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Existence of Entropy Solutions to a Doubly Nonlinear Integro-Differential Equation

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Abstract

We consider a class of doubly nonlinear history-dependent problems associated with the equation $\partial_t k * (b(v) - b(v_0)) = \operatorname{div} a(x, Dv) + f$. Our assumptions on the kernel k include the case $k(t) = t^{-\alpha}/\Gamma(1-\alpha)$, in which case the left-hand side becomes the fractional derivative of order $\alpha \in (0, 1)$ in the sense of Riemann-Liouville. Existence of entropy solutions is established for general L^1-data and Dirichlet boundary conditions. Uniqueness of entropy solutions has been shown in a previous work.

1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^d, d \geq 1$. The present work is concerned with the existence of solutions to the history-dependent initial boundary

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value problem

$$\begin{aligned} \partial_t(k*(b(v)-b(v_0))) - \operatorname{div} a(x,Dv) &= f \quad \text{in } Q_T \\ v &= 0 \quad \text{on } \Sigma. \end{aligned} \tag{1.1}$$

Here, $T>0, Q_T := (0,T)\times\Omega$ is the space-time cylinder, $\Sigma := (0,T)\times\partial\Omega$, where $\partial\Omega$ denotes the boundary of Ω, Dv stands for the gradient of v with respect to the spatial variable, $k \in L^1_{loc}(\mathbb{R}^+)$ is a singular kernel and $k*v$ denotes the convolution on the positive half-line with respect to the time variable,

$$(k*v)(t) := \int_0^t k(t-s)v(s)\,\mathrm{d}s, \quad t>0.$$

We consider the above problem for L^1-data, i.e.,

$$f \in L^1(Q_T), \quad \text{and} \quad v_0 : \Omega \to \overline{\mathbb{R}} \text{ is measurable with } b(v_0) = u_0 \in L^1(\Omega).$$

The kernel k is assumed to be of type $\mathcal{PC}$, i.e., k is nonnegative, nonincreasing, and there exists a kernel $l \in L^1_{loc}(\mathbb{R}^+)$ such that $(k*l)(t) = 1$ for every $t>0$.

We will further assume that the kernel k satisfies additional technical conditions which are introduced in the next section.

However, our assumptions on k cover the case of a fractional derivative in time, i.e., $k(t) = t^{-\alpha}/\Gamma(1-\alpha), \alpha \in (0,1)$. In this case, the integro-differential operator in (1.1) becomes the fractional derivative of order α in the sense of Riemann-Liouville. Recall that the latter is, for sufficiently smooth v, defined as

$$\partial_t^\alpha v(t) := \partial_t \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} v(s)\,\mathrm{d}s, \quad 0<\alpha<1. \tag{1.2}$$

The problem (1.1) then interpolates between the elliptic and the parabolic problem. Note that both limiting cases, the purely elliptic case with $\alpha = 0, k \equiv 1$, and the purely parabolic case with $\alpha = 1, k = \delta_0$ are not considered in the present work.

Another example of a kernel which satisfies our assumptions is given by the time-fractional case with exponential weight, i.e., $k(t) = t^{-\alpha}e^{-\mu t}/\Gamma(1-\alpha), \alpha \in (0,1), \mu > 0$.

We assume further that the function $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is Carathéodory, i.e., $a(\cdot,\xi) : \Omega \to \mathbb{R}^d$ is measurable for all $\xi \in \mathbb{R}^d$, and $a(x,\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ is a continuous vector field for almost every $x \in \Omega$. Moreover, for some $p > 1$, and $p' := p/(p-1)$ we assume that a is monotone, coercive and satisfies a growth condition, i.e.,

(A1) $(a(x,\xi) - a(x,\zeta)) \cdot (\xi - \zeta) \geq 0 \quad \forall \xi, \zeta \in \mathbb{R}^d, \xi \neq \zeta$ and almost every $x \in \Omega$,

(A2) $\exists c > 0$ such that $a(x,\xi) \cdot \xi \geq c|\xi|^p, \quad \forall \xi \in \mathbb{R}^d$ and almost every $x \in \Omega$,

(A3) $\exists \Lambda > 0, j \in L^{p'}(\Omega)$ such that $|a(x,\xi)| \leq \Lambda(j(x) + |\xi|^{p-1}) \quad \forall \xi \in \mathbb{R}^d$ and almost every $x \in \Omega$.

Concerning the function $b : \mathbb{R} \to \mathbb{R}$, we assume that

(B1) b is continuous, strictly increasing, and satisfies the condition $b(0) = 0$,

e.g., $b(r) = \arctan(r), b(r) = |r|^{m-2}r, m > 1$. Note that we do not allow b to be partially constant, in which case (1.1) partially degenerates to an elliptic problem.

Time fractional diffusion equations of order $0 < \alpha < 1$ can be used to model anomalous diffusion, see [19] and the references therein for more information. Another application of (1.1) is the transport of fluids in porous media with memory. In some geothermal areas, the fluids may precipitate minerals in the pores of the medium, thus diminishing their size. The decrease of permeability leads to a history dependence which according to [6] can be represented by a fractional derivative in time.

Let us recall, see e.g. [2, Appendix I], that even for the elliptic problem with $k \equiv 1$ one cannot expect existence of a weak solution for general L^1-data. Moreover, even if there exists a weak solution, this solution is in general not unique, see [20],[21]. Concerning the history-dependent problem (1.1), it has been shown in [16, Section 3.1] that the problems of nonexistence and nonuniqueness of weak solutions to the elliptic problem carry over to the time-fractional case with $k(t) = t^{-\alpha}/\Gamma(1-\alpha)$.

In order to overcome these problems, the notion of entropy solution has been introduced by V. Jakubowski et al. in [17]. Note that (1.1) is a special case of the problems considered in the cited article. The authors prove uniqueness of entropy solution but existence is only shown in the case $b = \mathrm{Id}$.

In the present article we will show existence of an entropy solution to the problem (1.1) for general b satisfying (B1). The main idea of our existence proof is a modification of the regularization method by R. Landes, see [18]. Recall that for $v \in L^p(0,T;W_0^{1,p}(\Omega))$ the regularization in time v_μ of v introduced by R. Landes is, for $\mu > 0$, defined as

$$v_\mu(t) := \frac{1}{\mu}\int_{-\infty}^{t} e^{\frac{1}{\mu}(s-t)} v(s)\,\mathrm{d}s, \quad 0 < t < T,$$

where v is extended by some $v_0 \in W_0^{1,p}(\Omega)$ for $s < 0$. It has been used to prove existence of solutions for a variety of problems, see e.g. [10] for a parabolic equation, or [1] for an elliptic-parabolic problem without history dependence. See also [3, 4] for more recent applications.

We introduce a different regularization in time which adapts to the nonlocal nature of the problem (1.1) and makes use of special properties of $\mathcal{PC}$-kernels, see Definition 2.2.

We conclude these introductory remarks by giving the definition of the entropy solution. For $K > 0$ we denote by T_K the truncation function defined by

$$T_K(r) := \min(-K, \max(r, K)), \quad r \in \mathbb{R}.$$

We will frequently use the notation $T_{K,L} := T_L - T_K$, for $L > K > 0$. Let $\mathrm{Lip}(\mathbb{R})$ be the set of all Lipschitz-continuous real-valued functions,

$$\mathcal{P} := \{S \in C^1(\mathbb{R}) : S'(t) \geq 0 \text{ for every } t > 0, \operatorname{supp} S' \text{ is compact, } S(0) = 0\}.$$

and define for $\varphi \in \mathbb{R}, S \in \mathcal{C}(\mathbb{R})$, and b satisfying (B1),

$$B_{S,\varphi}(r) := \int_0^r S(\varrho - \varphi)\,\mathrm{d}b(\varrho), \quad r \in \mathbb{R}. \tag{1.3}$$

Definition 1.1. For $n \in \mathbb{N}$ let $k_{1,n} := (k-n)^+$ and $k_{2,n} := k - k_{1,n}$. A measurable function $v : Q_T \to \overline{\mathbb{R}}$ is called entropy solution to the problem (1.1) if $b(v) \in L^1(0,T;L^1(\Omega)), T_K(v) \in L^p(0,T;W_0^{1,p}(\Omega))$ for all $K > 0$, and there holds

$$\begin{aligned} &-\int_{Q_T} k_{1,n} * (B_{S,\varphi}(v) - B_{S,\varphi}(v_0))\xi_t + \int_{Q_T} \partial_t[k_{2,n} * (b(v) - b(v_0))]S(v-\varphi)\xi \\ &+\int_{Q_T} a(x,Dv)\cdot DS(v-\varphi)\xi \leq \int_{Q_T} fS(v-\varphi)\xi \end{aligned} \tag{1.4}$$

for all $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \xi \in \mathcal{D}([0,T)), \xi \geq 0, S \in \mathcal{P}$, and all $n \in \mathbb{N}$.

Note that the uniqueness proof of V. Jakubowski et al. is based on the stronger assumption that the inequality (1.4) holds true for all nonnegative, nonincreasing functions $k_1, k_2 \in L^1_{\mathrm{loc}}([0,\infty))$ satisfying $k_1 + k_2 = k$, and $k_2(0^+) < \infty$, see [17, Definition 1]. However, by going through the cited uniqueness proof, one verifies that the entropy solution remains unique if the entropy inequality is only valid for all $k_{1,n}, k_{2,n}, n \in \mathbb{N}$.

In the next section we give a short introduction to kernels of type $\mathcal{PC}$. The main emphasis lies on the fundamental identity for integro-differential operators, which plays a crucial role in the existence proof. Moreover, some examples of $\mathcal{PC}$-kernels are given which satisfy our additional assumptions. Section 3 is concerned with the existence of entropy solutions. In particular, we show that the generalized solution of the associated abstract Volterra equation is an entropy solution.

2 Kernels of Type $\mathcal{PC}$

Kernels of type $\mathcal{PC}$ have been introduced by R. Zacher in [25]. In the first part of this section we recap the basic properties of those kernels. See also [22],[23], and [24] for a similar exposition. The second part of this section is devoted to examples of $\mathcal{PC}$-kernels satisfying our additional assumptions.

Definition 2.1. A kernel $k \in L^1_{\mathrm{loc}}([0,\infty))$ is called to be of type $\mathcal{PC}$ if it is nonnegative, nonincreasing, and there exists a kernel $l \in L^1_{\mathrm{loc}}([0,\infty))$ such that

$$(k * l)(t) = 1 \quad \forall\, t > 0. \tag{2.1}$$

In this case, we write $(k,l) \in \mathcal{PC}$.

For $(k,l) \in \mathcal{PC}$ we denote by r_λ, s_λ the solutions to the scalar-valued Volterra equations

$$s_\lambda(t) + \lambda^{-1}(l * s_\lambda)(t) = 1, \quad t > 0, \quad \lambda > 0, \tag{2.2}$$

and

$$r_\lambda(t) + \lambda^{-1}(l * r_\lambda)(t) = \lambda^{-1} l(t), \quad t > 0, \quad \lambda > 0. \tag{2.3}$$

Note that the variation of constants formula for Volterra integral equations [14, Part 1, Chapter 2, Theorem 3.5] yields

$$s_\lambda(t) = 1 - \int_0^t r_\lambda(\tau)\,\mathrm{d}\tau, \quad t > 0, \quad \lambda > 0. \tag{2.4}$$

Let X be a real Banach space, $(k,l) \in \mathcal{PC}$, and define

$$W^{1,p}_{0=0}(0,T;X) := \{v \in W^{1,p}(0,T;X) : v(0) = 0\}, \quad 1 \leq p < \infty.$$

Then the operator L defined by

$$Lu = \partial_t(k * u), \quad D(L) = \{u \in L^p(0,T;X) : k * u \in W^{1,p}_{0=0}(0,T;X)\} \tag{2.5}$$

is known to be m-accretive in $L^p(0,T;X)$, see [7, Theorem 3.1]. Its resolvent

$J_\lambda^L = (I+\lambda L)^{-1}, \lambda > 0$, and its Yosida approximation $L_\lambda = LJ_\lambda^L, \lambda > 0$, can be written in the form

$$J_\lambda^L v = r_\lambda * v, \quad L_\lambda v = \partial_t(k_\lambda * v), \tag{2.6}$$

where $k_\lambda := \lambda^{-1} s_\lambda$, see [22, Theorem 2.1]. Note that (2.6) entails $k * r_\lambda = k_\lambda$. By [8, Theorem 2.2], we see that $(k,l) \in \mathcal{PC}$ implies that l is completely positive. In particular, s_λ and thus k_λ are nonnegative and nonincreasing [8, Proposition 2.1]. Moreover, it follows from (2.4) that $s_\lambda, k_\lambda \in W^{1,1}(0,T), s_\lambda' = -r_\lambda$, and $\|r_\lambda\|_{L^1(0,T)} \leq 1$.
For $v \in L^p(0,T;X)$ we have

$$(k * l * v)(t) = 1 * v = \int_0^t v(s)\,\mathrm{d}s \in W^{1,p}_{0=0}(0,T;X), \quad t > 0,$$

which entails $l * v \in D(L)$ for every $v \in L^p(0,T;X)$. In view of (2.6) it follows

$$L_\lambda(l*v) = \partial_t(k_\lambda * l * v) = \partial_t(k * r_\lambda * l * v) = r_\lambda * v \to L(l*v) = v$$

in $L^p(0,T;X)$ as $\lambda \to 0$, which shows $r_\lambda * v \to v$ in $L^p(0,T;X)$ for any $v \in L^p(0,T;X)$. In particular,

$$k_\lambda = r_\lambda * k \to k \quad \text{in } L^1(0,T), \tag{2.7}$$

as $\lambda \to 0$.

We are now in a position to introduce our modification of the time regularization from R. Landes.

Definition 2.2. Let X be a real Banach space, X^* its dual, and $1 \le p' < \infty$. For $v \in L^{p'}(0,T;X^*)$ let $v_\mu \in L^{p'}(0,T;X^*)$ be defined by

$$v_\mu(t) = \int_t^T r_\mu(\tau - t) v(\tau)\,\mathrm{d}\tau, \quad t \in (0,T), \quad \mu > 0. \tag{2.8}$$

In the sequel the index μ is used in this meaning only.

Note that if $p' = p/(p-1)$, then $v_\mu = J_\mu^{L^*} v$, where $L^* : D(L^*) \subset L^{p'}(0,T;X^*) \to L^{p'}(0,T;X^*)$ is the adjoint of the operator L defined in (2.5). As a consequence, we have $v_\mu \to v$ in $L^{p'}(0,T;X^*)$ for any $v \in L^{p'}(0,T;X^*)$ as $\mu \to 0$. The following lemma is needed in our proof of existence.

Lemma 2.3. *Let $\mu > 0$, and assume that $v \in L^\infty(Q_T)$. Then $v_\mu \in L^\infty(Q_T)$ and*

$$\|v_\mu\|_{L^\infty(Q_T)} \le \|v\|_{L^\infty(Q_T)}.$$

Proof. Since $L^\infty(Q_T) \subset L^p(Q_T) \cong L^p(0,T;L^p(\Omega))$ for any $p \in [1,\infty)$ we have $v_\mu \in L^p(0,T;L^p(\Omega))$ for all $p \in [1,\infty)$. If we set $r_\mu(t) := 0$ if $t \notin (0,T)$ and define $r_\mu^\vee(t) := r_\mu(-t)$ it follows

$$v_\mu(t) = \int_{\mathbb{R}} r_\mu^\vee(t-\tau) v(\tau)\,\mathrm{d}\tau.$$

Thus, Young's inequality and $\|r_\mu\|_{L^1(0,T)} \le 1$ yield

$$\|v_\mu\|_{L^p(0,T;L^p(\Omega))} \le \|r_\mu\|_{L^1(0,T)}\|v\|_{L^p(0,T;L^p(\Omega))} \le \|v\|_{L^p(0,T;L^p(\Omega))} \tag{2.9}$$

for all $p \in [1,\infty)$. Identifying $L^p(0,T;L^p(\Omega))$ with $L^p(Q_T)$ and passing to the limit with $p \to \infty$ yields the result. $\square$

We next formulate the fundamental identity for integro-differential operators of the form $\partial_t(k * u)$.

Lemma 2.4. *Let $T > 0$ and U be an open subset of $\mathbb{R}$. Let further $k \in W^{1,1}(0,T), H \in C^1(U)$, and $v \in L^1(0,T)$ with $v(t) \in U$ for almost all $t \in (0,T)$. Suppose that the functions $H(v), H'(v)v$ and $H'(v)(k * v)$ belong to $L^1(0,T)$. Then we have for almost all $t \in (0,T)$*

$$\begin{aligned} H'(v(t))\partial_t(k * v)(t) &= \partial_t(k * H(v))(t) + (H'(v(t))v(t) - H(v(t)))k(t) \\ &\quad + \int_0^t (H(v(t-s)) - H(v(t)) - H'(v(t))[v(t-s) - v(t)])[-k'(s)]\,\mathrm{d}s. \end{aligned} \tag{2.10}$$

Taking into account that $\partial_t(k * u) = k(0)u + k' * u$ and $\partial_t(k * H(u)) = k(0)H(u) + k' * H(u)$, the fundamental identity follows from a straightforward computation. Note that the last term on the right-hand side is nonnegative in case H is convex and k is nonincreasing so that (2.10) reads as

$$H'(v(t))\partial_t(k * v)(t) \ge \partial_t(k * H(v))(t) + (H'(v(t))v(t) - H(v(t)))k(t).$$

Due to a lack of regularity, the fundamental identity is not applicable to kernels of type $\mathcal{PC}$. However, if $k(t) = \Gamma(1-\alpha)^{-1}t^{-\alpha}, \alpha \in (0,1)$, and $H(r) = \frac{1}{2}r^2$, a similar inequality is also valid, see [11, Lemma 3.1]. The following lemmas are an immediate consequence of (2.10) and will frequently be used in the sequel.

Lemma 2.5. *Let $v \in L^\infty(0,T), k$ a $\mathcal{PC}$-kernel, and, for $\lambda > 0$, let k_λ be the kernel of the Yosida approximation of the operator defined in (2.5), and assume that b satisfies (B1). Then we have for all $K > 0$ and almost all $t > 0$*

$$
\begin{aligned}
&\partial_t[k_\lambda * b(v)](t) T_K(v(t)) \\
&= \partial_t[k_\lambda * \int_0^v T_K(\sigma)\,\mathrm{d}b(\sigma)](t) + [T_k(v(t))b(v(t)) - \int_0^{v(t)} T_K(\sigma)\,\mathrm{d}b(\sigma)]k_\lambda(t) \\
&\quad + \int_0^t [\int_{v(t)}^{v(t-s)} T_K(\sigma)\,\mathrm{d}b(\sigma) - T_k(v(t))(b(v(t-s)) - b(v(t)))][-k_\lambda'(s)]\,\mathrm{d}s \\
&\geq \partial_t[k_\lambda * \int_0^v T_K(\sigma)\,\mathrm{d}b(\sigma)](t).
\end{aligned}
\tag{2.11}
$$

Proof. Let $K > 0$, and $u \in \mathrm{ran}(b)$ which is open, as b is strictly increasing. We define $H(u) := \int_0^u T_K \circ b^{-1}(\sigma)\,\mathrm{d}\sigma$. Consequently, $H(b(v(t))) = \int_0^{v(t)} T_K(\sigma)\,\mathrm{d}b(\sigma)$ and $H'(b(v(t)) = T_K(v(t))$ for a.e. $t > 0$. Since $k_\lambda \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^+)$, we may apply the fundamental identity (2.10) which shows the first equality. The monotonicity of $T_K \circ b^{-1}$ entails that H is convex. Since k_λ is nonincreasing, it therefore follows that the last term in (2.11) is nonnegative. Due to the monotonicity of b and the normalization condition $b(0) = 0$, we may conclude that the second term on the right in (2.11) is nonnegative as well, which concludes the proof of the lemma. $\square$

Lemma 2.6. *Let the assumptions of Lemma 2.5 hold. Then*

$$S(v(t)-\varphi)\partial_t[k_\lambda * (b(v)-b(v_0))](t) \geq \partial_t[k_\lambda * \int_{v_0}^{v} S(\sigma-\varphi)\,\mathrm{d}b(\sigma)](t)$$

for all $S \in \mathcal{P}$, and all $\varphi, v_0 \in \mathbb{R}$.

Proof. For $S \in \mathcal{P}$ and $\varphi, v_0 \in \mathbb{R}$ we define $H(u) := \int_0^u S \circ (b^{-1}(\sigma)-\varphi)\mathrm{d}\sigma - \int_0^{v_0} S(\sigma-\varphi)\,\mathrm{d}b(\sigma), u \in \mathrm{ran}(b)$. The fundamental identity yields

$$\begin{aligned}
S(v(t)-\varphi)\partial_t[k_\lambda * (b(v)-b(v_0))](t) &= \partial_t[k_\lambda * \int_{v_0}^{v(t)} S(\sigma-\varphi)\,\mathrm{d}b(\sigma) \\
&+ \big[S(v(t)-\varphi)(b(v(t))-b(v_0)) - \int_{v_0}^{v(t)} S(\sigma-\varphi)\,\mathrm{d}b(\sigma)\big]k_\lambda(t) \\
&+ \int_0^t \big[\int_{v(t)}^{v(t-s)} S(\sigma-\varphi)\,\mathrm{d}b(\sigma) \\
&\quad - S(v(t)-\varphi)(b(v(t-s))-b(v(t)))\big]\big[-k_\lambda'(s)\big]\,\mathrm{d}s
\end{aligned} \tag{2.12}$$

for almost every $t \in (0,T)$. Since $u \mapsto \int_0^u S \circ (b^{-1}(\sigma)-\varphi)\mathrm{d}\sigma$ is convex and k_λ is nonnegative and nonincreasing, we may conclude that the last two terms are nonnegative and the lemma is proven. $\square$

We next formulate our assumptions on the kernel k which are needed in our proof of existence. Let k be a kernel of type $\mathcal{PC}$ and for $\lambda > 0$ let k_λ be the kernels which arise in the characterisation (2.6) of the Yosida approximation. We assume that k satisfies the following conditions.

(K1) There exist constants $C_1, C_2 > 0$ such that

$$0 \leq k_\lambda(t) \leq C_1 k(t) + C_2, \quad \lambda > 0, \quad t \in (0,T).$$

(K2) $k \in \mathrm{AC}_{\mathrm{loc}}((0,T])$ and there exist constants $C_1, C_2 > 0$ such that

$$0 \leq -k_\lambda'(t) \leq -C_1 k'(t) + C_2, \quad \lambda > 0, \quad t \in (0,T).$$

Moreover, the convergence $k_\lambda'(t) \to k'(t)$ as $\lambda \to 0$ holds for almost every $t \in (0,T)$.

We conclude this section by studying some examples of $\mathcal{PC}$-kernels satisfying (K1) and (K2).

Example 2.7. The most prominent example of a pair $(k,l) \in \mathcal{PC}$ is given by

$$k(t) = g_{1-\alpha}(t), \quad \text{and} \quad l(t) = g_\alpha(t), \quad t > 0,$$

where $\alpha \in (0,1)$ and g_β denotes the Riemann-Liouville kernel

$$g_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad t > 0, \quad \beta > 0. \tag{2.13}$$

In this case, the operator defined by (2.5) describes the fractional time derivative of order α in the sense of Riemann-Liouville. Applying the Laplace transform to equation (2.2) we get

$$\widehat{s}_\lambda(\tau) = \frac{\tau^{\alpha-1}}{\tau^\alpha + \lambda^{-1}}, \quad \operatorname{Re} \tau > 0, \quad \lambda > 0, \tag{2.14}$$

which shows $s_\lambda(t) = E_\alpha(-\lambda^{-1}t^\alpha)$, where E_α is the Mittag-Leffler function given by $E_\alpha(t) = \sum_{n=0}^\infty \frac{t^n}{\Gamma(\alpha n+1)}$, see e.g. [15, Section 7]. Using another known representation of the Mittag-Leffler function [15, Section 7], it follows

$$s_\lambda(t) = \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty e^{-rt} \frac{\lambda^{-1} r^{\alpha-1}}{\lambda^{-2} + r^{2\alpha} + 2\lambda^{-1} r^\alpha \cos(\alpha\pi)} \, \mathrm{d}r, \quad t > 0, \quad \lambda > 0,$$

which can also be found in the monograph of J. Prüss [9, table 4.1]. Thus,

$$k_\lambda(t) = \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty e^{-rt} \frac{r^{\alpha-1}}{1 + \lambda^2 r^{2\alpha} + 2\lambda r^\alpha \cos(\alpha\pi)} \, \mathrm{d}r, \quad t > 0, \quad \lambda > 0,$$

and

$$k'_\lambda(t) = -\frac{\sin(\alpha\pi)}{\pi} \int_0^\infty e^{-rt} \frac{r^{\alpha}}{1 + \lambda^2 r^{2\alpha} + 2\lambda r^\alpha \cos(\alpha\pi)} \, \mathrm{d}r, \quad t > 0, \quad \lambda > 0.$$

Using the identity $\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin(\alpha\pi)}$ we see that

$$k(t) = \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty e^{-rt} r^{\alpha-1} \, \mathrm{d}r, \quad k'(t) = -\frac{\sin(\alpha\pi)}{\pi} \int_0^\infty e^{-rt} r^{\alpha} \, \mathrm{d}r \quad t > 0.$$

Thus, $k'_\lambda(t) \to k'(t)$ for all $t > 0$ as $\lambda \to 0$. If $\alpha \in (0, \frac{1}{2}]$, it easily follows that

$0 \leq k_\lambda(t) \leq k(t)$ and $0 \leq -k_\lambda'(t) \leq -k'(t)$ for all $t > 0$ and all $\lambda > 0$. If $\alpha \in (\frac{1}{2}, 1)$, a simple study of functions shows

$$1 + \lambda^2 r^{2\alpha} + 2\lambda r^\alpha \cos(\alpha\pi) \geq 1 - \cos^2(\alpha\pi), \quad \lambda > 0, \quad r > 0,$$

which shows that

$$0 \leq k_\lambda(t) \leq \frac{1}{1 - \cos^2(\alpha\pi)} k(t), \quad 0 \leq -k_\lambda'(t) \leq -\frac{1}{1 - \cos^2(\alpha\pi)} k'(t)$$

for all $\lambda > 0$ and all $t > 0$.

Example 2.8. Another example for a pair $(k, l) \in \mathcal{PC}$ is the time-fractional case with exponential weight, i.e.,

$$k(t) = g_{1-\alpha}(t)e^{-\mu t}, \quad l(t) = g_\alpha(t)e^{-\mu t} + \mu(1 * [g_\alpha(\cdot)e^{-\mu \cdot}])(t), \quad t > 0,$$

where $\mu > 0$ and $\alpha \in (0, 1)$. If we apply the Laplace transform on (2.2) we may conclude that

$$\widehat{s}_\lambda(z) = \frac{1}{z + \lambda^{-1}(z + \mu)^{1-\alpha}}, \quad \mathrm{Re}(z) > 0, \quad \lambda > 0.$$

Note that $\widehat{s}_\lambda$ is holomorphic on $\mathbb{C} \setminus ((-\infty, -\mu] \cup \{-w_\lambda\})$, where $-w_\lambda \in (-\mu, 0)$ denotes the unique solution of $\lambda z + (z + \mu)^{1-\alpha} = 0$. We will show that s_λ is given by

$$
\begin{aligned}
s_\lambda(t) =& \frac{\sin(\alpha\pi)}{\pi}\int_\mu^\infty e^{-rt}\frac{\lambda^{-1}(r-\mu)^{\alpha-1}}{\lambda^{-2}+2r\lambda^{-1}(r-\mu)^{\alpha-1}\cos(\alpha\pi)+r^2(r-\mu)^{2(\alpha-1)}}\,\mathrm{d}r \\
&+\frac{(\mu-\omega_\lambda)^\alpha}{\lambda^{-1}(1-\alpha)+(\mu-\omega_\lambda)^\alpha}e^{-\omega_\lambda t}, \quad t>0, \quad \lambda>0.
\end{aligned}
$$

To this end, we modify the proof of Lemma 2.1 in [5]. Let $\delta > 0$ such that $\omega_\lambda + \delta < \mu$ and $\omega_\lambda - \delta > 0$. Cauchy's integral formula yields

$$
\widehat{s}_\lambda(p) = \frac{1}{2\pi i}\int_{C_{\delta,R}^{\omega_\lambda,\phi}} \frac{\widehat{s}_\lambda(z)}{z-p}\,\mathrm{d}z, \quad \mathrm{Re}(p) > 0, \tag{2.15}
$$

where p is assumed to be inside the contour $C_{\delta,R}^{\omega_\lambda,\phi}$ made of two line segments

$$
I^\pm := \{z-\omega_\lambda \in \mathbb{C} : \delta \le |z| \le R, \arg(z) = \pm\phi\}, \quad \phi \in (\frac{\pi}{2},\pi),
$$

and two arcs

$$
\begin{aligned}
C_\delta &:= \{z-\omega_\lambda \in \mathbb{C} : |z| = \delta, |\arg(z)| \le \phi\}, \quad \phi \in (\frac{\pi}{2},\pi),\\
C_R &:= \{z-\omega_\lambda \in \mathbb{C} : |z| = R, |\arg(z)| \le \phi\}, \quad \phi \in (\frac{\pi}{2},\pi).
\end{aligned}
$$

Note that for $z = re^{i\phi}, r > 0, \phi \in (-\pi,\pi)$, we have

$$
|z+\lambda^{-1}(z+\mu)^{1-\alpha}| \ge |r-\lambda^{-1}|re^{i\phi}+\mu|^{1-\alpha}|, \quad \lambda > 0,
$$

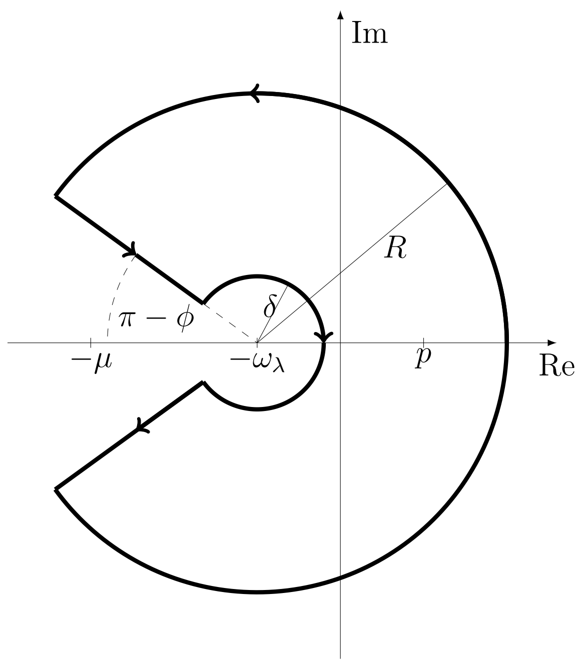

Figure 1: The contour $C^{\omega_\lambda,\phi}_{\delta,R}$

which implies for fixed $\lambda > 0$

$$|\widehat{s}_\lambda(z)| \leq \frac{1}{r - \lambda^{-1}|r+\mu|^{1-\alpha}} \tag{2.16}$$

for all $r > 0$ sufficiently large. Thus, it follows $\widehat{s}_\lambda(z) \to 0$ uniformly in ϕ as $|z| \to \infty$. As a consequence, we may pass to the limit in (2.15) with $R \to \infty$ to obtain

$$\widehat{s}_\lambda(p) = \frac{1}{2\pi i} \int_{C^{\omega_\lambda,\phi}_{\delta,\infty}} \frac{\widehat{s}_\lambda(z)}{z-p}\, \mathrm{d}z, \quad \mathrm{Re}(p) > 0.$$

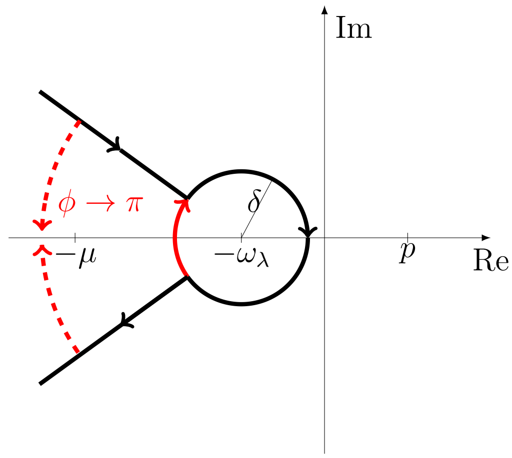

Figure 2: The contour $C_{\delta,\infty}^{\omega_\lambda,\phi}$ and γ_δ

Consequently,

$$\widehat{s}_\lambda(p) = \frac{1}{2\pi i}\int_\delta^\infty \frac{e^{i\phi}\widehat{s}_\lambda(-\omega_\lambda + re^{i\phi})}{p+\omega_\lambda - re^{i\phi}} - \frac{e^{-i\phi}\widehat{s}_\lambda(-\omega_\lambda + re^{-i\phi})}{p+\omega_\lambda - re^{-i\phi}}\,\mathrm{d}r \\ + \frac{1}{2\pi i}\int_{-\phi}^{\phi} \frac{\delta i e^{i\sigma}\widehat{s}_\lambda(-\omega_\lambda + \delta e^{i\sigma})}{p+\omega_\lambda - \delta e^{i\sigma}}\,\mathrm{d}\sigma.$$

Note that the inequality (2.16) entails that the first integral converges absolutely. As $\mathrm{Re}(p) > 0$ it follows $\mathrm{Re}(p+\omega_\lambda - re^{\pm i\phi}) > 0$ and $\mathrm{Re}(p+\omega_\lambda - \delta e^{i\sigma}) > 0$ for all $\phi \in (\frac{\pi}{2}, \pi)$ and all $\sigma \in (-\pi, \pi)$. Thus,

$$\frac{1}{p+\omega_\lambda - re^{\pm i\phi}} = \int_0^\infty e^{-t(p+\omega_\lambda - re^{\pm i\phi})}\,\mathrm{d}t,$$

and

$$\frac{1}{p+\omega_\lambda-\delta e^{i\sigma}}=\int_0^\infty e^{-t(p+\omega_\lambda-\delta e^{i\sigma})}\,\mathrm{d}t.$$

By Fubini's theorem, we may therefore conclude that

$$\widehat{s}_\lambda(p)=\int_0^\infty f_\lambda(t)e^{-pt}\,\mathrm{d}t, \tag{2.17}$$

where

$$\begin{aligned}f_\lambda(t)=&\frac{1}{2\pi i}\int_\delta^\infty e^{i\phi+t(-\omega_\lambda+re^{i\phi})}\widehat{s}_\lambda(-\omega_\lambda+re^{i\phi})\\&\qquad-e^{-\phi i+t(-\omega_\lambda+re^{-i\phi})}\widehat{s}_\lambda(-\omega_\lambda+re^{-i\phi})\,\mathrm{d}r\\&+\frac{1}{2\pi i}\int_{-\phi}^{\phi}\delta ie^{i\sigma+t(-\omega_\lambda+\delta e^{i\sigma})}\widehat{s}_\lambda(-\omega_\lambda+\delta e^{i\sigma})\,\mathrm{d}\sigma.\end{aligned}$$

Since

$$(z+\mu)^{1-\alpha}=|z+\mu|^{1-\alpha}e^{(1-\alpha)\arg(z+\mu)i},\quad z\in\mathbb{C}\setminus(-\infty,-\mu],$$

we obtain that $\widehat{s}_\lambda(\overline{z})=\overline{\widehat{s}_\lambda(z)}$ for all $z\in\mathbb{C}\setminus(-\infty,-\mu]$. Consequently,

$$\widehat{s}_\lambda(-\omega_\lambda+re^{i\phi})=\overline{\widehat{s}_\lambda(-\omega_\lambda+re^{-i\phi})},\quad r>\delta,\quad \phi\in(\frac{\pi}{2},\pi),$$

which implies that

$$\begin{aligned} f_\lambda = -\frac{1}{\pi}\int_\delta^\infty \operatorname{Im}(e^{-i\phi+t(-\omega_\lambda+re^{-i\phi})}\widehat{s}_\lambda(-\omega_\lambda+re^{-i\phi}))\,\mathrm{d}r \\ +\frac{1}{2\pi i}\int_{-\phi}^{\phi} \delta i e^{i\sigma+t(-\omega_\lambda+\delta e^{i\sigma})}\widehat{s}_\lambda(-\omega_\lambda+\delta e^{i\sigma})\,\mathrm{d}\sigma. \end{aligned} \tag{2.18}$$

If we assume for the moment that the limit

$$F_\lambda(r) := \lim_{\phi\to\pi} \widehat{s}_\lambda(-w_\lambda + re^{-i\phi})$$

exists for almost every $r > 0$ and take into account that $(r,\phi) \mapsto \widehat{s}_\lambda(-\omega_\lambda + re^{-i\phi})$ is bounded on $(\delta,\infty)\times(-\pi,\pi)$, we may pass to the limit in (2.18) with $\phi \to \pi$ to obtain

$$f_\lambda(t) = \frac{1}{\pi}\int_\delta^\infty e^{-t(r+\omega_\lambda)}\operatorname{Im}(F_\lambda(r))\,\mathrm{d}r + \frac{e^{-t\omega_\lambda}}{2\pi i}\int_{\gamma_\delta} e^{tz}\widehat{s}_\lambda(-\omega_\lambda+z)\,\mathrm{d}z, \tag{2.19}$$

where γ_δ denotes the circle with radius δ and centre $-\omega_\lambda$, see Figure 2. By the definition of ω_λ, the second integrand has an isolated singularity at $z = 0$. The residual theorem yields

$$\begin{aligned} &\frac{e^{-t\omega_\lambda}}{2\pi i}\int_{\gamma_\delta} e^{tz}\widehat{s}_\lambda(-\omega_\lambda+z)\,\mathrm{d}z \\ &\quad = e^{-\omega_\lambda t}\operatorname{res}_z \widehat{s}_\lambda(-\omega_\lambda+\cdot) = \frac{(\mu-\omega_\lambda)^\alpha}{\lambda^{-1}(1-\alpha)+(\mu-\omega_\lambda)^\alpha}e^{-\omega_\lambda t}. \end{aligned} \tag{2.20}$$

Since

$$\widehat{s}_\lambda(z) = \frac{(z+\mu)^{\alpha-1}}{z(z+\mu)^{\alpha-1}+\lambda^{-1}}, \quad z \in \mathbb{C} \setminus ((-\infty,-\mu] \cup \{-\omega_\lambda\}), \tag{2.21}$$

and

$$\lim_{\phi\to\pi}(-\omega_\lambda + re^{-i\phi}+\mu)^{\alpha-1} = \begin{cases} (\mu-\omega_\lambda-r)^{\alpha-1}, & 0<r<\mu-\omega_\lambda \\ |\mu-\omega_\lambda-r|^{\alpha-1}e^{(1-\alpha)\pi i}, & r>\mu-\omega_\lambda \end{cases}$$

it follows that $\lim_{\phi\to\pi}\widehat{s}_\lambda(-\omega_\lambda+re^{-i\phi})$ exists for almost all $r>0$ and

$$\begin{aligned}&\operatorname{Im}(F_\lambda(r))\\ &\quad= \begin{cases} 0, & r<\mu-\omega_\lambda \\ \frac{\sin(\alpha\pi)\lambda^{-1}(r-\mu+\omega_\lambda)^{\alpha-1}}{\lambda^{-2}+2(r+\omega_\lambda)\lambda^{-1}(r-\mu+\omega_\lambda)^{\alpha-1}\cos(\alpha\pi)+(r+\omega_\lambda)^2(r-\mu+\omega_\lambda)^{2(\alpha-1)}}, & r>\mu-\omega_\lambda. \end{cases}\end{aligned}$$

If we combine this result with (2.19),(2.20) and (2.17), we obtain

$$\begin{aligned}s_\lambda(t) &= \frac{(\mu-\omega_\lambda)^\alpha}{\lambda^{-1}(1-\alpha)+(\mu-\omega_\lambda)^\alpha}e^{-\omega_\lambda t}\\ &\quad+\frac{\sin(\alpha\pi)}{\pi}\int_\mu^\infty e^{-rt}\frac{\lambda^{-1}(r-\mu)^{\alpha-1}}{\lambda^{-2}+2r\lambda^{-1}(r-\mu)^{\alpha-1}\cos(\alpha\pi)+r^2(r-\mu)^{2(\alpha-1)}}\,\mathrm{d}r.\end{aligned}$$

The above formula implies the asymptotic behavior already discovered in [24] but, to the best of our knowledge, the explicit representation of s_λ is a new result. Since $k_\lambda = \lambda^{-1} s_\lambda$, it follows

$$\begin{aligned} k_\lambda(t) =& \frac{(\mu-\omega_\lambda)^\alpha}{(1-\alpha)+\lambda(\mu-\omega_\lambda)^\alpha} e^{-\omega_\lambda t} \\ &+ \frac{\sin(\alpha\pi)}{\pi} \int_\mu^\infty e^{-rt} \frac{(r-\mu)^{\alpha-1}}{1+2r\lambda(r-\mu)^{\alpha-1}\cos(\alpha\pi)+\lambda^2 r^2 (r-\mu)^{2\alpha-2}} \, \mathrm{d}r. \end{aligned}$$

and

$$\begin{aligned} k_\lambda'(t) =& -\omega_\lambda \cdot \frac{(\mu-\omega_\lambda)^\alpha}{(1-\alpha)+\lambda(\mu-\omega_\lambda)^\alpha} e^{-\omega_\lambda t} \\ &- \frac{\sin(\alpha\pi)}{\pi} \int_\mu^\infty e^{-rt} \frac{r(r-\mu)^{\alpha-1}}{1+2r\lambda(r-\mu)^{\alpha-1}\cos(\alpha\pi)+\lambda^2 r^2 (r-\mu)^{2\alpha-2}} \, \mathrm{d}r. \end{aligned}$$

As before, the identity $\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin(\alpha\pi)}$ shows that

$$k(t) = \frac{\sin(\alpha\pi)}{\pi} \int_\mu^\infty e^{-rt} (r-\mu)^{\alpha-1} \, \mathrm{d}r,$$

and

$$k'(t) = -\frac{\sin(\alpha\pi)}{\pi} \int_\mu^\infty e^{-rt} r (r-\mu)^{\alpha-1} \, \mathrm{d}r$$

for all $t > 0$. Since $-\omega_\lambda \in (-\mu, 0)$ was defined as the unique solution of $\lambda z + (z+\mu)^{1-\alpha} = 0$, we see that $\omega_\lambda \to \mu$ as $\lambda \to 0$. It follows

$$k_\lambda'(t) \to -\frac{\sin(\alpha\pi)}{\pi} \int_\mu^\infty e^{-rt} r (r-\mu)^{\alpha-1} \, \mathrm{d}r = k'(t)$$

for every $t > 0$. Moreover, if $\alpha \in (0, \frac{1}{2}]$, we see that

$$0 \le k_\lambda(t) \le k(t) + \frac{\mu^\alpha}{1-\alpha}, \quad \text{and} \quad 0 \le -k_\lambda'(t) \le -k'(t) + \frac{\mu^{1+\alpha}}{1-\alpha}$$

for all $t > 0$ and all $\lambda > 0$. If $\alpha \in (\frac{1}{2}, 1)$, it follows

$$0 \le k_\lambda(t) \le \frac{1}{1-\cos^2(\alpha\pi)} k(t) + \frac{\mu^\alpha}{1-\alpha},$$

and

$$0 \le -k_\lambda'(t) \le -\frac{1}{1-\cos^2(\alpha\pi)} k'(t) + \frac{\mu^{1+\alpha}}{1-\alpha}$$

for all $\lambda > 0$ and all $t > 0$. In particular, the kernel $k(t) = g_{1-\alpha}(t)e^{-\mu t}$ satisfies the conditions (K1) and (K2) for all $\alpha \in (0,1)$ and all $\mu \ge 0$.

3 Existence

This section is devoted to the existence of entropy solutions to the problem (1.1). To this end, we define the operator $A_\infty^b \subset L^1(\Omega) \times L^1(\Omega)$ by

$$\begin{aligned}(b(v), w) \in A_\infty^b \Leftrightarrow\ & v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), w \in L^1(\Omega), \\ & \text{and} \int_\Omega a(x, Dv) \cdot D\phi = \int_\Omega w\phi \\ & \text{for all } \phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).\end{aligned}$$

It is well known that A^b_∞ is an accretive operator in $L^1(\Omega)$ and that its graph closure $A_b := \overline{A^b_\infty}$ in $L^1(\Omega) \times L^1(\Omega)$ is a possibly multivalued m-accretive operator in $L^1(\Omega)$. According to [2], the operator A_b can be characterized by

$$
\begin{aligned}
(b(v), w) \in A_b \Leftrightarrow b(v), w \in L^1(\Omega), T_K(v) \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \text{ for all } K > 0, \\
\text{and } \int_\Omega a(x, Dv) \cdot DT_K(v - \phi) \le \int_\Omega w T_K(v - \phi) \\
\text{for all } \phi \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega).
\end{aligned}
\tag{3.1}
$$

If we take $\phi = T_l(v) \pm h(v)\xi \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega), l > 0$, as a test function in (3.1), where $h \in W^{1,\infty}(\mathbb{R})$ is compactly supported, $\xi \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$, and pass to the limit first with $l \to \infty$ and then with $K \to \infty$, we see that $(b(v), w) \in A_b$ implies

$$
\int_\Omega a(x, Dv) \cdot D(h(v)\xi) = \int_\Omega w h(v)\xi, \tag{3.2}
$$

see also [12]. Since the above equation holds true for all $h \in W^{1,\infty}(\mathbb{R})$ compactly supported, and all $\xi \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$, we may choose $\xi = S(v - \varphi)$, where $S \in \mathcal{P}, \varphi \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$, and $h = h_l$, where

$$
h_l(r) = \min((l + 1 - |r|)^+, 1), \quad r \in \mathbb{R}, \quad l > 0.
$$

Passing to the limit with $l \to \infty$ in (3.2) then shows that $(b(v), w) \in A_b$ implies

$$\int_\Omega a(x, Dv) \cdot DS(v - \varphi) = \int_\Omega wS(v - \varphi)\xi \tag{3.3}$$

for all $S \in \mathcal{P}$ and all $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Recall that by [13, Theorem 1], the abstract Volterra equation

$$\partial_t[k * (u - b(v_0))](t) + A_b(u(t)) \ni f(t) \tag{3.4}$$

admits for $b(u_0) \in \overline{D(A_b)}$ and $f \in L^1(0, T; L^1(\Omega))$ a unique generalized solution $u \in L^1(0, T; L^1(\Omega))$. For the special case $b = \mathrm{Id}$ it has been shown in [17, Theorem 7] that this generalized solution is an entropy solution to (1.1). It is a priori not clear in which sense the generalized solution satisfies (1.1) for general b . Note that $b = \mathrm{Id}$ entails that A_b is m-completely accretive, in which case the generalized solution satisfies the equation for sufficiently smooth data in the sense of distributions, a fact on which the cited existence proof relies on.

In order to overcome this difficulty, let $(f_\lambda)_{\lambda>0} \subset L^\infty(Q_T), (v_0^\lambda)_{\lambda>0} \subset L^\infty(\Omega)$, such that

$$\begin{aligned} f_\lambda \to f \qquad & \text{in } L^1(Q_T), \\ b(v_0^\lambda) \to b(v_0) \quad & \text{in } L^1(\Omega), \end{aligned}$$

as $\lambda \to 0$. We consider the approximating problem

$$L_\lambda(u_\lambda - b(v_0^\lambda))(t) + A_b(u_\lambda(t)) \ni f_\lambda(t), \quad u(0) = b(v_0^\lambda), \quad t \in (0,T), \lambda > 0, \tag{3.5}$$

where L_λ is the Yosida approximation of the operator L defined in (2.5) with $X = L^1(\Omega)$ and $p > 1$. According to [13, Theorem 5] the problem (3.5) admits a unique strong solution $u_\lambda = b(v_\lambda)$, i.e., $u_\lambda \in L^1(0,T;L^1(\Omega))$, and there exists a function $w_\lambda \in L^1(0,T;L^1(\Omega))$ such that $w_\lambda(t) \in A_b(u_\lambda(t))$ for almost every $t \in (0,T)$ and

$$L_\lambda(u_\lambda - b(v_0^\lambda))(t) + w_\lambda(t) = f_\lambda(t) \tag{3.6}$$

for almost every $t \in (0,T)$. Moreover, $u_\lambda \to u \in L^1(0,T;L^1(\Omega))$, where u is the generalized solution to the problem (1.1). The main result of this paper is the following.

Theorem 3.1. *Let $v_0 : \Omega \to \mathbb{R}$ be measurable with $u_0 = b(v_0) \in L^1(\Omega)$ and $f \in L^1(Q_T)$. Then the generalized solution u to (3.4) is of the form $u = b(v)$, where v is an entropy solution to (1.1).*

Proof. Let $u_\lambda = b(v_\lambda)$ be the strong solution to (3.5) and u be the generalized solution to (3.4). Our first aim is to show the existence of a measurable function $v : Q_T \to \mathbb{R}$ satisfying $b(v) = u$ and, selecting a subsequence if necessary, $v_\lambda \to v$ a.e. on Q_T. Since $u_\lambda \to u$ in $L^1(Q_T)$ we can extract a subsequence, still denoted by u_λ, such that $u_\lambda = b(v_\lambda) \to u$ a.e. on Q_T. Let $(t,x) \in Q_T$ such that $u_\lambda(t,x) \to u(t,x)$ and assume that $u(t,x) \in \operatorname{ran}(b) = (b_-, b^+)$, where $b_-, b^+ \in \overline{\mathbb{R}}, b_- < b^+$. Since b is strictly increasing and continuous, it follows $v_\lambda(t,x) \to v(t,x)$ for some $v(t,x) \in \mathbb{R}$ satisfying $b(v(t,x)) = u(t,x)$. If $u(t,x) \notin \operatorname{ran}(b)$, we may assume without loss of generality that $u(t,x) = b^+$ which clearly implies $v_\lambda(t,x) \to \infty =: v(t,x)$. It remains to show that $|v(t,x)| < \infty$ for almost every $(t,x) \in Q_T$. To this end, we choose for $K > 0$ the truncation $T_K(v_\lambda(t))$ as a test function in (3.5) to find

$$\begin{aligned}&\int_\Omega \partial_t[k_\lambda * (b(v_\lambda) - b(v_0^\lambda))](t) T_K(v_\lambda(t)) \\ &\qquad + \int_\Omega a(x, Dv_\lambda(t)) \cdot DT_K(v_\lambda(t)) \leq \int_\Omega f_\lambda(t) T_K(v_\lambda(t)),\end{aligned} \tag{3.7}$$

for almost every $t \in (0,T)$. Here, we used the representation (2.6) of the Yosida approximation and the characterisation (3.1) of A_b. If we apply Lemma 2.5 to the first term in (3.7), integrate in time over $(0,T)$, taking into account the coercivity condition (A2) of a, this results in

$$\begin{aligned}&\int_\Omega [k_\lambda * \int_0^{v_\lambda} T_K(\sigma)\,\mathrm{d}b(\sigma)](T) \\ &\quad + \int_{Q_T} [T_k(v_\lambda(t)) b(v_\lambda(t)) - \int_0^{v_\lambda(t)} T_K(\sigma)\,\mathrm{d}b(\sigma)] k_\lambda(t) \\ &\quad + \int_{Q_T} \int_0^t [\int_{v_\lambda(t)}^{v_\lambda(t-s)} T_K(\sigma)\,\mathrm{d}b(\sigma) \\ &\qquad\qquad - T_k(v_\lambda(t))(b(v_\lambda(t-s)) - b(v_\lambda(t)))][-k_\lambda'(s)]\,\mathrm{d}s \\ &\quad + c\int_{Q_T} |DT_k(v_\lambda(t))|^p \leq K\|f_\lambda\|_{L^1(Q_T)} + K\int_{Q_T} k_\lambda(t)|b(v_0^\lambda)| \leq KC\end{aligned} \tag{3.8}$$

for some constant $C = C(f, k, v_0, b) > 0$ independent of λ. Since all terms on the left-hand side in (3.8) are nonnegative, it follows by Poincaré's inequality that

$$|\{|v_\lambda| \geq K\}| \leq \frac{1}{K^p} \int_{Q_T} |T_K(v_\lambda)|^p \leq \frac{C}{K^p} \int_{Q_T} |DT_K(v_\lambda)|^p \leq CK^{1-p},$$

where C is a positive constant independent of λ. Passing to the limit with $\lambda \to 0$ and using $v_\lambda \to v$ a.e. on Q_T yield

$$|\{|v| \geq K\}| \leq CK^{1-p} \overset{K\to\infty}{\longrightarrow} 0,$$

which shows that $|\{|v| = \infty\}| = 0$. Consequently, $|v| < \infty$ a.e. on Q_T and therefore $b(v(t,x)) = u(t,x)$ for almost every $(t,x) \in Q_T$.

By (3.8), we see that $(DT_K(v_\lambda))_{\lambda>0}$ is a bounded sequence in $L^p(Q_T)^d$. Thus, $(T_K(v_\lambda))_{\lambda>0}$ is a bounded sequence in $L^p(0,T;W_0^{1,p}(\Omega))$, and we may assume that there exists $v_K \in L^p(0,T;W_0^{1,p}(\Omega))$ such that $T_K(v_\lambda) \rightharpoonup v_K$ weakly in $L^p(0,T;W_0^{1,p}(\Omega))$ as $\lambda \to 0$ along an appropriately chosen subsequence. Since $T_K(v_\lambda) \to T_K(v)$ a.e. on Q_T, we obtain $v_K = T_K(v)$ and therefore $T_K(v_\lambda) \rightharpoonup T_K(v)$ weakly in $L^p(0,T;W_0^{1,p}(\Omega))$ as $\lambda \to 0$. Moreover, the growth condition (A3) entails that $(a(x, DT_K(v_\lambda)))_{\lambda>0}$ is a bounded sequence in $L^{p'}(Q_T)^d$. Thus, there exist $\chi_K \in L^{p'}(Q_T)^d$ and a subsequence, still denoted the same way, such that

$$a(x, DT_K(v_\lambda)) \rightharpoonup \chi_K$$

weakly in $L^{p'}(Q_T)^d$. Our next aim is to prove that $\chi_K = a(x, DT_K(v))$. For $l > 0$ we denote by h_l the function defined by

$$h_l(u) = \min((l + 1 - |u|)^+, 1), \quad u \in \mathbb{R}.$$

We are going to show that, up to subsequences,

$$\begin{aligned} \liminf_{l\to\infty} \liminf_{\mu\to 0} \liminf_{\lambda\to 0} \int_{Q_\tau} &\partial_t[k_\lambda * (b(v_\lambda) - b(v_0^\lambda))] \\ &\times (T_K(v_\lambda) - h_l(v_\lambda)T_K(v)_\mu) \geq 0 \end{aligned} \tag{3.9}$$

for almost every $\tau \in (0, T)$. Here, $T_K(v)_\mu \in L^p(0, T; W_0^{1,p}(\Omega))$ is the time regularization of $T_K(v)$ introduced in Definition 2.2, and $Q_\tau := (0, \tau) \times \Omega$. As $k_\lambda \to k$ in $L^1(0, T)$ and $b(v_0^\lambda) \to b(v_0)$ in $L^1(\Omega)$ it follows $k_\lambda b(v_0^\lambda) \to kb(v_0)$ in $L^1(Q_T)$. By Lemma 2.3, we see that $T_K(v_\lambda) - h_l(v_\lambda)T_K(v)_\mu$ is uniformly bounded by $2K$ and since it converges a.e. on Q_T towards $T_K(v) - h_l(v)T_K(v)_\mu$ we may conclude that

$$\begin{aligned} &\int_{Q_\tau} k_\lambda b(v_0^\lambda)[T_K(v_\lambda) - h_l(v_\lambda)T_K(v)_\mu] \\ &\qquad \overset{\lambda\to 0}{\longrightarrow} \int_{Q_\tau} kb(v_0)[T_K(v) - h_l(v)T_K(v)_\mu] \end{aligned}$$

for every $\tau \in (0,T)$. Since $T_K(v)_\mu \to T_K(v)$ in $L^p(0,T;W_0^{1,p}(\Omega))$, we obtain

$$\int_{Q_\tau} kb(v_0)[T_K(v) - h_l(v)T_K(v)_\mu] \overset{\mu\to 0}{\longrightarrow} \int_{Q_\tau} kb(v_0)[T_K(v) - h_l(v)T_K(v)]$$

for every $\tau \in (0,T)$. As $h_l(u) \to 1$ for every $u \in \mathbb{R}$, we see by Lebesgue's theorem that

$$\int_{Q_\tau} kb(v_0)[T_K(v) - h_l(v)T_K(v)] \overset{l\to\infty}{\longrightarrow} 0$$

for every $\tau \in (0,T)$. Thus, it remains to show that

$$\liminf_{l\to\infty} \liminf_{\mu\to 0} \liminf_{\lambda\to 0} \int_{Q_\tau} \partial_t(k_\lambda * b(v_\lambda))(T_K(v_\lambda) - h_l(v_\lambda)T_K(v)_\mu) \geq 0$$

for almost every $\tau \in (0,T)$. If we apply the fundamental identity (2.10) to $\partial_t(k_\lambda * b(v_\lambda))h_l(v_\lambda)$, and, for some $\tau > 0$, integrate the resulting equality over

Q_τ we get

$$
\begin{aligned}
&-\int_{Q_\tau} \partial_t(k_\lambda * b(v_\lambda))h_l(v_\lambda)T_K(v)_\mu \\
&= -\int_{Q_\tau} \partial_t(k_\lambda * \int_0^{v_\lambda} h_l(\sigma)\,\mathrm{d}b(\sigma))T_K(v)_\mu \\
&\quad -\int_{Q_\tau} \big[h_l(v_\lambda)b(v_\lambda) - \int_0^{v_\lambda} h_l(\sigma)\,\mathrm{d}b(\sigma)\big]k_\lambda T_K(v)_\mu \\
&\quad -\int_{Q_\tau}\int_0^t \big[\int_{v_\lambda(t)}^{v_\lambda(t-s)} h_l(\sigma)\,\mathrm{d}b(\sigma) \\
&\qquad\qquad - h_l(v_\lambda(t))(b(v_\lambda(t-s)) - b(v_\lambda(t)))\big]\big[-k'_\lambda(s)\big]\,\mathrm{d}s\, T_K(v)_\mu \\
&=: -I^1_{\lambda,\mu,l} - I^2_{\lambda,\mu,l} - I^3_{\lambda,\mu,l}.
\end{aligned}
\tag{3.10}
$$

Define, for $l \in \mathbb{R}$, the functions $T^+_{l,l+1}(r) := T_{l,l+1}(\max(r,0))$, and $T^-_{l,l+1}(r) := T_{l,l+1}(-\max(-r,0))$. Since $h_l(\sigma) = T^-_{l,l+1}(\sigma) - T^+_{l,l+1}(\sigma) + 1$ it follows

$$
\begin{aligned}
I^2_{\lambda,\mu,l} = &\int_{Q_\tau} \big[T^-_{l,l+1}(v_\lambda)b(v_\lambda) - \int_0^{v_\lambda} T^-_{l,l+1}(\sigma)\,\mathrm{d}b(\sigma)\big]k_\lambda T_K(v)_\mu \\
&-\int_{Q_\tau} \big[T^+_{l,l+1}(v_\lambda)b(v_\lambda) - \int_0^{v_\lambda} T^+_{l,l+1}(\sigma)\,\mathrm{d}b(\sigma)\big]k_\lambda T_K(v)_\mu
\end{aligned}
$$

and

$$\begin{aligned}
I^3_{\lambda,\mu,l} = \int_{Q_\tau} \int_0^t \Big[\int_{v_\lambda(t)}^{v_\lambda(t-s)} & T^-_{l,l+1}(\sigma)\,\mathrm{d}b(\sigma) \\
& - T^-_{l,l+1}(v_\lambda(t))(b(v_\lambda(t-s)) - b(v_\lambda(t)))\Big]\big[-k'_\lambda(s)\big]\,\mathrm{d}s T_K(v)_\mu \\
- \int_{Q_\tau} \int_0^t \Big[\int_{v_\lambda(t)}^{v_\lambda(t-s)} & T^+_{l,l+1}(\sigma)\,\mathrm{d}b(\sigma) \\
& - T^+_{l,l+1}(v_\lambda(t))(b(v_\lambda(t-s)) - b(v_\lambda(t)))\Big]\big[-k'_\lambda(s)\big]\,\mathrm{d}s T_K(v)_\mu.
\end{aligned}$$

Here, we used that

$$b(v_\lambda) - \int_0^{v_\lambda} 1\,\mathrm{d}b(\sigma) = 0$$

and

$$\int_{v_\lambda(t)}^{v_\lambda(t-s)} 1\,\mathrm{d}b(\sigma) - (b(v_\lambda(t-s)) - b(v_\lambda(t))) = 0.$$

We will show that

$$\limsup_{l\to\infty} \limsup_{\mu\to 0} \limsup_{\lambda\to 0} (I^2_{\lambda,\mu,l} + I^3_{\lambda,\mu,l}) = 0.$$

For $l > 0$ we choose $T_{l,l+1}(v_\lambda)$ as a test function in (3.5). If we use that

$T_{l,l+1}(v_\lambda) = T_1(v_\lambda - T_l(v_\lambda))$, it follows by (3.1), (A2), and Lemma 2.5 that

$$
\begin{aligned}
& c\int_{Q_\tau} |DT_{l,l+1}(v_\lambda)|^p + \int_{Q_\tau} \big[T_{l,l+1}(v_\lambda)b(v_\lambda) - \int_0^{v_\lambda} T_{l,l+1}(\sigma)\,\mathrm{d}b(\sigma)\big]k_\lambda \\
&\quad + \int_{Q_\tau}\int_0^t \big[\int_{v_\lambda(t)}^{v_\lambda(t-s)} T_{l,l+1}(\sigma)\,\mathrm{d}b(\sigma) \\
&\qquad\qquad + T_{l,l+1}(v_\lambda(t))(b(v_\lambda(t-s)) - b(v_\lambda(t)))\big]\big[-k_\lambda'(s)\big]\,\mathrm{d}s \\
& \le \int_{Q_\tau\cap\{|v_\lambda|>l\}} |f_\lambda| + \int_{Q_\tau\cap\{|v_\lambda|>l\}} k_\lambda |b(v_0^\lambda)|
\end{aligned}
\tag{3.11}
$$

for every $\tau \in (0,T)$. Since $v_\lambda \to v$ a.e. on Q_T and $|v| < \infty$ a.e. on Q_T it follows

$$
\lim_{l\to\infty}\limsup_{\lambda\to 0} \int_{Q_\tau\cap\{|v_\lambda|>l\}} |f_\lambda| + \int_{Q_\tau\cap\{|v_\lambda|>l\}} k_\lambda |b(v_0^\lambda)| = 0.
$$

As all terms on the left-hand side in (3.11) are nonnegative, we may therefore conclude that, for every $\tau \in (0,T)$,

$$
\begin{aligned}
& \lim_{l\to\infty}\limsup_{\lambda\to 0} \int_{Q_\tau} |DT_{l,l+1}(v_\lambda)|^p = 0, \qquad (3.12) \\
& \lim_{l\to\infty}\limsup_{\lambda\to 0} \int_{Q_\tau} \big[T_{l,l+1}(v_\lambda)b(v_\lambda) - \int_0^{v_\lambda} T_{l,l+1}(\sigma)\,\mathrm{d}b(\sigma)\big]k_\lambda = 0, \\
& \lim_{l\to\infty}\limsup_{\lambda\to 0} \int_{Q_\tau}\int_0^t \big[\int_{v_\lambda(t)}^{v_\lambda(t-s)} T_{l,l+1}(\sigma)\,\mathrm{d}b(\sigma) \\
&\qquad\qquad + T_{l,l+1}(v_\lambda(t))(b(v_\lambda(t-s)) - b(v_\lambda(t)))\big]\big[-k_\lambda'(s)\big]\,\mathrm{d}s = 0.
\end{aligned}
$$

It is easily seen that the corresponding results hold true in case $T_{l,l+1}$ is replaced by $T^{\pm}_{l,l+1}$. In view of Lemma 2.3, it therefore follows

$$\limsup_{l\to\infty}\limsup_{\mu\to 0}\limsup_{\lambda\to 0}(I^2_{\lambda,\mu,l}+I^3_{\lambda,\mu,l})=0.$$

It remains to show that

$$\liminf_{l\to\infty}\liminf_{\mu\to 0}\liminf_{\lambda\to 0}\int_{Q_\tau}\partial_t(k_\lambda * b(v_\lambda))T_K(v_\lambda)-I^1_{\lambda,\mu,l}\geq 0,$$

for almost every $\tau\in(0,T)$. If we integrate (2.11) over Q_τ for some $\tau\in(0,T)$, we obtain

$$\begin{aligned}
&\int_{Q_\tau}\partial_t(k_\lambda * b(v_\lambda))T_K(v_\lambda)\\
&\quad=\int_\Omega[k_\lambda * \int_0^{v_\lambda}T_K(\sigma)\,\mathrm{d}b(\sigma)](\tau)\\
&\qquad+\int_{Q_\tau}\big[T_K(v_\lambda(t))b(v_\lambda(t))-\int_0^{v_\lambda(t)}T_K(\sigma)\,\mathrm{d}b(\sigma)\big]k_\lambda(t)\,\mathrm{d}t\\
&\qquad+\int_{Q_\tau}\int_0^t\big[\int_{v_\lambda(t)}^{v_\lambda(t-s)}T_K(\sigma)\,\mathrm{d}b(\sigma)\\
&\qquad\qquad-T_K(v_\lambda(t))(b(v_\lambda(t-s))-b(v_\lambda(t)))\big]\big[-k'_\lambda(s)\big]\,\mathrm{d}s\mathrm{d}t.
\end{aligned}\tag{3.13}$$

As it is our intention to pass to the limit with $\lambda\to 0$ in the preceding equality, note that the convergence $v_\lambda\to v$ a.e. on Q_T implies $\int_0^{v_\lambda}T_K(\sigma)\,\mathrm{d}b(\sigma)\to\int_0^{v}T_K(\sigma)\,\mathrm{d}b(\sigma)$ a.e. on Q_T. By the definition of the truncation functions T_K, we have $|\int_0^{v_\lambda}T_K(\sigma)\,\mathrm{d}b(\sigma)|\leq K|b(v_\lambda)|$, and since $b(v_\lambda)\to b(v)$ in $L^1(Q_T)$, we may suppose at least for a subsequence, still denoted the same way, that

$(b(v_\lambda))_{\lambda>0}$ is dominated by an $L^1(Q_T)-$function. By Lebesgue's theorem, we obtain

$$\int_0^{v_\lambda} T_K(\sigma)\,\mathrm{d}b(\sigma) \to \int_0^{v} T_K(\sigma)\,\mathrm{d}b(\sigma) \tag{3.14}$$

in $L^1(Q_T)$, and

$$\int_\Omega \int_0^{v_\lambda} T_K(\sigma)\,\mathrm{d}b(\sigma) \to \int_\Omega \int_0^{v} T_K(\sigma)\,\mathrm{d}b(\sigma)$$

in $L^1(0,T)$. As $k_\lambda \to k$ in $L^1(0,T)$, we may therefore conclude by Young's inequality that

$$\int_\Omega [k_\lambda * \int_0^{v_\lambda} T_K(\sigma)\,\mathrm{d}b(\sigma)](\cdot) \stackrel{\lambda\to 0}{\longrightarrow} \int_\Omega [k * \int_0^{v} T_K(\sigma)\,\mathrm{d}b(\sigma)](\cdot).$$

in $L^1(0,T)$ and, selecting a subsequence if necessary, a.e. on $(0,T)$. Note that the convergence (3.14) remains true, if the function T_K is replaced by any $S \in \mathcal{P}$. In particular, we have

$$\int_0^{v_\lambda} S(\sigma-\varphi)\,\mathrm{d}b(\sigma) \stackrel{\lambda\to 0}{\longrightarrow} \int_0^{v} S(\sigma-\varphi)\,\mathrm{d}b(\sigma) = B_{S,\varphi}(v) \text{ in } L^1(Q_T) \tag{3.15}$$

for all $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and all $S \in \mathcal{P}$. In order to pass to the limit in the remaining integrals of (3.13), note that the nonnegativity of all terms in

(3.8) entails

$$
\begin{aligned}
&\int_{Q_\tau} \big[T_K(v_\lambda(t))b(v_\lambda(t)) - \int_0^{v_\lambda(t)} T_K(\sigma)\,\mathrm{d}b(\sigma))\big]k_\lambda(t)\,\mathrm{d}t \\
&\quad + \int_{Q_\tau}\int_0^t \big[\int_{v_\lambda(t)}^{v_\lambda(t-s)} T_K(\sigma)\,\mathrm{d}b(\sigma) \\
&\qquad\qquad - T_K(v_\lambda(t))(b(v_\lambda(t-s)) - b(v_\lambda(t)))\big]\big[-k_\lambda'(s)\big]\,\mathrm{d}s\mathrm{d}t \le C
\end{aligned}
$$

for all $\tau \in (0,T)$ and some constant $C = C(K)$ independent of λ. As the kernel k satisfies (K1) and (K2), the lemma of Fatou yields

$$
\big[T_K(v)b(v) - \int_0^v T_K(\sigma)\,\mathrm{d}b(\sigma)\big]k \in L^1(Q_T), \tag{3.16}
$$

and

$$
\begin{aligned}
\chi_{(0,t)}\big[\int_{v(t)}^{v(t-s)} T_K(\sigma)\,\mathrm{d}b(\sigma) - T_K(v(t))(b(v(t-s)) - b(v(t)))\big]\big[-k'(s)\big] \\
\in L^1((0,T)\times Q_t).
\end{aligned} \tag{3.17}
$$

Moreover, we may pass to the limit in (3.13) with $\lambda \to 0$ which leads to

$$
\begin{aligned}
\liminf_{\lambda\to 0} \int_{Q_\tau} & \partial_t(k_\lambda * b(v_\lambda))T_K(v_\lambda) \\
\geq & \int_\Omega [k * \int_0^v T_K(\sigma)\,\mathrm{d}b(\sigma)](\tau) \\
& + \int_{Q_\tau} \big[T_K(v(t))b(v(t)) - \int_0^{v(t)} T_K(\sigma)\,\mathrm{d}b(\sigma)\big] k(t) \\
& + \int_{Q_\tau} \int_0^t \big[\int_{v(t)}^{v(t-s)} T_K(\sigma)\,\mathrm{d}b(\sigma) \\
& \qquad - T_K(v(t))(b(v(t-s)) - b(v(t)))\big] \big[-k'(s)\big]\,\mathrm{d}s
\end{aligned}
\tag{3.18}
$$

for almost every $\tau \in (0,T)$. On the other hand, the compact support of h_l implies that

$$
\partial_t(k_\lambda * \int_0^{v_\lambda} h_l(\sigma)\,\mathrm{d}b(\sigma)) \in L^{p'}(0,T;W^{-1,p'}(\Omega)).
$$

Thus,

$$
\begin{aligned}
I^1_{\lambda,\mu,l} &= \int_{Q_\tau} \partial_t(k_\lambda * \int_0^{v_\lambda} h_l(\sigma)\,\mathrm{d}b(\sigma))T_K(v)_\mu \\
&= \langle L_\lambda \int_0^{v_\lambda} h_l(\sigma)\,\mathrm{d}b(\sigma), J_\mu^{L^*} T_K(v)\rangle_{L^{p'}(0,\tau;W^{-1,p'}(\Omega))\times L^p(0,\tau;W_0^{1,p}(\Omega))} \\
&= \langle L_\mu J_\lambda^L \int_0^{v_\lambda} h_l(\sigma)\,\mathrm{d}b(\sigma), T_K(v)\rangle_{L^{p'}(0,\tau;W^{-1,p'}(\Omega))\times L^p(0,\tau;W_0^{1,p}(\Omega))}
\end{aligned}
\tag{3.19}
$$

for every $\tau \in (0,T)$. Here, we used that $v \in D(L)$ entails $J_\mu^L Lv = LJ_\mu^L v$ for any $\mu > 0$. In particular,

$$J_\mu^L L_\lambda v = L J_\mu^L J_\lambda^L v = L_\mu J_\lambda^L v$$

for all $v \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ and all $\lambda, \mu > 0$. As h_l is compactly supported, we may conclude that $\left(\int_0^{v_\lambda} h_l(\sigma)\,\mathrm{d}b(\sigma)\right)_{\lambda>0}$ is uniformly bounded. By Lebesgue's theorem, it follows

$$\int_0^{v_\lambda} h_l(\sigma)\,\mathrm{d}b(\sigma) \to \int_0^{v} h_l(\sigma)\,\mathrm{d}b(\sigma) \quad \text{in } L^{p'}(Q_T).$$

Since J_λ^L is a bounded operator satisfying $J_\lambda^L v \to v$ in $L^{p'}(0,T;W^{-1,p'}(\Omega))$ for all $v \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ as $\lambda \to 0$, the continuous embedding $L^{p'}(\Omega) \hookrightarrow W^{-1,p'}(\Omega)$ yields

$$J_\lambda^L \int_0^{v_\lambda} h_l(\sigma)\,\mathrm{d}b(\sigma) \to \int_0^{v} h_l(\sigma)\,\mathrm{d}b(\sigma) \quad \text{in} \quad L^{p'}(0,T;W^{-1,p'}(\Omega)).$$

Thus, the continuity of the Yosida approximation L_μ implies

$$\begin{aligned} I^1_{\lambda,\mu,l} &\xrightarrow{\lambda\to 0} \langle L_\mu \int_0^v h_l(\sigma)\,\mathrm{d}b(\sigma), T_K(v)\rangle_{L^{p'}(0,\tau;W^{-1,p'}(\Omega))\times L^p(0,\tau;W_0^{1,p}(\Omega))} \\ &= \int_{Q_\tau} \partial_t(k_\mu * \int_0^v h_l(\sigma)\,\mathrm{d}b(\sigma))T_K(v) \end{aligned}$$

for every $\tau \in (0,T)$. As it is our intention to apply the fundamental identity (2.10) to the term $\partial_t(k_\mu * \int_0^v h_l(\sigma)\,\mathrm{d}b(\sigma))T_K(v)$, we define $g_l(u) :=$

$\int_0^u h_l(\sigma)\,\mathrm{d}b(\sigma), u \in \mathbb{R}$ and

$$H(u) := \int_0^u T_K \circ g_l^{-1}(\sigma)\,\mathrm{d}\sigma, \quad u \in \operatorname{ran}(g_l).$$

Using the transformation $\sigma \mapsto g_l(\sigma)$, we see that

$$H(g_l(v(t))) = \int_0^{v(t)} T_K(\sigma)\,\mathrm{d}g_l(\sigma), \quad t \in (0,T).$$

By the definition of g_l, the measure dg_l is absolutely continuous with respect to db and the Radon-Nikodym derivative is given by h_l. Thus, the fundamental identity yields

$$\begin{aligned}
&-\int_{Q_\tau} \partial_t(k_\mu * \int_0^v h_l(\sigma)\,\mathrm{d}b(\sigma))T_K(v)\\
&\quad= -\int_\Omega (k_\mu * \int_0^v T_K(\sigma)h_l(\sigma)\,\mathrm{d}b(\sigma))(\tau)\\
&\quad- \int_{Q_\tau} \big[T_K(v(t)) \int_0^{v(t)} h_l(\sigma)\,\mathrm{d}b(\sigma) - \int_0^{v(t)} T_K(\sigma)h_l(\sigma)\,\mathrm{d}b(\sigma)\big]k_\mu(t)\\
&\quad- \int_{Q_\tau} \int_0^t \big[\int_{v(t)}^{v(t-s)} T_K(\sigma)h_l(\sigma)\,\mathrm{d}b(\sigma)\\
&\qquad\qquad - T_K(v(t)) \int_{v(t)}^{v(t-s)} h_l(\sigma)\,\mathrm{d}b(\sigma)\big]\big[-k_\mu'(s)\big]\mathrm{d}s.
\end{aligned} \tag{3.20}$$

Since $k_\mu \to k$ in $L^1(0,T)$, it follows from Young's inequality that

$$\int_\Omega [k_\mu * \int_0^v T_K(\sigma)h_l(\sigma)\,\mathrm{d}b(\sigma)] \xrightarrow{\mu\to 0} \int_\Omega [k * \int_0^v T_K(\sigma)h_l(\sigma)\,\mathrm{d}b(\sigma)]$$

in $L^1(0,T)$, and, selecting a subsequence if necessary, a.e. on $(0,T)$. Since $h_l(u) \to 1$ for every $u \in \mathbb{R}$, and $0 \le h_l \le 1$, we see that

$$\int_0^v T_K(\sigma) h_l(\sigma)\,\mathrm{d}b(\sigma) \to \int_0^v T_K(\sigma)\,\mathrm{d}b(\sigma)$$

a.e. on Q_T. Since the function b satisfies (B1) and $0 \le h_l \le 1$, it follows

$$0 \le \int_0^v T_K(\sigma) h_l(\sigma)\,\mathrm{d}b(\sigma) \le \int_0^v T_K(\sigma)\,\mathrm{d}b(\sigma)$$

a.e. on Q_T. Since the kernel k is nonnegative, Lebesgue's theorem implies

$$\int_\Omega [k * \int_0^v T_K(\sigma) h_l(\sigma)\,\mathrm{d}b(\sigma)] \overset{l\to\infty}{\longrightarrow} \int_\Omega [k * \int_0^v T_K(\sigma)\,\mathrm{d}b(\sigma)].$$

in $L^1(0,T)$ and, selecting a subsequence if necessary, a.e. on $(0,T)$. In order to pass to the limit in the remaining integrals of (3.20), note that the condition (B1) and $0 \le h_l \le 1$ also imply

$$\begin{aligned} 0 &\le T_K(v) \int_0^v h_l(\sigma)\,\mathrm{d}b(\sigma) - \int_0^v T_K(\sigma) h_l(\sigma)\,\mathrm{d}b(\sigma) \\ &\le T_K(v) b(v) - \int_0^v T_K(\sigma)\,\mathrm{d}b(\sigma) \end{aligned}$$

a.e. on Q_T, and

$$0 \le \int_{v(t)}^{v(t-s)} T_K(\sigma) h_l(\sigma)\,\mathrm{d}b(\sigma) - T_K(v(t)) \int_{v(t)}^{v(t-s)} h_l(\sigma)\,\mathrm{d}b(\sigma)$$
$$\le \int_{v(t)}^{v(t-s)} T_K(\sigma)\,\mathrm{d}b(\sigma) - T_K(v(t))(b(v(t-s)) - b(v(t)))$$

a.e. on $(0,T) \times Q_t$. By (3.16) and (3.17) we have

$$\big[T_K(v)b(v) - \int_0^v T_K(\sigma)\,\mathrm{d}b(\sigma)\big]k \in L^1(Q_T),$$

and

$$\chi_{(0,t)}\big[\int_{v(t)}^{v(t-s)} T_K(\sigma)\,\mathrm{d}b(\sigma) - T_K(v(t))(b(v(t-s)) - b(v(t)))\big]\big[-k'(s)\big]$$
$$\in L^1((0,T) \times Q_t).$$

Since k_μ is nonnegative, nonincreasing and satisfies (K1) and (K2), we may therefore apply Lebesgue's theorem to obtain

$$\lim_{l\to\infty}\lim_{\mu\to 0}\int_{Q_\tau}\big[T_K(v(t))\int_0^{v(t)} h_l(\sigma)\,\mathrm{d}b(\sigma)$$
$$- \int_0^{v(t)} T_K(\sigma)h_l(\sigma)\,\mathrm{d}b(\sigma)\big]k_\mu(t)$$
$$= \int_{Q_\tau}\big[T_K(v(t))b(v(t)) - \int_0^{v(t)} T_K(\sigma)\,\mathrm{d}b(\sigma)\big]k(t)$$

for every $\tau \in (0,T)$, and

$$\begin{aligned}
&\lim_{l\to\infty}\lim_{\mu\to 0}\int_{Q_\tau}\int_0^t \Big[\int_{v(t)}^{v(t-s)} T_K(\sigma)h_l(\sigma)\,\mathrm{d}b(\sigma) \\
&\qquad\qquad\qquad - T_K(v(t))\int_{v(t)}^{v(t-s)} h_l(\sigma)\,\mathrm{d}b(\sigma)\Big]\big[-k'_\mu(s)\big]\mathrm{d}s \\
&\quad = \int_{Q_\tau}\int_0^t \Big[\int_{v(t)}^{v(t-s)} T_K(\sigma)\,\mathrm{d}b(\sigma) \\
&\qquad\qquad\qquad - T_K(v(t))(b(v(t-s))-b(v(t)))\Big]\big[-k'(s)\big]\mathrm{d}s.
\end{aligned}$$

for every $\tau \in (0,T)$. Thus, passing to the limit in (3.19) yields

$$\begin{aligned}
&\lim_{l\to\infty}\lim_{\mu\to 0}\lim_{\lambda\to 0} I^1_{\lambda,\mu,l} \\
&\quad = \int_\Omega [k * \int_0^v T_K(\sigma)\,\mathrm{d}b(\sigma)](\tau) \\
&\qquad + \int_{Q_\tau} \big[T_K(v(t))b(v(t)) - \int_0^{v(t)} T_K(\sigma)\,\mathrm{d}b(\sigma))\big]k(t) \\
&\qquad + \int_{Q_\tau}\int_0^t \Big[\int_{v(t)}^{v(t-s)} T_K(\sigma)\,\mathrm{d}b(\sigma) \\
&\qquad\qquad\qquad - T_K(v(t))(b(v(t-s))-b(v(t)))\Big]\big[-k'(s)\big]\,\mathrm{d}s
\end{aligned} \tag{3.21}$$

for almost every $\tau \in (0,T)$. By (3.18) and (3.21), we obtain

$$\liminf_{l\to\infty}\liminf_{\mu\to 0}\liminf_{\lambda\to 0}\int_{Q_\tau} \partial_t(k_\lambda * b(v_\lambda))T_K(v_\lambda) - I^1_{\lambda,\mu,l} \geq 0$$

for almost every $\tau \in (0,T)$ which concludes the proof of (3.9).

Fix $\tau > 0$ such that (3.9) holds. If we choose $T_K(v_\lambda) - h_l(v_\lambda)T_K(v)_\mu$ as a test function in (3.5), and integrate over Q_τ, we obtain by using (3.1) and (3.2)

$$\begin{aligned}\int_{Q_\tau} &\partial_t[k_\lambda * (b(v_\lambda) - b(v_0^\lambda))](T_K(v_\lambda) - h_l(v_\lambda)T_K(v)_\mu) \\ &\quad + \int_{Q_\tau} a(x, Dv_\lambda) \cdot D[T_K(v_\lambda) - h_l(v_\lambda)T_K(v)_\mu] \\ &\leq \int_{Q_\tau} f_\lambda(T_K(v_\lambda) - h_l(v_\lambda)T_K(v)_\mu).\end{aligned}$$

Since

$$\lim_{l\to\infty}\lim_{\mu\to 0}\lim_{\lambda\to 0}\int_{Q_\tau} f_\lambda(T_K(v_\lambda) - h_l(v_\lambda)T_K(v)_\mu) = 0,$$

we see that (3.9) entails

$$\limsup_{l\to\infty}\limsup_{\mu\to 0}\limsup_{\lambda\to 0}\int_{Q_\tau} a(x, Dv_\lambda) \cdot D[T_K(v_\lambda) - h_l(v_\lambda)T_K(v)_\mu] \leq 0. \tag{3.22}$$

As $|h_l'(u)| = 1$ if $|u| \in (l, l+1)$ and $h'(u) = 0$ if $|u| > l+1$ or $|u| < l$, we have

$$\begin{aligned}\int_{Q_\tau} &a(x, Dv_\lambda) \cdot D[(h_l(v_\lambda) - 1)T_K(v_\lambda)] \\ &\leq \int_{Q_\tau} (h_l(v_\lambda) - 1)a(x, Dv_\lambda) \cdot DT_K(v_\lambda) \\ &\qquad + \int_{Q_\tau\cap\{l<|v_\lambda|<l+1\}} |a(x, Dv_\lambda) \cdot Dv_\lambda T_K(v_\lambda)|.\end{aligned}$$

Note that the first integral equals zero if $l > K$. If we apply the growth condition (A3) and Hölder's inequality, it follows

$$\int_{Q_\tau \cap \{l<|v_\lambda|<l+1\}} |a(x, Dv_\lambda) \cdot Dv_\lambda T_K(v_\lambda)| \le C \int_{Q_\tau \cap \{l<|v_\lambda|<l+1\}} |Dv_\lambda|^p,$$

for some constant $C = C(j, K) > 0$ independent of λ. Consequently, (3.12) shows that the inequality (3.22) implies

$$\limsup_{l\to\infty} \limsup_{\mu\to 0} \limsup_{\lambda\to 0} \int_{Q_\tau} a(x, Dv_\lambda) \cdot D[h_l(v_\lambda)(T_K(v_\lambda) - T_K(v)_\mu)] \le 0. \tag{3.23}$$

The growth condition (A3), Lemma 2.3 and Hölder's inequality imply the existence of a constant $C = C(j, K)$ such that

$$\begin{aligned} -\int_{Q_\tau} & h_l'(v_\lambda) a(x, Dv_\lambda) \cdot Dv_\lambda[(T_K(v_\lambda) - T_K(v)_\mu)] \\ & \le C \int_{Q_\tau \cap \{l<|v_\lambda|<l+1\}} |Dv_\lambda|^p. \end{aligned}$$

According to (3.23), it follows

$$\limsup_{l\to\infty} \limsup_{\mu\to 0} \limsup_{\lambda\to 0} \int_{Q_\tau} h_l(v_\lambda) a(x, Dv_\lambda) \cdot D[T_K(v_\lambda) - T_K(v)_\mu] \le 0. \tag{3.24}$$

Since $h_l(v_\lambda) = 0$ on $\{|v_\lambda| > l+1\}$, we see that

$$\begin{aligned}
\int_{Q_\tau \cap \{|v_\lambda|>K\}} & h_l(v_\lambda) a(x, Dv_\lambda) \cdot DT_K(v)_\mu \\
&= \int_{Q_\tau \cap \{|v_\lambda|>K\}} h_l(v_\lambda) a(x, DT_{l+1}(v_\lambda)) \cdot DT_K(v)_\mu \\
&= \int_{Q_\tau \cap \{|v_\lambda|>K\} \cap \{|v| \neq K\}} h_l(v_\lambda) a(x, DT_{l+1}(v_\lambda)) \cdot DT_K(v)_\mu \\
&\quad + \int_{Q_\tau \cap \{|v_\lambda|>K\} \cap \{|v| = K\}} h_l(v_\lambda) a(x, DT_{l+1}(v_\lambda)) \cdot DT_K(v)_\mu.
\end{aligned} \tag{3.25}$$

As $\mathbb{1}_{\{|v_\lambda|>K\}} \to \mathbb{1}_{\{|v|>K\}}$ a.e. on $\{|v| \neq K\}$ and $a(x, DT_{l+1}(v_\lambda)) \rightharpoonup \chi_{l+1}$ in $L^{p'}(Q_\tau)^d$ the first integral on the right hand side converges to

$$\int_{Q_\tau \cap \{|v|>K\}} h_l(v) \chi_{l+1} \cdot DT_K(v)_\mu$$

as $\lambda \to 0$. Since $\mathbb{1}_{\{|v_\lambda|>K\}} a(x, DT_{l+1}(v_\lambda))$ is bounded in $L^{p'}(Q_\tau \cap \{|v| = K\})^d$ there exists $\tilde{\chi}_{l+1} \in L^{p'}(Q_\tau \cap \{|v| = K\})^d$ and a subsequence, still indicated the same way, such that $\mathbb{1}_{\{|v_\lambda|>K\}} a(x, DT_{l+1}(v_\lambda)) \rightharpoonup \tilde{\chi}_{l+1}$ in $L^{p'}(Q_\tau \cap \{|v| = K\})^d$. Thus, the second integral on the right hand side in (3.25) converges to

$$\int_{Q_\tau \cap \{|v|=K\}} h_l(v) \tilde{\chi}_{l+1} \cdot DT_K(v)_\mu,$$

as $\lambda \to 0$. Since $T_K(v)_\mu \to T_K(v)$ in $L^p(0, T; W_0^{1,p}(\Omega))$ as $\mu \to 0$ it follows from (3.25) that

$$\lim_{\mu\to0}\lim_{\lambda\to0}\int_{Q_\tau\cap\{|v_\lambda|>K\}} h_l(v_\lambda)a(x,Dv_\lambda)\cdot DT_K(v)_\mu$$
$$=\int_{Q_\tau\cap\{|v|>K\}} h_l(v)\chi_{l+1}\cdot DT_K(v)+\int_{Q_\tau\cap\{|v|=K\}} h_l(v)\tilde{\chi}_{l+1}\cdot DT_K(v)=0,$$

where we used that $DT_K(v) = 0$ on $\{|v| \geq K\}$. Consequently, (3.24) is equivalent to

$$\limsup_{l\to\infty}\limsup_{\mu\to0}\limsup_{\lambda\to0}\int_{Q_\tau} h_l(v_\lambda)a(x,DT_K(v_\lambda))\cdot D[(T_K(v_\lambda)-T_K(v)_\mu)]\leq 0.$$

If $l > K$, we have $h_l(v_\lambda) = 1$ on $\{|v_\lambda| \leq K\}$ and we may conclude that

$$\limsup_{\mu\to0}\limsup_{\lambda\to0}\int_{Q_\tau} a(x,DT_K(v_\lambda))\cdot D[(T_K(v_\lambda)-T_K(v)_\mu)]\leq 0.$$

Since $DT_K(v)_\mu \to DT_K(v)$ in $L^p(Q_T)^d$ and $a(x, DT_K(v_\lambda)) \rightharpoonup \chi_K$ in $L^{p'}(Q_T)^d$, it follows that

$$\limsup_{\lambda\to0}\int_{Q_\tau} a(x,DT_K(v_\lambda))\cdot DT_K(v_\lambda)\leq\int_{Q_\tau}\chi_K\cdot DT_K(v). \tag{3.26}$$

The monotonicity condition (A1) entails

$$\int_{Q_\tau}(a(x,DT_K(v_\lambda))-a(x,H))\cdot(DT_K(v_\lambda)-H)\geq 0$$

for all $H \in L^p(Q_\tau)^d$ and all $\lambda > 0$. If we pass to the limit with $\lambda \to 0$ in the preceding inequality, using $DT_K(v_\lambda) \rightharpoonup DT_K(v)$ in $L^p(Q_T)^d$ and (3.26), we obtain

$$\int_{Q_\tau} (\chi_K - a(x,H)) \cdot (DT_K(v) - H) \geq 0$$

for all $H \in L^p(Q_\tau)^d$. As the operator $\mathcal{A} : L^p(Q_\tau)^d \to L^{p'}(Q_\tau)^d$ defined by $\mathcal{A}v = a(\cdot, v)$ is maximal monotone, it follows $\chi_K = a(x, DT_K(v))$ in $L^{p'}(Q_\tau)^d$. Since the inquality (3.9) holds true for almost every $\tau \in (0,T)$, and $a(x, DT_K(v)) \rightharpoonup \chi_K$ in $L^{p'}(Q_T)$, we may conclude that $\chi_K = a(x, DT_K(v))$ in $L^{p'}(Q_T)^d$.

We are now in a position to prove that $u = b(v)$ is an entropy solution to (1.1). Let $S \in \mathcal{P}, \xi \in \mathcal{D}([0,T)), \xi \geq 0, \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. If we choose $S(v_\lambda - \varphi)\xi$ as a test function in (3.5) and integrate in time over $(0,T)$, we obtain

$$\int_{Q_T} \partial_t[k_\lambda * (b(v_\lambda) - b(v_0^\lambda))] S(v_\lambda - \varphi)\xi \\ + \int_{Q_T} a(x, Dv_\lambda) \cdot DS(v_\lambda - \varphi)\xi = \int_{Q_T} f_\lambda S(v_\lambda - \varphi)\xi,$$

where we used (3.2). For $n \in \mathbb{N}$ we define $k_{1,\lambda,n} := (k_\lambda - n)^+$ and $k_{2,\lambda,n} := k_\lambda - k_{1,\lambda,n}$. Note that $k_{1,\lambda,n} + k_{2,\lambda,n} = k_\lambda$ and that $k_\lambda \to k$ in $L^1(0,T)$ implies $k_{1,\lambda,n} \to k_{1,n}$ and $k_{2,\lambda,n} \to k_{2,n}$ in $L^1(0,T)$, where $k_{1,n}, k_{2,n}$ are as in the definition of the entropy solution. If we apply Lemma 2.6 to $k_{1,\lambda,n}$ it follows

$$\begin{aligned}
&-\int_{Q_T} k_{1,\lambda,n} * \int_{v_0^\lambda}^{v_\lambda} S(\sigma-\varphi)\,\mathrm{d}b(\sigma)\xi_t \\
&\quad+\int_{Q_T} \partial_t[k_{2,\lambda,n} * (b(v_\lambda)-b(v_0^\lambda))]S(v_\lambda-\varphi)\xi \\
&\quad+\int_{Q_T} a(x,Dv_\lambda)\cdot DS(v_\lambda-\varphi)\xi \le \int_{Q_T} f_\lambda S(v_\lambda-\varphi)\xi.
\end{aligned} \tag{3.27}$$

By (3.15), we have $\int_0^{v_\lambda} S(\sigma-\varphi)\,\mathrm{d}b(\sigma) \to \int_0^v S(\sigma-\varphi)\,\mathrm{d}b(\sigma) = B_{S,\varphi}(v)$ in $L^1(Q_T)$ and $\int_0^{v_0^\lambda} S(\sigma-\varphi)\,\mathrm{d}b(\sigma) \to \int_0^{v_0} S(\sigma-\varphi)\,\mathrm{d}b(\sigma)$ in $L^1(\Omega)$. Thus,

$$-\int_{Q_T} k_{1,\lambda,n} * \int_{v_0^\lambda}^{v_\lambda} S(\sigma-\varphi)\,\mathrm{d}b(\sigma)\xi_t \xrightarrow{\lambda\to 0} -\int_{Q_T} [k_{1,n} * (B_{S,\varphi}(v)-B_{S,\varphi}(v_0))\xi_t.$$

The triangle inequality yields

$$\begin{aligned}
&\|\partial_t k_{2,\lambda,n} * (b(v_\lambda)-b(v_0^\lambda)) - \partial_t k_{2,n} * (b(v)-b(v_0))\|_{L^1(Q_T)} \\
&\quad\le \|\partial_t k_{2,\lambda,n} * [(b(v_\lambda)-b(v_0^\lambda)) - (b(v)-b(v_0))]\|_{L^1(Q_T)} \\
&\qquad+ \|\partial_t k_{2,\lambda,n} * (b(v)-b(v_0)) - \partial_t k_{2,n} * (b(v)-b(v_0))\|_{L^1(Q_T)} \\
&\quad=: A_\lambda^n + B_\lambda^n.
\end{aligned}$$

As $k_{2,\lambda,n} \in W^{1,1}(0,T)$ satisfies $0 \le k_{2,\lambda,n}(0) \le n$ for all $\lambda > 0$ and $n \in \mathbb{N}$, it follows by Young's inequality that

$$\begin{aligned}
A_\lambda^n &\le n\|b(v_\lambda)-b(v_0^\lambda)-(b(v)-b(v_0))\|_{L^1(Q_T)} \\
&\quad+ \|k'_{2,\lambda,n}\|_{L^1(0,T)}\|b(v_\lambda)-b(v_0^\lambda)-(b(v)-b(v_0))\|_{L^1(Q_T)}
\end{aligned}$$

for all $\lambda > 0$ and $n \in \mathbb{N}$. Since $k'_{2,\lambda,n}$ is nonnegative and nonincreasing, we have

$$\|k'_{2,\lambda,n}\|_{L^1(0,T)} \leq k_{2,\lambda,n}(0) \leq n$$

for all $\lambda > 0$ and $n \in \mathbb{N}$, which shows that $A^n_\lambda \to 0$ as $\lambda \to 0$. Moreover, we see that $\sup_{\lambda>0}(\mathrm{var}_{[0,T]} k_{2,\lambda,n}) \leq n$ for every $n \in \mathbb{N}$. Thus, we may conclude by [13, Lemma 3.4] that $B^n_\lambda \to 0$ as $\lambda \to 0$. It follows $\partial_t k_{2,\lambda,n} * [b(v_\lambda) - b(v_0^\lambda)] \to \partial_t k_{2,n} * [b(v) - b(v_0)]$ in $L^1(Q_T)$ as $\lambda \to 0$. Since $S \in \mathcal{P}$ is a bounded, continuous function, we therefore obtain

$$\begin{aligned}\int_{Q_T} &\partial_t[k_{2,\lambda,n} * (b(v_\lambda) - b(v_0^\lambda))] S(v_\lambda - \varphi)\xi \\ &\xrightarrow{\lambda\to 0} \int_{Q_T} \partial_t[k_{2,n} * (b(v) - b(v_0))] S(v - \varphi)\xi.\end{aligned}$$

In order to pass to the limit in the third integral on the left-hand side in (3.27), let $R > 0$ such that $\operatorname{supp} S' \subset [-R, R]$ and define $K := \|\varphi\|_\infty + R$.

By the monotonicity assumption (A1) it follows

$$\begin{aligned}
&\int_{Q_T} a(x, Dv_\lambda) \cdot DS(v_\lambda - \varphi)\xi \\
&\qquad = \int_{Q_T} a(x, DT_K(v_\lambda)) \cdot D[T_K(v_\lambda) - DT_K(v)]S'(v_\lambda - \varphi)\xi \\
&\qquad\quad + \int_{Q_T} a(x, DT_K(v_\lambda)) \cdot DT_K(v)S'(v_\lambda - \varphi)\xi \\
&\qquad\quad - \int_{Q_T} a(x, DT_K(v_\lambda)) \cdot D\varphi S'(v_\lambda - \varphi)\xi \\
&\qquad \geq \int_{Q_T} a(x, DT_K(v)) \cdot D[T_K(v_\lambda) - DT_K(v)]S'(v_\lambda - \varphi)\xi \\
&\qquad\quad + \int_{Q_T} a(x, DT_K(v_\lambda)) \cdot D[T_K(v) - \varphi]S'(v_\lambda - \varphi)\xi.
\end{aligned}$$

As S' is for every $S \in \mathcal{P}$ a bounded, continuous function, the convergence $v_\lambda \to v$ almost everywhere on Q_T and Lebesgue's theorem imply

$$D[T_K(v) - \varphi]S'(v_\lambda - \varphi) \to DS(v - \varphi) \quad \text{in } L^p(Q_T)^d$$

and

$$a(x, DT_K(v))S'(v_\lambda - \varphi) \to a(x, DT_K(v))S'(v - \varphi) \text{ in } L^{p'}(Q_T)^d.$$

Since $a(x, DT_K(v_\lambda)) \rightharpoonup a(x, DT_K(v))$ weakly in $L^{p'}(Q_T)^d$ and $DT_K(v_\lambda) \rightharpoonup DT_K(v)$ weakly in $L^p(Q_T)^d$, we may pass to the limit in the preceding inequality to obtain

$$\liminf_{\lambda\to 0}\int_{Q_T} a(x,Dv_\lambda)\cdot DS(v_\lambda-\varphi)\xi \geq \int_{Q_T} a(x,Dv)\cdot DS(v-\varphi)\xi.$$

Since, by Lebesgue's theorem,

$$\int_{Q_T} f_\lambda S(v_\lambda-\varphi)\xi \xrightarrow{\lambda\to 0} \int_{Q_T} fS(v-\varphi)\xi,$$

we may conclude that v is an entropy solution to (1.1). □

References

[1] K. Ammar and P. Wittbold. Existence of renormalized solutions of degenerate elliptic-parabolic problems. *Proc. Roy. Soc. Edinburgh Sect. A*, 133(3):477–496, 2003.

[2] Ph. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, and J.L. Vázquez. An L^1-theory of existence and uniqueness of solutions of non-linear elliptic equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 22(2):241–273, 1995.

[3] D. Blanchard, F. Petitta, and H. Redwane. Renormalized solutions of nonlinear parabolic equations with diffuse measure data. *Manuscripta Math.*, 141(3-4):601–635, 2013.

[4] D. Blanchard and H. Redwane. Existence of a solution for a class of

parabolic equations with three unbounded nonlinearities, natural growth terms and L^1 data. *Arab J. Math. Sci.*, 20(2):157–176, 2014.

[5] A. V. Bobylev and C. Cercignani. The inverse Laplace transform of some analytic functions with an application to the eternal solutions of the Boltzmann equation. *Appl. Math. Lett.*, 15(7):807–813, 2002.

[6] M. Caputo. Diffusion of fluids in porous media with memory. *Geothermics*, 28(1):113 – 130, 1999.

[7] Ph. Clément and E. Mitidieri. Qualitative properties of solutions of Volterra equations in Banach spaces. *Israel J. Math.*, 64(1):1–24, 1988.

[8] Ph. Clément and J. A. Nohel. Asymptotic behavior of solutions of nonlinear Volterra equations with completely positive kernels. *SIAM J. Math. Anal.*, 12(4):514–535, 1981.

[9] Ph. Clément and J. Prüss. Completely positive measures and Feller semigroups. *Math. Ann.*, 287(1):73–105, 1990.

[10] A. Dall'Aglio and L. Orsina. Nonlinear parabolic equations with natural growth conditions and L^1-data. *Nonlinear Anal.*, 27(1):59 – 73, 1996.

[11] J. I. Díaz, T. Pierantozzi, and L. Vázquez. Finite time extinction for nonlinear fractional evolution equations and related properties. *Electr. J. Differ. Equ.*, 239:1–13, August 2016.

[12] J. Droniou and A. Prignet. Equivalence between entropy and renormalized solutions for parabolic equations with smooth measure data. *NoDEA Nonlinear Differential Equations Appl.*, 14(1-2):181–205, 2007.

[13] G. Gripenberg. Volterra integro-differential equations with accretive nonlinearity. *J. Differential Equations*, 60(1):57 – 79, 1985.

[14] G. Gripenberg, S.-O. Londen, and O. J. Staffans. *Volterra integral and functional equations*. Encyclopedia of mathematics and its applications. Cambridge University Press, Cambridge, England, New York, 1990.

[15] H. J. Haubold, A. M. Mathai, and R. K. Saxena. Mittag-Leffler functions and their applications. *J. Appl. Math.*, page 51 pp., 2011.

[16] V. G. Jakubowski. *Nonlinear elliptic-parabolic integro-differential equations with L^1-data: existence, uniqueness, asymptotics*. PhD thesis, Universität Duisburg-Essen, 2001.

[17] V. G. Jakubowski and P. Wittbold. On a nonlinear elliptic-parabolic integro-differential equation with L^1-data. *J. Differential Equations*, 197(2):427–445, 2004.

[18] R. Landes. On the existence of weak solutions of perturbated systems with critical growth. *J. Reine Angew. Math.*, 393:21–38, 1989.

[19] R. Metzler and J. Klafter. The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Phys. Rep.*, 339(1):77, 2000.

[20] A. Prignet. Remarks on existence and uniqueness of solutions of elliptic problems with right-hand side measures. *Rend. Mat. Appl. (7)*, 15(3):321–337, 1995.

[21] J. Serrin. Pathological solutions of elliptic differential equations. *Ann. Scuola Norm. Sup. Pisa (3)*, 18:385–387, 1964.

[22] V. Vergara and R. Zacher. Lyapunov functions and convergence to steady state for differential equations of fractional order. *Math. Z.*, 259(2):287–309, 2008.

[23] V. Vergara and R. Zacher. A priori bounds for degenerate and singular evolutionary partial integro-differential equations. *Nonlinear Anal.*, 73(11):3572 – 3585, 2010.

[24] V. Vergara and R. Zacher. Optimal decay estimates for time-fractional and other nonlocal subdiffusion equations via energy methods. *SIAM J. Math. Anal.*, 47(1):210–239, 2015.

[25] R. Zacher. Weak solutions of abstract evolutionary integro-differential equations in Hilbert spaces. *Funkcial. Ekvac.*, 52(1):1–18, 2009.