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Preprint 2017-07

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Preprint 2017-02 April 2017

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Strain gradient visco-plasticity with dislocation densities contributing to the energy

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April 18, 2017

Abstract: We consider the energetic description of a visco-plastic evolution and derive an existence result. The energies are convex, but not necessarily quadratic. Our model is a strain gradient model in which the curl of the plastic strain contributes to the energy. Our existence results are based on a time-discretization, the limit procedure relies on Helmholtz decompositions and compensated compactness.

Key-words: visco-plasticity, strain gradient plasticity, energetic solution, div-curl lemma

MSC: 74C10, 49J45

1 Introduction

The quasi-stationary evolution of a visco-plastic body is analyzed in an energetic approach. We use the framework of infinitesimal plasticity with an additive decomposition of the strain. The equations are described with three functionals, the elastic energy, the plastic energy, and the dissipation. The three functionals are assumed to be convex, but not necessarily quadratic. In this sense, we study a three-fold non-linear system.

Our interest is to include derivatives of the plastic strain p in the free energy. We are therefore dealing with a problem in the context of strain gradient plasticity. Of particular importance are contributions of $\mathrm{curl}(p)$ to the plastic energy, since this term measures the density of dislocations. Attributing an energy to plastic deformations means that hardening of the material is modelled. Since derivatives of p contribute to the energy, the model introduces a length scale in the plasticity problem; this is desirable for the explanation of some experimental results.

We treat a model that was introduced in [16], with analysis available in [11], [21], and [22]. We discuss the literature below in Section 1.2. The model is entirely based on energies and is thermodynamically consistent. Our main result regards well-posedness of the system. We use the framework of energetic solutions to derive an existence result.

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1.1 Model and main results

We denote the deformation by u and decompose the gradient into an elastic and a plastic part, $\nabla u = e + p$. We do not use the symmetrization in the decomposition, but note that only the symmetric part $\mathrm{sym}(e)$ contributes to the elastic energy. The elastic energy $\mathcal{W}_e$ describes the elastic response of the material and the plastic energy $\mathcal{W}_p$ describes hardening effects and gradient plasticity. The precise form that we study in this contribution is given in (2.2)–(2.6); in a special case (setting $H_e = 0$, $H_p = 0$, $r = 2$), the two energies read

$$\mathcal{W}_e(\nabla u, p) = \int_\Omega Q(\mathrm{sym}(\nabla u - p))\,, \tag{1.1}$$

$$\mathcal{W}_p(p) = \int_\Omega |\,\mathrm{curl}(p)|^2 + \delta|\nabla p|^2\,, \tag{1.2}$$

where Q associates to a symmetric matrix an elastic energy, and $\delta \geq 0$ is a real parameter. Below, we write the elastic energy density in the form $W_e(F,p) = Q(\mathrm{sym}(F - p))$. The two energies are accompanied by a dissipation rate functional $\mathcal{R}$ with convex dual $\mathcal{R}^*$. The latter is used to express the flow rule of the plastic strain. We do not consider positively 1-homogeneous functionals $\mathcal{R}$ in this work; we hence treat here a visco-plastic model and not a rate-independent model.

We use the following variables: The deformation u with the two parts e and p of the gradient. The stress σ depends on elastic deformations and is, as usual, given by the functional derivative of the elastic energy. The gradient plasticity model uses one additional variable, the back-stress variable Σ. The back-stress Σ controls the evolution of the plastic strain and is given by a functional derivative of the total free energy $\mathcal{W} = \mathcal{W}_e + \mathcal{W}_p$ with respect to p. In its strong form, the plastic evolution problem reads

$$-\nabla \cdot \sigma = f\,, \tag{1.3}$$

$$\sigma = \mathrm{sym}\ \nabla_F W_e(\nabla u, p)\,, \tag{1.4}$$

$$-\Sigma \in \partial_p \mathcal{W}(\nabla u, p)\,, \tag{1.5}$$

$$\partial_t p \in \partial \mathcal{R}^*(\Sigma)\,. \tag{1.6}$$

The variational structure of the system can be made even more apparent by writing the two equations (1.3)–(1.4) equivalently as $f \in \partial_u \mathcal{W}(\nabla u, p)$.

In our results, we treat more general energies than those of (1.1)–(1.2). The additional term $H_p(p)$ of (2.4) allows to introduce more general hardening laws. The term $H_e(\nabla^s u)$ of (2.2) allows to associate an infinite energy to deformations with self-penetration (we emphasize that we do not introduce growth assumptions for H_p and H_e).

Main results. Our main results are existence theorems for the above system of equations. A major difficulty in the analysis is the non-linear character of the two equations (1.4) and (1.5) (in combination with the flow rule (1.6), which is always non-linear). Furthermore, the plastic energy contains the quantity $\mathrm{curl}(p)$; this means that the back-stress Σ contains the contribution $\mathrm{curl}(\mathrm{curl}(p))$.

The proofs rely on an energetic formulation, avoiding the deformation variable u. This is possible through the use of a marginal energy, compare for example [24] and [19]. This concept makes the gradient flow structure of the problem even more apparent. We construct approximate solutions through a time discretization of the problem, solving a stationary variational problem in each time step. At this point, the explicit time dependence of the energies must be treated with care, since the time dependence involves the deformation u. The limit procedure relies on (compensated) compactness properties of the sequence of approximate solutions. In the case $\delta = 0$ we need Helmholtz decompositions and the div-curl lemma in order to perform the limit procedure. The limit functions are shown to be energetic solutions to the system.

In our first theorem, we treat the case $\delta > 0$ and quite general energies. In this case, the plastic energy provides estimates for all derivatives of p and hence compactness of approximating sequences. Theorem 2.5 makes the following statement precise: Given a load f, an initial datum p_0, a time horizon $T > 0$, and $\Omega \subset \mathbb{R}^3$, there exists an energetic solution to system (1.3)–(1.6) on $\Omega \times (0, T)$.

Our second theorem treats the case that the plastic energy contains $\operatorname{curl}(p)$, but not the full gradient of p (i.e.: $\delta = 0$). In this case, only estimates for certain derivatives of p are at our disposal and space-time L^2-compactness of approximate solutions cannot be expected. Compensated compactness nevertheless allows to derive our second existence result; loosely speaking, the control of $\operatorname{curl}(p)$ is dual to the control of $\nabla \cdot \sigma$. While quite general energies are treated in Theorem 2.5, we can deal with the case $\delta = 0$ only for certain energies, essentially those of (1.1)–(1.2). Theorem 2.6 states the existence of solutions for $\delta = 0$.

1.2 Discussion and comparison with the literature

On the plasticity model. The importance of strain gradient models to describe the plastic deformation of metal is well-known, we mention [9] for an early model and the discussion of experiments with thin copper wires. For a comparison of different approaches, see [5]. The physical basis of a strain gradient model is a higher order contribution to the energy: Kröner's formula uses the curl of the plastic strain to measure the dislocation density (cp. [29] for a recent discussion). Hence $\operatorname{curl}(p)$ contributes via the self-energy of dislocations to the total energy [23]. For an analytical investigation of the energy based on single crystal plasticity we refer to [7]. In comparison to our model we note that the energy contribution of formula (2.4) in [7] is an L^1-norm of the curl, and not a squared contribution as in (1.2). We regard the energy of (1.2) as an approximation that regularizes the mathematically derived (single crystal) L^1-type energy.

The strain gradient model of this work appears e.g. in [16]. Writing their variables h^p, α^p, σ^{dis} as p, $\operatorname{curl}(p)$, Σ, our flow rule (1.6) appears as equation (12), the subdifferential description of $-\Sigma$ in (1.5) appears in Remark 2.2 of [16]. A first mathematical discussion of the resulting system was performed in [21]. Existence results were shown in [11] and [22] in the case of quadratic energies as in (1.1)–(1.2). Of a more general nature is the approach of [19], which allows to treat also non-convex energies; we give a more detailed comparison below.

In the context of finite strain elastoplasticity (i.e. with a multiplicative decomposition of the gradient) we are only aware of models that take the full gradient of p into account in the energy. An existence result for the single time-step in this situation is derived in [18]. The time continuous problem was solved in [20] for multiplicative visco-plasticity, and in [15] for multiplicative gradient plasticity. We recall that the full gradient of p was used in these contributions and note that quite restrictive growth conditions must be imposed on the energy contributions.

Let us conclude this discussion of the plasticity model with a remark concerning the non-quadratic character of the energies (and plasticity models in general). Plasticity models with an additive decomposition of the strain are sometimes criticized for the following reason: The assumption of an infinitesimal deformation does not fit well with the non-linear character of the plasticity system (in particular, of the flow rule), since the non-linear character of a system becomes relevant only at finite deformations. But the difficulty can be reconciled with a proper rescaling of the system: In the case of small deformations we consider configurations of the form $\Phi = \mathrm{id} + \varepsilon\, u$, where $\varepsilon > 0$ is a small parameter and u (a quantity of order 1) is the rescaled deformation. If, in this scaling, a non-linear function $\mathcal{F}$ of the physical equations is of the form $\mathcal{F}(\Phi) = \mathcal{G}((\nabla\Phi - \mathrm{id})/\varepsilon)$, then the quantity u is described by a non-linear system that involves $\mathcal{G}$—even in the limit $\varepsilon \to 0$.

For the above reasons, we are convinced that non-quadratic energies should be considered in a plasticity problem, even if the framework of infinitesimal deformations is used. For the general setting of plasticity models see [1] and [13], we also mention [8] for critical comments concerning current finite strain plasticity models.

Methods of proof and relations to other mathematical results. We derive existence results with the help of a time discretization. The time discrete solutions are found with variational arguments. The energies provide a priori estimates for the sequence of approximate solutions and we obtain easily the existence of (weak) limits. These are our candidates for a solution. The main task is to perform the limit procedure, i.e. to show that the weak limits provide a solution.

For the limit procedure, we work in the setting of energetic solutions [17]. In this approach, a weak solution is defined as a tuple of functions that satisfies two relations: a stability property in every time instance t and a (time integrated) energy inequality (compare our conditions (S_1) and (E_1) in Definition 2.2). Since our model uses additionally the back-stress variable, we have to accompany the solution concept with condition (F_1) to relate Σ to the other quantities. We find it actually helpful to work with an even more condensed system of equations in which the deformation u is not used explicitely, see conditions (F_2) and (E_2) in Definition 2.3. The limit procedure makes use of lower semicontinuity of functionals and of compactness properties.

We already mentioned the existence results of [11] and [22]. These results are also based on time discretizations of the problem. Since only quadratic energies are studied in both [11] and [22], the two relations (1.4) and (1.5) are linear in their case. This allows for a much more direct derivation of the limit equations.

Let us compare our results once more with those of [19]. Their system can be related to our model by replacing their variables Φ, z, I, $\mathcal{E}^1$, I^2, ψ by our variables u,

p, W, W_p, W_e, R. We have less restrictive assumptions on Q (compare the square-root growth assumption on D_zW in (W_3) of [19]). Another difference is that in [19] the values of z (our p) are confined to a compact set by using an indicator function in the energy. Furthermore, their exponent r in $|\nabla p|^r$ must be larger than the space dimension. We finally recall that [19] always uses the full gradient of $z \sim p$, while we treat also the case $\delta = 0$. On the other hand: We demand the convexity of all energies. In this point the contribution [19] treats a much more general setting.

For other results containing the full gradient (the case $\delta > 0$), we mention the book by Roubicek [25] and note that the explicit time dependence of the energy (that is present in our model due to the dependence of $\mathcal{E}_1$ on f) is not covered in his result ($\partial_t f \equiv 0$ in our setting). The same remark is valid concerning the abstract result of Colli and Visintin [6].

The remainder of this article is organized as follows. In Section 2 we state the model in a precise way and formulate the assumptions on the data. In Section 3 we show a stability property of our solution concept: Under certain assumptions, sequences of approximate solutions converge to solutions. In Section 4 we perform the time discretization and construct approximate solutions. The stability results of Section 3 can be applied and yield that the limit functions provide a solution to the time-continuous model.

2 Model equations and assumptions

For a domain $\Omega \subset \mathbb{R}^3$ and a space-time cylinder $\Omega_T := \Omega \times (0,T)$ we denote the deformation by $u : \Omega_T \to \mathbb{R}^3$ and the plastic strain by $p : \Omega_T \to \mathbb{R}^{3\times 3}$. Following [16] and [21], we do not impose that the plastic strain is symmetric; to incorporate rotational invariance of the energy, the elastic energy depends only on $e = \mathrm{sym}(\nabla u - p)$. We write here $\mathbb{R}^{3\times 3}_s$ for the space of symmetric $n \times n$ matrices and denote the projection onto symmetric matrices by $\mathrm{sym} : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}_s$, $F \mapsto \frac{1}{2}(F + F^T)$. We use also the notation $\nabla^s u := \mathrm{sym}\,\nabla u$ such that $\nabla^s u = e + \mathrm{sym}(p)$.

Despite its importance in mechanics, for simplicity of the presentation, we do not incorporate the decomposition into spherical and deviatoric parts in our model.

We use an energetic approach and formulate the plasticity equations with energies and dissipation rate functionals. We use an elastic energy $\mathcal{W}_e$, a plastic energy $\mathcal{W}_p$, and the total energy $\mathcal{W} = \mathcal{W}_e + \mathcal{W}_p$. The energies are based on energy density functions. We use the following four functions:

$$\begin{aligned} &Q : \mathbb{R}^{3\times 3}_s \to \mathbb{R},\ e \mapsto Q(e)\,, &&R : \mathbb{R}^{3\times 3} \to \mathbb{R},\ q \mapsto R(q)\,,\\ &H_e : \mathbb{R}^{3\times 3}_s \to \mathbb{R},\ \nabla^s u \mapsto H_e(\nabla^s u), &&H_p : \mathbb{R}^{3\times 3} \to \mathbb{R},\ p \mapsto H_p(p)\,. \end{aligned} \tag{2.1}$$

The elastic energy density $W_e : \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3} \to \mathbb{R}$ is

$$W_e(\nabla u, p) := Q(\mathrm{sym}(\nabla u - p)) + H_e(\nabla^s u)\,. \tag{2.2}$$

We are interested in a strain gradient plasticity model. To introduce derivatives of p in the energies, we use a factor $\delta \geq 0$ and an exponent $r \in \mathbb{R}$, $r > \frac{6}{5}$. We furthermore consider the curl of $p = (p_{i,j})_{i,j}$, which we define row-wise:

$(\operatorname{curl} p)_{k,\cdot} := \operatorname{curl}(p_{k,\cdot})$. With this definition, every smooth function $\varphi : \Omega \to \mathbb{R}^3$ satisfies $(\operatorname{curl} \nabla\varphi)_{k,\cdot} = \operatorname{curl}((\nabla\varphi)_{k,\cdot}) = \operatorname{curl} \nabla\varphi_k = 0$. For functions $u : \Omega \to \mathbb{R}^3$, $p : \Omega \to \mathbb{R}^{3\times 3}$ we define the energies

$$\mathcal{W}_e(\nabla u, p) := \int_\Omega W_e(\nabla u, p)\,, \tag{2.3}$$

$$\mathcal{W}_p(p) := \int_\Omega \left\{ H_p(p) + |\operatorname{curl} p|^2 + \delta |\nabla p|^r \right\}\,, \tag{2.4}$$

$$\mathcal{R}(q) := \int_\Omega R(q)\,, \tag{2.5}$$

and use the two convex duals R^* and $\mathcal{R}^*$. The total energy is

$$\mathcal{W}(\nabla u, p) := \mathcal{W}_e(\nabla u, p) + \mathcal{W}_p(p)\,. \tag{2.6}$$

General assumptions

We next collect our assumptions on the energy functionals, on initial and boundary conditions, and on the applied loads.

Assumption 2.1 (Energy and dissipation functional)**.** *Let $Q : \mathbb{R}^{3\times 3}_s \to [0,\infty)$ and $R, R^* : \mathbb{R}^{3\times 3} \to [0,\infty)$ be convex and continuous, and let $H_e : \mathbb{R}^{3\times 3}_s \to [0,\infty]$ and $H_p : \mathbb{R}^{3\times 3} \to [0,\infty]$ be convex, proper and lower-semicontinuous, with* $0 \in \operatorname{dom}(H_e)$.

We assume that Q, R, R^* have quadratic growth: For $0 < c < C$ holds

$$c|\zeta|^2 \le Q(\zeta) \le C|\zeta|^2\,, \tag{2.7}$$

for all matrices $\zeta \in \mathbb{R}^{3\times 3}_s$, and

$$c|\xi|^2 \le R(\xi) \quad \textit{and} \quad R^*(\xi) \le C|\xi|^2\,, \tag{2.8}$$

$$c|\xi|^2 \le R^*(\xi) \quad \textit{and} \quad R(\xi) \le C|\xi|^2\,, \tag{2.9}$$

for all matrices $\xi \in \mathbb{R}^{3\times 3}$.

Function spaces and boundary conditions. We always assume that $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz-domain. In the case $\delta = 0$, we assume additionally that Ω is simply connected with connected boundary, and that $\partial\Omega$ is of class $C^{1,1}$ (this assumption could be replaced by a convexity requirement). We always assume that we are given a relatively open non-empty subset $\Gamma_D \subset \partial\Omega$.

To impose a Dirichlet boundary condition on Γ_D we introduce the function space $H^1_D(\Omega;\mathbb{R}^3)$ of functions in $H^1(\Omega)$ with vanishing trace on Γ_D. Its dual space is denoted as $H^{-1}_D(\Omega;\mathbb{R}^3)$. For the prescribed load we assume

$$f \in L^2(0,T; L^2(\Omega;\mathbb{R}^3)) \cap H^1(0,T; H^{-1}_D(\Omega;\mathbb{R}^3))\,. \tag{2.10}$$

For p we consider an initial condition

$$p_0 \in L^2(\Omega;\mathbb{R}^{3\times 3})\,. \tag{2.11}$$

We denote by $H_0(\Omega,\mathrm{curl})$ the space of all $p \in L^2(\Omega;\mathbb{R}^{3\times 3})$ that have zero tangential trace $p \times \nu$ (where ν is the exterior normal on $\partial\Omega$) in the sense that

$$\int_\Omega \big(\varphi : \mathrm{curl}\, p - \mathrm{curl}\,\varphi : p\big) \;=\; 0 \qquad \text{for all } \varphi \in C^1(\bar\Omega;\mathbb{R}^{3\times 3})\,. \tag{2.12}$$

We equip this space with the inner product

$$(p,q)_{H_0(\Omega,\mathrm{curl})} \;=\; (p,q)_{L^2(\Omega;\mathbb{R}^{3\times 3})} + (\mathrm{curl}\, p, \mathrm{curl}\, q)_{L^2(\Omega;\mathbb{R}^{3\times 3})}$$

and the induced norm $\|\cdot\|_{H_0(\Omega,\mathrm{curl})}$. Then $H_0(\Omega,\mathrm{curl})$ is a Hilbert space and coincides with the completion of $C_c^\infty(\Omega;\mathbb{R}^{3\times 3})$ with respect to $\|\cdot\|_{H_0(\Omega,\mathrm{curl})}$, see [12, Theorem 2.6].

We note that Assumption 2.1 implies that $\mathcal{R}, \mathcal{R}^* : L^2(\Omega;\mathbb{R}^{3\times 3}) \to [0,\infty)$ and the function $\mathcal{Q} : L^2(\Omega;\mathbb{R}^{3\times 3}_s) \to [0,\infty)$ of (4.8) are continuous.

The strong formulation. In the strong formulation, we seek functions u, σ, p, Σ that satisfy the non-linear system (1.3)–(1.6) in a classical sense. Let us collect the equations in the special case $H_e \equiv 0$, $H_p \equiv 0$, using $e = \mathrm{sym}(\nabla u - p)$. Equation (1.4) becomes $\sigma = \mathrm{sym}\, \nabla_F Q(e)$, the standard equation for the stress. We note that, strictly speaking, this formula does not need the symbol "sym": The gradient of Q on the space of symmetric matrices is automatically a symmetric matrix. In the same setting, (1.5) becomes $\Sigma = \nabla_F Q(\nabla^s u - p) - \nabla_p \mathcal{W}_p(p) = \sigma - Lp$ with the positive differential operator L which reads $Lp = 2\,\mathrm{curl}\,\mathrm{curl}(p) - \delta\Delta_r p$.

Weak formulation in the variables (u, p, Σ)

Definition 2.2 (Weak solution in primary variables)**.** *We call* (u,p,Σ) *a weak solution of* (1.3)–(1.6) *(with boundary conditions) iff the following holds:*

(R_1) Regularity and boundary conditions:

$$u \in L^2(0,T;H^1_D(\Omega;\mathbb{R}^3))\,, \quad \Sigma \in L^2(0,T;L^2(\Omega;\mathbb{R}^{3\times 3}))\,, \tag{2.13}$$

$$p \in H^1(0,T;L^2(\Omega;\mathbb{R}^{3\times 3})) \cap L^2(0,T;H_0(\Omega,\mathrm{curl}))\,, \tag{2.14}$$

$$p \in L^\infty(0,T;W^{1,r}(\Omega)) \quad \text{if } \delta > 0\,, \tag{2.15}$$

and $p|_{t=0} = p_0$.

(S_1) Pointwise energy minimization: Equations (1.3)–(1.4) *are satisfied in the following sense: for a.e.* $t \in (0,T)$ *there holds*

$$\int_\Omega W_e(\nabla u(t), p(t)) - \int_\Omega f(t)\cdot u(t) \le \int_\Omega W_e(\nabla\varphi, p(t)) - \int_\Omega f(t)\cdot\varphi \tag{2.16}$$

for every $\varphi \in H^1_D(\Omega;\mathbb{R}^3)$.

(F_1) Back-stress variable: Instead of (1.5) *we demand (here, the dual is formed with respect to the variable p, the argument* ∇u *is fixed): Almost everywhere in* $(0,T)$ *holds*

$$\mathcal{W}(\nabla u, p) + \mathcal{W}^*(\nabla u, -\Sigma) = \langle -\Sigma, p\rangle\,. \tag{2.17}$$

(E_1) *Energy inequality: Instead of the flow rule* (1.6) *we demand that for a.e.* $t \in (0,T)$ *holds*

$$\left[\mathcal{W}(\nabla u(s), p(s)) - \int_\Omega f(s)\cdot u(s)\right]_{s=0}^t + \int_0^t \{\mathcal{R}(\partial_t p(s)) + \mathcal{R}^*(\Sigma(s))\}\, ds$$
$$\leq - \int_0^t \langle \partial_t f(s), u(s)\rangle\, ds\,. \tag{2.18}$$

Weak formulation in the variables (p,Σ)

Given an external load $f \in H_D^{-1}(\Omega;\mathbb{R}^3)$ we define two marginal functionals that assign to $p \in L^2(\Omega;\mathbb{R}^{3\times 3})$ an energy:

$$\mathcal{E}_1(p;f) := \inf\left\{\mathcal{W}_e(\nabla\varphi, p) - \langle f,\varphi\rangle \,\middle|\, \varphi \in H_D^1(\Omega;\mathbb{R}^3)\right\}\,, \tag{2.19}$$
$$\mathcal{E}(p;f) := \mathcal{E}_1(p;f) + \mathcal{W}_p(p)\,. \tag{2.20}$$

We denote by $\mathfrak{E}(p,f)$ the set of all $u \in H_D^1(\Omega;\mathbb{R}^3)$ that attain the infimum in (2.19). This set is non-empty, see Lemma A.1.

With the marginal functionals $\mathcal{E}_1$ and $\mathcal{E}$ we can give an alternative formulation of the equations in the variables (p,Σ), avoiding u. This reduction is based on the fact that requirement (S_1) is equivalent to: For almost every $t \in (0,T)$ holds the energy minimization property

$$\int_\Omega W_e(\nabla u(t), p(t)) - \int_\Omega f(t)\cdot u(t) = \mathcal{E}_1(p(t); f(t))\,. \tag{2.21}$$

Definition 2.3 (Weak solution of the condensed system)**.** *We call* (p,Σ) *a weak solution of* (1.3)–(1.6) *(with boundary conditions) iff the following holds:*

(R_2) *On* p *and* Σ *we demand the properties of Item* (R_1)*.*

(F_2) *Back-stress variable: Almost everywhere in* $(0,T)$ *holds*

$$\mathcal{E}\big(p;f\big) + \mathcal{E}^*\big(-\Sigma; f\big) = \langle -\Sigma, p\rangle\,. \tag{2.22}$$

(E_2) *Energy inequality: For almost all* $t \in (0,T)$ *holds*

$$\begin{aligned}\big[\mathcal{E}\big(p(s); f(s)\big)\big]_{s=0}^t + \int_0^t \{\mathcal{R}(\partial_t p(s)) + \mathcal{R}^*(\Sigma(s))\}\, ds \\ \leq - \int_0^t \inf_{\tilde u \in \mathfrak{E}(p(s),f(s))} \big\langle \partial_t f(s), \tilde u\big\rangle\, ds\,.\end{aligned} \tag{2.23}$$

The (negative of the) integrand on the right-hand side of (2.23) corresponds to a generalized time-derivative of the functional $\varphi \mapsto \mathcal{W}(\nabla\varphi, p(s)) - \langle f(s),\varphi\rangle$, compare Example 3 in the introduction in [19].

Formally, the three solution concepts are equivalent. Rigorous statements are collected in the following proposition.

Proposition 2.4 (Equivalence of solution concepts)**.** *The following holds:*

1. *Let* (p, Σ) *be a solution according to Definition 2.3. Furthermore, let us assume that for almost all* $t \in (0,T)$ *the set* $\mathfrak{E}(p(t), \Sigma(t))$ *consists of a unique element* $u(t)$*. Then* $u \in L^2(0,T; H^1_D(\Omega;\mathbb{R}^3))$ *holds and the triple* (u,p,Σ) *is a solution according to Definition 2.2.*

2. *Let* (u,p,Σ) *be a weak solution according to Definition 2.2. Let* Q*,* H_e*, and* H_p *be Fréchet-differentiable and let the solution have the regularity* $\partial_t p \in L^2(0,T; L^2(\Omega;\mathbb{R}^{3\times3}))$*,* $p \in L^2(0,T;H^2(\Omega;\mathbb{R}^{3\times3}))$*,* $u \in L^2(0,T;H^2(\Omega;\mathbb{R}^3))$*, and* $\Sigma \in L^2(0,T;H^1(\Omega;\mathbb{R}^{3\times3}))$*. Then, with* σ *defined by* (1.4)*,* (u,σ,p,Σ) *is a strong solution to* (1.3)*–*(1.6)*.*

3. *Let* (u,σ,p,Σ) *be a strong solution according to* (1.3)*–*(1.6) *and let* Q*,* H_e*,* H_p *be Fréchet-differentiable. Then* (u,p,Σ) *is a weak solution according to Definition 2.2.*

The proof of Proposition 2.4 is given in Appendix A. We note that the assumption on the uniqueness of the minimizer in Item 1 is only used to ensure that there exists a measurable selection $t \mapsto u(t) \in \mathfrak{E}(p(t), \Sigma(t))$.

Main results. We now formulate our main results concerning the existence of energetic solutions.

Theorem 2.5 (Existence result for $\delta > 0$)**.** *Let Assumption 2.1 on the energies be satisfied and let* f *and* p_0 *be as in* (2.10)*–*(2.11)*. For* $\delta > 0$*, there exists a weak solution* (p,Σ) *to system* (1.3)*–*(1.6) *in the sense of Definition 2.3.*

In the case $\delta = 0$ we must restrict ourselfes to energies with $H_e \equiv 0$ and $H_p \equiv 0$.

Theorem 2.6 (Existence result for $\delta = 0$)**.** *Let Assumption 2.1 on the energies and let the assumptions on* f*,* p_0*, and* Ω *be satisfied. We consider the case* $\delta = 0$ *and the energies of* (2.3)*–*(2.6) *with* $H_e \equiv 0$ *and* $H_p \equiv 0$*. There exists a weak solution* (p,Σ) *to system* (1.3)*–*(1.6) *in the sense of Definition 2.3.*

3 Stability results

This section is devoted to stability results for the plasticity system in the weak form: We show that, for convergent sequences of approximate solutions, the limit functions are solutions to the plasticity system (1.3)–(1.6).

We work with two families of functions, one denoted with a hat and the other denoted with an overbar. Later on, the first will be a sequence that is obtained by a piecewise affine interpolation of a time-discrete approximate solution, the latter will be the corresponding piecewise constant interpolant.

3.1 Stability for $\delta > 0$

The boundedness and the convergence properties of the sequences are summarized in the following assumption.

Assumption 3.1 (Convergence properties, $\delta > 0$)**.** *We consider a sequence of approximate solutions $(\bar{p}^N, \bar{\Sigma}^N)_N$ together with a sequence of loads $(\bar{f}^N)_N$ that satisfy*

$$\bar{p}^N \rightharpoonup p \quad \text{in } L^r(0,T;W^{1,r}(\Omega,\mathbb{R}^{3\times3}) \cap H_0(\Omega,\mathrm{curl}))\,, \tag{3.1}$$
$$\bar{p}^N \to p \quad \text{in } L^2(0,T;L^2(\Omega,\mathbb{R}^{3\times3}))\,, \tag{3.2}$$
$$\bar{\Sigma}^N \rightharpoonup \Sigma \quad \text{in } L^2(0,T;L^2(\Omega,\mathbb{R}^{3\times3}))\,, \tag{3.3}$$
$$\bar{f}^N \to f \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^3))\,, \tag{3.4}$$

as $N \to \infty$. We furthermore assume that there are approximations $(\hat{p}^N)_N$ and $(\hat{f}^N)_N$ that satisfy, as $N \to \infty$,

$$\hat{p}^N \rightharpoonup p \quad \text{in } H^1(0,T;L^2(\Omega,\mathbb{R}^{3\times3}))\,, \tag{3.5}$$
$$\hat{p}^N \to p \quad \text{in } L^2(0,T;L^2(\Omega,\mathbb{R}^{3\times3}))\,, \tag{3.6}$$
$$\hat{f}^N \to f \quad \text{in } H^1(0,T;H^{-1}_D(\Omega;\mathbb{R}^3))\,. \tag{3.7}$$

Moreover, we assume that the two sequences $(\hat{p}^N)_N$ and $(\bar{p}^N)_N$ are bounded in the space $L^\infty(0,T;H_0(\Omega,\mathrm{curl}) \cap W^{1,r}(\Omega;\mathbb{R}^{3\times3}))$ and that $(\bar{f}^N)_N$ is bounded in $L^\infty(0,T;H^{-1}_D(\Omega))$. Regarding initial values we assume that $\hat{p}^N(0) = \bar{p}^N(0) = p(0) = p_0$ and $\hat{f}^N(0) = \bar{f}^N(0) = f(0)$ holds for all $N \in \mathbb{N}$. Finally, to avoid issues regarding measurability, we assume that $\bar{p}^N$, $\bar{f}^N$, and $\partial_t \hat{f}^N$ are simple functions for all $N \in \mathbb{N}$.

The next assumption expresses that the functions are approximate solutions.

Assumption 3.2 (Approximate solution properties)**.** *Let $(\bar{p}^N, \bar{\Sigma}^N)_N$ together with $(\hat{p}^N)_N$ and $(\bar{f}^N)_N$, $(\hat{f}^N)_N$ be sequences as in Assumption 3.1. We assume that these functions are approximate solutions in the following sense:*

1. *Relation* (2.22) *of Item (F$_2$) is approximately satisfied: For almost every $t \in (0,T)$ holds*
$$\mathcal{E}\big(\bar{p}^N(t);\bar{f}^N(t)\big) + \mathcal{E}^*\big(-\bar{\Sigma}^N(t);\bar{f}^N(t)\big) \le \langle -\bar{\Sigma}^N(t), \bar{p}^N(t)\rangle + g_N(t)\,, \tag{3.8}$$
where the error functions g_N satisfy $\int_0^T |g_N(t)|\,dt \to 0$ as $N \to \infty$.

2. *The energy inequality* (2.23) *of Item (E$_2$) is approximately satisfied: For almost every $t \in (0,T)$, as $N \to \infty$, there holds*
$$\mathcal{E}\big(\bar{p}^N(s);\bar{f}^N(s)\big)\Big|_{s=0}^t + \int_0^t \big\{\mathcal{R}(\partial_t \hat{p}^N) + \mathcal{R}^*(\bar{\Sigma}^N)\big\}\,ds$$
$$\le -\int_0^t \inf_{\tilde{u}^N \in \mathfrak{E}(\bar{p}^N(s),\bar{f}^N(s))} \big\langle \partial_t \hat{f}^N(s), \tilde{u}^N\big\rangle\,ds + o(1)\,. \tag{3.9}$$

Proposition 3.3. *Let Assumptions 3.1 and 3.2 be satisfied. Then the pair (p,Σ) is a weak solution of the original problem in the sense of Definition 2.3.*

Proof. As weak limits, the functions p and Σ are in the appropriate function spaces of Item (R_2): $\Sigma \in L^2(0,T;L^2(\Omega;\mathbb{R}^{3\times 3}))$, $p \in H^1(0,T;L^2(\Omega;\mathbb{R}^{3\times 3})) \cap L^2(0,T;H_0(\Omega,\mathrm{curl}))$, $p \in L^\infty(0,T;W^{1,r}(\Omega))$. The properties of Items (F_2) and (E_2) are proved in the subsequent lemmas. Together, (R_2), (F_2), (E_2) imply that (p,Σ) is a weak solution. □

Lemma 3.4 (Item (F_2)). *Let Assumptions 3.1 and 3.2 be satisfied. Then Item (F_2) holds for the limit functions (p,Σ).*

Proof. We use an arbitrary non-negative function $\theta \in C^\infty(0,T;\mathbb{R})$ and consider inequality (3.8) in the integrated form

$$0 \geq \liminf_{N\to\infty} \int_0^T \theta(t)\Big[\mathcal{E}\big(\bar{p}^N(t);\bar{f}^N(t)\big) + \mathcal{E}^*\big(-\bar{\Sigma}^N(t);\bar{f}^N(t)\big) + \langle \bar{\Sigma}^N(t), \bar{p}^N(t)\rangle\Big]\,dt\,.$$

By the definition of the convex dual $\mathcal{E}^*$, for an arbitrary $\eta \in W^{1,r}(\Omega;\mathbb{R}^{3\times 3}) \cap H_0(\Omega,\mathrm{curl})$, we have

$$0 \geq \liminf_{N\to\infty} \int_0^T \theta(t)\Big[\mathcal{E}\big(\bar{p}^N(t);\bar{f}^N(t)\big) + \langle -\bar{\Sigma}^N(t), \eta - \bar{p}^N(t)\rangle - \mathcal{E}\big(\eta;\bar{f}^N(t)\big)\Big]\,dt\,.$$

We can exploit the strong convergence of $\bar{p}^N$ and $\bar{f}^N$ and the Lipschitz property of Item 6 of Lemma A.1 to conclude the convergence of the last two terms. In the first term we use once more the Lipschitz property of $\mathcal{E}_1$ (we exploit the bounds for $\bar{f}^N \in L^\infty(0,T;H_D^{-1}(\Omega))$ and $\bar{p}^N \in L^\infty(0,T;L^2(\Omega))$ to control the Lipschitz constant) and the lower semicontinuity of $\mathcal{W}_p$ and obtain

$$0 \geq \int_0^T \theta(t)\Big[\mathcal{E}\big(p(t);f(t)\big) + \langle -\Sigma(t), \eta - p(t)\rangle - \mathcal{E}\big(\eta;f(t)\big)\Big]\,dt\,. \tag{3.10}$$

In order to localize in t, we proceed as follows. We consider a countable dense subset $\big\{\big(\eta_i, \mathcal{W}_p(\eta_i)\big) : \eta_i \in \mathrm{dom}(\mathcal{W}_p), i \in \mathbb{N}\big\}$ of graph $\mathcal{W}_p|_{\mathrm{dom}(\mathcal{W}_p)}$, which is separable as a subset of the separable space $L^2(\Omega;\mathbb{R}^{3\times 3}) \times \mathbb{R}$.

Since θ was arbitrary, we deduce from (3.10) that there exists an exceptional set of time instances $B \subset (0,T)$ of measure zero such that, for all $t \in (0,T)\setminus B$ and all $i \in \mathbb{N}$,

$$0 \geq \mathcal{E}\big(p(t);f(t)\big) + \langle -\Sigma(t), \eta_i - p(t)\rangle - \mathcal{E}\big(\eta_i;f(t)\big)\,. \tag{3.11}$$

For an arbitrary $\eta \in \mathrm{dom}(\mathcal{W}_p)$ there exists a subsequence $i \to \infty$ (not relabeled) such that $\eta_i \to \eta$ in $L^2(\Omega;\mathbb{R}^{3\times 3})$ and $\mathcal{W}_p(\eta_i) \to \mathcal{W}_p(\eta)$. Using the first property, we can pass to the limit in the second term on the right hand side of (3.11) and also, by the 2-growth of Q, in $\mathcal{E}_1(\eta_i;f(t))$. Using the second property yields, for all $t \in (0,T)\setminus B$,

$$0 \geq \mathcal{E}\big(p(t);f(t)\big) + \langle -\Sigma(t), \eta - p(t)\rangle - \mathcal{E}\big(\eta;f(t)\big)\,,$$

first for all $\eta \in \mathrm{dom}(\mathcal{W}_p)$ and then, since $\mathcal{E}(\cdot, f(t))$ is infinite outside $\mathrm{dom}(\mathcal{W}_p)$, for all $\eta \in L^2(\Omega;\mathbb{R}^{3\times 3})$. Taking the supremum over all $\eta \in L^2(\Omega;\mathbb{R}^{3\times 3})$ yields

$$\langle -\Sigma(t), p(t)\rangle \geq \mathcal{E}\big(p(t);f(t)\big) + \mathcal{E}^*\big(-\Sigma(t);f(t)\big)$$

for all $t \in (0,T)\setminus B$, and hence the claim. □

Lemma 3.5 (Reconstruction of displacement fields)**.** *Let Assumption 3.1 be satisfied. There exists a sequence* $(\bar{u}^N)_N$ *in* $L^2(0,T;H^1_D(\Omega;\mathbb{R}^3))$ *such that*

$$\begin{aligned} \bar{u}^N(t) &\in \mathfrak{E}(\bar{p}^N(t);\bar{f}^N(t)) \quad \text{and} \\ \langle \partial_t \hat{f}^N(t), \bar{u}^N(t)\rangle &= \inf_{\tilde{u}\in\mathfrak{E}(\bar{p}^N(t),\bar{f}^N(t))} \langle \partial_t \hat{f}^N(t), \tilde{u}\rangle \end{aligned} \tag{3.12}$$

for every $N\in\mathbb{N}$ *and almost every* $t\in(0,T)$*. Furthermore, there exists a function* $u\in L^2(0,T;H^1_D(\Omega;\mathbb{R}^3))$ *and a subsequence* $N\to\infty$ *such that*

$$\bar{u}^N \rightharpoonup u \quad \text{in } L^2(0,T;H^1_D(\Omega;\mathbb{R}^3)) \quad \text{as } N\to\infty, \tag{3.13}$$

$$u(t) \in \mathfrak{E}(p(t);f(t)) \quad \text{for almost every } t\in(0,T). \tag{3.14}$$

Proof. By Lemma A.1, Item 4, and the continuity of $\langle \partial_t \hat{f}^N(t),\cdot\rangle$ under weak convergence in $H^1_D(\Omega;\mathbb{R}^3)$, for any $N\in\mathbb{N}$ and almost every $t\in[0,T]$, there exists a function $\bar{u}^N(t)\in\mathfrak{E}(\bar{p}^N(t);\bar{f}^N(t))$ that satisfies the minimizing property (3.12). Note that we can choose simple functions $t\mapsto\bar{u}^N(t)$; in particular, the functions are measurable. Lemma A.1, Item 4, and Assumption 3.1 imply that $(\bar{u}^N)_N$ is uniformly bounded in $L^2(0,T;H^1_D(\Omega;\mathbb{R}^3))$, hence we deduce the existence of $u\in L^2(0,T;H^1_D(\Omega;\mathbb{R}^3))$ such that (3.13) holds.

We next want to verify the minimizing property of the limit function u. Let $\varphi\in H^1_D(\Omega;\mathbb{R}^3)$ be arbitrary. For any non-negative $\theta\in C^\infty_c(0,T;\mathbb{R})$ there holds

$$\begin{aligned} &\int_0^T \theta(t)\Big(\int_\Omega W_e(\nabla u(t),p(t)) - \int_\Omega f(t)\cdot u(t)\Big)dt \\ &\quad\le \liminf_N \int_0^T \theta(t)\Big(\int_\Omega W_e(\nabla \bar{u}^N(t),\bar{p}^N(t)) - \int_\Omega \bar{f}^N(t)\cdot \bar{u}^N(t)\Big)dt \\ &\quad\le \liminf_N \int_0^T \theta(t)\Big(\int_\Omega W_e(\nabla\varphi,\bar{p}^N(t)) - \int_\Omega \bar{f}^N(t)\cdot\varphi\Big)dt \\ &\quad= \int_0^T \theta(t)\Big(\int_\Omega W_e(\nabla\varphi,p(t)) - \int_\Omega f(t)\cdot\varphi\Big)dt\,, \end{aligned}$$

where we used convexity of W_e and weak convergence of $\bar{u}^N$ in the first inquality, the minimization property of $\bar{u}^N$ in the second inequality, and the strong convergence of $\bar{p}^N$ together with the 2-growth of Q in the last equality. Since θ was arbitrary, this shows, for almost every $t\in(0,T)$,

$$\int_\Omega W_e(\nabla u(t),p(t)) - \int_\Omega f(t)\cdot u(t) \le \int_\Omega W_e(\nabla\varphi - p(t)) - \int_\Omega f(t)\cdot\varphi\,. \tag{3.15}$$

We next argue as in the proof of Lemma 3.4 and show that the last inequality holds for almost all $t\in(0,T)$ and all $\varphi\in H^1_D(\Omega;\mathbb{R}^3)$ (i.e.: the set of admissible t's can be chosen independent of φ). We define the functional $\mathcal{H}_e : H^1_D(\Omega;\mathbb{R}^3)\to[0,\infty]$,

$$\mathcal{H}_e(\varphi) = \int_\Omega H_e(\nabla^s\varphi(x))\,dx\,,$$

and choose a dense subset $A = \big\{\big(\varphi_i, \mathcal{H}_e(\varphi_i)\big) \,:\, \varphi_i \in \mathrm{dom}(\mathcal{H}_e), i \in \mathbb{N}\big\}$ of $\mathrm{graph}\,\mathcal{H}_e|_{\mathrm{dom}(\mathcal{H}_e)} \subset H^1_D(\Omega;\mathbb{R}^3) \times \mathbb{R}$. We deduce that there exists a set $B \subset (0,T)$ with $|B| = 0$ such that (3.15) holds for any $\varphi \in \{\varphi_i \,:\, i \in \mathbb{N}\}$ and any $t \in (0,T) \setminus B$. Again by the 2-growth of Q and the density of $A \subset \mathrm{graph}\,\mathcal{H}_e|_{\mathrm{dom}(\mathcal{H}_e)}$ we deduce that (3.15) holds for any $\varphi \in H^1_D(\Omega;\mathbb{R}^3)$ and any $t \in (0,T) \setminus B$. This provides (3.14). □

Lemma 3.6 (Item (E_2))**.** *Let Assumptions 3.1 and 3.2 be satisfied. Then* (2.23) *of Item (E_2) holds for the limit functions* (p, Σ).

Proof. We choose a sequence $(\bar{u}^N)_N$ with $\bar{u}^N(t) \in \mathfrak{E}(\bar{p}^N(t); \bar{f}^N(t))$ as in Lemma 3.5; in particular, we obtain a weak limit $u \in L^2(0,T; H^1_D(\Omega;\mathbb{R}^3))$ such that (3.13) holds.

From (3.2) we deduce that, for almost all $t \in (0,T)$, the sequence $(\bar{p}^N(t))_N$ is strongly convergent in $L^2(\Omega)$ to the limit $p(t)$. We can additionally assume that the sequence $(\bar{p}^N(t))_N$ is uniformly bounded in $H_0(\Omega,\mathrm{curl}) \cap W^{1,r}(\Omega;\mathbb{R}^{3\times 3})$. For such a t we have the weak convergence

$$\bar{p}^N(t) \rightharpoonup p(t) \quad \text{in } H_0(\Omega,\mathrm{curl}) \cap W^{1,r}(\Omega;\mathbb{R}^{3\times 3})\,. \tag{3.16}$$

The approximate solution property (3.9) and property (3.12) of $\bar{u}^N(t)$ yield

$$\begin{aligned}\mathcal{E}\big(\bar{p}^N(s); \bar{f}^N(s)\big)\Big|_{s=0}^{t} &+ \int_0^t \mathcal{R}(\partial_t \hat{p}^N(s)) + \mathcal{R}^*(\bar{\Sigma}^N(s))\,ds \\ &\le -\int_0^t \big\langle \partial_t \hat{f}^N(s), \bar{u}^N(s)\big\rangle\,ds + o(1)\,.\end{aligned} \tag{3.17}$$

The first term on the left-hand side is $\mathcal{E}_1\big(\bar{p}^N(t); \bar{f}^N(t)\big) + \mathcal{W}_p(\bar{p}^N(t))$. Using the lower semi-continuity of Lemma A.1, Item 7, the convergence (3.16), and the convexity of $\mathcal{W}_p$ we deduce the lower semicontinuity of this term in the limit,

$$\mathcal{E}\big(p(s); f(s)\big)\Big|_{s=0}^{t} \le \liminf_{N\to\infty} \mathcal{E}\big(\bar{p}^N(s); \bar{f}^N(s)\big)\Big|_{s=0}^{t}$$

for almost all $t \in (0,T)$, where we exploited that the initial values are fixed. By the convergences (3.3) and (3.5), the growth assumptions (2.8) and (2.9), and the convexity of $\mathcal{R}$ and $\mathcal{R}^*$, we also have

$$\int_0^t \mathcal{R}(\partial_t p(s)) + \mathcal{R}^*(\Sigma(s))\,ds \le \liminf_{N\to\infty} \int_0^t \mathcal{R}(\partial_t \hat{p}^N(s)) + \mathcal{R}^*(\bar{\Sigma}^N(s))\,ds\,.$$

Finally, (3.7) and (3.13) imply the convergence of the right-hand side of (3.17). We therefore obtain

$$\begin{aligned}&\mathcal{E}\big(p(s); f(s)\big)\Big|_{s=0}^{t} + \int_0^t \mathcal{R}(\partial_t p(s)) + \mathcal{R}^*(\Sigma(s))\,ds \\ &\quad\le \liminf_{N\to\infty} \Big(\mathcal{E}\big(\bar{p}^N(s); \bar{f}^N(s)\big)\Big|_{s=0}^{t} + \int_0^t \mathcal{R}(\partial_t \hat{p}^N(s)) + \mathcal{R}^*(\bar{\Sigma}^N(s))\,ds\Big) \\ &\quad\le -\int_0^t \big\langle \partial_t f(s), u(s)\big\rangle\,ds \le -\int_0^t \inf_{\tilde{u}\in\mathfrak{E}(p(s),f(s))} \big\langle \partial_t f(s), \tilde{u}\big\rangle\,ds\,,\end{aligned}$$

and have verified (2.23) for the limit functions. □

3.2 Stability for $\delta = 0$

In this section, we consider the case without gradient-term in the plastic energy, i.e. $\delta = 0$ in $\mathcal{W}_p$ of (2.4). We can treat this case only under further structural assumptions on the energy: We demand $H_e \equiv 0$ and $H_p \equiv 0$. Our aim is to show a stability result that replaces Proposition 3.3 in the case $\delta = 0$. The existence result of Theorem 2.6 will be a consequence of the stability property.

We proceed along the lines of the case $\delta > 0$. We start with our assumptions on the approximate solution sequence. The main difference is that only a weak convergence of the sequences $(\hat{p}^N)_N$ and $(\bar{p}^N)_N$ can be assumed.

Assumption 3.7 (Convergence properties, $\delta = 0$)**.** *We consider a sequence of approximate solutions $(\bar{p}^N, \bar{\Sigma}^N)_N$ together with a sequence of loads $(\bar{f}^N)_N$ that satisfy*

$$\bar{p}^N \rightharpoonup p \quad \text{in } L^2(0,T;H_0(\Omega,\mathrm{curl}))\,, \tag{3.18}$$

$$\bar{\Sigma}^N \rightharpoonup \Sigma \quad \text{in } L^2(0,T;L^2(\Omega,\mathbb{R}^{3\times3}))\,, \tag{3.19}$$

$$\bar{f}^N \to f \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^3))\,, \tag{3.20}$$

as $N \to \infty$. For approximations $(\hat{p}^N)_N$ and $(\hat{f}^N)_N$ we assume, as $N \to \infty$,

$$\hat{p}^N \rightharpoonup p \quad \text{in } H^1(0,T;L^2(\Omega,\mathbb{R}^{3\times3}))\,, \tag{3.21}$$

$$\hat{f}^N \to f \quad \text{in } H^1(0,T;H^{-1}_D(\Omega;\mathbb{R}^3))\,. \tag{3.22}$$

We additionally assume: $(\hat{p}^N)_N$ and $(\bar{p}^N)_N$ are bounded in $L^\infty(0,T;H_0(\Omega,\mathrm{curl}))$, $(\nabla\cdot\bar{\Sigma}^N)_N$ is bounded in $L^2(0,T;L^2(\Omega))$, $(\bar{f}^N)_N$ is bounded in $L^\infty(0,T;H^{-1}_D(\Omega))$. Finally, we impose a weak regularity of $\bar{p}^N$ in time: We demand

$$\sup_N \left\|\bar{p}^N(.+\rho) - \bar{p}^N(.)\right\|^2_{L^2(0,T-\rho;L^2(\Omega))} \to 0 \text{ as } \rho \to 0\,. \tag{3.23}$$

Regarding initial values we assume that $\hat{p}^N(0) = \bar{p}^N(0) = p(0) = p_0$ and $\hat{f}^N(0) = \bar{f}^N(0) = f(0)$ holds for all $N \in \mathbb{N}$. We assume that $\bar{p}^N$, $\bar{f}^N$, and $\partial_t \hat{f}^N$ are simple functions for all $N \in \mathbb{N}$.

In the case $\delta = 0$ we will impose the same approximate solution properties as in the case $\delta > 0$, i.e. those of Assumption 3.2. We find that the analog of Proposition 3.3 holds.

Proposition 3.8. *Let Assumptions 3.2 and 3.7 be satisfied. Then the limiting pair (p, Σ) is a weak solution of the original problem in the sense of Definition 2.3.*

Proof. Step 1: Item (R_2). As weak limits, the functions p and Σ are in the function spaces of Item (R_2): $\Sigma \in L^2(0,T;L^2(\Omega;\mathbb{R}^{3\times3}))$, $p \in H^1(0,T;L^2(\Omega;\mathbb{R}^{3\times3})) \cap L^2(0,T;H_0(\Omega,\mathrm{curl}))$.

Step 2: Item (F_2). The proof of Item (F_2) is analogous to the case $\delta > 0$, see Lemma 3.4. The only difference regards the limit procedure of the product term, leading to (3.10): For arbitrary $\theta \in C^\infty_c((0,T);\mathbb{R})$ we claim

$$\int_0^T \theta(t)\langle\bar{\Sigma}^N(t), \bar{p}^N(t)\rangle\,dt \to \int_0^T \theta(t)\langle\Sigma(t), p(t)\rangle\,dt \tag{3.24}$$

as $N \to \infty$. This limit is a consequence of the global div-curl lemma. The *global* div-curl lemma yields not only the distributional convergence of a product of weakly convergent sequences, but also the convergence of the integral of the product. For a global div-curl lemma, one always has to make use of boundary conditions. In our case, we know that tangential components of $\bar{p}^N$ vanish on the boundary by $\bar{p}^N(t) \in H_0(\Omega, \mathrm{curl})$, see (2.12). For a proof of the global div-curl lemma without t-dependence see e.g. Lemma 6.1 in [27].

For the case with t-dependence as in (3.24), we argue as follows. We use a small parameter $\rho > 0$ and a smooth sequence of symmetric mollifiers $\varphi_\rho : \mathbb{R} \to \mathbb{R}$. Functions that are defined on the interval $(0, T)$ are always identified with their trivial extension to all of $\mathbb{R}$. We claim that the following modification of (3.24) is valid:

$$\int_0^T \theta(t)\langle(\varphi_\rho * \bar{\Sigma}^N)(t), \bar{p}^N(t)\rangle\, dt \to \int_0^T \theta(t)\langle(\varphi_\rho * \Sigma)(t), p(t)\rangle\, dt\,. \tag{3.25}$$

Step 2a: Verification of (3.25). The sequence $\bar{p}^N$ is bounded in $L^\infty(0, T; L^2(\Omega))$ and, by (3.23), pre-compact in $L^2(0, T; H^{-1}(\Omega))$ [28, Theorem 1]. We therefore have $\bar{p}^N(t) \to p(t)$ in $H^{-1}(\Omega)$ for almost every $t \in (0, T)$. By the boundedness assumptions, this yields $\bar{p}^N(t) \rightharpoonup p(t)$ in $L^2(\Omega)$ with $\mathrm{curl}\, \bar{p}^N(t)$ bounded in $L^2(\Omega)$ for almost every t.

The function $\varphi_\rho * \bar{\Sigma}^N$ is of class $C^1([0, T], L^2(\Omega))$ with bounds that are independent of N. We therefore have $(\varphi_\rho * \bar{\Sigma}^N)(t) \rightharpoonup (\varphi_\rho * \Sigma)(t)$ in $L^2(\Omega)$ for every $t \in (0, T)$. Furthermore, $\nabla \cdot (\varphi_\rho * \bar{\Sigma}^N)(t)$ is bounded in $L^2(\Omega)$. The time-independent global div-curl lemma can be applied to the above functions and provides $\theta(t)\langle(\varphi_\rho * \bar{\Sigma}^N)(t), \bar{p}^N(t)\rangle \to \theta(t)\langle(\varphi_\rho * \Sigma)(t), p(t)\rangle$ for almost every t. Dominated convergence implies (3.25).

Step 2b: Verification of (3.24). Concerning the right hand sides of (3.24) and (3.25) we observe that $\varphi_\rho * \Sigma \to \Sigma$ holds in $L^2(0, T; L^2(\Omega))$.

Concerning the left hand sides, we observe

$$\begin{aligned}
&\int_0^T \big\langle \bar{\Sigma}^N(t), \varphi_\rho * (\theta\bar{p}^N)(t) - (\theta\bar{p}^N)(t)\big\rangle\, dt \\
&\quad = \int_{\mathbb{R}}\int_{\mathbb{R}} \big\langle \bar{\Sigma}^N(t), \varphi_\rho(s)\big((\theta\bar{p}^N)(t-s) - (\theta\bar{p}^N)(t)\big)\big\rangle\, dt\, ds \\
&\quad \le \int_{\mathbb{R}} \|\bar{\Sigma}^N\|_{L^2(0,T;L^2(\Omega))} \|(\theta\bar{p}^N)(.-s) - \theta\bar{p}^N\|_{L^2(0,T;L^2(\Omega))}\varphi_\rho(s)\, ds\,.
\end{aligned}$$

For a small parameter $\rho > 0$, the right-hand side is small, uniformly in N:

$$\begin{aligned}
&\sup_{|s|<\rho} \|(\theta\bar{p}^N)(.-s) - \theta\bar{p}^N\|_{L^2(0,T;L^2(\Omega))} \\
&\quad \le 2\|\theta\|_{C^1([0,T])}\Big(\sup_{|s|<\rho} \|\bar{p}^N(.-s) - \bar{p}^N\|_{L^2(0,T;L^2(\Omega))} + \sqrt{\rho}\|\bar{p}^N\|_{L^2(0,T;L^2(\Omega))}\Big)\,.
\end{aligned}$$

The smallness of the first contribution is guaranteed by the compactness property (3.23), the second contribution is small by the factor $\sqrt{\rho}$. We obtain that the convergence (3.25) provides the convergence (3.24).

Step 3: Construction of displacement fields. In the case $\delta > 0$, the energy minimizing displacement fields have been constructed in Lemma 3.5. The argument is similar in the case $\delta = 0$, we sketch here how the four-line calculation in the proof of Lemma 3.5 must be altered.

We assume that a minimizing sequence $\bar{u}^N$ is already chosen. Let $\varphi \in L^2(0,T;H^1_D(\Omega;\mathbb{R}^3))$ be an arbitrary test-function. We claim that we can find a sequence $(\varphi^N)_N$ with $\varphi^N \in L^2(0,T;H^1_D(\Omega;\mathbb{R}^3))$ such that

$$\varphi^N \rightharpoonup \varphi \text{ in } L^2(0,T;H^1(\Omega)) \quad \text{and} \quad \nabla\varphi^N - \bar{p}^N \to \nabla\varphi - p \text{ in } L^2(0,T;L^2(\Omega))\,. \tag{3.26}$$

Step 3a: Conclusion using (3.26). Let us assume that we have a sequence $(\varphi^N)_N$ satisfying (3.26). We can calculate, using first the minimality of $\bar{u}^N$, then the definition of W_e and $H_e \equiv 0$, and finally the convergence properties of the different sequences:

$$\begin{aligned}
&\liminf_N \int_0^T \theta(t) \left(\int_\Omega W_e(\nabla\bar{u}^N(t), \bar{p}^N(t)) - \int_\Omega \bar{f}^N(t)\cdot\bar{u}^N(t) \right) dt \\
&\quad\le \liminf_N \int_0^T \theta(t) \left(\int_\Omega W_e(\nabla\varphi^N(t), \bar{p}^N(t)) - \int_\Omega \bar{f}^N(t)\cdot\varphi^N(t) \right) dt \\
&\quad= \liminf_N \int_0^T \theta(t) \left(\int_\Omega Q(\mathrm{sym}(\nabla\varphi^N(t) - \bar{p}^N(t))) - \int_\Omega \bar{f}^N(t)\cdot\varphi^N(t) \right) dt \\
&\quad= \int_0^T \theta(t) \left(\int_\Omega Q(\mathrm{sym}(\nabla\varphi(t) - p(t))) - \int_\Omega f(t)\cdot\varphi(t) \right) dt\,.
\end{aligned}$$

In the last step we exploited (3.26) and the 2-growth of Q from (2.7). The rest of the proof is as in the case $\delta > 0$.

Step 3b: Construction of φ^N satisfying (3.26). We obtain the sequence $(\varphi^N)_N$ from the Helmholtz decomposition in space of the function

$$q^N := \nabla\varphi + \bar{p}^N - p\,. \tag{3.27}$$

The Helmholtz decomposition of $q^N(t)$ provides a gradient-potential $\varphi^N(t)$ and a curl-potential $\Psi^N(t)$ such that

$$q^N(t) = \nabla\varphi^N(t) + \mathrm{curl}\;\Psi^N(t)\,. \tag{3.28}$$

Since q^N is a matrix field $\Omega \to \mathbb{R}^{3\times3}$, we apply the usual Helmholtz decomposition for vector fields $\Omega \to \mathbb{R}^3$ to each row of q^N. This yields the desired result, since the k-th row of $\nabla\varphi$ is $\nabla\varphi_k$, and the k-th row of curl Ψ is $\mathrm{curl}\,\Psi_{k,\cdot}$.

On the potentials, we impose $\nabla\cdot\Psi^N(t) = 0$ (for each row $\Psi^N_{k,\cdot}$), and the following boundary conditions: On Γ_D, we demand that the normal component of $\Psi^N(t)$ vanishes and $\varphi^N(t)|_{\Gamma_D} = 0$, hence $\varphi^N(t) \in H^1_D(\Omega)$. On $\partial\Omega \setminus \Gamma_D$, we demand that tangential components of $\Psi^N(t)$ vanish. The existence of the two potentials together with $H^1(\Omega)$-estimates is guaranteed by the Helmholtz decomposition result, see e.g. Theorem 4.2 of [27] and, for mixed boundary conditions, [4].

By the boundedness of the potentials, we may assume the weak convergence of $\varphi^N, \nabla\varphi^N, \Psi^N, \nabla\Psi^N$ in the space $L^2(0,T;L^2(\Omega))$. The weak limits of the potentials provide a Helmholtz decomposition of the weak limit of q^N, which is $\nabla\varphi$.

Uniqueness of the Helmholtz decomposition implies $\varphi^N \rightharpoonup \varphi$ and and $\Psi^N \rightharpoonup 0$ in $L^2(0,T;H^1(\Omega))$. Relation (3.26) is verified once we show $q^N - \nabla\varphi^N \to 0$ strongly in $L^2(0,T;L^2(\Omega))$ or, equivalently,

$$\xi^N := \text{curl } \Psi^N \to 0 \text{ in } L^2(0,T;L^2(\Omega))\,. \tag{3.29}$$

For the subsequent argument we observe that $q^N(t)$ and $\nabla\varphi^N(t)$ have vanishing tangential components on Γ_D. This implies that also $\xi^N(t) = \text{curl } \Psi^N(t)$ has vanishing tangential components on Γ_D.

The boundedness of $\Psi^N \in L^2(0,T;H^1(\Omega))$ implies the spatial regularity of this sequence. We furthermore know that, for small $|\rho|$, $\rho \in \mathbb{R}$, differences $\bar{p}^N(.+\rho) - \bar{p}^N(.)$ are small in $L^2(0,T;L^2(\Omega))$, independent of N, by (3.23). The same is true for $\nabla\varphi$ and p. Since the Helmholtz decomposition yields a continuous linear map $q^N(t) \mapsto \Psi^N(t)$ from $L^2(\Omega)$ to $H^1(\Omega)$, this implies that also the sequence Ψ^N has small differences $\Psi^N(.+\rho) - \Psi^N(.)$ in $L^2(0,T;H^1(\Omega))$. The Fréchet-Kolmogorov compactness criterion yields the strong convergence $\Psi^N \to 0$ in $L^2(0,T;L^2(\Omega))$.

We can now conclude (3.29). We first note that the sequence $\text{curl } \xi^N = \text{curl}\,(q^N - \nabla\varphi^N) = \text{curl}\,(\nabla\varphi + \bar{p}^N - p - \nabla\varphi^N) = \text{curl}\,(\bar{p}^N - p)$ is bounded in $L^2(0,T;L^2(\Omega))$. We can therefore calculate in the limit $N \to \infty$

$$\int_0^T \int_\Omega |\xi^N|^2 = \int_0^T \int_\Omega \xi^N \cdot \text{curl } \Psi^N = \int_0^T \int_\Omega \text{curl } \xi^N \cdot \Psi^N \to 0\,.$$

In this calculation, the boundary conditions for ξ^N and Ψ^N allow the integration by parts. We have obtained the strong convergence (3.29).

Step 4: Item (E$_2$). The proof of Item (E$_2$) is essentially as in the case $\delta > 0$, we only need to replace the weak convergence (3.16) by the weak convergence in $H_0(\Omega, \text{curl})$. □

4 The time-stepping scheme

4.1 Solutions to the discrete problem and estimates

In this section we construct time-discrete approximations of the system (1.3)–(1.6). With a number $N \in \mathbb{N}$ of time steps of length $\tau := \frac{T}{N}$ we discretize the interval $[0,T]$ with

$$0 = t_0 < t_1 < \ldots < t_N = T, \quad t_k := k\,\tau, \quad k = 0,\ldots,N.$$

We use a variational scheme to obtain a familiy $(p_k)_{1\le k\le N}$ in the state space

$$X_\delta := \begin{cases} W_0^{1,r}(\Omega,\mathbb{R}^{3\times 3}) & \text{if } \delta > 0\,, \\ H_0(\Omega,\text{curl}) & \text{if } \delta = 0\,. \end{cases}$$

The functions $p_k \in X_\delta$ shall be approximations of the solution values $p(t_k)$. For $k = 0$, we use the initial data p_0 as the value in $t_0 = 0$. The loads are discretized with time averages as

$$f_k := \frac{1}{\tau}\int_{t_{k-1}}^{t_k} f(s)\,ds \quad \text{for } k = 2,\ldots,N, \quad \text{and } \; f_0 := f_1 := f(0)\,. \tag{4.1}$$

We note that (2.10) yields a constant $\Lambda_f > 0$, independent of N, such that

$$\sum_{k=0}^{N} \tau \|f_k\|^2_{L^2(\Omega;\mathbb{R}^3)} + \max_{1\le k\le N} \|f_k\|^2_{H_D^{-1}(\Omega;\mathbb{R}^3)} + \sum_{k=1}^{N} \tau \Big\| \frac{f_k - f_{k-1}}{\tau} \Big\|^2_{H_D^{-1}(\Omega;\mathbb{R}^3)} \le \Lambda_f^2 \,. \quad (4.2)$$

Lemma 4.1 (Existence of time-discrete approximations)**.** *For all $k = 1, \dots, N$ there exists a pair $(p_k, \Sigma_k) \in X_\delta \times L^2(\Omega; \mathbb{R}^{3\times3})$ such that*

$$\mathcal{E}(p_k; f_k) + \mathcal{E}^*(-\Sigma_k; f_k) \;=\; \langle -\Sigma_k, p_k \rangle \,, \quad (4.3)$$

and

$$\mathcal{E}(p_k; f_k) + \tau \mathcal{R}\Big(\frac{p_k - p_{k-1}}{\tau}\Big) + \tau \mathcal{R}^*(\Sigma_k) \;\le\; \mathcal{E}(p_{k-1}; f_{k-1}) - \langle f_k - f_{k-1}, u_{k-1} \rangle \quad (4.4)$$

holds for any $u_{k-1} \in \mathfrak{E}(p_{k-1}, f_{k-1})$. Furthermore, we have

$$\Sigma_k \;\in\; \partial \mathcal{R}\Big(\frac{p_k - p_{k-1}}{\tau}\Big) \,.$$

Proof. We define a functional $\mathcal{R}_\tau$ by setting

$$\mathcal{R}_\tau(q) \;:=\; \int_\Omega \tau R\left(\frac{q(x)}{\tau}\right) dx \,,$$

and define $\mathcal{G}_k : L^2(\Omega, \mathbb{R}^{3\times3}) \to \mathbb{R}$ by

$$\mathcal{G}_k(p) \;:=\; \mathcal{E}(p; f_k) + \mathcal{R}_\tau(p - p_{k-1}) \qquad \text{if } p \in X_\delta \,, \quad (4.5)$$

and $\mathcal{G}_k(p) := +\infty$ if $p \in L^2(\Omega, \mathbb{R}^{3\times3}) \setminus X_\delta$. In order to construct approximations $(p_k)_k$, $1 \le k \le N$, we use the following scheme: In every time step we minimize, given $p_{k-1} \in X_\delta$, the functional $\mathcal{G}_k$.

Step 1: Existence of minimizers. We treat here the case $\delta = 0$, for $\delta > 0$ the proof is easily adapted. By Lemma A.1 Items 1 and 6 (convexity and Lipschitz property of $\mathcal{E}_1$) and Assumption 2.1 (convexity and growth condition), $\mathcal{G}_k$ is convex and lower semi-continuous. Using the lower bound (A.2) for the energy together with (4.2) and the growth condition (2.8) on $\mathcal{R}$ we find, for any $p \in X_\delta$,

$$\mathcal{G}_k(p) \;\ge\; -\|p\|^2_{L^2(\Omega,\mathbb{R}^{3\times3})} - C\Lambda_f^2 + \frac{1}{\tau} c_R \|p - p_{k-1}\|^2_{L^2(\Omega,\mathbb{R}^{3\times3})} \,.$$

This implies that $\mathcal{G}_k$ is coercive for any $\tau < c_R$. The direct method of the Calculus of Variations implies that a minimizer $p_k \in Xp$ of $\mathcal{G}_k$ exists.

Step 2: The minimizing property of p_k, the continuity of $\mathcal{R}_\tau$ and subdifferential calculus for convex functions imply that

$$0 \;\in\; \partial \mathcal{G}_k(p_k) \;=\; \partial \mathcal{E}(p_k; f_k) + \partial \mathcal{R}_\tau(p_k - p_{k-1}) \,.$$

Therefore, there exists $\Sigma_k \in L^2(\Omega; \mathbb{R}^{3\times3})$ with

$$-\Sigma_k \;\in\; \partial \mathcal{E}(p_k; f_k) \,, \quad (4.6)$$

$$\Sigma_k \;\in\; \partial \mathcal{R}_\tau(p_k - p_{k-1}) \;=\; \partial \mathcal{R}\Big(\frac{p_k - p_{k-1}}{\tau}\Big) \,. \quad (4.7)$$

Subdifferential calculus provides that (4.6) implies (4.3). We now use (4.7) and the Fenchel equality, then (4.6) and the defining property of the subdifferential. In the last equality, we re-order terms and add a zero.

$$\begin{aligned}
\tau\mathcal{R}\Big(\frac{p_k - p_{k-1}}{\tau}\Big) + \tau\mathcal{R}^*(\Sigma_k) \;&=\; \tau\Big\langle \Sigma_k, \frac{p_k - p_{k-1}}{\tau}\Big\rangle \\
&\le\; \mathcal{E}(p_{k-1}; f_k) - \mathcal{E}(p_k; f_k) \\
&=\; \mathcal{E}(p_{k-1}; f_{k-1}) - \mathcal{E}(p_k; f_k) + \mathcal{E}(p_{k-1}; f_k) - \mathcal{E}(p_{k-1}; f_{k-1}) \,.
\end{aligned}$$

Item 5 of Lemma A.1 yields for any $u_{k-1} \in \mathfrak{E}(p_{k-1}, f_{k-1})$

$$\tau\mathcal{R}\Big(\frac{p_k - p_{k-1}}{\tau}\Big) + \tau\mathcal{R}^*(\Sigma_k) \;\le\; \mathcal{E}(p_{k-1}; f_{k-1}) - \mathcal{E}(p_k; f_k) - \langle f_k - f_{k-1}, u_{k-1}\rangle \,.$$

This proves (4.4). $\square$

In the following lemma, we treat the case $\delta = 0$ and hence $H_e \equiv 0$ and $H_p \equiv 0$. In this setting, we deduce that the divergence of Σ_k is controlled. Let us provide the idea of the argument: We have calculated in the introduction for this case $\Sigma = \sigma - 2\,\mathrm{curl}\,\mathrm{curl}(p)$. Since the divergence of a curl vanishes, we can expect $-\nabla\cdot\Sigma = -\nabla\cdot\sigma = f$, which is controlled. We use $\mathcal{Q} : L^2(\Omega; \mathbb{R}^{3\times 3}_s) \to \mathbb{R}$, defined by

$$\mathcal{Q}(e) \;:=\; \int_\Omega Q(e)\, dx \,. \tag{4.8}$$

Since we assume $H_e \equiv 0$ the elastic energy of an arbitrary deformation $\varphi \in H^1_D(\Omega, \mathbb{R}^3)$ and of an arbitrary plastic contribution $q \in L^2(\Omega, \mathbb{R}^{3\times 3})$ is given by

$$\mathcal{W}_e(\nabla\varphi, q) \;=\; \mathcal{Q}(\mathrm{sym}(\nabla\varphi - q)) \;=\; \int_\Omega Q(\mathrm{sym}(\nabla\varphi - q))\, dx \,.$$

Lemma 4.2. *Assume $\delta = 0$ with $H_e \equiv 0$ and $H_p \equiv 0$. Then, for any $1 \le k \le N$, there holds*

$$-\nabla\cdot\Sigma_k \;=\; f_k \,. \tag{4.9}$$

Proof. We analyze the distribution $\eta := -\Sigma_k - 2\,\mathrm{curl}\,\mathrm{curl}(p_k)$. From (4.3) we know $-\Sigma_k \in \partial\mathcal{E}(p_k; f_k)$. Subdifferential calculus (e.g. [3, Theorem 9.5.4]) yields $\eta \in \partial\mathcal{E}_1(p_k; f_k)$. The definition of the subdifferential yields

$$\mathcal{E}_1(p_k + \nabla\psi; f_k) \;\ge\; \mathcal{E}_1(p_k; f_k) + \langle \eta, \nabla\psi\rangle \tag{4.10}$$

for all $\psi \in H^1_D(\Omega, \mathbb{R}^3)$. We can evaluate the left hand side, arguing with $\tilde\varphi = \varphi - \psi$,

$$\begin{aligned}
\mathcal{E}_1(p_k + \nabla\psi; f_k) \;&=\; \inf_{\varphi \in H^1_D(\Omega,\mathbb{R}^3)} \Big(\mathcal{Q}(\mathrm{sym}(\nabla\varphi - p_k - \nabla\psi)) - \langle\varphi, f_k\rangle\Big) \\
&=\; \inf_{\tilde\varphi \in H^1_D(\Omega,\mathbb{R}^3)} \Big(\mathcal{Q}(\mathrm{sym}(\nabla\tilde\varphi - p_k)) - \langle\tilde\varphi, f_k\rangle\Big) - \langle\psi, f_k\rangle \\
&=\; \mathcal{E}_1(p_k; f_k) - \langle\psi, f_k\rangle \,.
\end{aligned}$$

Inserting into (4.10) yields

$$0 \;\ge\; \langle\eta, \nabla\psi\rangle + \langle\psi, f_k\rangle \qquad \text{for all } \psi \in H^1_D(\Omega, \mathbb{R}^3)\,,$$

and hence $-\nabla\cdot\eta + f_k = 0$. By definition of η, and since the divergence of a curl vanishes, this yields the claim of (4.9). $\square$

Lemma 4.3 (A priori bounds for the time-discrete solutions)**.** *Let the load f satisfy* (2.10) *and let $(f_k)_k$ be defined by* (4.1) *such that* (4.2) *is satisfied. Then there exists a constant $C = C(\Lambda_f, \|p_0\|_{L^2(\Omega,\mathbb{R}^{3\times 3})})$, independent of N, such that the sequence of time-discrete solutions satisfies the a priori estimate*

$$\max_k \mathcal{W}(\nabla u_k, p_k) + \sum_k \tau \left\{ \mathcal{R}\Big(\frac{p_k - p_{k-1}}{\tau}\Big) + \mathcal{R}^*(\Sigma_k) \right\} \le C\,, \tag{4.11}$$

where $u_k \in \mathfrak{E}(p_k, f_k)$, $1 \le k \le N$, are chosen arbitrarily. In particular, we have

$$\max_{1\le k\le N} \|p_k\|^2_{H_0(\Omega,\mathrm{curl})} + \max_{1\le k\le N} \int_\Omega \delta |\nabla p_k|^r + \sum_{1\le k\le N} \tau \int_\Omega \left\{ \Big| \frac{p_k - p_{k-1}}{\tau} \Big|^2 + |\Sigma_k|^2 \right\} \le C\,. \tag{4.12}$$

Proof. We choose $1 \le k_0 \le N$ arbitrary and take the sum of (4.4) over $k = 1, \dots, k_0$. We obtain

$$\begin{aligned} &\mathcal{E}(p_{k_0}; f_{k_0}) + \sum_{k=1}^{k_0} \tau \left\{ \mathcal{R}\Big(\frac{p_k - p_{k-1}}{\tau}\Big) + \mathcal{R}^*(\Sigma_k) \right\} \\ &\quad\le \mathcal{E}(p_0; f_0) - \sum_{k=1}^{k_0} \int_\Omega u_{k-1} \cdot (f_k - f_{k-1})\,, \end{aligned} \tag{4.13}$$

where the functions $u_{k-1} \in \mathfrak{E}(p_{k-1}, f_{k-1})$, $k = 1, \dots, k_0$, are arbitrary.

We estimate the second term on the right-hand side of (4.13). By our choice $f_0 = f_1$, using the bound (4.2) on $(f_k)_k$, we obtain for arbitrary $\lambda > 0$

$$\begin{aligned} &\Big| \sum_{k=1}^{k_0} \int_\Omega u_{k-1} \cdot (f_k - f_{k-1})\,dx \Big| \\ &\quad\le \Big(\sum_{k=2}^{k_0} \tau \Big\| \frac{f_k - f_{k-1}}{\tau} \Big\|^2_{H_D^{-1}(\Omega;\mathbb{R}^3)} \Big)^{\frac12} \Big(\sum_{k=2}^{k_0} \tau \|u_{k-1}\|^2_{H^1(\Omega;\mathbb{R}^3)} \Big)^{\frac12} \\ &\quad\le \frac{\Lambda_f^2}{4\lambda} + \lambda\, k_0\, \tau \max_{1\le k\le k_0-1} \|u_k\|^2_{H^1(\Omega;\mathbb{R}^3)} \\ &\quad\le \lambda T \max_{1\le k\le k_0-1} \|p_k\|^2_{L^2(\Omega;\mathbb{R}^{3\times3})} + C(T, \lambda, \Lambda_f)\,, \end{aligned} \tag{4.14}$$

where we have used the estimate for u_k from (A.3) in the last step. We can now obtain an N-independent bound for energies. We use (A.2) in the first inequality, then (4.13) together with (4.14), with the parameter $\lambda > 0$ unchanged,

$$\begin{aligned} &\frac12 \mathcal{W}_e(\nabla u_{k_0}, p_{k_0}) + \mathcal{W}_p(p_{k_0}) + \sum_{k=1}^{k_0} \tau \left\{ \mathcal{R}\Big(\frac{p_k - p_{k-1}}{\tau}\Big) + \mathcal{R}^*(\Sigma_k) \right\} \\ &\quad\le \mathcal{E}(p_{k_0}; f_{k_0}) + \sum_{k=1}^{k_0} \tau \left\{ \mathcal{R}\Big(\frac{p_k - p_{k-1}}{\tau}\Big) + \mathcal{R}^*(\Sigma_k) \right\} + \lambda \|p_{k_0}\|^2 + C \\ &\quad\le \mathcal{E}(p_0; f_0) + \lambda(T+1) \max_{1\le k\le k_0} \|p_k\|^2_{L^2(\Omega;\mathbb{R}^{3\times3})} + C\,, \end{aligned}$$

where $C = C(T, c_Q, \lambda, \Omega, \Gamma_D, \Lambda_f)$. This implies

$$\max_{1\le k\le N} \mathcal{W}(\nabla u_k, p_k) + \sum_{k=1}^{N} \tau \left\{ \mathcal{R}\Big(\frac{p_k - p_{k-1}}{\tau}\Big) + \mathcal{R}^*(\Sigma_k) \right\}$$
$$\le\ 3\mathcal{E}(p_0; f_0) + 3\lambda(T+1) \max_{1\le k\le N} \|p_k\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})} + C\,. \tag{4.15}$$

Our next aim is to find bounds for $(p_k)_k$. We consider an arbitrary $k \le N$ and start with the elementary triangle inequality

$$\|p_k\|_{L^2(\Omega;\mathbb{R}^{3\times 3})}\ \le\ \sum_{j=1}^{k} \|p_j - p_{j-1}\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} + \|p_0\|_{L^2(\Omega;\mathbb{R}^{3\times 3})}\,.$$

Taking the square and using the coercivity (2.8) of $\mathcal{R}$, we obtain

$$\begin{aligned}
\|p_k\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})}\ &\le\ 2k \sum_{j=1}^{k} \|p_j - p_{j-1}\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})} + 2\|p_0\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})}\\
&\le\ \frac{2k\tau^2}{c_R} \sum_{j=1}^{k} \mathcal{R}\Big(\frac{p_j - p_{j-1}}{\tau}\Big) + 2\|p_0\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})}\\
&\le\ C(T, c_R) \sum_{j=1}^{N} \tau\mathcal{R}\Big(\frac{p_j - p_{j-1}}{\tau}\Big) + 2\|p_0\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})}\,.
\end{aligned} \tag{4.16}$$

Hence, by choosing $\lambda > 0$ sufficiently small in (4.15), we obtain

$$\max_{1\le k\le N} \mathcal{W}(\nabla u_k, p_k) + \frac{1}{2}\sum_{k=1}^{N} \tau \left\{ \mathcal{R}\Big(\frac{p_k - p_{k-1}}{\tau}\Big) + \mathcal{R}^*(\Sigma_k) \right\}\ \le\ C\,, \tag{4.17}$$

with $C = C(T, c_R, c_Q, \Omega, \Gamma_D, \Lambda_f, p_0)$, and hence (4.11). Using (2.8) and (4.16) we deduce (4.12). $\square$

4.2 The time-continuous limit

The next step in our construction is to introduce interpolations of the time-discrete approximate solutions. For $N \in \mathbb{N}$, we have constructed time-discrete values $p_k = p_k^N \in X$ and $\Sigma_k = \Sigma_k^N \in L^2(\Omega, \mathbb{R}^{3\times 3})$, see Lemma 4.1. We define the piecewise constant and left-continuous interpolation $\bar{p}^N : [0,T] \to X$ and the piecewise affine and continuous interpolation $\hat{p}^N : [0,T] \to X$ by

$$\bar{p}^N(t) := p_k \qquad \text{for } t \in (t_{k-1}, t_k], 1 \le k \le N, \qquad \bar{p}^N(0) := p_0$$
$$\hat{p}^N(t) := (1-\mu)p_{k-1} + \mu p_k \quad \text{for } t = (1-\mu)t_{k-1} + \mu t_k,\ \mu \in [0,1], 1 \le k \le N\,.$$

Similarly, we define $\bar{\Sigma}^N$ and $\bar{f}^N$ as the piecewise constant left-continuous interpolations of $(\Sigma_k^N)_k$ and $(f_k^N)_k$. The function $\hat{f}^N$ is defined slightly differently, namely with a time-shift (we set $f_{N+1} := f_N$):

$$\hat{f}^N(t) := (1-\mu)f_k + \mu f_{k+1} \quad \text{for } t = (1-\mu)t_{k-1} + \mu t_k,\ \mu \in [0,1], 1 \le k \le N\,.$$

By the previous results we easily obtain a priori bounds for $\bar{p}^N$, $\bar{\Sigma}^N$, $\partial_t \hat{p}^N$.

Lemma 4.4 (Estimates for the interpolations)**.** *Under Assumption 2.1 there exists $C > 0$ independent of N such that the piecewise constant functions satisfy*

$$\|\bar{p}^N\|_{L^\infty(0,T;H_0(\Omega,\mathrm{curl}))} + \|\bar{\Sigma}^N\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{3\times 3}))} + \sqrt[r]{\delta}\|\nabla\bar{p}^N\|_{L^\infty(0,T;L^r(\Omega;\mathbb{R}^{3\times 3}))} \le C\,. \tag{4.18}$$

The piecewise affine functions satisfy

$$\|\hat{p}^N\|_{H^1(0,T;L^2(\Omega;\mathbb{R}^{3\times 3}))} + \|\hat{p}^N\|_{L^\infty(0,T;H_0(\Omega,\mathrm{curl}))} + \sqrt[r]{\delta}\|\nabla\hat{p}^N\|_{L^\infty(0,T;L^r(\Omega;\mathbb{R}^{3\times 3}))} \;\le\; C\,. \tag{4.19}$$

We have the time-shift property

$$\sup_N \left\|\bar{p}^N(.+\rho) - \bar{p}^N(.)\right\|^2_{L^2(0,T-\rho;L^2(\Omega;\mathbb{R}^{3\times 3}))} \to 0 \quad \textit{as} \quad \rho \to 0\,. \tag{4.20}$$

Finally, in the case $\delta = 0$, for $H_e \equiv 0$ and $H_p \equiv 0$, with Λ_f from (4.2)*:*

$$\|\nabla\cdot\bar{\Sigma}^N\|_{L^2(0,T;L^2(\Omega,\mathbb{R}^3))} \;\le\; \Lambda_f\,. \tag{4.21}$$

Proof. Lemma 4.3 provides the estimates (4.18) and (4.19) for $\bar{p}^N$, $\bar{\Sigma}^N$, and $\hat{p}^N$. Lemma 4.2 provides (4.21) for $\nabla\cdot\bar{\Sigma}^N$.

To prove (4.20) we first calculate for $\rho = \tau_j := j\tau$, $j \in \mathbb{N}$,

$$\begin{aligned}\int_0^{T-\rho} \|\bar{p}^N(t+\rho) - \bar{p}^N(t)\|^2_{L^2(\Omega)}\,dt \;&=\; \sum_{k=1}^{N-j}\tau\int_\Omega |p_{k+j}-p_k|^2 \\ = \sum_{k=1}^{N-j}\tau\int_\Omega \Big|\sum_{i=k+1}^{k+j}(p_i - p_{i-1})\Big|^2 &\le \sum_{k=1}^{N-j}\tau j\sum_{i=k+1}^{k+j}\tau^2\int_\Omega\Big|\frac{p_i-p_{i-1}}{\tau}\Big|^2 \;\le\; C\rho^2\,,\end{aligned}$$

where we used the a priori estimate (4.12) in the last step.

We now consider an arbitrary shift $\rho \in (\tau_j, \tau_{j+1}]$, $0 < j \in \mathbb{N}$. For t lying in an interval $(t_k, t_{k+1}]$, the number $t+\rho$ lies either in the interval $(t_{k+j}, t_{k+j+1}]$ or in the interval $(t_{k+j+1}, t_{k+j+2}]$. We hence obtain with a triangle inequality

$$|\bar{p}^N(t+\rho) - \bar{p}^N(t)| \le |\bar{p}^N(t+\tau_{j+1}) - \bar{p}^N(t)| + |\bar{p}^N(t+\tau_{j+2}) - \bar{p}^N(t)|\,.$$

The above inequalities yield (4.20) for $\bar{p}^N$. □

Lemma 4.4 implies some compactness properties.

Lemma 4.5 (Convergence of time-discrete approximations)**.** *There exists a subsequence $N \to \infty$ (not relabeled) and functions $p \in L^\infty(0,T;H_0(\Omega,\mathrm{curl})) \cap H^1(0,T;L^2(\Omega;\mathbb{R}^{3\times 3}))$ and $\Sigma \in L^2(0,T;L^2(\Omega;\mathbb{R}^{3\times 3}))$ such that*

$$\bar{p}^N \overset{*}{\rightharpoonup} p \quad \textit{in } L^\infty(0,T;H_0(\Omega,\mathrm{curl}))\,, \tag{4.22}$$

$$\hat{p}^N \overset{*}{\rightharpoonup} p \quad \textit{in } L^\infty(0,T;H_0(\Omega,\mathrm{curl}))\,, \tag{4.23}$$

$$\partial_t\hat{p}^N \rightharpoonup \partial_t p \quad \textit{in } L^2(0,T;L^2(\Omega,\mathbb{R}^{3\times 3}))\,, \tag{4.24}$$

$$\bar{\Sigma}^N \rightharpoonup \Sigma \quad \textit{in } L^2(0,T;L^2(\Omega,\mathbb{R}^{3\times 3}))\,. \tag{4.25}$$

For $\delta > 0$ we have additionally $p \in L^\infty(0,T;W^{1,r}(\Omega,\mathbb{R}^{3\times 3}))$ and, for $s = \frac{3r}{3-r} > 2$ and arbitrary $1 \le q < \infty$

$$\bar{p}^N \stackrel{*}{\rightharpoonup} p \quad \text{in } L^\infty(0,T;W^{1,r}(\Omega,\mathbb{R}^{3\times 3}))\,, \tag{4.26}$$

$$\hat{p}^N \stackrel{*}{\rightharpoonup} p \quad \text{in } L^\infty(0,T;W^{1,r}(\Omega,\mathbb{R}^{3\times 3}))\,, \tag{4.27}$$

$$\bar{p}^N \to p \quad \text{in } L^q(0,T;L^s(\Omega,\mathbb{R}^{3\times 3}))\,, \tag{4.28}$$

$$\hat{p}^N \to p \quad \text{in } L^q(0,T;L^s(\Omega,\mathbb{R}^{3\times 3}))\,. \tag{4.29}$$

Moreover, we have

$$\bar{f}^N \to f \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^3))\,, \tag{4.30}$$

$$\hat{f}^N \to f \quad \text{in } H^1(0,T;H^{-1}_D(\Omega;\mathbb{R}^3))\,. \tag{4.31}$$

Proof. The a priori estimates (4.18) and (4.19) allow to select a subsequence $N \to \infty$ and to find limits p and Σ as in the claim of the Lemma, such that (4.23), (4.24) and (4.25) hold. Furthermore, for a limit function $\bar{p} \in L^\infty(0,T;H_0(\Omega,\mathrm{curl}))$, there holds $\bar{p}^N \stackrel{*}{\rightharpoonup} \bar{p}$ in $L^\infty(0,T;H_0(\Omega,\mathrm{curl}))$.

The sequence $\hat{p}^N$ can be regarded as a sequence in the space $L^2(0,T;H^{-1}_D(\Omega))$. The Lions–Aubin Lemma [25, Lemma 7.7] implies that $(\hat{p}^N)_N$ is pre-compact in this space. We therefore have the strong convergence $\hat{p}^N \to p$ in $L^2(0,T;H^{-1}_D(\Omega))$. The strong convergence allows to compare the two interpolations, see [14, Lemma 3.2] or [26, Lemma 11.3]; we conclude that $\bar{p}^N$ has the same limit and converges also strongly, $\bar{p}^N \to \bar{p} = p$ in $L^2(0,T;H^{-1}_D(\Omega))$. In particular, we find (4.22).

In case that $\delta > 0$ we can apply (4.18) and conclude that $\nabla\hat{p}^N$ is uniformly bounded in $L^\infty(0,T;L^r(\Omega;\mathbb{R}^3))$. Since the only constant function in $W^{1,r}(\Omega;\mathbb{R}^3) \cap H_0(\Omega,\mathrm{curl})$ is zero, by [2, Sec. 6.16] a Poincaré inequality holds in this space and we deduce that $\hat{p}^N$ is uniformly bounded in $L^\infty(0,T;W^{1,r}(\Omega;\mathbb{R}^3))$. We conclude by a similar identification argument as above that (4.26) and (4.27) hold.

The estimate (4.19) together with the Lions–Aubin Lemma implies the compactness of the sequence $(\hat{p}^N)_N$ in $L^q(0,T;L^s(\Omega,\mathbb{R}^{3\times 3}))$ and hence (4.29). Comparison of the two interpolations (as above, using [14] or [26]) yields (4.28).

The statements on $\bar{f}^N$ and $\hat{f}^N$ are elementary approximation properties for discretizations of a given function f. $\square$

Proof of Theorem 2.5 and Theorem 2.6. The discrete solution property (4.3) yields that almost everywhere in $(0,T)$

$$\mathcal{E}(\bar{p}^N;\bar{f}^N) + \mathcal{E}^*(-\bar{\Sigma}^N;\bar{f}^N) \;=\; \langle -\bar{\Sigma}^N, \bar{p}^N\rangle\,. \tag{4.32}$$

The energy inequality for the time-discrete approximate solutions was derived in (4.13). We claim that that inequality implies, for almost every $t \in (0,T)$,

$$\begin{aligned} &\mathcal{E}(\bar{p}^N;\bar{f}^N)(t) + \int_0^t \left\{\mathcal{R}\big(\partial_t\hat{p}^N\big) + \mathcal{R}^*(\bar{\Sigma}^N)\right\}\,ds \\ &\quad\le\; \mathcal{E}(p_0;f_0) - \int_0^t \inf_{\tilde{u}^N\in\mathfrak{C}(\bar{p}^N(s);\bar{f}^N(s))} \langle\partial_t\hat{f}^N(s),\tilde{u}^N\rangle\,ds + o(1) \end{aligned} \tag{4.33}$$

as $N \to \infty$.

Indeed, let $u_k \in \mathfrak{E}(p_k, f_k)$ be chosen arbitrarily. We consider first a time instance $t = t_{k_0} \in \tau\mathbb{N}$. For such a time instance t, the relation (4.33) coincides with (4.13) except for the very last time interval; the difference of the right hand sides is $\int_\Omega u_{k_0} \cdot (f_{k_0+1} - f_{k_0})$.

Let us now consider an arbitrary time instance. For $t \in (t_{k_0-1}, t_{k_0}]$ the left hand side of (4.33) is not larger than the left hand side of (4.13), since R and R^* are non-negative. The difference of the right hand sides is

$$\frac{t - t_{k_0-1}}{\tau} \int_\Omega u_{k_0} \cdot (f_{k_0+1} - f_{k_0}) = o(1)\,,$$

for almost every t. The smallness is a consequence of the following facts: (i) uniform bound $t - t_{k_0-1} \le \tau$, (ii) uniform (in k_0, N) boundedness of u_{k_0} in $H^1_D(\Omega)$, and (iii) the continuous embedding of $H^1(0,T;H^{-1}(\Omega))$ in $C^{0,\alpha}([0,T];H^{-1}(\Omega))$ for all $0 < \alpha < \frac{1}{2}$ and the uniform (in k_0, N) estimate

$$\|f_{k+1} - f_k\|_{H^{-1}(\Omega)} = \left\| \frac{1}{\tau} \int_{t_k}^{t_{k+1}} \big(f(s) - f(s-\tau)\big)\, ds \right\|_{H^{-1}(\Omega)} \le \|f\|_{C^{0,\alpha}([0,T];H^{-1}(\Omega))} \tau^\alpha.$$

The two relations (4.32) and (4.33) imply that the approximate solutions satisfy the solution properties of Assumption 3.2.

Lemma 4.5 provides the convergence properties of Assumptions 3.1 and 3.7 in the two cases $\delta > 0$ and $\delta = 0$. We emphasize that, in the case $\delta = 0$, with (4.20) we have verified (3.23) and with (4.21) we have verified the boundedness of $\nabla \cdot \bar\Sigma^N$. We can apply the stability results of Propositions 3.3 and 3.8. They yield that (p, Σ) is a generalized solution of the visco-plasticity system. □

A The marginal functional

We discuss some properties of the marginal functional $\mathcal{E}_1$.

Lemma A.1. *The functional $\mathcal{E}_1(\cdot; f) : L^2(\Omega;\mathbb{R}^{3\times 3}) \to \mathbb{R}$ from* (2.19) *has the following properties.*

1. *For any $f \in H^{-1}_D(\Omega;\mathbb{R}^3)$ the functional $\mathcal{E}_1(\cdot; f)$ is convex.*

2. *For any $p \in L^2(\Omega;\mathbb{R}^{3\times 3})$ and any $f \in H^{-1}_D(\Omega;\mathbb{R}^3)$ we have*

$$\mathcal{E}_1(p; f) \le C(1 + \|p\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})})\,. \tag{A.1}$$

3. *For any $p \in L^2(\Omega;\mathbb{R}^{3\times 3})$, $f \in H^{-1}_D(\Omega;\mathbb{R}^3)$, $u \in \mathfrak{E}(p,f)$, and $\lambda > 0$ there exist $c_\lambda = c_\lambda(C_Q, \Omega, \Gamma_D) > 0$ and $C_\lambda = C_\lambda(C_Q, \Omega, \Gamma_D)$ such that*

$$\mathcal{E}_1(p; f) \ge \frac{1}{2}\mathcal{W}_e(\nabla u, p) + c_\lambda \|u\|^2_{H^1_D(\Omega;\mathbb{R}^3)} - \lambda \|p\|^2_{L^2(\Omega)} - C_\lambda \|f\|^2_{H^{-1}_D(\Omega;\mathbb{R}^3))}\,. \tag{A.2}$$

4. *For any $p \in L^2(\Omega;\mathbb{R}^{3\times 3})$ and any $f \in H_D^{-1}(\Omega;\mathbb{R}^3)$ there exists an $u \in \mathfrak{E}(p,f)$. Any $u \in \mathfrak{E}(p,f)$ satisfies*

$$\|u\|_{H^1_D(\Omega;\mathbb{R}^3)} \le C\big(1 + \|p\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} + \|f\|_{H_D^{-1}(\Omega;\mathbb{R}^3)}\big)\,. \tag{A.3}$$

5. *For any $p \in L^2(\Omega;\mathbb{R}^{3\times 3})$, any $f, g \in H_D^{-1}(\Omega;\mathbb{R}^3)$ and any $u \in \mathfrak{E}(p,g)$ we have*

$$\mathcal{E}_1(p;f) - \mathcal{E}_1(p;g) \le -\langle f-g, u\rangle\,. \tag{A.4}$$

6. *The map $\mathcal{E}_1 : L^2(\Omega;\mathbb{R}^{3\times 3}) \times H_D^{-1}(\Omega;\mathbb{R}^3) \to \mathbb{R}$ is locally Lipschitz continuous. More precisely, for any Λ_0 there exists $C(\Lambda_0, Q)$ such that*

$$\big|\mathcal{E}_1(p;f) - \mathcal{E}_1(q;g)\big| \le C(\Lambda_0,Q)\Big(\|p-q\|_{L^2(\Omega)} + \|f-g\|_{H_D^{-1}(\Omega)}\Big)$$

for any f, g, p, q with

$$\|f\|_{H_D^{-1}(\Omega;\mathbb{R}^3)} + \|g\|_{H_D^{-1}(\Omega;\mathbb{R}^3)} + \|p\|_{L^2(\Omega)} + \|q\|_{L^2(\Omega)} \le \Lambda_0\,. \tag{A.5}$$

7. *For any sequence $(p_j, f_j)_j$ and any (p,f) in $L^2(\Omega;\mathbb{R}^{3\times 3}) \times H_D^{-1}(\Omega;\mathbb{R}^3)$ with $p_j \rightharpoonup p$ in $L^2(\Omega;\mathbb{R}^{3\times 3})$ and $f_j \to f$ in $H_D^{-1}(\Omega;\mathbb{R}^3)$ we have*

$$\mathcal{E}_1(p;f) \le \liminf_{j\to\infty} \mathcal{E}_1(p_j;f_j)\,. \tag{A.6}$$

Proof. In the following, for any $p \in L^2(\Omega;\mathbb{R}^{3\times 3})$, we use the shortcut $p^s := \mathrm{sym}(p)$.
Item 1: Let $p, q \in L^2(\Omega;\mathbb{R}^{3\times 3})$ and $0 < \lambda < 1$ be arbitrary. From the convexity of Q and H_e we deduce

$$\begin{aligned}
&(1-\lambda)\mathcal{E}_1(p;f) + \lambda\mathcal{E}_1(q;f)\\
&= \inf_{\varphi,\psi \in H^1_D(\Omega;\mathbb{R}^3)} \Big[\int_\Omega \Big((1-\lambda)W_e(\nabla\varphi,p) + \lambda W_e(\nabla\psi,q)\Big) - \Big\langle f, ((1-\lambda)\varphi + \lambda\psi)\Big\rangle\Big]\\
&\ge \inf_{\varphi,\psi \in H^1_D(\Omega;\mathbb{R}^3)} \Big[\int_\Omega \Big(Q\big((1-\lambda)\left(\nabla^s\varphi - p^s\right) + \lambda\left(\nabla^s\psi - q^s\right)\big)\\
&\qquad\qquad + H_e\big((1-\lambda)\nabla^s\varphi + \lambda\nabla^s\psi\big)\Big) - \Big\langle f, \big((1-\lambda)\varphi + \lambda\psi\big)\Big\rangle\Big]\\
&= \inf_{\tilde\varphi \in H^1_D(\Omega;\mathbb{R}^3)} \Big[\int_\Omega \Big(Q\big(\nabla^s\tilde\varphi - ((1-\lambda)p^s + \lambda q^s)\big)\big) + H_e\big(\nabla^s\tilde\varphi\big)\Big) - \langle f, \tilde\varphi\rangle\Big]\\
&= \mathcal{E}_1\big((1-\lambda)p + \lambda q; f\big)\,.
\end{aligned}$$

This proves the convexity of $\mathcal{E}_1(\cdot;f)$.
Item 2: We use $u \equiv 0$ as a competitor to estimate the energy from above. The growth assumption on Q implies

$$\mathcal{E}_1(p;f) \le \int_\Omega \big(Q(\mathrm{sym}\ p) + H_e(0)\big) \le C(Q,H_e)\,(1 + \|p\|^2_{L^2(\Omega)}).$$

Item 3: In any Hilbert space and for any $\mu > 0$ holds $\frac{1}{1+\mu}\|a\|^2 - 2\langle a,b\rangle + (1+\mu)\|b\|^2 \ge 0$; this inequality can be rearranged as $\|a-b\|^2 \ge \frac{\mu}{1+\mu}\|a\|^2 - \mu\|b\|^2$. Applying the latter inequality with $\mu = \frac{2\lambda}{c_Q}$, using the growth assumptions on Q and Korn's inequality [26, Korollar 25.6], we deduce that for any $\varphi \in H^1_D(\Omega;\mathbb{R}^3)$

$$\begin{aligned}
&\int_\Omega Q(\mathrm{sym}(\nabla\varphi - p)) - 2\big|\langle f,\varphi\rangle\big| \\
&\quad\geq c_Q\|\mathrm{sym}(\nabla\varphi - p)\|^2_{L^2(\Omega)} - 2\|f\|_{H^{-1}_D(\Omega;\mathbb{R}^3)}\|\varphi\|_{H^1_D(\Omega;\mathbb{R}^3)} \\
&\quad\geq \frac{2c_Q\lambda}{c_Q+2\lambda}\|\nabla^s\varphi\|^2_{L^2(\Omega)} - 2\lambda\|p\|^2_{L^2(\Omega)} - \frac{c_Q\lambda}{c_Q+2\lambda}\|\nabla^s\varphi\|^2_{L^2(\Omega)} \\
&\qquad - C(C_Q,\lambda,\Omega,\Gamma_D)\|f\|^2_{H^{-1}_D(\Omega;\mathbb{R}^3)} \\
&\quad\geq \frac{c_Q\lambda}{c_Q+2\lambda}\|\nabla^s\varphi\|^2_{L^2(\Omega)} - 2\lambda\|p\|^2_{L^2(\Omega)} - C(C_Q,\lambda,\Omega,\Gamma_D)\|f\|^2_{H^{-1}_D(\Omega;\mathbb{R}^3)}\,. \qquad (A.7)
\end{aligned}$$

Now consider any $u \in \mathfrak{E}(p,f)$. The definition of $\mathcal{E}_1$ yields $\mathcal{E}_1(p;f) \geq \mathcal{W}_e(\nabla u,p) - \langle f,u\rangle$ and we obtain from $H_e \geq 0$ and (A.7)

$$\begin{aligned}
&\mathcal{E}_1(p;f) - \frac{1}{2}\mathcal{W}_e(\nabla u,p) \\
&\quad\geq \frac{1}{2}\int_\Omega Q(\mathrm{sym}(\nabla u - p)) - \langle f,u\rangle \\
&\quad\geq \frac{c_Q\lambda}{2(c_Q+2\lambda)}\|\nabla^s u\|^2_{L^2(\Omega)} - \lambda\|p\|^2_{L^2(\Omega)} - C(C_Q,\lambda,\Omega,\Gamma_D)\|f\|^2_{H^{-1}_D(\Omega;\mathbb{R}^3)}\,.
\end{aligned}$$

By Korn's inequality, this yields (A.2).
Item 4: We consider the functional $\tilde{\mathcal{E}} : H^1_D(\Omega;\mathbb{R}^3) \to \mathbb{R}\cup\{+\infty\}$, given by

$$\tilde{\mathcal{E}}(\varphi) := \int_\Omega W_e(\nabla\varphi,p) - \big\langle f,\varphi\big\rangle\,.$$

The properties of Q and H_e, in particular, their convexity, imply that $\tilde{\mathcal{E}}$ is lower semicontinuous with respect to weak convergence in $H^1_D(\Omega;\mathbb{R}^3)$. By (A.7), the functional $\tilde{\mathcal{E}}$ is also coercive. The direct method of the Calculus of Variations ensures the existence of a minimizer $u \in H^1_D(\Omega;\mathbb{R}^3)$ of $\tilde{\mathcal{E}}$, hence the existence of $u \in \mathfrak{E}(p,f)$.

The estimates (A.1) and (A.2) yield (A.3).
Item 5: By definition of $\mathcal{E}_1$ and by the minimization property $u \in \mathfrak{E}(p,g)$ we have

$$\begin{aligned}
\mathcal{E}_1(p;f) - \mathcal{E}_1(p,g) &\leq \mathcal{W}_e(\nabla u,p) - \langle f,u\rangle - \Big(\mathcal{W}_e(\nabla u,p) - \langle g,u\rangle\Big) \\
&= -\langle f-g,u\rangle\,.
\end{aligned}$$

Item 6: We consider a bound $\Lambda_0 > 0$ and study functions $f,g \in H^{-1}_D(\Omega;\mathbb{R}^3)$ and $p,q \in L^2(\Omega;\mathbb{R}^{3\times3})$ satisfying (A.5). We deduce from Item 4 that for $\Lambda = C(1+\Lambda_0)$

$$\mathcal{E}_1(p;f) = \inf_{\varphi\in H^1_D(\Omega;\mathbb{R}^3),\|\varphi\|\leq\Lambda}\int_\Omega W_e(\nabla\varphi,p) - \big\langle f,\varphi\big\rangle\,,$$

and accordingly for $\mathcal{E}_1(q;g)$.

We use once more the functional $\mathcal{Q}$ of (4.8). By the quadratic growth assumption (2.7) on Q, this functional is Lipschitz continuous on $B(0,\Lambda_0+\Lambda) \subset L^2(\Omega;\mathbb{R}^{3\times3}_s)$

with some Lipschitz constant $L > 0$ depending only on $\Lambda_0 + \Lambda$, see e.g. [10, Theorem 4.47]. Therefore, for any $\varphi \in H^1_D(\Omega;\mathbb{R}^3)$ with $\|\varphi\|_{H^1_D(\Omega;\mathbb{R}^3)} \leq \Lambda$ we have

$$\begin{aligned}\Big| \int_\Omega Q(\mathrm{sym}(\nabla\varphi - p)) - \langle f, \varphi\rangle - \Big(\int_\Omega Q(\mathrm{sym}(\nabla\varphi - q)) - \langle g, \varphi\rangle\Big)\Big| \\ \leq\ L\|p-q\|_{L^2(\Omega)} + \Lambda\|f-g\|_{H^{-1}_D(\Omega;\mathbb{R}^3)} .\end{aligned}$$

This implies

$$\begin{aligned}&\mathcal{E}_1(p;f) - \mathcal{E}_1(q;g) \\ &\quad\leq \sup_{\varphi\in H^1_D(\Omega;\mathbb{R}^3), \|\varphi\|\leq\Lambda} \Big(\int_\Omega W_e(\nabla\varphi, p) - \langle f,\varphi\rangle - \Big[\int_\Omega W_e(\nabla\varphi, q) - \langle g, \varphi\rangle\Big]\Big) \\ &\quad= \sup_{\varphi\in H^1_D(\Omega;\mathbb{R}^3), \|\varphi\|\leq\Lambda} \Big(\int_\Omega Q(\mathrm{sym}(\nabla\varphi - p)) - Q(\mathrm{sym}(\nabla\varphi - q)) - \langle f - g, \varphi\rangle\Big) \\ &\quad\leq\ L\|p-q\|_{L^2(\Omega)} + \Lambda\|f-g\|_{H^{-1}_D(\Omega;\mathbb{R}^3)} ,\end{aligned}$$

and similarly

$$\mathcal{E}_1(p;f) - \mathcal{E}_1(q;g) \geq\ -L\|p-q\|_{L^2(\Omega)} - \Lambda\|f-g\|_{H^{-1}_D(\Omega;\mathbb{R}^3)} .$$

These inequalities prove the Lipschitz-continuity of $\mathcal{E}_1$.
Item 7: The functional $\mathcal{E}_1(\cdot; f)$ is lower semicontinuous under weak convergence in $L^2(\Omega;\mathbb{R}^3)$ because of Lipschitz continuity and the convexity of Item 1. Since the sequence $(p_j, f_j)_j$ is uniformly bounded in $L^2(\Omega;\mathbb{R}^3) \times H^{-1}_D(\Omega;\mathbb{R}^3)$ we deduce from Item 5

$$\begin{aligned}\liminf_{j\to\infty} \mathcal{E}_1(p_j; f_j) \ &\geq\ \liminf_{N\to\infty} \mathcal{E}_1(p_j; f) + \liminf_{j\to\infty} \Big(\mathcal{E}_1(p_j; f_j) - \mathcal{E}_1(p_j; f)\Big) \\ &\geq\ \mathcal{E}_1(p; f) - \limsup_{j\to\infty} C\|f_j - f\|_{H^{-1}_D(\Omega;\mathbb{R}^3)} =\ \mathcal{E}_1(p; f) .\end{aligned}$$

This concludes the proof of the lemma. □

We can now show the equivalence of the solution concepts.

Proof of Proposition 2.4. Item 1: Let (p, Σ) be a solution according to Definition 2.3. Our aim is to show that there exists a solution (u, p, Σ) according to Definition 2.2.

The existence statement before (A.3) yields that, for almost every $s \in (0, T)$, there exists $u(s) \in \mathfrak{E}(p(s), f(s))$.

We claim that the map $s \mapsto u(s)$ is measurable. We choose sequences $(p^N)_N$, $(f^N)_N$ of simple functions $p^N : (0, T) \to L^2(\Omega, \mathbb{R}^{3\times3})$ and $f^N : (0, T) \to H^{-1}_D(\Omega, \mathbb{R}^3)$ such that $p^N(t) \to p(t)$ in $L^2(\Omega, \mathbb{R}^{3\times3})$ and $f^N(t) \to f(t)$ in $H^{-1}_D(\Omega, \mathbb{R}^3)$ for almost all $t \in (0, T)$. For $t \in (0, T)$ we choose an element $u^N(t) \in \mathfrak{E}(p^N(t), f^N(t))$, such that $u^N : (0, T) \to H^1_D(\Omega, \mathbb{R}^3)$ is a simple function. By the uniform boundedness and lower-semicontinuity properties (A.3) and (A.6) we deduce that $u^N(t) \rightharpoonup u(t)$ in $H^1_D(\Omega, \mathbb{R}^3)$ for almost all $t \in (0, T)$. Hence, with respect to the weak topology in $H^1_D(\Omega, \mathbb{R}^3)$, the function $s \mapsto u(s)$ can pointwise almost everywhere be approximated by simple functions. The Pettis measurability theorem [10, Theorem 2.104] yields that $u : (0,T) \to H^1_D(\Omega, \mathbb{R}^3)$ is measurable.

The estimate (A.3) implies $u \in L^2(0,T; H^1_D(\Omega;\mathbb{R}^3))$, hence (u,p,Σ) meets the regularity requirements (R_1) of Definition 2.2. The stability property (S_1) is a consequence of $u(t) \in \mathfrak{E}(p(t), f(t))$ for almost all $t \in (0,T)$.

We next prove that (F_2) implies (F_1). We consider arbitrary functions $f \in H^{-1}_D(\Omega)$, $p, q \in L^2(\Omega;\mathbb{R}^{3\times 3})$, and minimizers $u \in \mathfrak{E}(p,f)$, $v \in \mathfrak{E}(q,f)$. By definition of the energies and by minimality, there holds

$$
\begin{aligned}
&\mathcal{E}(p;f) - \mathcal{E}(q;f)\\
&\quad = \mathcal{E}_1(p;f) - \mathcal{E}_1(q;f) + \mathcal{W}_p(p) - \mathcal{W}_p(q)\\
&\quad = \mathcal{W}_e(\nabla u, p) - \langle f, u\rangle - \mathcal{W}_e(\nabla v, q) + \langle f, v\rangle + \mathcal{W}_p(p) - \mathcal{W}_p(q)\\
&\quad \geq \mathcal{W}_e(\nabla u, p) - \mathcal{W}_e(\nabla u, q) + \mathcal{W}_p(p) - \mathcal{W}_p(q)\\
&\quad = \mathcal{W}(\nabla u, p) - \mathcal{W}(\nabla u, q)\,.
\end{aligned}
$$

Hence, for arbitrary $\Sigma \in L^2(\Omega;\mathbb{R}^{3\times 3})$,

$$
\begin{aligned}
\mathcal{E}(p;f) + \mathcal{E}^*(-\Sigma;f) &= \sup_{q\in L^2(\Omega;\mathbb{R}^{3\times 3})} \Big(\mathcal{E}(p;f) - \mathcal{E}(q;f) + \langle -\Sigma, q\rangle\Big)\\
&\geq \sup_{q\in L^2(\Omega;\mathbb{R}^{3\times 3})} \Big(\mathcal{W}(\nabla u, p) - \mathcal{W}(\nabla u, q) + \langle -\Sigma, q\rangle\Big)\\
&= \mathcal{W}(\nabla u, p) + \mathcal{W}^*(\nabla u, -\Sigma)\,.
\end{aligned}
$$

Inserting into property (F_2), we find that for almost all $t \in (0,T)$

$$\mathcal{W}(\nabla u, p) + \mathcal{W}^*(\nabla u, -\Sigma) \leq \langle -\Sigma, p\rangle\,.$$

Since the opposite inequality always holds, we obtain equality and hence (F_1).

In order to conclude (E_1), we first note that $u(t) \in \mathfrak{E}(p(t), f(t))$ implies

$$\mathcal{E}(p(t); f(t)) = \mathcal{W}(\nabla u(t), p(t)) - \langle f(t), u(t)\rangle\,.$$

With this equality, (E_2) implies (E_1); the infimum in (2.23) is attained by the uniqueness assumption.

Item 2: To prove that regular weak solutions (u,p,Σ) are strong solutions, we first observe that the stability property (S_1) implies (under the additional regularity assumptions) the Euler–Lagrange equation

$$0 = -\nabla\cdot(\mathrm{sym}\nabla_F W_e(\nabla u(t), p(t))) - f(t)$$

for almost all $t \in (0,T)$. Choosing σ as in (1.4), this is the balance of forces (1.3). The back-stress relation (F_1) implies directly (1.5).

It remains to derive the flow rule (1.6). We write the first term on the left hand side of (E_1) as an integral over its time derivative

$$\frac{d}{ds}\left(\mathcal{W}(\nabla u(s), p(s)) - \int_\Omega f(s)\cdot u(s)\right)$$

$$
\begin{aligned}
&= \big\langle \partial_t p(s), \nabla_p \mathcal{W}(\nabla u(s), p(s)) \big\rangle - \langle \partial_t f(s), u(s) \rangle \\
&= -\big\langle \partial_t p(s), \Sigma(s) \big\rangle - \langle \partial_t f(s), u(s) \rangle \,,
\end{aligned} \tag{A.8}
$$

where terms containing $\partial_t u(s)$ cancel by balance of forces. Inserting into (E_1) yields

$$
\int_0^T \big\langle \partial_t p(s), -\Sigma(s) \big\rangle + \mathcal{R}(\partial_t p(s)) + \mathcal{R}^*(\Sigma(s))\, ds \;\le\; 0\,.
$$

By definition of $\mathcal{R}^*$, the integrand is nonnegative, hence

$$
\big\langle \partial_t p(s), -\Sigma(s) \big\rangle + \mathcal{R}(\partial_t p(s)) + \mathcal{R}^*(\Sigma(s)) \;=\; 0
$$

for almost all $s \in (0,T)$. This yields (1.6).

Item 3: We now consider a strong solution (u, σ, p, Σ) to (1.3)–(1.6). By the balance of forces (1.3) and (1.4), $u(t)$ is a critical point of the convex map $\varphi \mapsto \mathcal{W}(\nabla\varphi, p(t)) - \langle f(t), \varphi \rangle$ for almost all $t \in (0,T)$. It hence satisfies the minimality (S_1). Property (1.5) of the back-stress Σ yields (F_1). Balance of forces (1.3) and (1.4) allows to calculate as in (A.8). We obtain, using (1.6),

$$
\begin{aligned}
&\frac{d}{ds}\left(\mathcal{W}(\nabla u(s), p(s)) - \int_\Omega f(s)\cdot u(s) \right) + \mathcal{R}(\partial_t p(s)) + \mathcal{R}^*(\Sigma(s)) \\
&\quad = -\langle \partial_t f(s), u(s) \rangle\,.
\end{aligned}
$$

Integrating this equality implies (E_1). □

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