theorem 4.1 and Theorem 4.3 and for all*

$$(F,G) \in \Big(\,\mathsf{L}^2_{>\frac{1}{2}}(\Omega) \times \mathsf{L}^2_{>\frac{1}{2}}(\Omega)\,\Big) \cap \mathcal{N}_{\mathsf{gen}}(\,\mathcal{M}-\omega\,)^{\perp}$$

the time-harmonic Maxwell system (1.1), (1.2) *has a unique radiating solution*

$$(E,H) \in \mathbf{R}_{<-\frac{1}{2},\Gamma_1}(\Omega) \times \mathbf{R}_{<-\frac{1}{2},\Gamma_2}(\Omega) \quad \textit{with} \quad \big(\,\Lambda_0 + \sqrt{\varepsilon_0\mu_0}\;\Xi\,\big)(E,H) \in \mathsf{L}^2_{>-\frac{1}{2}}(\Omega) \times \mathsf{L}^2_{>-\frac{1}{2}}(\Omega)\,.$$

The corresponding solution operator is continuous and will be denoted by $\mathcal{L}_{\Lambda,\omega}$.

5. Low Frequency Asymptotics $\omega \to 0$

In order to discuss the low frequency asymptotics we first have to ensure that $\sigma_{\mathsf{gen}}(\mathcal{M})$ does not accumulate at zero. For that we show an estimate emerging from a representation formula for the homogeneous, isotropic whole space problem, i.e., $\Omega = \mathbb{R}^3$ and $\Lambda = \Lambda_0$.

Proposition 5.1. $\Omega = \mathbb{R}^3$, $\Lambda = \Lambda_0$ *and* $\omega \in \mathbb{C}_+ \setminus \{0\}$, *it holds*

$$\mathcal{N}_{\mathsf{gen}}(\,\mathcal{M}-\omega\,) = \{0\}\,.$$

Thus the solution operator $\mathcal{L}_{\Lambda_0,\omega}$ *is well defined for all* $(F,G) \in \mathsf{L}^2_{>\frac{1}{2}} \times \mathsf{L}^2_{>\frac{1}{2}}$.

Proof. Let $(E,H) \in \mathcal{N}_{\mathsf{gen}}(\,\mathcal{M}-\omega\,)$. By Theorem 4.3 (i) and the differential equation we have

$$(E,H) \in (\,\mathbf{R} \cap {}_0\mathsf{D}\,) \times (\,\mathbf{R} \cap {}_0\mathsf{D}\,) \qquad \text{with} \qquad \mathrm{M}\,(E,H) = -i\omega\Lambda_0(E,H)\,.$$

Hence, by [9, Lemma 4.2], $(E,H) \in (\,\mathbf{H}^k \cap {}_0\mathsf{D}\,) \times (\,\mathbf{H}^k \cap {}_0\mathsf{D}\,)$ for all $k \in \mathbb{N}_0$ and we obtain

$$\Delta\,(E,H) = \mathrm{M}^2(E,H) = -\omega^2\varepsilon_0\mu_0\,(E,H)\,.$$

In other words, $(E,H) \in \mathbf{H}^2 \times \mathbf{H}^2$ satisfies the Helmholtz-equation with right hand side zero. If $\omega \in \mathbb{C} \setminus \mathbb{R}$ we are done, since $\Delta : \mathbf{H}^2 \subset \mathsf{L}^2 \longrightarrow \mathsf{L}^2$ is selfadjoint and therefore $\sigma(\Delta) \subset \mathbb{R}$, yielding $(E,H) = (0,0)$. For $\omega \in \mathbb{R} \setminus \{0\}$ the assertion follows using the Rellich estimate (cf. [12], p. 59) and the unique continuation principle. $\square$

Now, let $\Omega = \mathbb{R}^3$, $\Lambda = \Lambda_0$, $\omega \in \mathbb{C}_+ \setminus \{0\}$, $(F,G) \in \mathring{\mathsf{C}}^\infty \times \mathring{\mathsf{C}}^\infty$, and let $(E,H) := \mathcal{L}_{\Lambda_0,\omega}(F,G)$ be the corresponding radiating solution. Again, by [9, Lemma 4.2] and the differential equation, we obtain

$$(E,H) \in \big(\,\mathbf{H}^2_{<-\frac{1}{2}} \cap \mathsf{C}^\infty\,\big) \times \big(\,\mathbf{H}^2_{<-\frac{1}{2}} \cap \mathsf{C}^\infty\,\big)$$

and

$$\big(\,\Delta + \varepsilon_0\mu_0\,\omega^2\,\big)\,(E,H) = (\,\mathrm{M} - i\omega\widetilde{\Lambda}_0\,)\,(F,G) - \frac{i}{\omega}\Lambda_0^{-1}(\nabla\operatorname{div}F, \nabla\operatorname{div}G) =: (\hat{F},\hat{G}) \in \mathring{\mathsf{C}}^\infty \times \mathring{\mathsf{C}}^\infty\,. \tag{5.1}$$

In fact, (E,H) is the unique radiating solution of the whole space problem (cf. [34, Section 4])

$$(E,H) \in \mathsf{H}^2_{<-\frac{1}{2}} \times \mathsf{H}^2_{<-\frac{1}{2}} ,$$
$$(\,\Delta + \omega^2 \varepsilon_0 \mu_0\,)\,(E,H) = (\hat{F},\hat{G}) ,$$
$$\exp\big(-i\omega\sqrt{\varepsilon_0\mu_0}\,r\big)\,(E,H) \in \mathsf{H}^1_{>-\frac{3}{2}} \times \mathsf{H}^1_{>-\frac{3}{2}} .$$

For non-real frequencies $\omega \in \mathbb{C}_+ \setminus \mathbb{R}$ this is trivial, because then [9, Lemma 4.2] yields $(E,H) \in \mathsf{H}^2 \times \mathsf{H}^2$ and the Laplacian is self-adjoint on $\mathsf{H}^2 \times \mathsf{H}^2$. For $\omega \in \mathbb{R} \setminus \{0\}$ the radiation condition (4.1) shows

$$(\xi \cdot E, \xi \cdot H) \in \mathsf{L}^2_{>-\frac{1}{2}} \times \mathsf{L}^2_{>-\frac{1}{2}}$$

and via the differential equation and the radiation condition we obtain

$$\operatorname{rot}\big(\exp\big(-i\omega\sqrt{\varepsilon_0\mu_0}\,r\big)\,E\big) = \exp\big(-i\omega\sqrt{\varepsilon_0\mu_0}\,r\big)\Big(F - i\omega\big(\,\varepsilon_0 E - \sqrt{\varepsilon_0\mu_0}\;\xi \times H\,\big)\Big) \in \mathsf{L}^2_{>-\frac{1}{2}} ,$$
$$\operatorname{div}\big(\exp\big(-i\omega\sqrt{\varepsilon_0\mu_0}\,r\big)\,E\big) = -i\exp\big(-i\omega\sqrt{\varepsilon_0\mu_0}\,r\big)\Big(\omega\sqrt{\varepsilon_0\mu_0}\;\xi \cdot E + (\omega\varepsilon_0)^{-1}\operatorname{div} F\Big) \in \mathsf{L}^2_{>-\frac{1}{2}} .$$

Analogously, we see the corresponding results for H. Hence, by [9, Lemma 4.2],

$$\exp\big(-i\omega\sqrt{\varepsilon_0\mu_0}\,r\big)\,(E,H) \in \mathsf{H}^1_{>-\frac{3}{2}} \times \mathsf{H}^1_{>-\frac{3}{2}} .$$

Thus, by [11, Theorem 4.27, Remark 4.28] we may describe (E,H) using the representation formula of the Helmholtz-equation, i.e.,

$$E = \phi_\omega \star \hat{F} := (\,\phi_\omega * \hat{F}_\ell\,)_{\ell=1,2,3} , \qquad\qquad H = \phi_\omega \star \hat{G} := (\,\phi_\omega * \hat{G}_\ell\,)_{\ell=1,2,3} ,$$

where $\phi_\omega = -(4\pi r)^{-1}\exp\big(-i\omega\sqrt{\varepsilon_0\mu_0}\,r\big)$ is the fundamental solution of the scalar Helmholtz-equation and $*$ denotes scalar convolution in $\mathbb{R}^3$. Then (5.1) yields

$$E = \phi_\omega \star \Big(-\operatorname{rot} G - i\omega\mu_0\,F - \frac{i}{\omega\varepsilon_0}\nabla\operatorname{div} F\Big), \qquad H = \phi_\omega \star \Big(\operatorname{rot} F - i\omega\varepsilon_0\,G - \frac{i}{\omega\mu_0}\nabla\operatorname{div} G\Big),$$

a representation formula for (E,H) provided $(F,G) \in \mathring{\mathsf{C}}^\infty \times \mathring{\mathsf{C}}^\infty$. Next we would like to allow more general right hand sides (F,G). For that we move some of the differential operators from F resp. G to ϕ_ω, illustrating the procedure for $\phi_\omega \star \operatorname{rot} F$ and $\phi_\omega \star \nabla\operatorname{div} F$.

As both fields F and G are compactly supported we do not have to worry about integrability of ϕ_ω at infinity. In U_1 we can estimate $|\phi_\omega| \leq c \cdot r^{-1}$ and $|\nabla\phi_\omega| \leq c \cdot r^{-2}$, hence $\phi_\omega, \nabla\phi_\omega \in \mathsf{L}^1(\mathrm{U}_1)$. Moreover, with $\tilde{\eta}$ (the cut-off function from above) we define for $n \in \mathbb{N}$ and fixed $x \in \mathbb{R}^3$ the functions

$$\eta_n(y) := \tilde{\eta}(n \cdot |x-y|).$$

Then $|\nabla\eta_n| \leq c \cdot |x-y|^{-1}$ holds uniformly in n, such that

$$|\,\eta_n \cdot \tau_x\phi_\omega\,| \leq c \cdot |x-y|^{-1} , \qquad |\,\partial_j\eta_n \cdot \tau_x\phi_\omega\,| \leq c \cdot |x-y|^{-2} , \qquad |\,\eta_n \cdot \partial_j(\tau_x\phi_\omega)\,| \leq c \cdot |x-y|^{-2} ,$$

where $\tau_x\phi_\omega(y) := \phi_\omega(x-y)$. Lebesgue's dominated convergence theorem shows

$$\big(\phi_\omega * \partial_j F_k\big)(x) = \lim_{n\to\infty}\langle\,\tau_x\phi_\omega\,,\overline{\partial_j(\eta_n F_k)}\,\rangle_{\mathsf{L}^2} = \lim_{n\to\infty}\langle\,\tau_x\partial_j\phi_\omega\,,\overline{\eta_n F_k}\,\rangle_{\mathsf{L}^2} = \big(\partial_j\phi_\omega * F_k\big)(x) ,$$

which yields $\phi_\omega \star \nabla\operatorname{div} F = \operatorname{div} F \star \nabla\phi_\omega$ and

$$-\phi_\omega \star \operatorname{rot} F = \begin{pmatrix} \phi_\omega * \partial_3 F_2 - \phi_\omega * \partial_2 F_3 \\ \phi_\omega * \partial_1 F_3 - \phi_\omega * \partial_3 F_1 \\ \phi_\omega * \partial_2 F_1 - \phi_\omega * \partial_1 F_2 \end{pmatrix} = \begin{pmatrix} F_2 * \partial_3\phi_\omega - F_3 * \partial_2\phi_\omega \\ F_3 * \partial_1\phi_\omega - F_1 * \partial_3\phi_\omega \\ F_1 * \partial_2\phi_\omega - F_2 * \partial_1\phi_\omega \end{pmatrix} =: F \circledast \nabla\phi_\omega .$$

Theorem 5.2. *Let $0 \neq \omega \in \mathrm{K} \Subset \mathbb{C}_+$ and $\varepsilon_0, \mu_0 \in \mathbb{R}_+$. Furthermore, let $1/2 < s < 3/2$, $t := s-2$, and $(F,G) \in \mathbf{D}_s \times \mathbf{D}_s$. Then for $(E,H) := \mathcal{L}_{\Lambda_0,\omega}(F,G)$ the representation formulas*

$$E = G \circledast \nabla\phi_\omega \;-\; i\omega\mu_0\,\phi_\omega \star F - \frac{i}{\omega\varepsilon_0}\operatorname{div} F \star \nabla\phi_\omega, \tag{5.2}$$

$$H = -F \circledast \nabla\phi_\omega \;-\; i\omega\varepsilon_0\,\phi_\omega \star G - \frac{i}{\omega\mu_0}\operatorname{div} G \star \nabla\phi_\omega \tag{5.3}$$

hold in the sense of L^2_t. *Moreover, there exist* $c>0$, *such that for all* $\omega\in\mathrm{K}\setminus\{0\}$ *and all* $(F,G)\in\mathbf{D}_s\times\mathbf{D}_s$

$$\|(E,H)\|_{\mathbf{R}_t}\leq c\Big(\|(F,G)\|_{\mathsf{L}^2_s}+\frac{1}{\omega}\|(\operatorname{div}F,\operatorname{div}G)\|_{\mathsf{L}^2_s}\Big).$$

Proof. Since $\mathring{\mathsf{C}}^\infty\subset\mathbf{D}_s$ is dense, we choose a sequence $\big((F_n,G_n)\big)_{n\in\mathbb{N}}\subset\mathring{\mathsf{C}}^\infty\times\mathring{\mathsf{C}}^\infty$ converging to (F,G) and define $(E_n,H_n)=\mathcal{L}_{\Lambda_0,\omega}(F_n,G_n)\in\mathsf{L}^2_t\times\mathsf{L}^2_t$. Then Remark 4.4 yields convergence of $\big((E_n,H_n)\big)_{n\in\mathbb{N}}$ to $(E,H)\in\mathbf{R}_t\times\mathbf{R}_t$ and as shown above, we may represent (E_n,H_n) by

$$E_n=G_n\circledast\nabla\phi_\omega\ -\ i\omega\mu_0\,\phi_\omega\star F_n-\frac{i}{\omega\varepsilon_0}\operatorname{div}F_n\star\nabla\phi_\omega,\tag{5.4}$$

$$H_n=-F_n\circledast\nabla\phi_\omega\ -\ i\omega\varepsilon_0\,\phi_\omega\star G_n-\frac{i}{\omega\mu_0}\operatorname{div}G_n\star\nabla\phi_\omega\,.\tag{5.5}$$

The involved convolution kernels essentially consist of ϕ_ω and $\partial_j\phi_\omega$, which can be estimated by

$$|\phi_\omega|\,,\,|\partial_j\phi_\omega|\leq c\cdot\Big(|x-y|^{-1}+|x-y|^{-2}\Big),\qquad(\,j=1,2,3\,).$$

Moreover, from [13, Lemma 1] we obtain that integral operators with kernels of the form $|x-y|^{\alpha-\beta-3}$ map L^2_α continuously to L^2_β, if $-3/2<\alpha<\beta<3/2$. Hence, by choosing

$$-3/2<t=s-2<\tilde{t}:=s-1<s<3/2\,,$$

we have

$$|x-y|^{-1}=|x-y|^{s-t-3}\qquad\text{resp.}\qquad|x-y|^{-2}=|x-y|^{s-\tilde{t}-3}\,,$$

and the right hand sides of (5.4) and (5.5) define bounded linear operators from L^2_s to L^2_t. Passing to the limit $n\longrightarrow\infty$ in (5.4),(5.5) we obtain the asserted representation formulas. By the continuity of the convolution operators we have the estimate

$$\|(E,H)\|_{\mathsf{L}^2_t}\leq c\Big(\|(F,G)\|_{\mathsf{L}^2_s}+|\omega|^{-1}\|(\operatorname{div}F,\operatorname{div}G)\|_{\mathsf{L}^2_s}\Big)$$

which holds uniformly in ω. Finally the differential equation yields the asserted estimate. □

A similar estimate also holds for radiating solutions in exterior weak Lipschitz domains.

Corollary 5.3. *Let* $1/2<s<3/2$, $t:=s-2$, *and let* ε,μ *be* $\kappa-\mathsf{C}^1-$*decaying with order* $\kappa>2$, *as well as let* $\mathrm{K}\Subset\mathbb{C}_+$. *Then there exist* $c,\delta>0$ *such that for all* $0\neq\omega\in\mathrm{K}$ *and*

$$(F,G)\in\big(\mathbf{D}_s(\Omega)\times\mathbf{D}_s(\Omega)\big)\cap\mathcal{N}_{\mathsf{gen}}(\mathcal{M}-\omega)^\perp$$

it holds

$$\|\mathcal{L}_{\Lambda,\omega}(F,G)\|_{\mathsf{L}^2_t(\Omega)}\leq c\Big(\|(F,G)\|_{\mathsf{L}^2_s(\Omega)}+\frac{1}{|\omega|}\|(\operatorname{div}F,\operatorname{div}G)\|_{\mathsf{L}^2_s(\Omega)}+\|\mathcal{L}_{\Lambda,\omega}(F,G)\|_{\mathsf{L}^2(\Omega_\delta)}\Big).$$

Moreover, by the differential equation the $\|\cdot\|_{\mathsf{L}^2_t(\Omega)}-$*norm on the left hand side can be replaced by* $\|\cdot\|_{\mathbf{R}_t(\Omega)}$.

Proof. Let $(E,H)=\mathcal{L}_{\Lambda,\omega}(F,G)$ (which exists by Remark 4.4) and $\tilde{r}>\hat{r}$ such that $\varepsilon,\mu\in\mathsf{C}^1(\check{\mathrm{U}}_{\tilde{r}})$. Then

$$(\tilde{E},\tilde{H}):=\eta_{\tilde{r}}(E,H)\in\mathbf{R}_{<-\frac{1}{2}}\times\mathbf{R}_{<-\frac{1}{2}}\,,$$

and as $(\,\mathrm{M}+i\omega\Lambda\,)(E,H)=(F,G)$ it holds

$$(\operatorname{div}\varepsilon E,\operatorname{div}\mu H)=-\frac{i}{\omega}(\operatorname{div}F,\operatorname{div}G)\in\mathsf{L}^2_s(\Omega)\times\mathsf{L}^2_s(\Omega)\,,\tag{5.6}$$

such that by Lemma 3.6 we even have

$$(E,H)\in\mathbf{H}^1_{<-\frac{1}{2}}(\operatorname{supp}\eta_{\tilde{r}})\times\mathbf{H}^1_{<-\frac{1}{2}}(\operatorname{supp}\eta_{\tilde{r}})\,,\qquad\text{especially}\qquad(\tilde{E},\tilde{H})\in\mathbf{H}^1_{<-\frac{1}{2}}\times\mathbf{H}^1_{<-\frac{1}{2}}\,.$$

Moreover, $(\tilde{E},\tilde{H})$ satisfies the radiation condition

$$\big(\,\Lambda_0+\sqrt{\varepsilon_0\mu_0}\,\Xi\,\big)(\tilde{E},\tilde{H})\in\mathsf{L}^2_{>-\frac{1}{2}}(\Omega)\times\mathsf{L}^2_{>-\frac{1}{2}}(\Omega)$$

and (as $\kappa > 2 \geq s + 1/2$) solves

$$\big(\mathrm{M} + i\omega\Lambda_0\big)\,(\tilde{E},\tilde{H}) = \mathrm{C}_{\mathrm{M},\eta_{\tilde{r}}}(E,H) - i\omega\big(\Lambda - \Lambda_0\big)(\tilde{E},\tilde{H}) + \eta_{\tilde{r}}(F,G) =: (\tilde{F},\tilde{G}) \in \mathbf{D}_s \times \mathbf{D}_s\,. \tag{5.7}$$

We obtain $(\tilde{E},\tilde{H}) = \mathcal{L}_{\Lambda_0,\omega}(\tilde{F},\tilde{G})$ and by Theorem 5.2 there exists $c > 0$ such that

$$\big\|\,(\tilde{E},\tilde{H})\,\big\|_{\mathsf{L}^2_t} \leq c\left(\big\|\,(\tilde{F},\tilde{G})\,\big\|_{\mathsf{L}^2_s} + \frac{1}{\omega}\big\|\,(\operatorname{div}\tilde{F},\operatorname{div}\tilde{G})\,\big\|_{\mathsf{L}^2_s}\right) \tag{5.8}$$

independent of ω, $(\tilde{F},\tilde{G})$ or $(\tilde{E},\tilde{H})$. Furthermore, (5.6) and the differential equation (5.7) show

$$\operatorname{div} F = i\omega\operatorname{div}\varepsilon E\,, \qquad \operatorname{div} G = i\omega\operatorname{div}\mu H\,, \qquad \text{in } \Omega\,, \tag{5.9}$$

$$\operatorname{div}\tilde{F} = i\omega\varepsilon_0\operatorname{div}\tilde{E}\,, \qquad \operatorname{div}\tilde{G} = i\omega\mu_0\operatorname{div}\tilde{H}\,, \qquad \text{in } \mathbb{R}^3\,, \tag{5.10}$$

such that combining (5.8) and (5.10) it holds

$$\begin{aligned}
\big\|\,(E,H)\,\big\|_{\mathsf{L}^2_t(\Omega)} &\leq c\left(\big\|\,(E,H)\,\big\|_{\mathsf{L}^2(\Omega_{2\tilde{r}})} + \big\|\,(\tilde{E},\tilde{H})\,\big\|_{\mathsf{L}^2_t(\check{\mathrm{U}}_{\tilde{r}})}\right)\\
&\leq c\left(\big\|\,(E,H)\,\big\|_{\mathsf{L}^2(\Omega_{2\tilde{r}})} + \big\|\,(\tilde{F},\tilde{G})\,\big\|_{\mathsf{L}^2_s} + \frac{1}{|\omega|}\big\|\,(\operatorname{div}\tilde{F},\operatorname{div}\tilde{G})\,\big\|_{\mathsf{L}^2_s}\right)\\
&\leq c\left(\big\|\,(E,H)\,\big\|_{\mathsf{L}^2_{s-\kappa}(\Omega)} + \big\|\,(F,G)\,\big\|_{\mathsf{L}^2_s(\Omega)} + \big\|\,(\operatorname{div}\varepsilon_0\tilde{E},\operatorname{div}\mu_0\tilde{H})\,\big\|_{\mathsf{L}^2_s}\right).
\end{aligned}$$

With (5.9) the last term on the right hand side can be estimated by

$$\begin{aligned}
&\big\|\,(\operatorname{div}\varepsilon_0\tilde{E},\operatorname{div}\mu_0\tilde{H})\,\big\|_{\mathsf{L}^2_s}\\
&\quad\leq c\Big(\,\|\,(E,H)\,\|_{\mathsf{L}^2(\Omega_{2\tilde{r}})} + \|\,(\operatorname{div}\varepsilon_0 E,\operatorname{div}\mu_0 H)\,\|_{\mathsf{L}^2_s(\operatorname{supp}\eta_{\tilde{r}})}\,\Big)\\
&\quad\leq c\Big(\,\|\,(E,H)\,\|_{\mathsf{L}^2(\Omega_{2\tilde{r}})} + \|\,(\operatorname{div}\varepsilon E,\operatorname{div}\mu H)\,\|_{\mathsf{L}^2_s(\operatorname{supp}\eta_{\tilde{r}})} + \|\,(\operatorname{div}\hat{\varepsilon}E,\operatorname{div}\hat{\mu}H)\,\|_{\mathsf{L}^2_s(\operatorname{supp}\eta_{\tilde{r}})}\,\Big)\\
&\quad\leq c\Big(\,\|\,(E,H)\,\|_{\mathsf{L}^2(\Omega_{2\tilde{r}})} + \frac{1}{|\omega|}\,\|\,(\operatorname{div}F,\operatorname{div}G)\,\|_{\mathsf{L}^2_s(\Omega)} + \|\,(E,H)\,\|_{\mathsf{H}^1_{s-\kappa-1}(\operatorname{supp}\eta_{\tilde{r}})}\,\Big)\,.
\end{aligned}$$

We end up with

$$\begin{aligned}
\big\|\,(E,H)\,\big\|_{\mathsf{L}^2_t(\Omega)} \leq c\Big(\,&\|\,(E,H)\,\|_{\mathsf{L}^2_{s-\kappa}(\Omega)} + \|\,(E,H)\,\|_{\mathsf{H}^1_{s-\kappa-1}(\operatorname{supp}\eta_{\tilde{r}})}\\
&+ \|\,(F,G)\,\|_{\mathsf{L}^2_s(\Omega)} + \frac{1}{|\omega|}\,\|\,(\operatorname{div}F,\operatorname{div}G)\,\|_{\mathsf{L}^2_s(\Omega)}\,\Big)
\end{aligned}$$

and the estimate from Lemma 3.6 as well as the differential equation together with (5.9) yield

$$\|\,(E,H)\,\|_{\mathsf{L}^2_t(\Omega)} \leq c\Big(\,\|\,(E,H)\,\|_{\mathsf{L}^2_{s-\kappa}(\Omega)} + \|\,(F,G)\,\|_{\mathsf{L}^2_s(\Omega)} + \frac{1}{|\omega|}\,\|\,(\operatorname{div}F,\operatorname{div}G)\,\|_{\mathsf{L}^2_s(\Omega)}\,\Big)\,.$$

Finally, as $\kappa > 2$ the assertion follows by

$$\|\,(E,H)\,\|^2_{\mathsf{L}^2_{s-\kappa}(\Omega)} \leq \|\,(E,H)\,\|^2_{\mathsf{L}^2(\Omega_\delta)} + \big(1+\delta^2\big)^{2-\kappa}\cdot\|\,(E,H)\,\|^2_{\mathsf{L}^2_t(\Omega)}\,,$$

choosing $\delta > \hat{r}$ big enough. □

Theorem 5.4. *Let $1/2 < s < 3/2$, $t := s-2$, and let ε,μ be $\kappa - \mathsf{C}^1-$decaying with order $\kappa > 2$, and let*

$$\mathfrak{B}_1(\Omega) = \{\mathcal{B}_{1,1},\dots,\mathcal{B}_{1,d_{1,2}}\} \subset \mathbf{R}_{\Gamma_1}(\Omega) \qquad \textit{resp.} \qquad \mathfrak{B}_2(\Omega) = \{\mathcal{B}_{2,1},\dots,\mathcal{B}_{2,d_{2,1}}\} \subset \mathbf{R}_{\Gamma_2}(\Omega)$$

be the sets from Theorem 3.11. Then:

(i) *$\sigma_{\mathsf{gen}}(\mathcal{M})$ has no accumulation point at zero. In particular, there exists some $\tilde{\omega} > 0$ such that*

$$\sigma_{\mathsf{gen}}(\mathcal{M}) \cap \mathbb{C}_{+,\tilde{\omega}} = \emptyset \qquad \textit{with} \qquad \mathbb{C}_{+,\tilde{\omega}} := \big\{\,\omega \in \mathbb{C}_+ \;:\; |\omega| \leq \tilde{\omega}\,\big\}\,.$$

(ii) *$\mathcal{L}_{\Lambda,\omega}$ is well defined on the whole of $\mathsf{L}^2_{>\frac{1}{2}}(\Omega) \times \mathsf{L}^2_{>\frac{1}{2}}(\Omega)$ for all $\omega \in \mathbb{C}_{+,\tilde{\omega}} \setminus \{0\}$.*

(iii) There exists a constant $c > 0$ such that

$$\|\,\mathcal{L}_{\Lambda,\omega}(F,G)\,\|_{\mathsf{L}^2_t(\Omega)} \leq c\Big(\,\|\,(F,G)\,\|_{\mathsf{L}^2_s(\Omega)} + \frac{1}{|\omega|}\,\|\,(\operatorname{div} F, \operatorname{div} G)\,\|_{\mathsf{L}^2_s(\Omega)} + \frac{1}{|\omega|}\sum_{\ell=1,\dots,d_{1,2}} |\langle\, F\,, \mathcal{B}_{1,\ell}\,\rangle_{\mathsf{L}^2(\Omega)}| + \frac{1}{|\omega|}\sum_{\ell=1,\dots,d_{2,1}} |\langle\, G\,, \mathcal{B}_{2,\ell}\,\rangle_{\mathsf{L}^2(\Omega)}|\Big)$$

holds for all $\omega \in \mathbb{C}_{+,\tilde{\omega}} \setminus \{0\}$ and $(F,G) \in \mathbf{D}_s(\Omega) \times \mathbf{D}_s(\Omega)$. Using the differential equation, the $\|\cdot\|_{\mathsf{L}^2_t(\Omega)}-$norm on the left hand side may be replaced by the natural norm in

$$\big(\,\mathbf{R}_{t,\Gamma_1}(\Omega) \cap \varepsilon^{-1}\mathbf{D}_{t,\Gamma_2}(\Omega)\,\big) \times \big(\,\mathbf{R}_{t,\Gamma_2}(\Omega) \cap \mu^{-1}\mathbf{D}_{t,\Gamma_1}(\Omega)\,\big)\,.$$

Proof. Assuming that zero is an accumulation point of $\sigma_{\mathsf{gen}}(\mathcal{M})$ there exist a sequence $(\omega_n)_{n\in\mathbb{N}} \subset \mathbb{R}\setminus\{0\}$ (cf. Theorem 4.3 (iii)) tending to zero and a sequence $\big((E_n,H_n)\big)_{n\in\mathbb{N}}$ with $(E_n,H_n) \in \mathcal{N}_{\mathsf{gen}}(\mathcal{M}-\omega_n)$ and

$$\|\,(E_n,H_n)\,\|_{\mathsf{L}^2_t(\Omega)} = 1 \qquad \text{for some} \qquad -3/2 < t < -1/2\,.$$

Using the differential equation we obtain $(E_n,H_n) \in \big(\,\mathbf{R}_{t,\Gamma_1}(\Omega)\cap\varepsilon^{-1}{}_0\mathsf{D}_{t,\Gamma_2}(\Omega)\,\big)\times\big(\,\mathbf{R}_{t,\Gamma_2}(\Omega)\cap\mu^{-1}{}_0\mathsf{D}_{t,\Gamma_1}(\Omega)\,\big)$ with

$$\|\,(\operatorname{rot} E_n, \operatorname{rot} H_n)\,\|_{\mathsf{L}^2_t(\Omega)} = |\omega_n|\cdot\|\,(E_n,H_n)\,\|_{\mathsf{L}^2_t(\Omega)} \xrightarrow{n\to\infty} 0\,.$$

Consequently $\big((E_n,H_n)\big)_{n\in\mathbb{N}}$ is bounded in

$$\big(\,\mathbf{R}_{t,\Gamma_1}(\Omega) \cap \varepsilon^{-1}\mathbf{D}_{t,\Gamma_2}(\Omega)\,\big) \times \big(\,\mathbf{R}_{t,\Gamma_2}(\Omega) \cap \mu^{-1}\mathbf{D}_{t,\Gamma_1}(\Omega)\,\big)\,.$$

Thus Weck's local selection theorem yields a subsequence $\big((E_{\pi(n)},H_{\pi(n)})\big)_{n\in\mathbb{N}}$ converging in $\mathsf{L}^2_{\tilde{t}}(\Omega)\times\mathsf{L}^2_{\tilde{t}}(\Omega)$ for all $\tilde{t} < t$. In particular, as $t > -3/2$ we may assume $t > \tilde{t} \geq -3/2$. Then $\big((E_{\pi(n)},H_{\pi(n)})\big)_{n\in\mathbb{N}}$ converges in $\big(\,\mathbf{R}_{\tilde{t},\Gamma_1}(\Omega)\cap\varepsilon^{-1}{}_0\mathsf{D}_{\tilde{t},\Gamma_2}(\Omega)\,\big)\times\big(\,\mathbf{R}_{\tilde{t},\Gamma_2}(\Omega)\cap\mu^{-1}{}_0\mathsf{D}_{\tilde{t},\Gamma_1}(\Omega)\,\big)$ to some

$$(E,H) \in {}_\varepsilon\mathcal{H}_{\tilde{t},\Gamma_1,\Gamma_2}(\Omega) \times {}_\mu\mathcal{H}_{\tilde{t},\Gamma_2,\Gamma_1}(\Omega) \overset{(3.13)}{=} {}_\varepsilon\mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega) \times {}_\mu\mathcal{H}_{\Gamma_2,\Gamma_1}(\Omega)\,.$$

In addition, the differential equation together with (3.15) yields

$$(E_{\pi(n)},H_{\pi(n)}) \in \mathfrak{B}_1(\Omega)^{\perp_\varepsilon} \times \mathfrak{B}_2(\Omega)^{\perp_\mu} \quad\Longrightarrow\quad (E,H) \in \mathfrak{B}_1(\Omega)^{\perp_\varepsilon} \times \mathfrak{B}_2(\Omega)^{\perp_\mu}\,.$$

Therefore by Theorem 3.11

$$(E,H) \in \big(\,{}_\varepsilon\mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega) \cap \mathfrak{B}_1(\Omega)^{\perp_\varepsilon}\,\big) \times \big(\,{}_\mu\mathcal{H}_{\Gamma_2,\Gamma_1}(\Omega) \cap \mathfrak{B}_2(\Omega)^{\perp_\mu}\,\big) = \{0\}\times\{0\}\,.$$

Finally Corollary 5.3 yields constants $c, \delta > 0$ independent of n such that

$$1 = \big\|\,(E_{\pi(n)},H_{\pi(n)})\,\big\|_{\mathsf{L}^2_t(\Omega)} \leq c\cdot\big\|\,(E_{\pi(n)},H_{\pi(n)})\,\big\|_{\mathsf{L}^2(\Omega_\delta)} \xrightarrow{n\to\infty} 0\,,$$

a contradiction which proves (i) resp. (ii). In order to prove (iii), we assume that the asserted estimate is wrong. Then we obtain sequences $(\omega_n)_{n\in\mathbb{N}} \subset \mathbb{C}_{+,\tilde{\omega}}\setminus\{0\}$ tending to zero and

$$\big((F_n,G_n)\big)_{n\in\mathbb{N}} \subset \mathbf{D}_s(\Omega)\times\mathbf{D}_s(\Omega) \qquad \text{with} \qquad \|\,\mathcal{L}_{\Lambda,\omega_n}(F_n,G_n)\,\|_{\mathsf{L}^2_t(\Omega)} = 1$$

such that

$$\|\,(F_n,G_n)\,\|_{\mathsf{L}^2_s(\Omega)} \xrightarrow{n\longrightarrow\infty} 0\,, \qquad |\omega_n|^{-1}\cdot\|\,(\operatorname{div} F_n, \operatorname{div} G_n)\,\|_{\mathsf{L}^2_s(\Omega)} \xrightarrow{n\longrightarrow\infty} 0\,,$$

and

$$|\omega_n|^{-1}\cdot|\langle\, F_n\,, \mathcal{B}_{1,\ell}\,\rangle_{\mathsf{L}^2(\Omega)}| \xrightarrow{n\longrightarrow\infty} 0\,, \qquad \ell = 1,\dots,d_{1,2}\,, \tag{5.11}$$

$$|\omega_n|^{-1}\cdot|\langle\, G_n\,, \mathcal{B}_{2,\ell}\,\rangle_{\mathsf{L}^2(\Omega)}| \xrightarrow{n\longrightarrow\infty} 0\,, \qquad \ell = 1,\dots,d_{2,1}\,. \tag{5.12}$$

As above, the differential equation shows $\big((E_n,H_n)\big)_{n\in\mathbb{N}}$ with $(E_n,H_n) := \mathcal{L}_{\Lambda,\omega_n}(F_n,G_n)$ is bounded in

$$\big(\,\mathbf{R}_{t,\Gamma_1}(\Omega) \cap \varepsilon^{-1}\mathbf{D}_{t,\Gamma_2}(\Omega)\,\big) \times \big(\,\mathbf{R}_{t,\Gamma_2}(\Omega) \cap \mu^{-1}\mathbf{D}_{t,\Gamma_1}(\Omega)\,\big)$$

and again Weck's local selection theorem provides a subsequence $\big((E_{\pi(n)},H_{\pi(n)})\big)_{n\in\mathbb{N}}$ converging in

$$\big(\,\mathsf{R}_{\tilde{t}}(\Omega)\cap\varepsilon^{-1}\mathsf{D}_{\tilde{t},\Gamma_2}(\Omega)\,\big)\times\big(\,\mathsf{R}_{\tilde{t},\Gamma_2}(\Omega)\cap\mu^{-1}\mathsf{D}_{\tilde{t},\Gamma_1}(\Omega)\,\big)$$

for all $-3/2\leq\tilde{t}<t$. We obtain

$$(E,H):=\lim_{n\to\infty}(E_{\pi(n)},H_{\pi(n)})\in{}_{\varepsilon}\mathcal{H}_{\tilde{t},\Gamma_1,\Gamma_2}(\Omega)\times{}_{\mu}\mathcal{H}_{\tilde{t},\Gamma_2,\Gamma_1}(\Omega)\overset{(3.13)}{=}{}_{\varepsilon}\mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega)\times{}_{\mu}\mathcal{H}_{\Gamma_2,\Gamma_1}(\Omega)\,.$$

Moreover, by (5.11) we compute for $\ell=1,\dots,d_{1,2}$

$$\begin{aligned}0\xleftarrow{n\longrightarrow\infty}\;&|\omega_n|^{-1}\cdot|\langle\,F_n\,,\mathcal{B}_{1,\ell}\,\rangle_{\mathsf{L}^2(\Omega)}|\\&=|\omega_n|^{-1}\cdot|\underbrace{\langle\,\operatorname{rot}H_n\,,\mathcal{B}_{1,\ell}\,\rangle_{\mathsf{L}^2(\Omega)}}_{0}+i\omega_n\langle\,\varepsilon E_n\,,\mathcal{B}_{1,\ell}\,\rangle_{\mathsf{L}^2(\Omega)}|\xrightarrow{n\longrightarrow\infty}|\langle\,\varepsilon E\,,\mathcal{B}_{1,\ell}\,\rangle_{\mathsf{L}^2(\Omega)}|\,,\end{aligned}$$

hence $E\in\mathfrak{B}_1(\Omega)^{\perp_\varepsilon}$ and with (5.12) analogously $H\in\mathfrak{B}_2(\Omega)^{\perp_\mu}$. Thus (E,H) must vanish and again Corollary 5.3 yields constants $c,\delta>0$ independent of n such that

$$\begin{aligned}1&=\|\,(E_n,H_n)\,\|_{\mathsf{L}^2_t(\Omega)}\\&\leq c\Big(\,\|\,(F_n,G_n)\,\|_{\mathsf{L}^2_s(\Omega)}+|\omega_n|^{-1}\,\|\,(\operatorname{div}F_n,\operatorname{div}G_n)\,\|_{\mathsf{L}^2_s(\Omega)}+\|\,(E_n,H_n)\,\|_{\mathsf{L}^2(\Omega_\delta)}\Big)\xrightarrow{n\longrightarrow\infty}0\,,\end{aligned}$$

a contradiction. □

We are ready to prove our main result:

Theorem 5.5. *Let ε,μ be $\kappa-\mathsf{C}^1-$decaying with order $\kappa>0$, $1/2<s<3/2$, $t:=s-2$, and let $\tilde{\omega}$ be the radius from Theorem 5.4. Then for $(\omega_n)_{n\in\mathbb{N}}\subset\mathbb{C}_{+,\tilde{\omega}}\setminus\{0\}$ tending to zero and*

$$\big((F_n,G_n)\big)_{n\in\mathbb{N}}\subset\mathsf{D}_{s,\Gamma_2}(\Omega)\times\mathsf{D}_{s,\Gamma_1}(\Omega)$$

such that

$$\begin{aligned}(F_n,G_n)&\xrightarrow{n\longrightarrow\infty}(F,G)&&\text{in}\quad\mathsf{L}^2_s(\Omega)\times\mathsf{L}^2_s(\Omega)\,,\\-i\omega_n^{-1}(\operatorname{div}F_n,\operatorname{div}G_n)&\xrightarrow{n\longrightarrow\infty}(f,g)&&\text{in}\quad\mathsf{L}^2_s(\Omega)\times\mathsf{L}^2_s(\Omega)\,,\\-i\omega_n^{-1}\langle\,F_n\,,\mathcal{B}_{1,\ell}\,\rangle_{\mathsf{L}^2(\Omega)}&\xrightarrow{n\longrightarrow\infty}\zeta_\ell&&\text{in}\quad\mathbb{C}\,,\qquad\ell=1,\dots,d_{1,2}\,,\\-i\omega_n^{-1}\langle\,G_n\,,\mathcal{B}_{2,\ell}\,\rangle_{\mathsf{L}^2(\Omega)}&\xrightarrow{n\longrightarrow\infty}\theta_\ell&&\text{in}\quad\mathbb{C}\,,\qquad\ell=1,\dots,d_{2,1}\,,\end{aligned}$$

the sequence $\big((E_n,H_n)\big)_{n\in\mathbb{N}}:=\big(\mathcal{L}_{\Lambda,\omega_n}(F_n,G_n)\big)_{n\in\mathbb{N}}$ of radiating solutions converges for all $\tilde{t}<t$ in

$$\big(\,\mathsf{R}_{\tilde{t},\Gamma_1}(\Omega)\cap\varepsilon^{-1}\mathsf{D}_{\tilde{t},\Gamma_2}(\Omega)\,\big)\times\big(\,\mathsf{R}_{\tilde{t},\Gamma_2}(\Omega)\cap\mu^{-1}\mathsf{D}_{\tilde{t},\Gamma_1}(\Omega)\,\big)$$

to the static solutions $(E,H)\in\big(\,\mathsf{R}_{-1,\Gamma_1}(\Omega)\cap\varepsilon^{-1}\mathsf{D}_{-1,\Gamma_2}(\Omega)\,\big)\times\big(\,\mathsf{R}_{-1,\Gamma_2}(\Omega)\cap\mu^{-1}\mathsf{D}_{-1,\Gamma_1}(\Omega)\,\big)$ of

$$\begin{aligned}\operatorname{rot}E&=G\,,&\operatorname{div}\varepsilon E&=f\,,&\langle\,E\,,\mathcal{B}_{1,\ell}\,\rangle_{\mathsf{L}^2_\varepsilon(\Omega)}&=\zeta_\ell&&(\,\ell=1,\dots,d_{1,2}\,)\,,\\\operatorname{rot}H&=F\,,&\operatorname{div}\mu H&=g\,,&\langle\,H\,,\mathcal{B}_{2,\ell}\,\rangle_{\mathsf{L}^2_\mu(\Omega)}&=\theta_\ell&&(\,\ell=1,\dots,d_{2,1}\,)\,.\end{aligned}$$

Proof. By Lemma 5.4 (*iii*) the sequence $\big((E_n,H_n)\big)_{n\in\mathbb{N}}$ is bounded in

$$\big(\,\mathsf{R}_{t,\Gamma_1}(\Omega)\cap\varepsilon^{-1}\mathsf{D}_{t,\Gamma_2}(\Omega)\,\big)\times\big(\,\mathsf{R}_{t,\Gamma_2}(\Omega)\cap\mu^{-1}\mathsf{D}_{t,\Gamma_1}(\Omega)\,\big)$$

and the differential equation yields

$$M(E_n,H_n)=(F_n,G_n)-i\omega_n\Lambda(E_n,H_n)\,,\qquad(\operatorname{div}\varepsilon E_n,\operatorname{div}\mu H_n)=-\frac{i}{\omega_n}(\operatorname{div}F_n,\operatorname{div}G_n)\,,$$

such that by assumption

$$\begin{aligned}(\operatorname{rot}E_n,\operatorname{rot}H_n)&\xrightarrow{n\longrightarrow\infty}(F,G)&&\text{in}\quad\mathsf{L}^2_t(\Omega)\times\mathsf{L}^2_t(\Omega)\,,\\(\operatorname{div}\varepsilon E_n,\operatorname{div}\mu H_n)&\xrightarrow{n\longrightarrow\infty}(f,g)&&\text{in}\quad\mathsf{L}^2_s(\Omega)\times\mathsf{L}^2_s(\Omega)\,.\end{aligned}$$

Moreover, for $\ell = 1, \dots, d_{1,2}$ we compute by (3.15)

$$\langle E_n , \mathcal{B}_{1,\ell} \rangle_{\mathsf{L}^2_\varepsilon(\Omega)} = -\frac{i}{\omega_n} \underbrace{\langle \operatorname{rot} H_n , \mathcal{B}_{1,\ell} \rangle_{\mathsf{L}^2(\Omega)}}_{=0} - \frac{i}{\omega_n} \langle F_n , \mathcal{B}_{1,\ell} \rangle_{\mathsf{L}^2(\Omega)} \xrightarrow{n\longrightarrow\infty} \zeta_\ell$$

and analogously $\langle H_n , \mathcal{B}_{2,\ell} \rangle_{\mathsf{L}^2_\mu(\Omega)} \xrightarrow{n\longrightarrow\infty} \theta_\ell$ for $\ell = 1, \dots, d_{2,1}$. By Weck's local selection theorem we may extract a subsequence $\big((E_{\pi(n)}, H_{\pi(n)})\big)_{n\in\mathbb{N}}$ with

$$(E_{\pi(n)}, H_{\pi(n)}) \xrightarrow{n\longrightarrow\infty} : (\tilde{E}, \tilde{H}) \qquad \text{in} \qquad \mathsf{L}^2_{\tilde{t}}(\Omega) \times \mathsf{L}^2_{\tilde{t}}(\Omega)$$

for all $-3/2 < \tilde{t} < t$. Then

$$(\tilde{E}, \tilde{H}) \in \big(\mathsf{R}_{>-\frac{3}{2},\Gamma_1}(\Omega) \cap \varepsilon^{-1}\mathsf{D}_{>-\frac{3}{2},\Gamma_2}(\Omega) \big) \times \big(\mathsf{R}_{>-\frac{3}{2},\Gamma_2}(\Omega) \cap \mu^{-1}\mathsf{D}_{>-\frac{3}{2},\Gamma_1}(\Omega) \big)$$

and $(\tilde{E}, \tilde{H})$ solves the electro-magneto static system

$$\begin{aligned}
\operatorname{rot} \tilde{E} &= G\,, & \operatorname{div} \varepsilon\tilde{E} &= f\,, & \langle \tilde{E} , \mathcal{B}_{1,\ell} \rangle_{\mathsf{L}^2_\varepsilon(\Omega)} &= \zeta_\ell & (\,\ell &= 1, \dots, d_{1,2}\,)\,,\\
\operatorname{rot} \tilde{H} &= F\,, & \operatorname{div} \mu\tilde{H} &= g\,, & \langle \tilde{H} , \mathcal{B}_{2,\ell} \rangle_{\mathsf{L}^2_\mu(\Omega)} &= \theta_\ell & (\,\ell &= 1, \dots, d_{2,1}\,)\,.
\end{aligned}$$

Finally, the difference $(e, h) := (E, H) - (\tilde{E}, \tilde{H})$ satisfies

$$(e, h) \in \big({}_\varepsilon\mathcal{H}_{>-\frac{3}{2},\Gamma_1,\Gamma_2}(\Omega) \cap \mathfrak{B}_1(\Omega)^{\perp_\varepsilon} \big) \times \big({}_\mu\mathcal{H}_{>-\frac{3}{2},\Gamma_2,\Gamma_1}(\Omega) \cap \mathfrak{B}_2(\Omega)^{\perp_\mu} \big)\,.$$

Hence, by (3.13) and Theorem 3.11 we have $(E, H) = (\tilde{E}, \tilde{H})$ and due to the uniqueness of the limit (E, H) even the whole sequence $\big((E_n, H_n)\big)_{n\in\mathbb{N}}$ must converge to (E, H) in $\mathsf{L}^2_{<t}(\Omega) \times \mathsf{L}^2_{<t}(\Omega)$. □

References

[1] S. Bauer, D. Pauly, and M. Schomburg. The Maxwell Compactness Property in Bounded Weak Lipschitz Domains with Mixed Boundary Conditions. *SIAM J. Math. Anal.*, 48(4):2912–2943, 2016.

[2] M. Costabel. A remark on the regularity of solutions of maxwell's equations on lipschitz domains. *Math. Methods Appl. Sci.*, 12(4):365–368, 1990.

[3] D. M. Eidus. The principle of limiting absorption. *Amer. Math. Soc. Transl. Ser. 2*, 47:157–191, 1965.

[4] D. M. Eidus. On the spectra and eigenfunctions of the Schrödinger and Maxwell operators. *Journal of Mathematical Analysis and Applications*, 106(2):540–568, 1985.

[5] D. M. Eidus. The limiting absorption and amplitude principles for the diffraction problem with two unbounded media. *Comm. Math. Phys.*, 107(1):29–38, 1986.

[6] P. Fernandes and G. Gilardi. Magnetostatic and Electrostatic Problems in Inhomogeneous Anisotropic Media with Irregular Boundary and Mixed Boundary Conditions. *Math. Models Methods Appl. Sci.*, 07(7):957–991, 1997.

[7] F. Jochmann. A compactness result for vector fields with divergence and curl in $L^2(\Omega)$ involving mixed boundary conditions. *Appl. Anal.*, 66(1):189–203, 1997.

[8] R. Kress. Potentialtheoretische Randwertprobleme bei Tensorfeldern beliebiger Dimension und beliebigen Ranges. *Arch. Ration. Mech. An.*, 47:59–80, 1972.

[9] P. Kuhn and D. Pauly. Regularity results for generalized electro-magnetic problems. *Analysis*, 30(3):225–252, 2010.

[10] R. Leis. Zur Theorie elektromagnetischer Schwingungen in anisotropen inhom. Medien. *Math. Z.*, 106:213–224, 1968.

[11] R. Leis. Aussenraumaufgaben in der Theorie der Maxwellschen Gleichungen. In *Topics in Analysis*, pages 237–247. Springer, Berlin, Heidelberg, 1974.

[12] R. Leis. *Initial Boundary Value Problems in Mathematical Physics.* Courier Corporation, 2013.

[13] R. C. McOwen. The behavior of the laplacian on weighted sobolev spaces. *Comm. Pure Appl. Math.*, 32(6):783–795, 1979.

[14] A. Milani and R. Picard. Decomposition theorems and their application to non-linear electro- and magneto-static boundary value problems. In S. Hildebrandt and R. Leis, editors, *Partial Differential Equations and Calculus of Variations*, volume 1357 of *Lecture Notes in Mathematics*, pages 317–340. Springer Berlin Heidelberg, 1988.

[15] C. Müller. Randwertprobleme der Theorie elektromagnetischer Schwingungen. *Math. Z.*, 56(3):261–270, 1952.

[16] C. Müller. On the behavior of the solutions of the differential equation $\Delta u = f(x, u)$ in the neighborhood of a point. *Comm. Pure Appl. Math.*, 7(3):505–515, 1954.

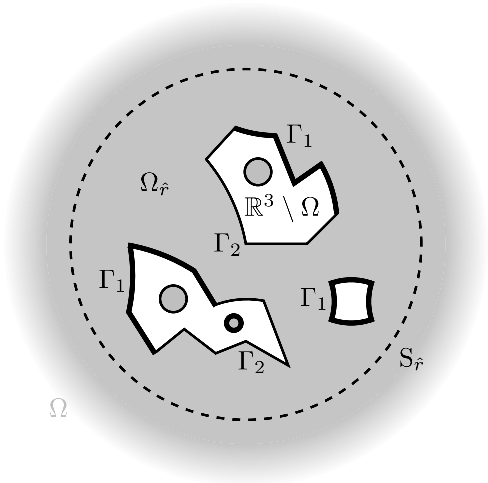
[17] F. Osterbrink and D. Pauly. Time-harmonic electro-magnetic scattering in exterior weak Lipschitz domains with mixed boundary conditions. In U. Langer, D. Pauly, and S. Repin, editors, *Maxwell's Equations: Analysis and Numerics*, volume 24 of *Radon Ser. Comput. Appl. Math.*, pages 341–382. De Gruyter, 2019.

[18] D. Pauly. *Niederfrequenzasymptotik der Maxwell-Gleichung im inhomogenen und anisotropen Außengebiet.* Dissertation, Universität Duisburg-Essen, Fakultät für Mathematik, 2003.

[19] D. Pauly. Low frequency asymptotics for time-harmonic generalized Maxwell equations in nonsmooth exterior domains. *Adv. Math. Sci. Appl*, 16(2):591–622, 2006.
[20] D. Pauly. Generalized electro-magneto statics in nonsmooth exterior domains. *Analysis (Munich)*, 27(4):425–464, 2007.
[21] D. Pauly. Complete low frequency asymptotics for time-harmonic generalized Maxwell equations in nonsmooth exterior domains. *Asymptotic Analysis*, 60(3):125–184, 2008.
[22] D. Pauly. Hodge-Helmholtz decompositions of weighted Sobolev spaces in irregular exterior domains with inhomogeneous and anisotropic media. *Math. Methods Appl. Sci.*, 31(13):1509–1543, 2008.
[23] D. Pauly. On polynomial and exponential decay of eigen-solutions to exterior boundary value problems for the generalized time-harmonic Maxwell system. *Asymptot. Anal.*, 79(1):133–160, 2012.
[24] D. Pauly. Solution Theory, Variational Formulations, and Functional a Posteriori Error Estimates for General First Order Systems with Applications to Electro-Magneto-Statics and More. *Numer. Funct. Anal. Optim.*, 0(0):01–97, 2019.
[25] D. Pauly and S. Repin. Functional a posteriori error estimates for elliptic problems in exterior domains. *J. Math. Sci.*, 162(3):393, 2009.
[26] R. Picard. Randwertaufgaben in der verallgemeinerten Potentialtheorie. *Math. Meth. Appl. Sci.*, 3(1):218–228, 1981.
[27] R. Picard. On the boundary value problems of electro- and magnetostatics. *Proc. Roy. Soc. Edinburgh Sect. A*, 92(1-2):165–174, 1982.
[28] R. Picard. An elementary proof for a compact imbedding result in generalized electromagnetic theory. *Math. Z.*, 187:151–164, 1984.
[29] R. Picard. Some decomposition theorems and their application to non-linear potential theory and Hodge theory. *Math. Meth. Appl. Sci.*, 12(1):35–52, 1990.
[30] R. Picard, N. Weck, and K. J. Witsch. Time-Harmonic Maxwell Equations in the Exterior of Perfectly Conducting, Irregular Obstacles. *Analysis (Munich)*, 21(3):231–264, 2001.
[31] C. Weber. A local compactness theorem for Maxwell's equations. *Math. Meth. Appl. Sci.*, 2(1):12–25, 1980.
[32] N. Weck. Maxwell's boundary value problem on Riemannian manifolds with nonsmooth boundaries. *J. Math. Anal. Appl.*, 46(2):410–437, 1974.
[33] N. Weck and K. J. Witsch. Complete low frequency analysis for the reduced wave equation with variable coefficients in three dimensions. *Comm. Partial Differential Equations*, 17(9):1619–1663, 1992.
[34] N. Weck and K. J. Witsch. Generalized linear elasticity in exterior domains. I: Radiation problems. *Math. Methods Appl. Sci.*, 20(17):1469–1500, 1997.
[35] N. Weck and K. J. Witsch. Generalized linear elasticity in exterior domains. II: low-frequency asymptotics. *Math. Methods Appl. Sci.*, 20(17):1501–1530, 1997.
[36] K. J. Witsch. A remark on a compactness result in electromagnetic theory. *Math. Meth. Appl. Sci.*, 16(2):123–129, 1993.

Appendix A. Proof of Theorem 3.11

Without loss of generality we concentrate on the construction of $\mathfrak{B}_1(\Omega)$ for $\gamma = \mathbb{1}$. As mentioned, the idea is to construct $\mathfrak{B}_1(\Omega)$ using a basis $\mathfrak{B}(\Omega_{\hat{r}})$ of $\mathcal{H}_{\Gamma_{\hat{r}},\Gamma_2}(\Omega_{\hat{r}})$, $\Gamma_{\hat{r}} := \Gamma_1 \cup \mathrm{S}_{\hat{r}}$. More precisely, we define



$$\mathfrak{B}_1(\Omega) := \big\{ \mathcal{E}_\Omega(\mathrm{B}) \,:\, B \in \mathfrak{B}(\Omega_{\hat{r}}) \big\} \subset {}_0\mathsf{R}_{\Gamma_1}(\Omega)\,,$$

where $\mathcal{E}_\Omega : \mathsf{L}^2(\Omega_{\hat{r}}) \longrightarrow \mathsf{L}^2(\Omega)$ extends functions resp. fields defined on $\Omega_{\hat{r}}$ by zero to Ω, and show the following:

Step 1: Choosing a basis $\mathfrak{B}(\Omega_{\hat{r}})$ of $\mathcal{H}_{\Gamma_{\hat{r}},\Gamma_2}(\Omega_{\hat{r}})$, extending the elements in $\mathfrak{B}(\Omega_{\hat{r}})$ by zero to Ω and projecting them onto $\mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega)$, we obtain a linearly independent subset of $\mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega)$,

Step 2: Choosing a basis $\mathfrak{B}(\Omega)$ of $\mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega)$, restricting the elements in $\mathfrak{B}(\Omega)$ to $\Omega_{\hat{r}}$ and projecting them onto $\mathcal{H}_{\Gamma_{\hat{r}},\Gamma_2}(\Omega_{\hat{r}})$, we obtain a linearly independent subset of $\mathcal{H}_{\Gamma_{\hat{r}},\Gamma_2}(\Omega_{\hat{r}})$.

Then, Step 1 and Step 2 already imply (cf. (3.14))

$$|\mathfrak{B}_1(\Omega)| = \dim \mathcal{H}_{\Gamma_{\hat{r}},\Gamma_2}(\Omega_{\hat{r}}) = \dim \mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega) = d_{1,2} < \infty\,.$$

Moreover, by Step 1 the projections of the elements in $\mathfrak{B}_1(\Omega)$ along $\overline{\nabla \mathsf{H}^1_{\Gamma_1}}(\Omega)$ are linearly independent and thus form a basis of the Dirichlet-Neumann fields $\mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega)$. Hence, it just remains to show:

$$\textbf{Step 3:}\quad \mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega) \cap \mathfrak{B}_1(\Omega)^\perp = \{0\}$$

Lemma A.1 (Step 1)**.** *Let* $\pi : {}_0\mathsf{R}_{\Gamma_1}(\Omega) \longrightarrow \mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega)$ *be the orthogonal projection given by*

$$ {}_0\mathsf{R}_{\Gamma_1}(\Omega) = \nabla\mathsf{H}^1_{-1,\Gamma_1}(\Omega) \oplus \mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega) \tag{A.1} $$

from Remark 3.10. Then the composition

$$ \pi \circ \mathcal{E}_\Omega : \mathcal{H}_{\Gamma_{\hat{r}},\Gamma_2}(\Omega_{\hat{r}}) \longrightarrow \mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega) $$

is injective.

Proof. Let $H \in \mathcal{H}_{\Gamma_{\hat{r}},\Gamma_2}(\Omega_{\hat{r}})$. Then $\mathcal{E}_\Omega(H) \in {}_0\mathsf{R}_{\Gamma_1}(\Omega)$ and with (A.1) we can decompose

$$ \mathcal{E}_\Omega(H) = \nabla w + \theta \in \nabla\mathsf{H}^1_{-1,\Gamma_1}(\Omega) \oplus \mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega)\,. $$

To show injectivity we assume $\theta = 0$. Then $\nabla w = \mathcal{E}_\Omega(H) = 0$ in $\breve{\mathrm{U}}_{\hat{r}}$. Thus w is constant in $\breve{\mathrm{U}}_{\hat{r}}$ and as $w \in \mathsf{H}^1_{-1,\Gamma_1}(\Omega)$ it has to vanish in $\breve{\mathrm{U}}_{\hat{r}}$, hence $w \in \mathbf{H}^1_{\Gamma_{\hat{r}}}(\Omega_{\hat{r}})$. By partial integration we conclude

$$ \| H \|^2_{\mathsf{L}^2(\Omega_{\hat{r}})} = \langle H \,, \nabla w \rangle_{\mathsf{L}^2(\Omega_{\hat{r}})} = -\langle \operatorname{div} H \,, w \rangle_{\mathsf{L}^2(\Omega_{\hat{r}})} = 0\,. $$

□

Lemma A.2 (Step 2)**.** *Let* $\pi : {}_0\mathsf{D}_{\Gamma_2}(\Omega_{\hat{r}}) \longrightarrow \mathcal{H}_{\Gamma_{\hat{r}},\Gamma_2}(\Omega_{\hat{r}})$ *be the orthogonal projection given by*

$$ {}_0\mathsf{D}_{\Gamma_2}(\Omega_{\hat{r}}) = \operatorname{rot}\mathbf{R}_{\Gamma_2}(\Omega_{\hat{r}}) \oplus \mathcal{H}_{\Gamma_{\hat{r}},\Gamma_2}(\Omega_{\hat{r}}) \tag{A.2} $$

from Lemma 3.2 (iii). Moreover, let $\mathcal{R}_{\Omega_{\hat{r}}} : \mathsf{L}^2(\Omega) \longrightarrow \mathsf{L}^2(\Omega_{\hat{r}})$ *be the operator restricting functions resp. fields on* Ω *to* $\Omega_{\hat{r}}$*. Then*

$$ \pi \circ \mathcal{R}_{\Omega_{\hat{r}}} : \mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega) \longrightarrow \mathcal{H}_{\Gamma_{\hat{r}},\Gamma_2}(\Omega_{\hat{r}}) $$

is injective.

Proof. Let $H \in \mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega)$. By (A.2), the restriction $\mathcal{R}_{\Omega_{\hat{r}}}(H) \in {}_0\mathsf{D}_{\Gamma_2}(\Omega_{\hat{r}})$ can be decomposed into

$$ \mathcal{R}_{\Omega_{\hat{r}}}(H) = \operatorname{rot} E + \theta \in \operatorname{rot}\mathbf{R}_{\Gamma_2}(\Omega_{\hat{r}}) \oplus \mathcal{H}_{\Gamma_{\hat{r}},\Gamma_2}(\Omega_{\hat{r}})\,. $$

To show injectivity we assume $\theta = 0$. In $\Omega_{\hat{r}}$ we have

$$ H = \mathcal{R}_{\Omega_{\hat{r}}}(H) = \operatorname{rot} E \quad \text{with} \quad E \in \mathbf{R}_{\Gamma_2}(\Omega_{\hat{r}})\,. \tag{A.3} $$

Furthermore, $H \in {}_0\mathsf{R}(\breve{\mathrm{U}}_{\hat{r}})$ and as the *Neumann-fields* $\mathcal{H}_{\emptyset,\mathrm{S}_{\hat{r}}}(\breve{\mathrm{U}}_{\hat{r}}) = {}_0\mathsf{R}(\breve{\mathrm{U}}(\hat{r})) \cap {}_0\mathsf{D}_{\mathrm{S}_{\hat{r}}}(\breve{\mathrm{U}}(\hat{r})) = \{0\}$ are trivial (the dimension is determined by the number of handles of $\breve{\mathrm{U}}_{\hat{r}}$, cf. [14, 27]) Lemma 3.9 yields

$$ {}_0\mathsf{R}(\breve{\mathrm{U}}_{\hat{r}}) = \nabla\mathsf{H}^1_{-1}(\breve{\mathrm{U}}_{\hat{r}}) \oplus \mathcal{H}_{\emptyset,\mathrm{S}_{\hat{r}}}(\breve{\mathrm{U}}_{\hat{r}}) = \nabla\mathsf{H}^1_{-1}(\breve{\mathrm{U}}_{\hat{r}})\,. $$

Thus, there exists $w \in \mathsf{H}^1_{-1}(\breve{\mathrm{U}}_{\hat{r}})$ such that $H = \nabla w$ in $\breve{\mathrm{U}}_{\hat{r}}$. Using a suitable extension operator (e.g., the one of Stein), we extend w to $\widehat{w} \in \mathsf{H}^1_{-1,\Gamma}(\Omega)$. Then $H - \nabla\widehat{w} \in {}_0\mathsf{R}_{\Gamma_1}(\Omega)$ with $H - \nabla\widehat{w} = 0$ in $\breve{\mathrm{U}}_{\hat{r}}$ and hence

$$ H - \nabla\widehat{w} \in {}_0\mathsf{R}_{\Gamma_{\hat{r}}}(\Omega_{\hat{r}})\,, \qquad \Gamma_{\hat{r}} = \Gamma_1 \cup \mathrm{S}_{\hat{r}}\,. \tag{A.4} $$

From (A.4) and (A.3) we conclude

$$ \begin{aligned} \| H \|^2_{\mathsf{L}^2(\Omega)} &= \langle H \,, H - \nabla\widehat{w} \rangle_{\mathsf{L}^2(\Omega)} + \langle H \,, \nabla\widehat{w} \rangle_{\mathsf{L}^2(\Omega)} \\ &= \langle \operatorname{rot} E \,, H - \nabla\widehat{w} \rangle_{\mathsf{L}^2(\Omega_{\hat{r}})} - \underbrace{\langle \operatorname{div} H \,, \widehat{w} \rangle_{\mathsf{L}^2(\Omega)}}_{=0} = \langle E \,, \operatorname{rot}(H - \nabla\widehat{w}) \rangle_{\mathsf{L}^2(\Omega_{\hat{r}})} = 0\,. \end{aligned} $$

□

Lemma A.3 (Step 3)**.** *Let* $\mathfrak{B}(\Omega_{\hat{r}})$ *be a basis of* $\mathcal{H}_{\Gamma_{\hat{r}},\Gamma_2}(\Omega_{\hat{r}})$ *and let* $\mathfrak{B}_1(\Omega)$ *be defined as above. It holds*

$$ \mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega) \cap \mathfrak{B}_1(\Omega)^\perp = \{0\}\,. $$

Proof. Let $H \in \mathcal{H}_{\Gamma_1,\Gamma_2}(\Omega) \cap \mathfrak{B}_1(\Omega)^{\perp}$. Then, for all $B \in \mathfrak{B}(\Omega_{\hat{r}})$ we have by definition of $\mathfrak{B}_1(\Omega)$

$$\langle\, \mathcal{R}_{\Omega_{\hat{r}}}(H)\,,B\,\rangle_{\mathsf{L}^2(\Omega_{\hat{r}})} = \langle\, H\,,\mathcal{E}_{\Omega}(B)\,\rangle_{\mathsf{L}^2(\Omega)} = 0\,,$$

and hence by (A.2)

$$\mathcal{R}_{\Omega_{\hat{r}}}(H) \in {}_0\mathsf{D}_{\Gamma_2}(\Omega_{\hat{r}}) \cap \mathcal{H}_{\Gamma_{\hat{r}},\Gamma_2}(\Omega_{\hat{r}})^{\perp} = \operatorname{rot} \mathbf{R}_{\Gamma_2}(\Omega_{\hat{r}})\,.$$

The assertion ($H = 0$) now follows by continuing as in the latter proof after (A.3). □

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