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Preprint 2020-03

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Abstract

Let A be an m-T-accretive operator in $L^1(\Omega)$ and B the realization of a subdifferential ∂j in $L^1(\Omega)$. In this paper we obtain sufficient conditions for the range condition $R(I+\lambda(A+B)) = L^1(\Omega)$ for $\lambda > 0$ to hold and investigate their necessity.
The central statement is that finiteness of the underlying measure space Ω and a maximum principle satisfied by the operator A ensure the validity of the preceding range condition if j is independent of the space variable $x \in \Omega$, or $j(\cdot, r) \in L^1(\Omega)$ for any $r \in \mathbb{R}$. Then we prove that the maximum principle can be replaced by the condition $D(B) \supseteq D(A)$ and the finiteness of Ω by σ-finiteness and a closedness requirement on A. We confirm these conclusions by some examples showing the necessity of the provided constraints. We apply the abstract results to a concrete nonlinear diffusion convection problem with absorption, including problems of obstacle type.

Keywords: T-accretive operator, subdifferential, perturbation, range condition, maximum principle
AMS subject classification: 47H06, 47H14, 47H04, 46T99

1 Introduction

The present paper treats a particular problem from perturbation theory of m-accretive operators in $L^1(\Omega)$. The question for m-accretivity of sums of m-accretive operators has been studied in the past by several authors in a more or less general context. Since m-accretivity of an operator A in a Banach space X is an essential criterion for the existence and uniqueness of mild solutions of the evolution problem

$$u' + Au \ni f, \quad u(0) = u_0 \in \overline{D(A)} \quad \text{for } f \in L^1([0,T]:X),$$

it is of particular interest to examine under which conditions a perturbed operator $A+B$ is still m-accretive. It is a well-known fact that the accretivity of A and B does not guarantee automatically the accretivity of $A+B$ unless the underlying Banach space possesses a certain geometrical structure like uniform smoothness for example. In general Banach spaces, however, the sum fails to be accretive for arbitrary accretive perturbations B. Moreover, even if not, the sum is not necessarily m-accretive. Thus, the main problem in perturbation theory is to determine useful conditions under which the

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sum of m-accretive operators satifies the range condition. Results in this direction have been established first by Lescarret [20], Browder [13] and Rockafellar [25]. For further important results see, among others, [11], [16], [18], [19] and [23].

In this paper we concentrate our attention on the Banach space $L^1(\Omega)$, which is not uniformly smooth. Since $L^1(\Omega)$ plays a prominent role in many applications, it is important to develop appropriate perturbation results also in this space. The problem of its lacking geometrical structure can be compensated by restricting to certain classes of m-accretive operators and perturbations.
In [4], [7] and [26] there have been obtained positive results for m-completely accretive operators in $L^1(\Omega)$, perturbed by subdifferential type operators. In view of this, we want to develop analogous results for m-T-accretive operators in $L^1(\Omega)$. In contrast to the completely accretive case, we are still confronted with the problem that the sum of T-accretive operators fails to be (T-)accretive in $L^1(\Omega)$.
Thus, the present problem is not actually the m-T-accretivity of the sum of two m-T-accretive operators A and B, but more generally, the verification of the range condition $R(I+\lambda(A+B)) = L^1(\Omega)$ for $\lambda > 0$, independently of the T-accretivity of $A+B$.
In the case of an arbitrary Banach space and an m-accretive operator A there already have been established perturbation results in [19] and [23] for single-valued and continuous perturbation operators B with $D(B) \supseteq \overline{D(A)}$, which have even been extended by Bothe in [8] to multi-valued upper-semicontinuous perturbations with convex compact values. We will discuss the relationship of our results with his ones in Section 6 of this paper.

In the first section of the paper we restrict ourselves not only to m-T-accretive operators A in $L^1(\Omega)$ satisfying a maximum principle and to a finite measure space Ω, but also to perturbations of subdifferential type $B \sim \partial j$. Although such sums are not T-accretive in general, we are able to prove that they satisfy the necessary range condition, i.e., $R(I+\lambda(A+B)) = L^1(\Omega)$ for $\lambda > 0$. Moreover, we also prove the above range condition for x-dependent functions j which satisfy the extra condition $j(\cdot,r) \in L^1(\Omega)$ for all $r \in \mathbb{R}$. In both cases the idea is to construct an adequate bi-monotone regularization $B_{\lambda,\nu}$ of the perturbation operator B, by means of Yosida approximations, such that the associated problem $u_{\lambda,\nu} + Au_{\lambda,\nu} + B_{\lambda,\nu}(u_{\lambda,\nu}) \ni f$ is solvable for all $f \in L^\infty(\Omega)$. Subsequently, we show that its solutions $u_{\lambda,\nu}$ converge to a solution of $u + Au + Bu \ni f$ for $f \in L^\infty(\Omega)$. Passing to $f \in L^1(\Omega)$ by bi-monotone approximation through $L^\infty(\Omega)$-functions, we finally prove the assertion. According to [26] we have additionally verified that in the general case, i.e., the function j does not satisfy the condition $j(\cdot,r) \in L^1(\Omega)$ for all $r \in \mathbb{R}$ the operator $A_j := \liminf_{\lambda\to 0} A + B_\lambda$ is an m-T-accretive extension of $A+B$, where A is an m-0-T-accretive operator in $L^1(\Omega)$.
In the following section we examine our assumptions for necessity. On the one hand, we state that the maximum principle can be replaced by a restriction $D(B) \supseteq D(A)$ on the domain of the perturbation operator. On the other hand, we will see that the finiteness of the measure space can be reduced to σ-finiteness if A is closed in a certain sense.
Then, we study an application of the abstract results to a nonlinear diffusion-convection problem with absorption, including problems of obstacle type.
The following section is dedicated to the study of the relation of our results with the results of Bothe. We verify that our result can not be deduced from the results in [8].
Finally, the last section presents open problems which will be studied in future work.

2 Preliminaries

In the following let (Ω, Σ, μ) be a σ-finite measure space and $L^p(\Omega) = L^p(\Omega, \Sigma, \mu)$, $1 \leq p \leq \infty$, be the classical Lebesgue space with the usual norm $\| \ \|_p$.
Furthermore, we will make use of the Banach space $L^{1\cap\infty}(\Omega) = L^1(\Omega) \cap L^\infty(\Omega)$, endowed with the norm $\|u\|_{1\cap\infty} = \max\{\|u\|_1, \|u\|_\infty\}$. Note that $L^{1\cap\infty}(\Omega)$ is a common subspace of $L^1(\Omega)$, $L^2(\Omega)$ and $L^\infty(\Omega)$ which is dense in $L^1(\Omega)$ and $L^2(\Omega)$.

According to [2], [5] and [14] we introduce the following definitions. An operator A in $L^1(\Omega)$ is a multivalued mapping $A : D(A) \to 2^{L^1(\Omega)}$ with effective domain $D(A) = \{u \in L^1(\Omega) \mid Au \neq \emptyset\}$ and range $R(A) = \bigcup_{u \in D(A)} Au$. Each operator A can be identified with its graph $\{(u,v) \mid u \in D(A),\, v \in Au\}$.
An operator A in $L^1(\Omega)$ is called *accretive* if all (u,v), $(\hat{u},\hat{v}) \in A$ verify

$$\|u - \hat{u}\|_1 \leq \|u - \hat{u} + \lambda(v - \hat{v})\|_1 \quad \text{for every } \lambda > 0,$$

or, equivalently, if there exists a $\kappa \in L^\infty(\Omega)$, $\kappa \in \operatorname{sign}(u - \hat{u})$ such that

$$\int_\Omega \kappa(v - \hat{v}) \geq 0. \tag{1}$$

Moreover, A is called *T-accretive* if for all (u,v), $(\hat{u},\hat{v}) \in A$ we have

$$\|(u - \hat{u})^+\|_1 \leq \|(u - \hat{u} + \lambda(v - \hat{v}))^+\|_1 \quad \text{for every } \lambda > 0,$$

or, equivalently, if there is a $\kappa \in L^\infty(\Omega)$, $\kappa \in \operatorname{sign}^+(u - \hat{u})$ such that (1) holds.
A is called 0-*T-accretive* if for all (u,v), $(\hat{u},\hat{v}) \in A$ the inequality (1) is satisfied for $\kappa = \operatorname{sign}_0^+(u - \hat{u})$. A is called s-T-accretive if for all $(u,v), (\hat{u},\hat{v}) \in A$ and for all $\kappa \in$ $sign^+(u - \hat{u})$

$$\int_\Omega \kappa(v - \hat{v}) \geq 0.$$

s-T-accretive operators are 0-T-accretive and 0-T-accretive operators are T-accretive which, in turn, are also accretive in $L^1(\Omega)$. Sums of 0-T-accretive operators are 0-T-accretive again. An accretive or T-accretive operator A is called *m-accretive* or *m-T-accretive*, respectively, if A verifies the range condition $R(I + \lambda A) = L^1(\Omega)$ for all $\lambda > 0$.
If A is accretive, then its *resolvents* $J_\lambda^A = (I + \lambda A)^{-1}$, $\lambda > 0$, are single-valued contractions and vice versa. A is T-accretive if and only if its resolvents are T-contractions, that is, J_λ^A is an order-preserving contraction for every $\lambda > 0$. The so-called *Yosida approximation* of an accretive operator A is the single-valued Lipschitz-continuous operator A_λ, defined by $A_\lambda = \lambda^{-1}(I - J_\lambda^A)$ for $\lambda > 0$. If A is m-accretive, then A_λ is also m-accretive and $\liminf_{\lambda \downarrow 0} A_\lambda = A$.

We consider the class

$$\mathscr{J}_0 = \{j : \mathbb{R} \to [0,\infty] \mid j \text{ is convex, lower-semicontinuous and } j(0) = 0\}$$

and define the *subdifferential* of $j \in \mathscr{J}_0$ as the multivalued mapping $\partial j : \mathbb{R} \to 2^{\mathbb{R}}$, satisfying

$$y \in \partial j(r) \quad \Longleftrightarrow \quad j(\xi) \geq j(r) + y(\xi - r) \quad \text{for all } \xi \in \mathbb{R}.$$

In addition we define the class $\mathscr{J}_0(\Omega)$, i.e., the class of functions $j : \Omega \times \mathbb{R} \to [0,\infty]$ satisfying *i)* $j(\cdot,r)$ is measurable in $x\in\Omega$ for all $r\in\mathbb{R}$,

ii) $j(x,\cdot)$ is convex, l.s.c. in $r\in\mathbb{R}$ for a.e. x in Ω,

iii) $j(\cdot,0)=0$ a.e. in Ω.

According to [24], in the case $j\in\mathscr{J}_0$ every subdifferential ∂j is a maximal monotone graph in $\mathbb{R}$, that is, for all $y\in\partial j(r)$, $\hat{y}\in\partial j(\hat{r})$ it holds that $(r-\hat{r})(y-\hat{y})\geq 0$ and ∂j has no proper monotone extension. If $j\in\mathscr{J}_0(\Omega)$ the subdifferential $\partial j(x,\cdot)$ is a maximal monotone graph in $\mathbb{R}$ for a.e. x in Ω. In the case $j\in\mathscr{J}_0$ the realization of the subdifferential ∂j in $L^1(\Omega)$ is an operator B in $L^1(\Omega)$ with

$$w\in Bu \quad\Longleftrightarrow\quad u,w\in L^1(\Omega) \text{ and } w(x)\in\partial j(u(x)) \text{ for a.e. } x\in\Omega.$$

and in the case $j\in\mathscr{J}_0(\Omega)$

$$w\in Bu \quad\Longleftrightarrow\quad u,w\in L^1(\Omega) \text{ and } w(x)\in\partial j(x,u(x)) \text{ for a.e. } x\in\Omega.$$

In both cases we write $B\sim\partial j$. Note that every $B\sim\partial j$ is m-0-T-accretive in $L^1(\Omega)$.

Following [4], we introduce further a relation on $L^p(\Omega)$, $1\leq p\leq\infty$, by means of the class $\mathscr{J}_0$ through

$$u<<v \quad\Longleftrightarrow\quad \int_\Omega j(u)\leq\int_\Omega j(v) \quad \text{for all } j\in\mathscr{J}_0.$$

It is clear that $u<<v$ implies $\|u\|_p\leq\|v\|_p$ for all $1\leq p\leq\infty$.

An operator A in $L^1(\Omega)$ is called *completely accretive* (ca for short) if all (u,v), $(\hat{u},\hat{v})\in A$ verify

$$u-\hat{u}<<u-\hat{u}+\lambda(v-\hat{v}) \quad \text{for } \lambda>0.$$

Moreover, ca operators in $L^1(\Omega)$ are T-accretive and also accretive in $L^1(\Omega)$. Thus, $B\sim\partial j$ is T-accretive in $L^1(\Omega)$ for any $j\in\mathscr{J}_0$ and $j\in\mathscr{J}_0(\Omega)$.
If A is *m-completely accretive* (mca for short) in $L^1(\Omega)$, i.e., A is ca and $R(I+\lambda A)=L^1(\Omega)$ for all $\lambda>0$, we denote by A^0u for $u\in D(A)$ the unique element in Au with minimal norm. In this case we have $\lim_{\lambda\downarrow 0}A_\lambda u=A^0u$ for every $u\in D(A)$.

3 Perturbation theorems

As already referred to in the introduction, it is a well-known fact that in general, the sum of accretive operators fails to be accretive again. Furthermore, the following example shows that even the sum of an m-T-accretive operator and a subdifferential type perturbation operator in $L^1(\Omega)$ does not necessarily have to be accretive.

Example 1: Let $\Omega=\{\omega_1,\omega_2\}$, endowed with the counting measure. Hence, $L^1(\Omega)$ can be identified with the Banach space $\mathbb{R}^2$, endowed with the norm $\|(x,y)\|_1=|x|+|y|$. Further, let $j\in\mathscr{J}_0$ such that

$$\partial j(r)=\begin{cases}2 & \text{for } r>0,\\ [-4,2] & \text{for } r=0,\\ -4 & \text{for } r<0\end{cases}$$

and $B \sim \partial j$ its m-T-accretive realization in $L^1(\Omega)$. Let A be the operator in $L^1(\Omega)$, defined by

$$D(A) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid x_1 \le 0, x_2 \ge 0 \right\}$$

and

$$A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{cases} \binom{0}{0} & \text{for } x_1 < 0, x_2 > 0, \\ \left\{ \lambda \binom{1}{0} \mid \lambda \in [0, x_2] \right\} \cup \left\{ \binom{x_2}{0} + \lambda \binom{1}{-1} \mid \lambda \ge 0 \right\} & \text{for } x_1 = 0, x_2 \ge 0, \\ \left\{ \lambda \binom{0}{-1} \mid \lambda \in [0, -x_1] \right\} \cup \left\{ \binom{0}{x_1} + \lambda \binom{1}{-1} \mid \lambda \ge 0 \right\} & \text{for } x_1 \le 0, x_2 = 0. \end{cases}$$

Using elementary but tedious distinction of cases, which we omit at this place, one can prove that A is indeed m-T-accretive in $L^1(\Omega)$. In order to show that the sum $A+B$ fails to be accretive, consider u, v, $w \in L^1(\Omega)$, given by

$$u = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, v = \begin{pmatrix} -4 \\ 2 \end{pmatrix}, w = \begin{pmatrix} 4 \\ -3 \end{pmatrix}.$$

Since $v \in Bu$, $w \in Au$, we obtain in consequence $v + w = -u \in (A+B)u$ and $0 \in A0$, $0 \in B0$ yields $0 \in (A+B)0$. Thus, we have

$$\|u - 0 + \lambda(-u - 0)\|_1 = (1-\lambda)\|u\|_1 < \|u\|_1 \quad \text{for } \lambda > 1$$

and therefore $A+B$ is not accretive.

Remark 1 *Note that a T-accretive operator A in $L^1(\Omega)$ is 0-T-accretive in $L^1(\Omega)$, iff for all $j \in \mathscr{J}_0(\Omega)$ the operator $A+B$ is T-accretive in $L^1(\Omega)$, where $B \sim \partial j$ is the induced operator in $L^1(\Omega)$ by the subdifferential ∂j (see [3]).*

Moreover, even if $A+B$ is T-accretive, we still do not know anything about the m-T-accretivity of $A+B$. The question under which conditions $A+B$, independently from its accretivity, verifies the range condition, leads us to the first main result of the present paper.

Theorem 1 *Let (Ω, Σ, μ) be a finite measure space and let A be an m-T-accretive operator in $L^1(\Omega)$ with $0 \in A0$, satisfying the maximum principle*

$$\|J_\lambda^A u\|_\infty \le \|u\|_\infty \quad \text{for } u \in L^1(\Omega), \lambda > 0. \tag{M$_0$}$$

Furthermore, let $\beta = \partial j$ for $j \in \mathscr{J}_0$ be a maximal monotone graph in $\mathbb{R}$ with realization $B \sim \beta$ in $L^1(\Omega)$.
Then

$$R(I + \lambda(A+B)) = L^1(\Omega) \quad \text{for } \lambda > 0.$$

If, in addition, A is 0-T-accretive or B is single-valued, then $A+B$ is m-T-accretive in $L^1(\Omega)$.

Proof. In the following we assume $\lambda = 1$. For $\lambda \ne 1$ the proof remains almost the same but for notational convenience we only give the proof for $R(I+A+B) = L^1(\Omega)$.

Step 1: Approximation of the maximal monotone graph.
We start by decomposing the monotone graph β into a positive and negative part as follows:

$$\beta_+(r) = \begin{cases} \beta(r) & \text{for } r > 0 \\ [0, \sup\beta(0)] & \text{for } r = 0 \\ 0 & \text{for } r < 0 \end{cases}, \qquad \beta_-(r) = \begin{cases} 0 & \text{for } r > 0 \\ [\inf\beta(0), 0] & \text{for } r = 0 \\ \beta(r) & \text{for } r < 0 \end{cases}.$$

Considering the realizations $B_+ \sim \beta_+$, $B_- \sim \beta_-$ in $L^1(\Omega)$ and their Yosida approximations $(B_+)_\lambda$, $(B_-)_\nu$ for λ, $\nu > 0$, we define a pointwise bi-monotone approximation of the m-T-accretive operator $B \sim \beta$ by

$$B_{\lambda,\nu}(u) = (B_+)_\lambda(u) + (B_-)_\nu(u) \quad \text{for } u \in L^1(\Omega).$$

Then

$$\lim_{\lambda,\nu\downarrow 0} B_{\lambda,\nu}(u) = B_+^0(u) + B_-^0(u) = B^0(u) \quad \text{for } u \in D(B)$$

and

$$\lim\inf_{\lambda,\nu\downarrow 0} B_{\lambda,\nu} = B_+ + B_- = B.$$

Obviously, $B_{\lambda,\nu}$ is single-valued, m-accretive and Lipschitz-continuous. The graph $\beta_{\lambda,\nu} : \mathbb{R} \to \mathbb{R}$, where $B_{\lambda,\nu} \sim \beta_{\lambda,\nu}$, is increasing and therefore we obtain for all u, $\hat{u} \in L^1(\Omega)$ and each $\kappa \in \text{sign}^+(u - \hat{u})$

$$\int_\Omega \kappa(\beta_{\lambda,\nu}(u) - \beta_{\lambda,\nu}(\hat{u}))\,\mathrm{d}x = \int_{\{u \geq \hat{u}\}} \beta_{\lambda,\nu}(u) - \beta_{\lambda,\nu}(\hat{u})\,\mathrm{d}x \geq 0$$

Consequently, $A + B_{\lambda,\nu}$ is T-accretive. Due to well-known results from pertubation theory, cf. e.g. [5], the Lipschitz-continuity of $B_{\lambda,\nu}$ yields the m-accretivity and hence the m-T-accretivity of $A + B_{\lambda,\nu}$ in $L^1(\Omega)$. Thus, $A + B_{\lambda,\nu}$ verifies the range condition

$$R(I + A + B_{\lambda,\nu}) = L^1(\Omega) \supseteq L^\infty(\Omega)$$

and the problem

$$u_{\lambda,\nu} + Au_{\lambda,\nu} + B_{\lambda,\nu}(u_{\lambda,\nu}) \ni f \tag{2}$$

admits in particular for every $f \in L^\infty(\Omega)$ a solution $u_{\lambda,\nu}$.

Step 2: Boundedness of $(u_{\lambda,\nu})_{\lambda,\nu}$ and $(B_{\lambda,\nu}(u_{\lambda,\nu}))_{\lambda,\nu}$.
We want to prove now that the maximum principle (M_0) and the m-T-accretivity of A imply the boundedness of the sequences $(u_{\lambda,\nu})_{\lambda,\nu}$ and $(B_{\lambda,\nu}(u_{\lambda,\nu}))_{\lambda,\nu}$ in all $L^p(\Omega)$-spaces, that is, for all $f \in L^\infty(\Omega)$ the following estimates hold:

$$\|u_{\lambda,\nu}\|_p \leq \|f\|_p \quad \text{and} \quad \|B_{\lambda,\nu}(u_{\lambda,\nu})\|_p \leq \|f\|_p \quad \text{for } 1 \leq p \leq \infty.$$

In a first step we show that $u \ll u + \gamma v$ for all $(u, v) \in A$ and $\gamma > 0$. For this purpose let $u \in D(J_\gamma^A) = L^1(\Omega)$ and let $w \in L^1(\Omega)$ with $w(x) = k > 0$ for a.e. $x \in \Omega$. Then the maximum principle (M_0) yields

$$J_\gamma^A w(x) \leq \|J_\gamma^A w(x)\|_\infty \leq \|w(x)\|_\infty = k \quad \text{a.e. on } \Omega.$$

Since A is T-accretive, this implies

$$\int_\Omega (J_\gamma^A u - k)^+ \leq \int_\Omega (J_\gamma^A u - J_\gamma^A w)^+ \leq \int_\Omega (u - w)^+ = \int_\Omega (u - k)^+.$$

Let $w \in L^1(\Omega)$ with $w(x) = -k$ for a.e. $x \in \Omega$. Then we have

$$\begin{aligned}\int_\Omega (J_\gamma^A u + k)^- &= \int_\Omega (-J_\gamma^A u - k)^+ \le \int_\Omega (-J_\gamma^A u + J_\gamma^A w)^+ \\ &\le \int_\Omega (w-u)^+ = \int_\Omega (u-w)^- = \int_\Omega (u+k)^-.\end{aligned}$$

Hence, according to [4], Lemma 1.3, it follows $J_\gamma^A u << u$ and consequently $u << u + \gamma v$ for all $(u,v) \in A$ and $\gamma > 0$. Following [4], Prop. 2.2, this is equivalent to $\int_\Omega p(u)v \ge 0$ for all $(u,v) \in A$ and $p \in \mathscr{P}_0 = \{p \in C^\infty(\mathbb{R}) \mid 0 \le p' \le 1, \operatorname{supp} p' \text{ is compact and } 0 \notin \operatorname{supp} p\}$. Due to (2), we obtain in particular for all $f \in L^\infty(\Omega)$

$$\int_\Omega p(u_{\lambda,\nu})(f - u_{\lambda,\nu} - B_{\lambda,\nu}(u_{\lambda,\nu})) \ge 0,$$

that is,

$$\int_\Omega p(u_{\lambda,\nu}) f \,\mathrm{d}x \ge \int_\Omega p(u_{\lambda,\nu}) u_{\lambda,\nu} \,\mathrm{d}x + \int_\Omega p(u_{\lambda,\nu}) \beta_{\lambda,\nu}(u_{\lambda,\nu}) \,\mathrm{d}x. \tag{3}$$

Since $\beta_{\lambda,\nu}$ and $p \in \mathscr{P}_0$ are both monotone and contain the origin, we have

$$\int_\Omega p(u_{\lambda,\nu}) u_{\lambda,\nu} \,\mathrm{d}x \ge 0 \quad \text{and} \quad \int_\Omega p(u_{\lambda,\nu}) \beta_{\lambda,\nu}(u_{\lambda,\nu}) \,\mathrm{d}x \ge 0. \tag{4}$$

The combination of (3) and (4) implies

$$u_{\lambda,\nu} << u_{\lambda,\nu} + \gamma(f - u_{\lambda,\nu}) = (1-\gamma) u_{\lambda,\nu} + \gamma f \quad \text{for all } \gamma > 0$$

and consequently $u_{\lambda,\nu} << f$ which yields $\|u_{\lambda,\nu}\|_p \le \|f\|_p$ for all $1 \le p \le \infty$.
Moreover, due to [4], Prop. 2.2, the inequalities (3) and (4) imply by approximation

$$\int_\Omega (\hat{p}(u_{\lambda,\nu})(f - \beta_{\lambda,\nu}(u_{\lambda,\nu})))^+ \,\mathrm{d}x \ge \int_\Omega (\hat{p}(u_{\lambda,\nu})(f - \beta_{\lambda,\nu}(u_{\lambda,\nu})))^- \,\mathrm{d}x \tag{5}$$

for all increasing functions $\hat{p} : \mathbb{R} \to \mathbb{R}$ with $\hat{p}(0) = 0$. Thus, this is also true for $\hat{p} = p \circ \beta_{\lambda,\nu}$, that is,

$$\int_\Omega (p(\beta_{\lambda,\nu}(u_{\lambda,\nu}))(f - \beta_{\lambda,\nu}(u_{\lambda,\nu})))^+ \,\mathrm{d}x \ge \int_\Omega (p(\beta_{\lambda,\nu}(u_{\lambda,\nu}))(f - \beta_{\lambda,\nu}(u_{\lambda,\nu})))^- \,\mathrm{d}x.$$

As a result of this, we obtain for all $\gamma > 0$

$$B_{\lambda,\nu}(u_{\lambda,\nu}) << B_{\lambda,\nu}(u_{\lambda,\nu}) + \gamma(f - B_{\lambda,\nu}(u_{\lambda,\nu})) = (1-\gamma) B_{\lambda,\nu}(u_{\lambda,\nu}) + \gamma f$$

and consequently $\|B_{\lambda,\nu}(u_{\lambda,\nu})\|_p \le \|f\|_p$ for all $1 \le p \le \infty$.

Step 3: Convergence of $(u_{\lambda,\nu})_{\lambda,\nu}$ and $(B_{\lambda,\nu}(u_{\lambda,\nu}))_{\lambda,\nu}$.
We show first the monotonicity of the approximative solutions $(u_{\lambda,\nu})_\lambda$ and $(u_{\lambda,\nu})_\nu$. For this purpose let $\nu > 0$ be fixed and $\lambda > \widetilde{\lambda} > 0$. For $f \in L^\infty(\Omega)$ consider the according equations

$$u_{\lambda,\nu} + A u_{\lambda,\nu} + B_{\lambda,\nu}(u_{\lambda,\nu}) \ni f \quad \text{and} \quad u_{\widetilde{\lambda},\nu} + A u_{\widetilde{\lambda},\nu} + B_{\widetilde{\lambda},\nu}(u_{\widetilde{\lambda},\nu}) \ni f.$$

Note that the second inclusion is equivalent to

$$u_{\widetilde{\lambda},\nu} + A u_{\widetilde{\lambda},\nu} + B_{\lambda,\nu}(u_{\widetilde{\lambda},\nu}) \ni f + (B_+)_\lambda(u_{\widetilde{\lambda},\nu}) - (B_+)_{\widetilde{\lambda}}(u_{\widetilde{\lambda},\nu}).$$

Then the T-accretivity of $A+B_{\lambda,\nu}$ yields

$$\|(u_{\widetilde{\lambda},\nu}-u_{\lambda,\nu})^+\|_1 \le \|((B_+)_\lambda(u_{\widetilde{\lambda},\nu})-(B_+)_{\widetilde{\lambda}}(u_{\widetilde{\lambda},\nu}))^+\|_1 = 0,$$

since $(B_+)_\lambda$ is increasing in $\lambda>0$. Consequently, we have $u_{\lambda,\nu}\ge u_{\widetilde{\lambda},\nu}$ a.e. in Ω and hence $(u_{\lambda,\nu})_\lambda$ is increasing in $\lambda>0$. In exactly the same way one concludes that the sequence $(u_{\lambda,\nu})_\nu$ is decreasing in $\nu>0$.
According to the monotone convergence theorem, there exist u_ν, $u\in L^1(\Omega)$ such that

$$u_{\lambda,\nu}\xrightarrow{\lambda\downarrow 0}u_\nu\xrightarrow{\nu\downarrow 0}u \quad \text{in } L^1(\Omega).$$

As $(u_{\lambda,\nu})_{\lambda,\nu}$ are uniformly bounded in $L^\infty(\Omega)$, and (Ω,Σ,μ) is a finite measure space, convergence in $L^2(\Omega)$ follows, i.e., we have

$$u_{\lambda,\nu}\xrightarrow{\lambda\downarrow 0}u_\nu\xrightarrow{\nu\downarrow 0}u \quad \text{in } L^1(\Omega) \text{ and } L^2(\Omega).$$

Moreover, due to the fact that $L^1(\Omega)$ is a separable Banach space and $(B_{\lambda,\nu}(u_{\lambda,\nu}))_{\lambda,\nu}$ is bounded in $L^\infty(\Omega)$, the theorem of Banach-Alaoglu implies that this sequence possesses a weak*-convergent subsequence in $L^\infty(\Omega)$, i.e., there exists $b_\nu\in L^\infty(\Omega)$ such that $B_{\lambda_k,\nu}(u_{\lambda_k,\nu})\rightharpoonup_* b_\nu$ for $\lambda_k\downarrow 0$.
Since this yields $\|b_\nu\|_\infty\le\liminf_{\lambda_k\downarrow 0}\|B_{\lambda_k,\nu}(u_{\lambda,\nu})\|_\infty\le\|f\|_\infty$, $(b_\nu)_\nu$ also possesses a weak*-convergent subsequence, that is, $b_{\nu_n}\rightharpoonup_* b\in L^\infty(\Omega)$ for some subsequence $\nu_n\downarrow 0$.
For notational convenience we identify in the sequel the subsequences $(B_{\lambda_k,\nu}(u_{\lambda_k,\nu}))_{\lambda_k}$ and $(b_{\nu_n})_{\nu_n}$ with $(B_{\lambda,\nu}(u_{\lambda,\nu}))_\lambda$ and $(b_\nu)_\nu$. Summarizing our results with the new notation, we obtain the weak*-convergence

$$B_{\lambda,\nu}(u_{\lambda,\nu})\overset{\lambda\downarrow 0}{\rightharpoonup_*}b_\nu\overset{\nu\downarrow 0}{\rightharpoonup_*}b \quad \text{in } L^\infty(\Omega).$$

Since (Ω,Σ,μ) is a finite measure space, $L^\infty(\Omega)\hookrightarrow L^2(\Omega)\hookrightarrow L^1(\Omega)$ where $\hookrightarrow$ denotes continuous injection, hence the weak*-convergence of $B_{\lambda,\nu}(u_{\lambda,\nu})$ in $L^\infty(\Omega)$ (resp. of $(b_\nu)_\nu$) implies its weak convergence in $L^2(\Omega)$ as well as in $L^1(\Omega)$ to the corresponding same limits. It remains to verify that $b\in B(u)$. For this purpose we prove in a first step that $b_\nu\in B_+(u_\nu)+(B_-)_\nu(u_\nu)$.
Since b_ν is the weak limit of $B_{\lambda,\nu}(u_{\lambda,\nu})=(B_+)_\lambda(u_{\lambda,\nu})+(B_-)_\nu(u_{\lambda,\nu})$ and the Lipschitz-continuity of $(B_-)_\nu$ already implies $(B_-)_\nu(u_{\lambda,\nu})\to(B_-)_\nu(u_\nu)$ for $\lambda\downarrow 0$ in $L^1(\Omega)$, it is only left to show that $b_\nu^+=\text{w-}L^2(\Omega)-\lim_{\lambda\downarrow 0}(B_+)_\lambda(u_{\lambda,\nu})\in B_+(u_\nu)$. More precisely, we want to show that $b_\nu^+(x)\in\beta_+(u_\nu(x))=\partial j_+(u_\nu(x))$ for a.e. $x\in\Omega$. According to the definition of $(\beta_+)_\lambda(u_{\lambda,\nu}(x))$, we have

$$(j_+)_\lambda(z)\ge(j_+)_\lambda(u_{\lambda,\nu}(x))+(\beta_+)_\lambda(u_{\lambda,\nu}(x))(z-u_{\lambda,\nu}(x)) \quad \text{for all } z\in\mathbb{R}.$$

Now, let $E\subseteq\Omega$ be a measurable set, $\chi_E\in L^1(\Omega)$ the characteristic function and $\lambda_0>0$ be fixed. For all $\lambda_0>\lambda>0$ it holds that

$$\int_\Omega(j_+)_\lambda(z)\chi_E\,\mathrm{d}x\ge\int_\Omega(j_+)_{\lambda_0}(u_{\lambda,\nu}(x))\chi_E\,\mathrm{d}x+\int_\Omega(\beta_+)_\lambda(u_{\lambda,\nu}(x))(z-u_{\lambda,\nu}(x))\chi_E\,\mathrm{d}x.$$

Then $(\beta_+)_\lambda(u_{\lambda,\nu})\rightharpoonup b_\nu^+$ and $u_{\lambda,\nu}\to u_\nu$ for $\lambda\downarrow 0$ in $L^2(\Omega)$ lead to

$$\int_E(\beta_+)_\lambda(u_{\lambda,\nu}(x))(z-u_{\lambda,\nu}(x))\,\mathrm{d}x\xrightarrow{\lambda\downarrow 0}\int_E b_\nu^+(x)(z-u_\nu(x))\,\mathrm{d}x.$$

Moreover, as $(j_+)_\lambda(z) \to j_+(z)$ and $(j_+)_{\lambda_0}(u_{\lambda,\nu}) \to (j_+)_{\lambda_0}(u_\nu)$ for $\lambda \downarrow 0$ in $L^1(\Omega)$, we have

$$\int_E j_+(z)\,\mathrm{d}x \geq \int_E (j_+)_{\lambda_0}(u_\nu(x))\,\mathrm{d}x + \int_E b_\nu^+(x)(z-u_\nu(x))\,\mathrm{d}x.$$

Letting $\lambda_0 \downarrow 0$, the monotone convergence theorem yields

$$\int_E j_+(z)\,\mathrm{d}x \geq \int_E j_+(u_\nu(x))\,\mathrm{d}x + \int_E b_\nu^+(x)(z-u_\nu(x))\,\mathrm{d}x$$

for all measurable sets $E \subseteq \Omega$. As a result we obtain for all $z \in \mathbb{R}$

$$j_+(z) \geq j_+(u_\nu(x)) + b_\nu^+(x)(z-u_\nu(x)) \quad \text{a.e. on } \Omega.$$

Thus, $b_\nu^+(x) \in \partial j_+(u_\nu(x))$ a.e on Ω and $b_\nu^+ \in B_+(u_\nu)$.
On this basis we are now able to prove $b \in B(u) = B_+(u) + B_-(u)$, where $b = \text{w-}\lim_{\nu\downarrow 0}(b_\nu^+ + b_\nu^-) = b^+ + b^-$ in $L^2(\Omega)$. The monotonicity of B_+ and $(u_\nu, b_\nu^+) \in B_+$ yield

$$0 \leq \langle y - b_\nu^+, w - u_\nu\rangle \xrightarrow{\nu\downarrow 0} \langle y - b^+, w - u\rangle \quad \text{for all } (w,y) \in B_+,$$

hence we can immediately deduce $b^+ \in B_+(u)$ from the maximality of B_+.
To complete the proof, we only need to show that $b^- \in B_-(u)$, or to be more precise, $b^-(x) \in B_-(u(x)) = \partial j_-(u(x))$ for a.e. $x \in \Omega$. Using the definition of $(\beta_-)_\nu(u_\nu(x))$, we obtain in analogy to the first part $b^-(x) \in \partial j_-(u(x))$ for a.e. $x \in \Omega$ and consequently $b = b^+ + b^- \in B(u)$.

Step 4: Transition from $A + B_{\lambda,\nu}$ to $A+B$.
In order to show that $R(I + A + B_{\lambda,\nu}) \supseteq L^\infty(\Omega)$ implies $R(I+A+B) \supseteq L^\infty(\Omega)$, let $f \in L^\infty(\Omega)$. Exploiting the maximal accretivity of A, we prove in a first step $f - u_\nu - b_\nu \in Au_\nu$. For this purpose we have to verify the accretivity of $A \cup \{(u_\nu, f - u_\nu - b_\nu)\}$, that is, for all $(z,w) \in A$ we have to show the existence of $\kappa_\nu \in \operatorname{sign}(u_\nu - z)$ satisfying

$$\int_\Omega \kappa_\nu(f - u_\nu - b_\nu - w) \geq 0.$$

Since A is accretive and $(u_{\lambda,\nu}, f - u_{\lambda,\nu} - B_{\lambda,\nu}(u_{\lambda,\nu})) \in A$, there is for every $(z,w) \in A$ a $\kappa_{\lambda,\nu} \in \operatorname{sign}(u_{\lambda,\nu} - z)$ with

$$\int_\Omega \kappa_{\lambda,\nu}(f - u_{\lambda,\nu} - B_{\lambda,\nu}(u_{\lambda,\nu}) - w) \geq 0. \tag{6}$$

According to step 3, we have the weak convergence

$$f - u_{\lambda,\nu} - B_{\lambda,\nu}(u_{\lambda,\nu}) - w \overset{\lambda\downarrow 0}{\rightharpoonup} f - u_\nu - b_\nu - w \quad \text{in } L^1(\Omega). \tag{7}$$

Since $\kappa_{\lambda,\nu} \in \operatorname{sign}(u_{\lambda,\nu} - z)$ is obviously bounded in $L^\infty(\Omega) \cong (L^1(\Omega))^*$, it follows that (a subsequence of) $(\kappa_{\lambda,\nu})_\lambda$ converges weakly* to a certain $\kappa_\nu \in \operatorname{sign}(u_\nu - z)$ in $L^\infty(\Omega)$ for $\lambda \downarrow 0$.
In order to prove the convergence of the integral in (6), we decompose Ω in the following way:

1. $\Omega_1 = \{x \in \Omega \mid u_\nu(x) > z(x)\}$
   Since $u_{\lambda,\nu} \downarrow u_\nu$ for $\lambda \downarrow 0$, we obtain $u_{\lambda,\nu}(x) \geq u_\nu(x) > z(x)$ on Ω_1 and therefore $\kappa_{\lambda,\nu}(x) = 1$ a.e on Ω_1. For this reason we even get the strong convergence $\kappa_{\lambda,\nu} \to \kappa_\nu = 1$ in $L^\infty(\Omega_1)$ which implies

$$\langle \kappa_{\lambda,\nu}, f - u_{\lambda,\nu} - B_{\lambda,\nu}(u_{\lambda,\nu}) - w\rangle_{L^\infty(\Omega_1),L^1(\Omega_1)} \xrightarrow{\lambda\downarrow 0} \int_{\Omega_1} \kappa_\nu(f - u_\nu - b_\nu - w).$$

2. $\Omega_2 = \{x \in \Omega \mid u_\nu(x) < z(x)\}$
Due to the fact, that for all $x \in \Omega_2$ there exists an index $N(x)$ with $u_\nu(x) \leq u_{\lambda,\nu}(x) < z(x)$ for $0 < \lambda < \lambda_{N(x)}$, it follows $\kappa_{\lambda,\nu}(x) = -1$ for $0 < \lambda < \lambda_{N(x)}$ and for a.e. $x \in \Omega_2$. Since $(\kappa_{\lambda,\nu})_\lambda$ is bounded in $L^\infty(\Omega)$ and (7) holds, we obtain therefore

$$\int_{\Omega_2} \kappa_{\lambda,\nu}(f - u_{\lambda,\nu} - B_{\lambda,\nu}(u_{\lambda,\nu}) - w) \xrightarrow{\lambda\downarrow 0} \int_{\Omega_2} \kappa_\nu(f - u_\nu - b_\nu - w) = \int_{\Omega_2} -(f - u_\nu - b_\nu - w).$$

3. $\Omega_3 = \{x \in \Omega \mid u_\nu(x) = z(x)\}$
As $u_{\lambda,\nu}(x) \geq u_\nu(x) = z(x)$ for a.e. $x \in \Omega_3$, it seems to be promising to decompose Ω_3 again in

$$\hat{\Omega}_3 = \{x \in \Omega \mid u_{\lambda,\nu}(x) > u_\nu(x) = z(x)\ \forall \lambda > 0\} \quad \text{and}$$
$$\widetilde{\Omega}_3 = \{x \in \Omega \mid \exists \text{ index } N(x) : u_{\lambda,\nu}(x) = u_\nu(x) = z(x)\ \forall 0 < \lambda < \lambda_{N(x)}\}.$$

   (a) On $\hat{\Omega}_3$, we are nearly in the same situation as on Ω_1 since $\kappa_{\lambda,\nu}(x) = 1$ for a.e. $x \in \hat{\Omega}_3$. Thus, we obtain the strong convergence of $(\kappa_{\lambda,\nu})_\lambda$ to $\kappa_\nu = 1$ in $L^\infty(\hat{\Omega}_3)$ which yields

$$\int_{\hat{\Omega}_3} \kappa_{\lambda,\nu}(f - u_{\lambda,\nu} - B_{\lambda,\nu}(u_{\lambda,\nu}) - w) \xrightarrow{\lambda\downarrow 0} \int_{\hat{\Omega}_3} \kappa_\nu(f - u_\nu - b_\nu - w).$$

   (b) On $\widetilde{\Omega}_3$, however, we are confronted with the problem that we are not able to make any statement about the strong convergence of $(\kappa_{\lambda,\nu})_\lambda$ in $L^\infty(\widetilde{\Omega}_3)$. Nevertheless, we have for a.e. $x \in \widetilde{\Omega}_3$ and for every $0 < \lambda < \lambda_{N(x)}$ the pointwise convergence

$$(\beta_+)_\lambda(u_{\lambda,\nu}(x)) = (\beta_+)_\lambda(u_\nu(x)) \xrightarrow{\lambda\downarrow 0} (\beta_+)^0(u_\nu(x)).$$

   Since $|(\beta_+)_\lambda(u_{\lambda,\nu}(x))| \leq \| f \|_\infty$ for a.e. x in $\tilde{\Omega}_3$, Lebesgue's dominated convergence theorem immediately yields also the strong convergence of $(\beta_+)_\lambda(u_{\lambda,\nu})$ in $L^1(\widetilde{\Omega}_3)$. Thus, we obtain the L^1-convergence

$$B_{\lambda,\nu}(u_{\lambda,\nu}) = (B_+)_\lambda(u_{\lambda,\nu}) + (B_-)_\nu(u_{\lambda,\nu}) \xrightarrow{\lambda\downarrow 0} (B_+)^0(u_\nu) + (B_-)_\nu(u_\nu) = b_\nu$$

   and as a result of this

$$\int_{\widetilde{\Omega}_3} \kappa_{\lambda,\nu}(f - u_{\lambda,\nu} - B_{\lambda,\nu}(u_{\lambda,\nu}) - w) \xrightarrow{\lambda\downarrow 0} \int_{\widetilde{\Omega}_3} \kappa_\nu(f - u_\nu - b_\nu - w).$$

Thus, we have proved the accretivity of $A \cup \{(u_\nu, f - u_\nu - b_\nu)\}$ which implies $f - u_\nu - b_\nu \in Au_\nu$ by virtue of the maximal accretivity of A.
In exactly the same way one concludes $f - u - b \in Au$.

Step 5: Transition from $L^\infty(\Omega)$ to $L^1(\Omega)$.
In the previous steps we demonstrated how the existence of solutions of the approximative equations

$$u_{\lambda,\nu} + Au_{\lambda,\nu} + B_{\lambda,\nu}(u_{\lambda,\nu}) \ni f$$

yields the solvability of

$$u + Au + b \ni f \quad \text{with} \quad b \in Bu. \tag{8}$$

for all $f \in L^\infty(\Omega)$. Now, we are going to show that (8) even admits for every $f \in L^1(\Omega)$ a solution. The idea is to approximate $f \in L^1(\Omega)$ through a sequence of specific $L^\infty(\Omega)$-functions. For this purpose consider the truncation map $T_m : \mathbb{R} \to \mathbb{R}$, $m \geq 0$, defined by

$$T_m(r) = \begin{cases} m & \text{for } r > m, \\ r & \text{for } |r| \leq m, \\ -m & \text{for } r < -m. \end{cases}$$

Due to dominated convergence, it holds that $T_m g \to g$ for $m \to \infty$ in $L^1(\Omega)$.
Further, we assign to every $f \in L^1(\Omega)$ a sequence $(f^{m,n})_{m,n} \subseteq L^\infty(\Omega)$, given by

$$f^{m,n} = T_m(f^+) - T_n(f^-) \quad \text{for } m,\, n \in \mathbb{N},$$

where $f^{m,n} \to f^+ - f^- = f$ in $L^1(\Omega)$ for $m, n \to \infty$. Moreover, since $f^{m,n} \in L^\infty(\Omega)$, there exists for all m, $n \in \mathbb{N}$ a solution $u^{m,n}$ of

$$u^{m,n} + Au^{m,n} + b^{m,n} \ni f^{m,n} \quad \text{with} \quad b^{m,n} \in Bu^{m,n},$$

with $u^{m,n} = L^1(\Omega)\text{-}\lim_{\lambda,\nu\to 0} u^{m,n}_{\lambda,\nu}$ and $b^{m,n} = \text{w*-}L^\infty(\Omega)\text{-}\lim_{\lambda,\nu\to 0} B_{\lambda,\nu}(u^{m,n}_{\lambda,\nu})$.
In order to prove their strong convergence in $L^1(\Omega)$, we check the sequences $(f^{m,n})_{m,n}$, $(u^{m,n}_{\lambda,\nu})_{m,n}$ and $(B_{\lambda,\nu}(u^{m,n}_{\lambda,\nu}))_{m,n}$ for monotonicity and boundedness in $L^1(\Omega)$. For this purpose let $m \geq \widetilde{m} > 0$ and n be fixed. Apparently, it holds that $T_m(f^+) \geq T_{\widetilde{m}}(f^+)$ and thus

$$f^{m,n} = T_m(f^+) - T_n(f^-) \geq T_{\widetilde{m}}(f^+) - T_n(f^-) = f^{\widetilde{m},n}.$$

For the monotonicity of $(u^{m,n}_{\lambda,\nu})_{m,n}$, consider the approximative equations

$$\begin{aligned} u^{m,n}_{\lambda,\nu} + Au^{m,n}_{\lambda,\nu} + B_{\lambda,\nu}(u^{m,n}_{\lambda,\nu}) &\ni f^{m,n} \quad \text{and} \\ u^{\widetilde{m},n}_{\lambda,\nu} + Au^{\widetilde{m},n}_{\lambda,\nu} + B_{\lambda,\nu}(u^{\widetilde{m},n}_{\lambda,\nu}) &\ni f^{\widetilde{m},n}. \end{aligned}$$

Then, by virtue of the T-accretivity of $A + B_{\lambda,\nu}$, we have

$$\|(u^{\widetilde{m},n}_{\lambda,\nu} - u^{m,n}_{\lambda,\nu})^+\|_1 \leq \|(f^{\widetilde{m},n} - f^{m,n})^+\|_1 = 0$$

and hence $u^{m,n}_{\lambda,\nu} \geq u^{\widetilde{m},n}_{\lambda,\nu}$ a.e. in Ω.
Thus, the monotonicity of $B_{\lambda,\nu}$ implies immediatly $B_{\lambda,\nu}(u^{m,n}_{\lambda,\nu}) \geq B_{\lambda,\nu}(u^{\widetilde{m},n}_{\lambda,\nu})$ a.e. on Ω. Since the passage to the weak*-limit in $L^\infty(\Omega)$ for λ, $\nu \to 0$ is order-preserving, we obtain $b^{m,n} \geq b^{\widetilde{m},n}$. In exactly the same way one proves that $f^{m,\widetilde{n}} \leq f^{m,\widetilde{n}}$, $u^{m,n}_{\lambda,\nu} \leq u^{m,\widetilde{n}}_{\lambda,\nu}$ and $b^{m,n} \leq b^{m,\widetilde{n}}$ for $n \geq \widetilde{n} > 0$ and m fixed.
Further, applying the result from step 2 to the present case yields

$$\|u^{m,n}_{\lambda,\nu}\|_1 \leq \|f^{m,n}\|_1 \quad \text{and} \quad \|B_{\lambda,\nu}(u^{m,n}_{\lambda,\nu})\|_1 \leq \|f^{m,n}\|_1$$

which lead to

$$\begin{aligned} \|u^{m,n}\|_1 &= \lim_{\lambda,\nu\to 0} \|u^{m,n}_{\lambda,\nu}\|_1 \leq \|f\|_1 \quad \text{and} \\ \|b^{m,n}\|_1 &\leq \liminf_{\lambda,\nu\to 0} \|B_{\lambda,\nu}(u^{m,n}_{\lambda,\nu})\|_1 \leq \|f\|_1, \end{aligned}$$

because of $|f^{m,n}| \to |f|$, $u_{\lambda,\nu}^{m,n} \to u^{m,n}$ and $B_{\lambda,\nu}(u_{\lambda,\nu}^{m,n}) \rightharpoonup b^{m,n}$ in $L^1(\Omega)$.
Hence, monotone convergence yields the following convergence behavior in $L^1(\Omega)$:

$$u^{m,n} \xrightarrow{m\to\infty} u^n \xrightarrow{n\to\infty} u \in L^1(\Omega)$$
$$b^{m,n} \xrightarrow{m\to\infty} b^n \xrightarrow{n\to\infty} b \in L^1(\Omega).$$

Since A and B are both m-accretive and therefore closed in $L^1(\Omega)$, we obtain $b \in Bu$ and consequently the desired result $f - u - b \in Au$.
Additionally, 0-T-accretivity of A implies T-accretivity of $A+B$, since B is also 0-T-accretive and hence also the sum. Moreover, if $B \sim \beta$ is single-valued, it follows immediately by the monotonicity of β, that B is s-T-accretive and therefore $A+B$ is T-accretive. $\square$

Theorem 2 *Under the assumptions on Ω and A of Theorem 1 and if $j \in \mathscr{J}_0(\Omega)$ satisfying the integrability condition $j(\cdot,r) \in L^1(\Omega)$ for all $r \in \mathbb{R}$, then the same conclusions as in Theorem 1 hold.*

Proof.
The proof goes along the same lines as the proof of Theorem 1, with the exception that, in the x - dependent case, we do no longer get the estimate
$\| B_{\lambda,\nu}(u_{\lambda,\nu}) \|_p \leq \| f \|_p$ for all $1 \leq p \leq \infty$ (see Step 2) but only the estimate
$\| B_{\lambda,\nu}(u_{\lambda,\nu}) \|_1 \leq \| f \|_1$ which is not sufficient to get weak convergence, say at least in L^1, but only (for a subsequence) weak*-convergence in $M_b(\Omega)$, the space of bounded Radon measures. Under the additional integrability condition $j(\cdot,r) \in L^1(\Omega)$ for all $r \in \mathbb{R}$, however, in Step 3, weak convergence of $(B_{\lambda,\nu}(u_{\lambda,\nu}))_{\lambda,\nu}$ in L^1 can be established and we can conclude as in the x-independet case. $\square$

Remark 2 *It is clear that, combining the arguments of the two preceding proofs, a corresponding perturbation result holds for subdifferential perturbations of the form $j = j_1 + j_2$ with $j_1 \in \mathscr{J}_0$, $j_2 \in \mathscr{J}_0(\Omega)$ s.t. the integrability condition is satisfied.*

As we have mentioned in the introduction the sum of *m*-accretive operators $A+B$ is not necessarily *m*-accretive again. Hence, we introduce the concept of generalized solutions.

Definition 1 *Let A be an m-T-accretive operator in $L^1(\Omega)$ and $B \sim \partial j$ the induced operator by the subdifferential ∂j in $L^1(\Omega)$ with $j \in \mathscr{J}_0(\Omega)$. Moreover, let $f \in L^1(\Omega)$. A function $u \in L^1(\Omega)$ is called generalized solution for the equation*

$$u + Au + Bu \ni f,$$

if for $u_\lambda := (I + A + B_\lambda)^{-1} f$ we have

$$\lim_{\lambda \to 0} u_\lambda = u \quad \text{in } L^1(\Omega).$$

Theorem 3 *For every $f \in L^1(\Omega)$ there exists a generalized solution $u \in L^1(\Omega)$ according to Definiton 1. (see [3])*

Note that the proof of the existence of a generalized solution in the case of a completely accretive operator A [Théorème 1.1, [26]], which is also T-accretive can be transfer to the proof of Theorem 3.

Theorem 4 *Let A be an m-0-T-accretive operator in $L^1(\Omega)$ and $B \sim \partial j$ the induced operator in $L^1(\Omega)$ by the subdifferential ∂j with $j \in \mathscr{J}_0(\Omega)$. Then, the operator $A_j := \liminf_{\lambda \to 0} A + B_\lambda$ is an m-T-accretive extension of $A+B$, i.e.,*

$$A + B \subseteq A_j.$$

Proof of Theorem 4 Note that it is sufficient to consider the case $j \in \mathscr{J}_0(\Omega)$ with

$$j(x,r) = 0$$

for all $r \leq 0$ a.e. in Ω and the case $j \in \mathscr{J}_0(\Omega)$ with

$$j(x,r) = 0$$

for all $r \geq 0$ a.e. in Ω. If

$$A + B_+ \subseteq A_{j_+}$$

with $B_+ \sim \partial j_+$, then we obtain

$$A + B = (A + B_+) + B_- \subseteq A_{j_+} + B_- \subseteq (A_{j_+})_{j_-}.$$

Moreover, we have

$$A_j = (A_{j_+})_{j_-} = (A_{j_-})_{j_+}.$$

(see for a proof in [3], [26] and [27]). Consequently, we may consider without loss of generality a function $j \in \mathscr{J}_0(\Omega)$ with

$$j(x,r) = 0$$

for all $r \leq 0$ a.e. in Ω. Let $(u, f) \in L^1(\Omega) \times L^1(\Omega)$ be a pair s.t.

$$u + Au + w \ni f, \tag{9}$$

where $w \in Bu$. For $m \in \mathbb{N}$ we define the function

$$j^m(x,r) := \int_0^r \inf(m, (\partial j)^0(x,s))ds$$

a.e. in Ω and for all $r \in \mathbb{R}$. We obtain $j^m \in \mathscr{J}_0(\Omega)$ and

$$\partial j^m(x,r) = \partial j(x,r) \wedge m$$

a.e. in Ω and for all $r \in \mathbb{R}$. According to (9) we have

$$u + Au + w \wedge m \ni f + (w \wedge m - w) \tag{10}$$

with $w(x) \wedge m \in \partial j^m(x, u(x))$ a.e. in Ω. In the following we denote with $B^m \sim \partial j^m$ the induced operator in $L^1(\Omega)$ by the subdifferential ∂j^m. Due to the fact that $j^m(\cdot, r) \in L^1(\Omega)$ for all $r \in \mathbb{R}$, according to Theorem 2 there exists a unique solution u_m for

$$u_m + Au_m + w_m \ni f \tag{11}$$

with $w_m \in B^m u_m$. The combination of (10), (11) and the accretivity of $A+B^m$ yields

$$\| u - u_m \|_{L^1(\Omega)} \leq \| f - (f + w \wedge m - w) \|_{L^1(\Omega)} = \| w \wedge m - w \|_{L^1(\Omega)} \to 0$$

as $m \to \infty$. Consequently we have

$$u_m \to u$$

in $L^1(\Omega)$ as $m \to \infty$. Thus, we obtain

$$A + B \subseteq \liminf_{m \to \infty} A + B^m.$$

Now, we are going to verify that

$$A + B^m = \liminf_{\lambda \downarrow 0} A + B^m_\lambda$$

with $B^m_\lambda \sim \partial(j_\lambda)^m$. For this, we consider for $\lambda > \tilde{\lambda} > 0$, $m \in \mathbb{N}$ the solutions $u^m_\lambda, u^m_{\tilde{\lambda}}$ of

$$\begin{aligned} u^m_\lambda + A u^m_\lambda + \partial(j_\lambda)^m(\cdot, u^m_\lambda) &\ni f \\ u^m_{\tilde{\lambda}} + A u^m_{\tilde{\lambda}} + \partial(j_{\tilde{\lambda}})^m(\cdot, u^m_{\tilde{\lambda}}) &\ni f. \end{aligned}$$

Since $A + \partial(j_\lambda)^m$ is T-accretive, we obtain for $\lambda > \tilde{\lambda} > 0$

$$u^m_\lambda \geq u^m_{\tilde{\lambda}}$$

a.e. in Ω. Moreover, we have for all $\lambda > 0$

$$\| u^m_\lambda \|_{L^1(\Omega)} \leq \| f \|_{L^1(\Omega)} .$$

For $\lambda \downarrow 0$ the monotone convergence theorem yields

$$u^m_\lambda \downarrow u^m \tag{12}$$

a.e. in Ω and in $L^1(\Omega)$. Furthermore, we have for all $\lambda > 0$ and for all $m \in \mathbb{N}$

$$\| \partial(j_\lambda)^m(\cdot, u^m_\lambda) \|_{L^1(\Omega)} \leq \| f \|_{L^1(\Omega)} .$$

Due to the fact that

$$\partial(j_\lambda)^m(x,r) = \partial j_\lambda(x,r) \wedge m \leq m$$

a.e. in Ω and for all $r \in \mathbb{R}$, we obtain for all $\lambda > 0$

$$\partial(j_\lambda)^m(\cdot, u^m_\lambda) \leq m$$

a.e. in Ω. According to the theorem of Dunford-Pettis and the theorem of Banach-Alaoglu-Bourbaki there exists a subsequence such that for $\lambda \downarrow 0$

$$\begin{aligned} &\partial(j_\lambda)^m(\cdot, u^m_\lambda) \rightharpoonup b^m \text{ in } L^1(\Omega) \text{ and} \\ &\partial(j_\lambda)^m(\cdot, u^m_\lambda) \rightharpoonup_* b^m \text{ in } L^\infty(\Omega). \end{aligned} \tag{13}$$

Now, we are going to prove that for $m \in \mathbb{N}$

$$b^m \in \partial j^m(\cdot, u^m)$$

a.e. in Ω. For this purpose let $E \subseteq \Omega$ be an arbitrary and measurable subset. Then, we have for all $\lambda_0 > \lambda > 0$ and for all $r \in \mathbb{R}$

$$\begin{aligned}\int_E (j_\lambda)^m(x,r) &\geq \int_E (j_\lambda)^m(x,u_\lambda^m(x)) + \partial(j_\lambda)^m(x,u_\lambda^m(x))(r-u_\lambda^m(x)) \\ &\geq \int_E (j_{\lambda_0})^m(x,u_\lambda^m(x)) + \partial(j_\lambda)^m(x,u_\lambda^m(x))(r-u_\lambda^m(x)), \qquad (14)\end{aligned}$$

where we use the fact that for all $\lambda_0 > \lambda > 0$

$$\partial j_{\lambda_0}(x,s) \leq \partial j_\lambda(x,s)$$

a.e. in Ω and for all $s \in \mathbb{R}$.

Moreover, the function $(j_\lambda)^m : \Omega \times \mathbb{R} \to [0,\infty]$ satisfies the Carathéodory condition. Hence, we have for $\lambda \downarrow 0$

$$(j_{\lambda_0})^m(\cdot,u_\lambda^m) \to (j_{\lambda_0})^m(\cdot,u^m)$$

a.e. in Ω. Due to the fact that

$$\partial j_\lambda(x,s) \geq 0$$

for all $s \geq 0$ and for $\lambda_0 > \lambda > 0$

$$u_{\lambda_0}^m \geq u_\lambda^m$$

a.e. in Ω we obtain

$$(j_{\lambda_0})^m(\cdot,u_\lambda^m) \leq (j_{\lambda_0})^m(\cdot,u_{\lambda_0}^m)$$

a.e. in Ω.

For $\lambda \downarrow 0$ Lebesgue's dominated convergence theorem immediately yields

$$(j_{\lambda_0})^m(\cdot,u_\lambda^m) \to (j_{\lambda_0})^m(\cdot,u^m) \qquad (15)$$

in $L^1(\Omega)$. For $\lambda \downarrow 0$ we obtain

$$\partial j_\lambda(x,s) \wedge m \uparrow (\partial j)^0(x,s) \wedge m$$

a.e. in Ω and for all $s \in \mathbb{R}$. Furthermore, we have for $\lambda_0 > \lambda > 0$

$$(j_{\lambda_0})^m(x,r) \leq (j_\lambda)^m(x,r) \leq j^m(x,r)$$

a.e. in Ω and for all $r \in \mathbb{R}$.

For $\lambda \downarrow 0$ the monotone convergence theorem yields

$$(j_\lambda)^m(x,r) \to j^m(x,r)$$

a.e. in Ω and for all $r \in \mathbb{R}$. According to (12) and (13) we obtain for $\lambda \downarrow 0$ in (14)

$$\int_E j^m(x,r) \geq \int_E (j_{\lambda_0})^m(x,u^m(x)) + b^m(x)(r-u^m(x))$$

for all $r \in \mathbb{R}$. For $\lambda_0 \downarrow 0$ we obtain again with the monotone convergence theorem

$$\int_E j^m(x,r) \geq \int_E j^m(x,u^m(x)) + b^m(x)(r-u^m(x))$$

for all $r \in \mathbb{R}$. Consequently, we obtain

$$b^m \in \partial j^m(\cdot, u^m)$$

a.e. in Ω. Since A is 0-T-accretive we have for all $(v,w) \in A$

$$\int_\Omega \kappa_\lambda (f - u_\lambda^m - \partial (j_\lambda)^m(\cdot, u_\lambda^m) - w) \geq 0$$

for all $\lambda > 0$, where

$$\kappa_\lambda = sign_0^+(u_\lambda^m - v).$$

Now, we are going to prove that

$$f - u^m - b^m \in Au^m.$$

For this purpose we are going to prove that for an arbitrarily chosen couple $(v,w) \in A$ there exists a function $\kappa \in L^\infty(\Omega)$ with $\kappa \in sign^+(u^m - v)$ such that for $\lambda \downarrow 0$

$$\int_\Omega \kappa_\lambda (f - u_\lambda^m - \partial (j_\lambda)^m(\cdot, u_\lambda^m) - w) \to \int_\Omega \kappa(f - u^m - b^m - w). \tag{16}$$

According to the theorem of Banach-Alaoglu-Bourbaki there exists a subsequence s.t. for $\lambda \downarrow 0$

$$\kappa_\lambda \rightharpoonup_* \kappa$$

in $L^\infty(\Omega)$ with $\kappa \in sign^+(u^m - v)$.

In order to prove (16), we decompose Ω in the following way:
1.) $\Omega_1 = \{x \in \Omega : u^m(x) > v(x)\}$
For $\lambda \downarrow 0$ we have $u_\lambda^m \downarrow u^m$. Thus, we obtain

$$u_\lambda^m(x) \geq u^m(x) > v(x)$$

in Ω_1. Due to the fact that $\kappa_\lambda = sign_0^+(u_\lambda^m - v)$ we have $\kappa_\lambda(x) = 1$ a.e. in Ω_1. Additionally, $\kappa \in sign^+(u^m - v)$, i.e., $\kappa(x) = 1$ a.e. in Ω_1 s.t.

$$\kappa_\lambda \to \kappa$$

in $L^\infty(\Omega_1)$ for $\lambda \downarrow 0$. The combination of

$$f - u_\lambda^m - \partial (j_\lambda)^m(\cdot, u_\lambda^m) - w \rightharpoonup f - u^m - b^m - w$$

in $L^1(\Omega)$ for $\lambda \downarrow 0$ and

$$\kappa_\lambda \to \kappa$$

in $L^\infty(\Omega_1)$ for $\lambda \downarrow 0$ yields

$$\langle \kappa_\lambda, f - u_\lambda^m - \partial (j_\lambda)^m(\cdot, u_\lambda^m) - w \rangle_{L^\infty(\Omega_1), L^1(\Omega_1)} \to \int_{\Omega_1} \kappa(f - u^m - b^m - w) = \int_{\Omega_1} f - u^m - b^m - w$$

for $\lambda \downarrow 0$.

2.) $\Omega_2 = \{x \in \Omega : u^m(x) < v(x)\}$
Due to the fact that for $\lambda \downarrow 0$ we have $u^m_\lambda \downarrow u^m$, for $x \in \Omega_2$ there exists an index $N(x) \in \mathbb{N}$ s.t. for all $0 < \lambda < \lambda_{N(x)}$

$$u^m(x) \leq u^m_\lambda(x) < v(x).$$

As a result of $\kappa_\lambda = sign_0^+(u_{\lambda,v} - v)$ and $\kappa \in sign^+(u^m - v)$ we obtain for $0 < \lambda < \lambda_{N(x)}$

$$|\kappa_\lambda(x) - \kappa(x)| = 0$$

and consequently for $\lambda \downarrow 0$

$$\kappa_\lambda(x) \to \kappa(x)$$

a.e. in Ω_2. Moreover, we have for $\lambda \downarrow 0$

$$f - u^m_\lambda - \partial(j_\lambda)^m(\cdot, u^m_\lambda) - w \rightharpoonup f - u^m - b^m - w$$

in $L^1(\Omega)$ and for all $\lambda > 0$

$$|f - u^m_\lambda - \partial(j_\lambda)^m(\cdot, u^m_\lambda)| \leq |f| + |h| + m$$

a.e. in Ω, where $h \in L^1(\Omega)$.

Thus, we obtain for $\lambda \downarrow 0$

$$0 = \int_{\Omega_2} \kappa_\lambda(f - u^m_\lambda - \partial(j_\lambda)^m(\cdot, u^m_\lambda) - w) \to \int_{\Omega_2} \kappa(f - u^m - b^m - w) = 0.$$

3.) $\Omega_3 = \{x \in \Omega : u^m(x) = v(x)\}$.
We decompose the set Ω_3 in the following way:

$$a)\ \hat{\Omega}_3 = \{x \in \Omega : u^m_\lambda(x) > u^m(x) = v(x)\ \forall \lambda > 0\}$$

und

$$b)\ \tilde{\Omega}_3 = \{x \in \Omega : \exists N(x) \in \mathbb{N} \text{ s.d. } u^m_\lambda(x) = u^m(x) = v(x)\ \forall 0 < \lambda < \lambda_{N(x)}\}.$$

$a)$ On the set $\hat{\Omega}_3$ we have $\kappa_\lambda(x) = 1$ a.e. in $\hat{\Omega}_3$. Going along the same lines as in 1.) we obtain for $\lambda \downarrow 0$

$$\int_{\hat{\Omega}_3} \kappa_\lambda(f - u^m_\lambda - \partial(j_\lambda)^m(\cdot, u^m_\lambda) - w) \to \int_{\hat{\Omega}_3} \kappa(f - u^m - b^m - w) = \int_{\hat{\Omega}_3} f - u^m - b^m - w.$$

$b)$ For all $0 < \lambda < \lambda_{N(x)}$ we have the pointwise convergence

$$\partial(j_\lambda)^m(x, u^m_\lambda(x)) = \partial(j_\lambda)^m(x, u^m(x)) \overset{\lambda\downarrow 0}{\to} (\partial j)^0(x, u^m(x)) \wedge m = \partial j^m(x, u^m(x))$$

a.e. in $\tilde{\Omega}_3$. Furthermore, we have for all $0 < \lambda < \lambda_{N(x)}$

$$|\partial(j_\lambda)^m(\cdot, u^m_\lambda)| = |\partial(j_\lambda)^m(\cdot, u^m)| \leq m$$

a.e. in $\tilde{\Omega}_3$. The dominated convergence theorem of Lebesgue implies immediately the strong convergence of $\partial(j_\lambda)^m(\cdot,u_\lambda^m)$ in $L^1(\tilde{\Omega}_3)$ s.t. for $\lambda\downarrow 0$ we have

$$\partial(j_\lambda)^m(\cdot,u_\lambda^m)\to b^m$$

in $L^1(\tilde{\Omega}_3)$. As a result of this we obtain for $\lambda\downarrow 0$

$$\int_{\tilde{\Omega}_3}\kappa_\lambda(f-u_\lambda^m-\partial(j_\lambda)^m(\cdot,u_\lambda^m)-w)\to\int_{\tilde{\Omega}_3}\kappa(f-u^m-b^m-w).$$

Consequently, the cases 1.), 2.) and 3.) imply the T-accretivity of the operator $A\cup\{(u^m,f-u^m-b^m)\}$ s.t. we obtain with the m-accretivity of A the desired result $f-u^m-b^m\in Au^m$. All in all, we have shown that

$$A+B^m=\liminf_{\lambda\downarrow 0}A+B_\lambda^m.$$

Furthermore, we have

$$\begin{aligned}
u=(I+A+B)^{-1}f&=\lim_{m\to\infty}(I+A+B^m)^{-1}f\\
&=\lim_{m\to\infty}\lim_{\lambda\downarrow 0}(I+A+B_\lambda^m)^{-1}f\\
&=\inf_{m\in\mathbb{N}}\inf_{\lambda>0}(I+A+B_\lambda^m)^{-1}f\\
&=\inf_{\lambda>0}\inf_{m\in\mathbb{N}}(I+A+B_\lambda^m)^{-1}f\\
&=\lim_{\lambda\downarrow 0}\lim_{m\to\infty}(I+A+B_\lambda^m)^{-1}f\\
&=\lim_{\lambda\downarrow 0}(I+A+B_\lambda)^{-1}f\\
&=(I+A_j)^{-1}f.
\end{aligned}$$

Hence, we obtain

$$A+B\subseteq A_j.$$

□

4 Necessity of conditions

In order to justify the necessity of the maximum principle in the preceding theorem, the following example demonstrates that there actually exist m-T-accretive operators, not satisfying (M_0), for which the conclusion of Theorem 1 fails to hold.

Example 2: Let $\Omega=(0,1)$ and consider the operator A in $L^1(0,1)$, defined by

$$A=\left\{\left(k\frac{1}{2\sqrt{x}},m\left(f-k\frac{1}{2\sqrt{x}}\right)\right)\middle|\int_0^1 f=k,k\in\mathbb{R},m>0\right\}.$$

Then A is m-0-T-accretive but does not verify the maximum principle.
In order to prove these assertions, let (u,v), $(\hat{u},\hat{v})\in A$ and $\kappa\in\operatorname{sign}_0^+(u-\hat{u})$. Since

$u > \hat{u}$ iff $k > \hat{k}$, it holds that $\{x \in (0,1) \mid u(x) > \hat{u}(x)\} = (0,1)$ and hence we obtain

$$\begin{aligned}\int_0^1 \kappa(v-\hat{v}) = \int_{[u>\hat{u}]} (v-\hat{v}) &= \int_0^1 \left(m\left(f - k\frac{1}{2\sqrt{x}}\right) - \hat{m}\left(\hat{f} - \hat{k}\frac{1}{2\sqrt{x}}\right)\right) \\ &= m\int_0^1 f - mk\int_0^1 \frac{1}{2\sqrt{x}} - \hat{m}\int_0^1 \hat{f} + \hat{m}\hat{k}\int_0^1 \frac{1}{2\sqrt{x}} \\ &= mk - mk\left[\sqrt{x}\right]_0^1 - \hat{m}\hat{k} + \hat{m}\hat{k}\left[\sqrt{x}\right]_0^1 = 0.\end{aligned}$$

Moreover, since by definition $((\int_0^1 f)\frac{1}{2\sqrt{x}}, f - (\int_0^1 f)\frac{1}{2\sqrt{x}}) \in A$, the equation $u + Au \ni f$ admits for every $f \in L^1(0,1)$ the solution $u = (\int_0^1 f)\frac{1}{2\sqrt{x}} \in D(A)$. Thus, A is m-0-T-accretive.
In order to demonstrate that A does not satisfy the maximum principle (M_0), we give a trivial counterexample:
If $m = k = 1$ and $f(x) = 1$ for a.e. $x \in (0,1)$, then we obtain $(\frac{1}{2\sqrt{x}}, f - \frac{1}{2\sqrt{x}}) \in A$, $J_1^A f = \frac{1}{2\sqrt{x}}$ and in consequence $\|J_1^A f\|_\infty = \sup_{x\in(0,1)} |\frac{1}{2\sqrt{x}}| > 1 = \|f\|_\infty$.
Furthermore, consider the monotone graph β in $\mathbb{R}$, given by $\beta(r) = r|r|$ for $r \in \mathbb{R}$, and its realization $B \sim \beta$ in $L^1(0,1)$

$$B = \{(u,v) \mid u, v \in L^1(0,1), v(x) = u(x)\,|u(x)| \text{ for a.e. } x \in (0,1)\}.$$

As already mentioned in the preliminaries, it is well-known that the realization of a monotone graph in $L^1(0,1)$ is m-0-T-accretive. Hence, also the sum $A + B$ is 0-T-accretive.
But $A + B$ is not m-0-T-accretive. To see this, let $g \in L^1(0,1)$. Since $D(A) \cap D(B) = \{0\}$, the inclusion $u + Au + Bu \ni g$ admits only the solution $u = 0$ which implies $mf \ni g$ with $m > 0$ and $\int_0^1 f = 0$. Thus, the inclusion is only solvable for those $g \in L^1(0,1)$ which also satisfy $\int_0^1 g = 0$ and therefore the range condition $R(I + A + B) = L^1(0,1)$ is not fulfilled.

With regard to the previous example, there arises the question which of the conditions in Theorem 1 are really indispensable for its conclusion and whether they can be substituted by other reasonable constraints.
As a first result, we claim that the maximum principle (M_0) can be replaced by a simple restriction on the domain of the pertubation operator $B \sim \partial j$.

Proposition 1 *The requirement* (M_0) *in Theorem 1 can be substituted by the condition that $D(A) \subseteq D(B)$.*

Proof. Indeed, (M_0) is used in step 2 to prove the boundedness and in the following the convergence of the sequences $(u_{\lambda,\nu})_{\lambda,\nu}$ and $(B_{\lambda,\nu}(u_{\lambda,\nu}))_{\lambda,\nu}$. But, as we will see below, the boundedness in $L^1(\Omega)$ as well as the convergence of $(u_{\lambda,\nu})_{\lambda,\nu}$ are just direct consequences of the accretivity of A.
Let $f \in L^\infty(\Omega)$. Since $0 \in A0$ and $f - B_{\lambda,\nu}(u_{\lambda,\nu}) - u_{\lambda,\nu} \in Au_{\lambda,\nu}$, there is a $\kappa_{\lambda,\nu} \in \operatorname{sign}(u_{\lambda,\nu})$ such that

$$\int_\Omega \kappa_{\lambda,\nu}(f - B_{\lambda,\nu}(u_{\lambda,\nu}) - u_{\lambda,\nu}) \geq 0.$$

Due to the monotonicity of $B_{\lambda,\nu}$, we have $\kappa_{\lambda,\nu} B_{\lambda,\nu}(u_{\lambda,\nu}) = |B_{\lambda,\nu}(u_{\lambda,\nu})|$ and hence we obtain the boundedness of $(u_{\lambda,\nu})_{\lambda,\nu}$ and $(B_{\lambda,\nu}(u_{\lambda,\nu}))_{\lambda,\nu}$ in $L^1(\Omega)$ by

$$\begin{aligned}\|B_{\lambda,\nu}(u_{\lambda,\nu})\|_1+\|u_{\lambda,\nu}\|_1 &= \int_\Omega |B_{\lambda,\nu}(u_{\lambda,\nu})|+\int_\Omega |u_{\lambda,\nu}|\\ &= \int_\Omega \kappa_{\lambda,\nu}B_{\lambda,\nu}(u_{\lambda,\nu})+\int_\Omega \kappa_{\lambda,\nu}u_{\lambda,\nu}\\ &\le \int_\Omega \kappa_{\lambda,\nu}f \le \int_\Omega |f| = \|f\|_1.\end{aligned}$$

Thus, in virtue of the monotone behavior of $(u_{\lambda,\nu})_{\lambda,\nu}$, the monotone convergence theorem yields

$$u_{\lambda,\nu} \xrightarrow{\lambda\downarrow 0} u_\nu \xrightarrow{\nu\downarrow 0} u \quad \text{in } L^1(\Omega).$$

In contrast, the verification of the weak convergence of $(B_{\lambda,\nu}(u_{\lambda,\nu}))_{\lambda,\nu}$ in $L^1(\Omega)$ without (M_0) will indeed require the new condition $D(A)\subseteq D(B)$. For this purpose we compare the solution $u_{\lambda,\nu}$ of

$$u_{\lambda,\nu}+Au_{\lambda,\nu}+B_{\lambda,\nu}(u_{\lambda,\nu}) \ni f$$

with the solutions v and w of

$$v+Av\ni -f^- \quad \text{and} \quad w+Aw\ni f^+.$$

In order to show that $v\le u_{\lambda,\mu}\le w$ holds a.e. on Ω, we consider the approximate perturbed inclusions

$$v_{\lambda,\nu}+Av_{\lambda,\nu}+B_{\lambda,\nu}(v_{\lambda,\nu})\ni -f^- \quad \text{and} \quad w_{\lambda,\nu}+Aw_{\lambda,\nu}+B_{\lambda,\nu}(w_{\lambda,\nu})\ni f^+.$$

Since $A+B_{\lambda,\mu}$ is T-accretive, we obtain on the one hand

$$\|(v_{\lambda,\nu}-u_{\lambda,\nu})^+\|_1 \le \|(-f^- -f)^+\|_1 = \|(-f^+)^+\|_1 = 0,$$

on the other hand

$$\|(u_{\lambda,\nu}-w_{\lambda,\nu})^+\|_1 \le \|(f-f^+)^+\|_1 = \|(-f^-)^+\|_1 = 0$$

and in consequence $v_{\lambda,\nu}\le u_{\lambda,\nu}\le w_{\lambda,\nu}$ a.e. on Ω.
Moreover, observe that since $0\in A0$ and $B_{\lambda,\nu}(0)=0$, $u_{\lambda,\nu}=0$ is an admissible solution for $u_{\lambda,\nu}+Au_{\lambda,\nu}+B_{\lambda,\nu}(u_{\lambda,\nu})\ni 0$. Applying this to the result above, yields $v_{\lambda,\nu}\le 0\le w_{\lambda,\nu}$ a.e. on Ω which implies $B_{\lambda,\nu}(v_{\lambda,\nu})\le 0\le B_{\lambda,\nu}(w_{\lambda,\nu})$. Now, due to the T-accretivity of A, we are able to compare the solutions v and w with $v_{\lambda,\nu}$ and $w_{\lambda,\nu}$, that is,

$$\|(v-v_{\lambda,\nu})^+\|_1 \le \|(-f^- -(-f^- -B_{\lambda,\nu}(u_{\lambda,\nu})))^+\|_1 = \|(B_{\lambda,\nu}(v_{\lambda,\nu}))^+\|_1 = 0$$

and

$$\|(w_{\lambda,\nu}-w)^+\|_1 \le \|((f^+ -B_{\lambda,\nu}(w_{\lambda,\nu})-f^+)^+\|_1 = \|(-B_{\lambda,\nu}(w_{\lambda,\nu}))^+\|_1 = 0.$$

As a result of this, we obtain $v\le v_{\lambda,\nu}\le 0\le w_{\lambda,\nu}\le w$ and hence finally the desired inequality $v\le u_{\lambda,\mu}\le w$ a.e. on Ω. Along with the monotonicity of $B_{\lambda,\nu}$ and $v, w\in D(A)\subseteq D(B)$, this leads to

$$\begin{aligned}&0\ge B_{\lambda,\nu}(u_{\lambda,\nu})\ge B_{\lambda,\nu}(v)=B_\nu(v)\downarrow_{\mu\downarrow 0} B^0(v) \quad \text{and}\\ &0\le B_{\lambda,\nu}(u_{\lambda,\nu})\le B_{\lambda,\nu}(w)=B_\lambda(w)\uparrow_{\lambda\downarrow 0} B^0(w).\end{aligned}$$

Since v, $w \in D(B) \subseteq L^1(\Omega)$ there exist g, $h \in L^1(\Omega)$ such that $g \in Bv$ and $h \in Bw$ and consequently by definition $\|B^0(v)\|_1 \le \|g\|_1 < \infty$ and $\|B^0(w)\|_1 \le \|h\|_1 < \infty$, the sequence $(B_{\lambda,\nu}(u_{\lambda,\nu}))_{\lambda,\nu}$ is enclosed by the L^1-functions $B^0(v)$ and $B^0(w)$. Thus, the theorem of Dunford-Pettis implies the weak relative compactness of $(B_{\lambda,\nu}(u_{\lambda,\nu}))_{\lambda,\nu}$ in $L^1(\Omega)$ and we obtain with the simplified notation

$$f - u_{\lambda,\nu} - B_{\lambda,\nu}(u_{\lambda,\nu}) - w \overset{\lambda\downarrow 0}{\rightharpoonup} f - u_\nu - b_\nu - w \quad \text{in } L^1(\Omega).$$

In exactly the same way as in the primary proof follows $f - u_\nu - b_\nu \in Au_\nu$ and $f - u - b \in Au$. □

Remark 3 *We present now a particular case of the preceding proposition. If we replace the maximum principle* (M_0) *by a specific linear growth restriction on the subdifferential pertubation term, that is,*

$$\exists C > 0 \quad |\beta^0(r)| \le C(1+|r|) \quad \forall r \in \mathbb{R}, \tag{W$_0$}$$

then the requirement $\overline{D(A)} \subseteq D(B)$ is trivially fulfilled. Indeed, since (W_0) *implies $\beta^0(u) \in L^1(\Omega)$ for all $u \in L^1(\Omega)$, it holds that $\beta^0(u) \in Bu$ for every $u \in L^1(\Omega)$ and in consequence $D(B) = L^1(\Omega) \supseteq \overline{D(A)}$.*

The second result in our discussion is that the measure space (Ω, Σ, μ) does not have to be necessarily finite if the m-T-accretive operator A satisfies a certain sort of closedness condition.

Proposition 2 *Let (Ω, Σ, μ) be an arbitrary σ-finite measure space. Then Theorem 1 holds if the operator A is additionally closed in the following sense:*

$$\left\{\begin{array}{c} (u_n, v_n) \in A \cap (L^\infty(\Omega) \times L^\infty(\Omega)), \\ (u_n)_n,\ (v_n)_n \text{ are bounded in } L^\infty(\Omega), \\ u_n \to u \text{ in } L^1(\Omega) \text{ for } n \to \infty, \\ v_n \rightharpoonup_* v \text{ in } L^\infty(\Omega) \text{ for } n \to \infty, \\ (u,v) \in L^{1\cap\infty}(\Omega) \times L^{1\cap\infty}(\Omega) \end{array}\right\} \Longrightarrow \quad (u,v) \in A. \tag{C$_0$}$$

Proof. The first problem we are confronted with, is that the inclusion $L^\infty(\Omega) \subseteq L^1(\Omega)$ does not have to hold for a possibly infinite measure space Ω. However, we can simply replace $L^\infty(\Omega)$ by the Banach space $L^{1\cap\infty}(\Omega)$ and thus the problem

$$u_{\lambda,\nu} + Au_{\lambda,\nu} + B_{\lambda,\nu}(u_{\lambda,\nu}) \ni f$$

admits in particular for every $f \in L^{1\cap\infty}(\Omega) \subseteq L^1(\Omega)$ a solution $u_{\lambda,\nu} \in L^{1\cap\infty}(\Omega)$.
We are going to show that $f - u_{\lambda,\nu} - B_{\lambda,\nu}(u_{\lambda,\nu}) \in Au_{\lambda,\nu}$ implies $f - u - b \in Au$ with $b \in Bu$ and using (C_0) instead of $|\Omega| < \infty$.
Since $L^{1\cap\infty}(\Omega)$ is a subspace of $L^p(\Omega)$ for $p = 1, 2, \infty$, we obtain in almost the same way as in the proof of Theorem 1 for every $f \in L^{1\cap\infty}(\Omega)$ the estimates

$$\|u_{\lambda,\nu}\|_p \le \|f\|_p \quad \text{and} \quad \|B_{\lambda,\nu}(u_{\lambda,\nu})\|_p \le \|f\|_p \quad \text{for } p = 1, 2, \infty.$$

Thus, the monotone convergence theorem and the Hölder inequality yield again the convergence of $(u_{\lambda,\nu})_{\lambda,\nu}$ as follows:

$$u_{\lambda,\nu} \xrightarrow{\lambda\downarrow 0} u_\nu \xrightarrow{\nu\downarrow 0} u \quad \text{in } L^1(\Omega) \text{ and } L^2(\Omega).$$

Moreover, according to the theorem of Banach-Alaoglu, $(u_{\lambda,\nu})_{\lambda,\nu}$ also converges weakly* in $L^\infty(\Omega)$, that is,
$$u_{\lambda,\nu} \overset{\lambda\downarrow 0}{\rightharpoonup}_* u_\nu \overset{\nu\downarrow 0}{\rightharpoonup}_* u \quad \text{in } L^\infty(\Omega).$$

Furthermore, the theorem of Eberlein-Smulyan, resp. of Banach-Alaoglu, yields the weak convergence in $L^2(\Omega)$, resp. the weak*-convergence in $L^\infty(\Omega)$, of $(B_{\lambda,\nu}(u_{\lambda,\nu}))_{\lambda,\nu}$, i.e.,
$$B_{\lambda,\nu}(u_{\lambda,\nu}) \overset{\lambda\downarrow 0}{\rightharpoonup} b_\nu \overset{\nu\downarrow 0}{\rightharpoonup} b \quad \text{in } L^2(\Omega),$$
as well as
$$B_{\lambda,\nu}(u_{\lambda,\nu}) \overset{\lambda\downarrow 0}{\rightharpoonup}_* b_\nu \overset{\nu\downarrow 0}{\rightharpoonup}_* b \quad \text{in } L^\infty(\Omega).$$
Since we have just shown that
$$f - u_{\lambda,\nu} - B_{\lambda,\nu}(u_{\lambda,\nu}) \overset{\lambda\downarrow 0}{\rightharpoonup}_* f - u_\nu - b_\nu \overset{\nu\downarrow 0}{\rightharpoonup}_* f - u - b \quad \text{in } L^\infty(\Omega),$$
the sequence $(u_{\lambda,\nu}, f - u_{\lambda,\nu} - B_{\lambda,\nu}(u_{\lambda,\nu}))_{\lambda,\nu} \in A$ complies with the requirements of the closedness criterion (C_0) which implies $f - u_\nu - b_\nu \in Au_\nu$. In an analogous manner one applies (C_0) to deduce $f - u - b \in Au$.
Using the weak-$L^2(\Omega)$- and weak*-$L^\infty(\Omega)$-convergence of $(B_{\lambda,\nu}(u_{\lambda,\nu}))_{\lambda,\nu}$, we obtain $b \in Bu$ in exactly the same way as in Theorem 1.
The remaining step, the transition from $L^{1\cap\infty}(\Omega)$ to $L^1(\Omega)$, proceeds almost like the transition from $L^\infty(\Omega)$ to $L^1(\Omega)$ in the primary proof.
To prove this, let $f \in L^1(\Omega)$. Since the approximative sequence $(f^{m,n})_{m,n\in\mathbb{N}} = (T_m(f^+) - T_n(f^-))_{m,n\in\mathbb{N}}$ is part of $L^{1\cap\infty}(\Omega)$, the problem
$$u^{m,n} + Au^{m,n} + b^{m,n} \ni f^{m,n}$$
admits therefore the solution $u^{m,n} = \text{w*-}L^\infty(\Omega)\text{-}\lim_{\lambda,\nu\to 0} u^{m,n}_{\lambda,\nu}$ where
$b^{m,n} = \text{w*-}L^\infty(\Omega)\text{-}\lim_{\lambda,\nu\to 0} B_{\lambda,\nu}(u^{m,n}_{\lambda,\nu}) \in Bu^{m,n}$.
Due to the monotone convergence, we obtain similarly $u^{m,n} \to u$ in $L^1(\Omega)$ for $m,n\to\infty$.
But, in contrast to the primary proof, we can not apply the monotone convergence theorem to $(b^{m,n})_{m,n}$ since in this case it is not generally bounded in $L^1(\Omega)$. However, in order to prove $f - u - b \in Au$, we can make use of the new condition (C_0) again.
First of all, note that $(u^{m,n})_{m,n}$ and $(f^{m,n} - u^{m,n} - b^{m,n})_{m,n}$ are bounded sequences in $L^\infty(\Omega)$ because of
$$\begin{aligned} \|u^{m,n}\|_\infty &\le \liminf_{\lambda,\nu\downarrow 0} \|u^{m,n}_{\lambda,\nu}\|_\infty \le \|f^{m,n}\|_\infty \le \|f\|_\infty, \\ \|b^{m,n}\|_\infty &\le \liminf_{\lambda,\nu\downarrow 0} \|B_{\lambda,\nu}(u^{m,n}_{\lambda,\nu})\|_\infty \le \|f^{m,n}\|_\infty \le \|f\|_\infty. \end{aligned}$$
In addition, the theorem of Banach-Alaoglu implies the weak*-convergence
$$f^{m,n} - u^{m,n} - b^{m,n} \overset{m\to\infty}{\rightharpoonup}_* f^n - u^n - b^n \overset{n\to\infty}{\rightharpoonup}_* f - u - b \quad \text{in } L^\infty(\Omega),$$
and thus (C_0) immediatly yields the desired result $f - u - b \in Au$.
It only remains to show that $b^n \in Bu^n$ and $b \in Bu$. For this purpose consider $b^{m,n} \in Bu^{m,n}$, or more precisely, $b^{m,n}(x) \in \partial j(u^{m,n}(x))$ for a.e. $x \in \Omega$. Thus, we have by definition
$$j(z) \ge j(u^{m,n}(x)) + b^{m,n}(x)(z - u^{m,n}(x)) \quad \text{for all } z \in \mathbb{R}.$$

Now, let $E \subseteq \Omega$ be an arbitrary measurable set and $\chi_E \in L^1(\Omega)$ its characteristic function. Then

$$\int_\Omega j(z)\chi_E \,\mathrm{d}x \geq \int_\Omega j(u^{m,n}(x))\chi_E \,\mathrm{d}x + \int_\Omega b^{m,n}(x)(z - u^{m,n}(x))\chi_E \,\mathrm{d}x. \qquad (17)$$

Since $b^{m,n} \rightharpoonup_* b^n$ in $L^\infty(\Omega)$ and $u^{m,n} \to u^n$ in $L^1(\Omega)$ for $m \to \infty$, we obtain also the convergence

$$\int_E b^{m,n}(x)(z - u^{m,n}(x))\,\mathrm{d}x \xrightarrow{n\to\infty} \int_E b^n(x)(z - u^n(x))\,\mathrm{d}x.$$

Moreover, due to the fact that the mapping $j \in \mathscr{J}_0$ is nonnegative and lower-semicontinuous, the lemma of Fatou implies

$$\liminf_{m\to\infty} \int_E j(u^{m,n}(x))\,\mathrm{d}x \geq \int_E \liminf_{m\to\infty} j(u^{m,n}(x))\,\mathrm{d}x \geq \int_E j(u^n(x))\,\mathrm{d}x.$$

Applying this to our integral inequality (17) for $m \to \infty$, leads to

$$\int_E j(z)\,\mathrm{d}x \geq \int_E j(u^n(x))\,\mathrm{d}x + \int_E b^n(x)(z - u^n(x))\,\mathrm{d}x$$

for all measurable sets $E \subseteq \Omega$ and consequently we have a.e. on Ω

$$j(z) \geq j(u^n(x)) + b^n(x)(z - u^n(x)) \quad \text{for all } z \in \mathbb{R}.$$

Thus, we obtain by definition $b^n(x) \in \partial j(u^n(x))$ a.e on Ω and hence $b^n \in Bu^n$.
To complete the proof one concludes $b \in Bu$ in exactly the same way. □

Remark 4 *Note that in the case $\Omega = \mathbb{R}^N$ the operator*

$$A := \{(u,v) \in L^{1\cap\infty}(\Omega) \times L^{1\cap\infty}(\Omega) : u \in W_0^{1,p}(\Omega),\ v = -(\Delta_p(u) + div\ F(u)) \ in\ D'(\Omega)\},$$

where $F : \mathbb{R} \to \mathbb{R}^N$ is locally LIPSCHITZ continuous with $|F(r)| \leq C|r|^{\frac{p}{p'}}$ for all $r \in \mathbb{R}$ and a constant $C > 0$ satisfies the condition (C_0).

5 Application to nonlinear diffusion-convection problem with absorption

We consider a bounded LIPSCHITZ domain $\Omega \subseteq \mathbb{R}^N$ with $N \geq 1$. Furthermore, let $1 < p < \infty$. We study the following nonlinear diffusion-convection problem with absorption

$$\hat{E}(j,f) \begin{cases} u - \Delta_p(u) - div\ F(u) + \partial j(\cdot,u) \ni f \text{ in } \Omega, \\ u_{|\partial\Omega} = 0, \end{cases}$$

where $f \in L^1(\Omega)$. The function $F : \mathbb{R} \to \mathbb{R}^N$ is locally LIPSCHITZ-continuous, i.e., for all $M > 0$ there exists a constant $L(M) > 0$ s.t. for all $r, \hat{r} \in \mathbb{R}$ with $|r| \leq M$ and $|\hat{r}| \leq M$ we have

$$|F(r) - F(\hat{r})| \leq L(M)|r - \hat{r}|.$$

Moreover, let $j \in \mathscr{J}_0(\Omega)$ which satisfies the integrability condition $j(\cdot,r) \in L^1(\Omega)$ for all $r \in \mathbb{R}$. We define the operator

$$A_0 := \{(u,v) \in L^\infty(\Omega) \times L^\infty(\Omega) : u \in W_0^{1,p}(\Omega),\ v = -(\Delta_p(u) + div\ F(u)) \text{ in } D'(\Omega)\}$$

which is the associated operator of

$$u \mapsto -\Delta_p(u) - div\ F(u).$$

in $L^\infty(\Omega)$.

Proposition 3 *The operator A_0 is 0-T-accretive in $L^1(\Omega)$ and satisfies the condition*

$$R(I + \lambda A_0) = L^\infty(\Omega)$$

for all $\lambda > 0$.

Corollary 1 *The operator $A := \overline{A_0}^{L^1(\Omega)}$ is m-T- accretive in $L^1(\Omega)$.*

Later, we verifiy the m-0-T-accretivity of the operator A. According to Theorem 4 we obtain that the solution of

$$u + Au + Bu \ni f$$

is also a unique generalized solution. For this purpose we introduce first the concept of entropy solutions and later the concept of renormalized solutions. It is well known that A can be characterized by means of entropy solution equivalently renormalized solution. (see [6] and [22])

Recall that a measurable function $u : \Omega \to \mathbb{R}$ is called entropy solution for $\hat{E}(j,f)$, if there exists a function $w \in L^1(\Omega)$ with

$$w(x) \in \partial j(x, u(x))$$

a.e. in Ω s.t.

$i)$ $T_k(u) \in W_0^{1,p}(\Omega)$ for all $k > 0$ and

$ii)$ for all $\varphi \in D(\Omega)$ and for all $k > 0$

$$\int_\Omega u\ T_k(u-\varphi) + |\nabla u|^{p-2}\nabla u\ \nabla T_k(u-\varphi) + F(u)\ \nabla T_k(u-\varphi) + w\ T_k(u-\varphi) \leq \int_\Omega f\ T_k(u-\varphi)$$

respectively a measurable function $u : \Omega \to \mathbb{R}$ is called renormalized solution for $\hat{E}(j,f)$, if

$i)$ $u \in L^1(\Omega)$ and $T_k(u) \in W_0^{1,p}(\Omega)$ for all $k > 0$.

$ii)$ For $k \to \infty$ we have

$$\int_{\{k<|u|<k+h\}} |\nabla u|^p \to 0$$

for all $h > 0$.

iii) For all $h \in C_c^1(\mathbb{R})$ and for all $\xi \in C_c^\infty(\Omega)$ we have

$$\int_\Omega uh(u)\xi + (|\nabla u|^{p-2}\nabla u + F(u))\nabla[h(u)\xi] + \int_\Omega wh(u)\xi = \int_\Omega fh(u)\xi$$

with a function $w \in L^1(\Omega)$ and $w(x) \in \partial j(x, u(x))$ a.e. in Ω.

It is well known that the notions coincide.

With the concept of renormalized solutions we obtain the following comparison principle s.t. we are able to verify the m-0-T-accretivity of the operator A.

Theorem 5 *Let $u, \tilde{u}$ be renormalized solutions of $\hat{E}(j,f)$ and $\hat{E}(j,\tilde{f})$ with $f, \tilde{f} \in L^1(\Omega)$. Then, we have*

$$\int_\Omega (u-\tilde{u})^+ \le \int_\Omega sign_0^+(u-\tilde{u})(f-\tilde{f}).$$

Sketch of proof. Consider

$$\int_\Omega uh(u)\xi + (|\nabla u|^{p-2}\nabla u + F(u))\nabla[h(u)\xi] = \int_\Omega fh(u)\xi \tag{18}$$

and

$$\int_\Omega \tilde{u}h(\tilde{u})\xi + (|\nabla \tilde{u}|^{p-2}\nabla \tilde{u} + F(\tilde{u}))\nabla[h(\tilde{u})\xi] = \int_\Omega fh(\tilde{u})\xi \tag{19}$$

for all $h \in C_c^1(\mathbb{R})$ and for all $\xi \in C_c^\infty(\Omega)$. In (18) using for $k, l > 0$

$$\xi = \frac{T_k}{k}(u - T_l(\tilde{u}))^+$$

and in (19) for $k, l > 0$

$$\xi = \frac{T_k}{k}(T_l(u) - \tilde{u})^+.$$

Consider for $m > 0$ the function $h = h_m$, where $h_m : \mathbb{R} \to \mathbb{R}_{\ge 0}$ is defined by

$$h_m(r) := \min((m - |r|)^+, 1).$$

Subtraction of (18) and (19) yields

$$A_{k,l}^h + B_{k,l}^h = C_{k,l}^h,$$

where

$$A_{k,l}^h := \int_\Omega uh(u)\frac{T_k}{k}(u - T_l(\tilde{u}))^+ - \int_\Omega \tilde{u}h(\tilde{u})\frac{T_k}{k}(T_l(u) - \tilde{u})^+,$$
$$C_{k,l}^h := \int_\Omega fh(u)\frac{T_k}{k}(u - T_l(\tilde{u}))^+ - \int_\Omega \tilde{f}h(\tilde{u})\frac{T_k}{k}(T_l(u) - \tilde{u})^+$$

and

$$B_{k,l}^h := \int_\Omega [|\nabla u|^{p-2}\nabla u + F(u)]\nabla[h(u)\frac{T_k}{k}(u - T_l(\tilde{u}))^+] - \int_\Omega [|\nabla \tilde{u}|^{p-2}\nabla \tilde{u} + F(\tilde{u})]\nabla[h(\tilde{u})\frac{T_k}{k}(T_l(u) - \tilde{u})^+].$$

The limits $l \to \infty$, $k \to 0^+$ and $m \to \infty$ yield with Lebesgue's theorem

$$A^h_{k,l} \to \int_\Omega (u-\tilde{u})^+,$$
$$B^h_{k,l} \to 0$$
and
$$C^h_{k,l} \to \int_\Omega (f-\tilde{f}) sign_0^+(u-\tilde{u}).$$

Hence, we obtain

$$\int_\Omega (u-\tilde{u})^+ \le \int_\Omega sign_0^+(u-\tilde{u})^+(f-\tilde{f}).$$

□

Corollary 2 *The operator $A = \overline{A_0}^{L^1(\Omega)}$ is m-0-T-accretive in $L^1(\Omega)$.*

Remark 5 *Note that the preceding results also hold for functions $j \in \mathscr{J}_0$ of obstacle type, e.g.*

$$j(r) := \begin{cases} +\infty & ,|r| > 1 \\ 0 & ,|r| \le 1. \end{cases}$$

Definition 2 *Let $j \in \mathscr{J}_0(\Omega)$ and $f \in L^1(\Omega)$. A function $u \in L^1(\Omega)$ is called generalized solution for $\hat{E}(j,f)$, if*

$$\lim_{\lambda \to 0} u_\lambda = u \text{ in } L^1(\Omega),$$

whereas $u_\lambda := (I + A + B_\lambda)^{-1} f$ with $A = \overline{A_0}^{L^1(\Omega)}$ and $B \sim \partial j$.

Theorem 6 *There exists a generalized solution for $\hat{E}(j,f)$.*

The following theorem is a characterization of bounded generalized solutions for $\hat{E}(j,f)$. The characterization is motivated by the results in [9]. Note that in the case $F \equiv 0$ in [28] bounded generalized solutions have been characterized in a similar way. Due to the fact that for the characterization we have used some results of capacity theory (see [17]) let us first introduce some further notations and recall some classical results of capacity theory.

Given a set $E \subseteq \Omega$, $c_{1,p}(E)$ denotes the capacity of E with respect to the norm of $W^{1,p}(\Omega)$. Recall that, if $O \subseteq \Omega$ is open, then

$$c_{1,p}(O) = \inf\{\| \varphi \|^p_{W^{1,p}(\Omega)} : \varphi \in W^{1,p}_0(\Omega), \varphi \ge \chi_O \text{ a.e. } \Omega\}.$$

It is well-known that any element $u \in W^{1,p}_0(\Omega)$ admits a unique (in the sense of quasi-everwhere (q.e.for short), i.e., unique up to a set of capacity 0) quasi-continuous representative, i.e., a function $\tilde{u} : \Omega \to \mathbb{R}$ such that $u = \tilde{u}$ a.e. on Ω and, for any $\varepsilon > 0$, there exists an open set $O_\varepsilon \subseteq \Omega$ with $c_{1,p}(O_\varepsilon) < \varepsilon$ and such that $\tilde{u}_{|\Omega \setminus O_\varepsilon}$ is continuous. Moreover, given $(u_n)_{n \in \mathbb{N}}, u \subseteq W^{1,p}_0(\Omega)$ with $u_n \to u$ in $W^{1,p}_0(\Omega)$, there exists a subsequence $(n_k)_k$ such that $\tilde{u}_{n_k} \to \tilde{u}$ q.e. Given $j \in \mathscr{J}_0(\Omega)$, we define

$$\mathscr{J} : W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \to [0,\infty]$$
$$u \mapsto \int_\Omega j(\cdot, u)$$

Note that $\mathscr{J}$ is convex, l.s.c. on $(W_0^{1,p}(\Omega)\cap L^\infty(\Omega), max\{\|\cdot\|_{1,p},\|\cdot\|_\infty\})$ with $\mathscr{J}(0)=0$. Moreover, $C:=\overline{D(\mathscr{J})}^{\|\cdot\|_{1,p}}$ is a convex bilateral set, i.e., C is a closed convex subset of $W_0^{1,p}(\Omega)$ satisfying

$$u,w\in C \Rightarrow u\wedge w, u\vee w\in C.$$

According to [1] there exist unique (in the sense q.e.) functions γ_-,γ_+ which are quasi-u.s.c. and quasi-l.s.c., respectively, such that

$$\overline{D(\mathscr{J})}^{\|\cdot\|_{1,p}} = \{u\in W_0^{1,p}(\Omega); \gamma_- \le \tilde{u} \le \gamma_+ \text{ q.e. on } \Omega\}.$$

Moreover, $\gamma_-(x)=\inf_n \tilde{u}_n(x)=\lim_n(\inf_{1\le k\le n}\tilde{u}_k(x))$ q.e. $x\in\Omega$ (respectively the corresponding analogue for γ_+) for any $\|\cdot\|_{1,p}$-dense sequence $(u_n)_n\subseteq D(\mathscr{J})$. Recall that the subdifferential $\mathscr{B}=\partial\mathscr{J}$ is the monotone operator $\mathscr{B}\subseteq(W_0^{1,p}(\Omega)\cap L^\infty(\Omega))\times(W^{-1,p'}(\Omega)+(L^\infty(\Omega))^*)$, given by

$$\begin{aligned} &T\in\mathscr{B}u \Leftrightarrow u\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega), T\in W^{-1,p'}(\Omega)+(L^\infty(\Omega))^*,\\ &\mathscr{J}(w)\ge\mathscr{J}(u)+\langle T,w-u\rangle \text{ for all } w\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega),\end{aligned}$$

where, here $\langle\cdot,\cdot\rangle$ denotes the duality between $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and its dual. Finally, given a bounded Radon measure $\mu\in\mathscr{M}_b(\Omega)$, we denote by μ_r and μ_s the regular and the singular part of μ, respectively; μ^+ and μ^- denotes the positive and the negative part of the Jordan decomposition of μ.

Theorem 7 *Let $j\in\mathscr{J}_0(\Omega)$, $f\in L^1(\Omega)$ and $u\in L^\infty(\Omega)$. Then, u is a generalized solution for $\hat{E}(j,f)$, iff $u\in W_0^{1,p}(\Omega)$ and there exists a bounded Radon measure $\mu\in M_b(\Omega)$ s.t.*

$$\begin{aligned} u-\Delta_p(u)-div\,F(u)+\mu=f \textit{ in } D'(\Omega)\\ \textit{with } \mu_r(x)\in\partial j(x,u(x))+\partial I_{[\gamma_-(x),\gamma_+(x)]}(u(x)) \textit{ a.e. in } \Omega,\\ \tilde{u}=\gamma_-\ \mu_s^- -a.e. \textit{ in } \Omega \textit{ and } \tilde{u}=\gamma_+\ \mu_s^+ -a.e. \textit{ in } \Omega.\end{aligned}$$

6 Related results

So far, we have studied a rather particular perturbation problem, namely the perturbation of m-T-accretive operators by operators of subdifferential type in $L^1(\Omega)$. It is also possible to achieve perturbation results for m-accretive operators in arbitrary Banach spaces X by providing more restrictions on the perturbation term. Consider for example the following result about continuous perturbations, which has been proved independently by Kobayashi [19] and Pierre [23] and generalizes a result of Barbu [2]:

> Let A be m-accretive and $F:\overline{D(A)}\to X$ be continuous such that $A+F$ is accretive. Then $A+F$ is m-accretive.

In [8] this result has been extended to even multivalued perturbations:

Theorem 8 (Bothe) *Let X be a real Banach space, $A: D(A)\subseteq X\to 2^X\setminus\emptyset$ be m-accretive and $F:\overline{D(A)}\to 2^X\setminus\emptyset$ be upper-semicontinuous with compact, convex values such that $A+F$ is accretive. Then $A+F$ is m-accretive.*

Here, an operator $F : D(F) \subseteq X \to 2^X \setminus \emptyset$ is called *upper-semicontinuous* (u.s.c. for short) if $F^{-1}(B) := \{x \in D(F) \mid F(x) \cap B \neq \emptyset\}$ is closed in $D(F)$, for all closed subsets $B \subseteq X$. In the case of F being single-valued, upper-semicontinuity coincides with the classical notion of continuity and hence Theorem 8 reduces to the result mentioned above.

Certainly, we are particulary interested in the question whether Theorem 8 will be applicable to our case as well. In other words, does the realization B of a subdifferential ∂j, $j \in \mathscr{J}_0$, in $L^1(\Omega)$ comply with the requirements of Theorem 8?
First, note that since the monotone graph ∂j in $\mathbb{R}$ possesses convex values and the convexity carries over to $L^1(\Omega)$, B also has convex values. But, B does not have compact values in general, as shown in the following example.

Example 3: Consider the monotone graph

$$\beta(r) = \begin{cases} \emptyset & \text{for } |r| > 1 \\ (-\infty, 0] & \text{for } r = -1 \\ 0 & \text{for } |r| < 1 \\ [0, \infty) & \text{for } r = 1 \end{cases}$$

and its realization $B \sim \beta$ in $L^1(\Omega)$ for a finite measure space (Ω, Σ, μ). Then B is obviously not bounded since $Bu = L^1(\Omega)_+$ for $u \equiv 1$. Thus, although every $B \sim \partial j$ has closed values, in this case, B does not possess compact and not even weakly compact values.

Consequently, since subdifferential type perturbations in $L^1(\Omega)$ do not have necessarily compact values in general, our result in Theorem 1 is not just a particular case of Theorem 8.

In Example 2 we already justified the necessity of the maximum principle (M_0) in Theorem 1. With regard to Bothe's result, it now remains to show that also the T-accretivity of A is indispensable for the conclusion of Theorem 1. For this purpose we would like to find an example of an m-accretive, but not T-accretive, operator A in $L^1(\Omega)$ and a subdifferential operator $B \sim \partial j$ with $D(B) \supseteq \overline{D(A)}$ such that $A + B$ is accretive but not m-accretive in $L^1(\Omega)$.
Unfortunately, we had difficulties not only to construct such an example but even to find an accretive operator in $L^1(\Omega)$ which is not already T-accretive. Consequently, the question arises whether there actually exist such operators in $L^1(\Omega)$. Since we could not find an answer to this in the literature, it is necessary to concern ourselves with this fundamental problem.

For this, we construct in the following example a linear single-valued accretive operator in $L^1(\Omega)$ which satisfies the maximum principle (M_0) but fails to be T-accretive.

Example 4: Let $\Omega = [0, 1]$ and consider the linear mapping defined by

$$S : L^1([0,1]) \to L^1([0,1]), \quad Su = -\int_0^1 u(x)\,\mathrm{d}x.$$

Then S is a contraction since

$$\|Su\|_1 = \int_0^1 \left| - \int_0^1 u(x)\,\mathrm{d}x \right| \mathrm{d}y = \left| \int_0^1 u(x)\,\mathrm{d}x \right| \leq \int_0^1 |u(x)|\,\mathrm{d}x = \|u\|_1.$$

However, S is not order-preserving and therefore not a T-contraction. In fact, for $u \equiv -1$ we get $Su = 1$ and thus $\|(Su)^+\|_1 = 1 > 0 = \|u^+\|_1$.
Now, according to [5], the contraction S induces the linear single-valued s-accretive operator $A = I - S$ in $L^1([0,1])$.
In order to verify (M_0), note that every $f \in L^1([0,1])$ can be approximated by appropriate continuous functions to which we restrict ourselves in the following. Thus, also $u = J_\lambda^A f$ is continuous for $\lambda > 0$ and takes therefore its maximum on $[0,1]$. That is, there is a $x_0 \in [0,1]$ such that either $u(x_0) = \|u\|_\infty$ or $u(x_0) = -\|u\|_\infty$. Consider further the equation

$$f = u + \lambda(I - S)u = (1+\lambda)u + \lambda \int_0^1 u(x)\,\mathrm{d}x \quad \text{for } \lambda > 0.$$

Since $-\|u\|_\infty \leq \int_0^1 u(x)\,\mathrm{d}x \leq \|u\|_\infty$, we obtain in the first case

$$f(x_0) = (1+\lambda)\|u\|_\infty + \lambda \int_0^1 u(x)\,\mathrm{d}x \geq (1+\lambda)\|u\|_\infty - \lambda\|u\|_\infty$$

and in the second case

$$f(x_0) = -(1+\lambda)\|u\|_\infty + \lambda \int_0^1 u(x)\,\mathrm{d}x \leq -(1+\lambda)\|u\|_\infty + \lambda\|u\|_\infty.$$

Hence

$$\|u\|_\infty \leq f(x_0) \leq \|f\|_\infty \quad \text{or} \quad -\|u\|_\infty \geq f(x_0) \geq -\|f\|_\infty$$

which yields in both cases the desired result $\|u\|_\infty \leq \|f\|_\infty$.
It remains to prove that A, however, is not T-accretive.
For this purpose consider $u \in D(A) = L^1([0,1])$ and $\kappa \in \operatorname{sign}^+(u)$, defined by

$$u(x) = \begin{cases} 1 & \text{for } x \in [0, \frac{1}{2}) \\ -10 & \text{for } x \in [\frac{1}{2}, 1] \end{cases}, \qquad \kappa(x) = \begin{cases} 1 & \text{for } x \in [0, \frac{1}{2}) \\ 0 & \text{for } x \in [\frac{1}{2}, 1]. \end{cases}$$

Then we obtain

$$\begin{aligned} \int_0^1 \kappa(I-S)u &= \int_0^1 \kappa(x) \left(u(x) + \int_0^1 u(y)\,\mathrm{d}y \right) \mathrm{d}x = \int_0^{\frac{1}{2}} \left(u(x) + \int_0^1 u(y)\,\mathrm{d}y \right) \mathrm{d}x \\ &= \int_0^{\frac{1}{2}} (u(x) - 4.5)\,\mathrm{d}x = -\int_0^{\frac{1}{2}} 3.5\,\mathrm{d}x < 0. \end{aligned}$$

Consequently, the accretive operator $A = I - S$ is not T-accretive in $L^1([0,1])$.

Although the operator A in Example 4 is not T-accretive, its lack of order structure can be compensated by its special structure $A = I - S$ and the fact that S is strongly continuous, i.e. $u_n \rightharpoonup u$ implies $Su_n \to Su$. Hence it will be impossible to find an appropriate operator $B \sim \partial j$ such that $A + B$ does not verify the range condition $R(I + A + B) = L^1([0,1])$. In fact, similarly to Theorem 1, one can prove the m-accretivity of $A + B$ via the convergence of the solutions of the regularized problem $u_\lambda + (I - S)u_\lambda + B_\lambda u_\lambda \ni f$, using the strong continuity of S.

7 Open problems

In Theorem 7 we were able to characterize bounded generalized solutions of $\hat{E}(j,f)$. It is an interesting question if there is a characterization of unbounded generalized solutions, i.e. $u \in L^1(\Omega)$.
Due to the fact that the p-Laplacian regularized the problem $\hat{E}(j,f)$ it would also be interesting to study

$$\tilde{E}(j,f)\begin{cases} u + div\, F(u) + \partial j(\cdot,u) \ni f \text{ in } \Omega, \\ u_{|\partial\Omega} = 0, \end{cases}$$

where $f \in L^1(\Omega)$, $j \in \mathscr{J}_0(\Omega)$ and $F : \mathbb{R} \to \mathbb{R}^N$ is a locally LIPSCHITZ-continuous function. This equation will be studied in future work. Note that certain unilateral problems for time-dependent conservation laws have already been considered in [21].

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