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MEAN CONVEXITY OF THE ZERO SET OF SYMMETRIC MINIMAL SURFACES

by

Tobias Tennstädt

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TOBIAS TENNSTÄDT

Abstract. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $\alpha > 0$ a given constant, and u a bounded local minimizer of the functional

$$\mathcal{F}(u) := \int_\Omega u^\alpha \sqrt{1+|Du|^2}\, dx$$

in the class $BV_+^{1+\alpha}(\Omega) := \{u \in L^\alpha(\Omega) : u \geq 0,\ u^{1+\alpha} \in BV(\Omega)\}$.

We show that minimizers are elements of $W^{1,1}_{loc}(\Omega)$ and that the coincidence set $\{u = 0\}$ is a set of locally finite perimeter in Ω with nonnegative inward mean curvature in the variational sense, i.e. is mean convex. In particular, we prove the inequality

$$\int_\Omega |D\chi_{\{u=0\}\cap E}| \leq \int_\Omega |D\chi_E| - \int_{E\cap\{u>0\}} \frac{\alpha}{u\sqrt{1+|Du|^2}}\, dx$$

for all sets $E \subset\subset \Omega$ of finite perimeter.

1. Introduction

In this paper we consider the functional

$$\mathcal{F}(u) := \int_\Omega u^\alpha \sqrt{1+|Du|^2}\, dx, \tag{1}$$

which was first investigated by J. Bemelmans and U. Dierkes [2] as the n-dimensional generalization of the catenary problem, that is to find radially symmetric minimal surfaces bounded by two disks in 3-dimensional space. It turns out that, for $\alpha \in \mathbb{N}$, $\mathcal{F}(u)$ is equal, up to a constant factor, to the area of the surface of revolution

$$\mathcal{M}_{rot} := \left\{(x, u(x)\omega) : x \in \Omega,\ \omega \in S^\alpha \subset \mathbb{R}^{\alpha+1}\right\} \subset \mathbb{R}^{n+\alpha+1}.$$

Therefore, solutions u, either local minimizers of $\mathcal{F}$ or positive solutions to the corresponding Euler equation

$$\operatorname{div}\left(\frac{u^\alpha Du}{\sqrt{1+|Du|^2}}\right) = \sqrt{1+|Du|^2}, \tag{2}$$

or equivalently

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \frac{\alpha}{u\sqrt{1+|Du|^2}}, \tag{3}$$

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Address: Fakultät für Mathematik, Universität Duisburg-Essen, 45127 Essen, Germany
Phone: +49 201 1837441
E-mail address: tobias.tennstaedt@uni-due.de.

are called *symmetric* or *singular* minimal surfaces respectively, cp. [7][8]. In addition to the results obtained in the paper [2], their properties were investigated extensively in several papers by U. Dierkes [3][4][5][6], and U. Dierkes in collaboration with G. Huisken [9][10].

The purpose of the present paper is to study the geometric and analytic properties of the boundary of the coincidence set $\{u = 0\}$ of a local minimizer u. Observe that here the natural free boundary condition

$$f_p(x, u, Du) \cdot Du = f(x, u, Du) \text{ on } \Omega \cap \partial\{u = 0\}$$

degenerates to the useless identity $0 = 0$ for an integrand like $f(x, z, p) = z^\alpha \sqrt{1 + |p|^2}$ as considered in our paper.

2. Definitions

In this section, let Ω be an open subset of $\mathbb{R}^n$.

The functional $\mathcal{F}$ may be defined in the class

$$BV_+^{1+\alpha}(\Omega) := \left\{u \in L^\alpha(\Omega) : u \geq 0,\ u^{1+\alpha} \in BV(\Omega)\right\}$$

by setting

$$\mathcal{F}(u) := \sup \left\{ \int_\Omega u^\alpha \phi_{n+1} + \frac{u^{1+\alpha}}{1+\alpha} \sum_{i=0}^{n} D_i \phi_i \, dx : \phi \in C_c^1(\Omega, \mathbb{R}^{n+1}),\ ||\phi||_{\infty,\Omega} \leq 1 \right\}.$$

We further recall the standard definition of mean convexity of a Caccioppoli set.

Definition 1 (Mean Convexity)**.** A Caccioppoli set $E \subset \Omega$ is called *mean convex*, if

$$\int_\Omega |D\chi_E| \leq \int_\Omega |D\chi_{E\cup F}| \tag{4}$$

for all Caccioppoli sets $F \subset\subset \Omega$.

Remark. Whenever $\partial E \in C^2$, (4) implies the everywhere nonnegativity of the inward mean curvature $H_{\partial E}$ of E. This can easily be verified by calculating the first variation of the area of ∂E (see e.g. [8]), which by (4) must be nonnegative.

However, mean convexity is not well defined by (4) in case E has only locally finite perimeter. Because we have to consider such sets in the following, we are going to use a different definition for local mean convexity.

Definition 2 (Local Mean Convexity)**.** A Caccioppoli set $E \subset \Omega$ is called *locally mean convex*, if

$$\int_\Omega |D\chi_{E\cap F}| \leq \int_\Omega |D\chi_F| \tag{5}$$

for all Caccioppoli sets $F \subset\subset \Omega$.

Remark. Notice that by virtue of the inequality

$$\int_\Omega |D\chi_{F\cup E}| + \int_\Omega |D\chi_{F\cap E}| \leq \int_\Omega |D\chi_F| + \int_\Omega |D\chi_E|$$

which holds for all Caccioppoli sets $E, F \subset \Omega$, local mean convexity follows from mean convexity. Also, whenever a locally mean convex set $E \subset\subset \Omega$ has finite perimeter, we may test (5) with $F \cup E$ to obtain (4), which means that E is mean convex in this case.

Furthermore we adopt the following notation.

Definition 3. Let $E \subset \Omega$ have locally finite perimeter. Then we call

(i) $\mathcal{F}E := \left\{ x \in \Omega : \left| \lim_{\rho\to 0} \frac{\int_{B_\rho} D\chi_E}{\int_{B_\rho} |D\chi_E|} \right| = 1 \right\}$ the reduced boundary of E,

(ii) $\operatorname{int}^* E := \left\{ x \in \Omega : \lim_{\rho\to 0} \frac{|E\cap B_\rho|}{|B_\rho|} = 1 \right\}$ the measure-theoretic interior of E,

(iii) $\operatorname{ext}^* E := \left\{ x \in \Omega : \lim_{\rho\to 0} \frac{|E\cap B_\rho|}{|B_\rho|} = 0 \right\}$ the measure-theoretic exterior of E, and

(iv) $\partial^* E := \Omega \setminus (\operatorname{int}^* E \cup \operatorname{ext}^* E)$ the measure-theoretic boundary of E.

For a detailed discussion of these sets and their properties we refer to the monographs [1] and [11].

3. Statement of main theorems

Let $\Omega \subset \mathbb{R}^n$ be a bounded open region, $\alpha > 0$ fixed, and $u \in BV_+^{1+\alpha}(\Omega)$ a bounded local minimizer of the functional $\mathcal{F}$ in the class $BV_+^{1+\alpha}(\Omega)$.

Remark. The existence of minimizers in this class with given boundary values $\psi \in L^{1+\alpha}(\partial\Omega)$ was proved in [2]. More precisely, it was shown that there exists a minimizer u of

$$\mathcal{F}(u) + \frac{1}{1+\alpha} \int_{\partial\Omega} |u^{1+\alpha} - \psi^{1+\alpha}|\, d\mathcal{H}^{n-1}$$

in the class $BV_+^{1+\alpha}(\Omega)$. There it was also shown that such minimizers obey a weak maximum principle, $||u||_{\infty,\Omega} \le ||\psi||_{\infty,\partial\Omega}$, which justifies our assumptions on the boundedness of local minimizers.

Then the following two theorems hold:

Theorem 1. *There exists a set $S \subset \Omega$ with $\dim_{\mathcal{H}}(S) \le n-6$ such that $u \in C^0(\Omega \setminus S)$, as well as an open set $R \subset \Omega$, which is identical to $\{u > 0\}$ up to a set of Lebesgue measure 0, such that $u \in C^\omega(R)$. In addition we have $u^{1+\alpha} \in W^{1,1}(\Omega)$ and also $u \in W^{1,1}_{loc}(\Omega)$.*

Theorem 2. *The set $\{u = 0\}$ has locally finite perimeter in Ω and fulfills the inequality*

$$\int_\Omega |D\chi_{\{u=0\}\cap E}| \le \int_\Omega |D\chi_E| - \int_{E\cap\{u>0\}} \frac{\alpha}{u\sqrt{1+|Du|^2}}\, dx \tag{6}$$

for all Caccioppoli sets $E \subset\subset \Omega$. If in addition $\{u = 0\} \subset\subset \Omega$ is known, we have

$$\int_\Omega |D\chi_{\{u=0\}}| \le \int_\Omega |D\chi_{\{u=0\}\cup E}| - \int_{E\cap\{u>0\}} \frac{\alpha}{u\sqrt{1+|Du|^2}}\, dx, \tag{7}$$

for all Caccioppoli sets $E \subset\subset \Omega$, i.e. $\{u = 0\}$ is mean convex in Ω.

Remark. Note that Theorem 2 in particular implies the local integrability of the mean curvature of the graph of the positive part of u,

$$\frac{\alpha\chi_{\{u>0\}}}{u\sqrt{1+|Du|^2}},$$

4. Proof of Theorem 1

The proof requires the following regularity theorem by Bemelmans and Dierkes [2].

Theorem 3 (Bemelmans, Dierkes)**.** *Let u be a local minimizer of $\mathcal{F}$. Then, setting $Q := \Omega \times \mathbb{R}_+$ and $U := \{y = (x,z) \in \Omega \times \mathbb{R} : z \le u(x)\}$, $\mathcal{F}U \cap \operatorname{int}(Q)$ is an analytic hypersurface in $\operatorname{int}(Q)$ and the set $\operatorname{sing}(\partial U) := \partial U \setminus \mathcal{F}U$ is compact in every half-space of the type $\{z > \delta\}$, $\delta > 0$. In addition, the following holds:*

(i) $\operatorname{sing}(\partial U) = \emptyset$ *if* $n \le 6$.
(ii) $\operatorname{sing}(\partial U)$ *is locally finite in* $\operatorname{int}(Q)$ *if* $n = 7$.
(iii) $\mathcal{H}^{n-7+\gamma}(\operatorname{sing}(\partial U)) = 0$ *for all* $\gamma > 0$ *if* $n > 7$.

The formulation of Theorem 3 given above somewhat differs from the one in [2] where without loss of generality only the case $\alpha = 1$ is considered.

In [2], the continuity of u in the case $n \le 6$ was immediately inferred from Theorem 3. For $n \ge 7$ we have

Proposition 1. *Let $u \in BV_+^{1+\alpha}(\Omega) \cap L^\infty(\Omega)$ be a bounded local minimizer of $\mathcal{F}$. Then there exists a set $S \subset \Omega$ with $\dim_{\mathcal{H}}(S) \le n-6$, such that $u \in C^0(\Omega \setminus S)$. Additionally, there exists an open set $R \subset \Omega$, with $|\{u > 0\}\Delta R| = 0$ and $u \in C^\omega(R)$. Finally we have $u^{1+\alpha} \in W^{1,1}(\Omega)$.*

Proof. We set

$$\Omega_\delta := \{x \in \Omega : \exists z > \delta\, ((x,z) \in \partial U)\}$$

and define

$$R_\delta := \{x \in \Omega_\delta : \forall z > \delta\, ((x,z) \in \partial U \Rightarrow (x,z) \in \mathcal{F}U)\}$$

and $S_\delta := \Omega_\delta \setminus R_\delta$. Then $\dim_{\mathcal{H}}(S_\delta) \le n-6$, because $S_\delta = \operatorname{proj}((\partial U \setminus \mathcal{F}U) \cap \{z > \delta\})$, where $\operatorname{proj}\colon (x,z) \mapsto x$ denotes the orthogonal projection of $\mathbb{R}^{n+1}$ onto $\mathbb{R}^n$.

At first we will show the following: If $x_0 \in R_\delta$, then there exists one and only one $z_0 > 0$ with $(x_0, z_0) \in \partial U$. Because if there existed another $z_1 > 0$ with $(x_0, z_1) \in \partial U$, then because of the definition of U the line $g = \overline{(x_0,z_0)(x_0,z_1)}$ would be a subset of ∂U and since $x_0 \in R_\delta$, $g \cap \{z > \delta\} \subset \mathcal{F}U$. But since $\mathcal{F}U$ is analytic, g must then extend to infinity, a contradiction to the boundedness of u. Thus there exists only one such z_0.

Now we define $R := R_0 = \bigcup_{\delta>0} R_\delta$ and set $S := S_0 = \operatorname{proj}((\partial U \setminus \mathcal{F}U) \cap \{z > 0\})$, so that $\Omega_0 = S \cup R$. In the following we will pick a representative of u that is uniquely defined in $\Omega \setminus S$ by the choice of U as a representative with the regularity properties of Theorem 3.

Next we show that u is continuous in R. Suppose on the contrary that for a point $x_0 \in R$,

$$\overline{u}(x_0) := \limsup_{x \to x_0} u(x) > \liminf_{x \to x_0} u(x) =: \underline{u}(x_0),$$

then also, because ∂U is closed, $(x_0, \overline{u}(x_0)) \in \partial U$ and $(x_0, \underline{u}(x_0)) \in \partial U$, in contradiction to what has been shown above, so that u is in fact continuous in x_0. Obviously we may use the same argument to prove the continuity of u in $\Omega \setminus \Omega_0$, since for every x_0 in this set only the point $(x_0, 0)$ lies in ∂U. On the whole we get $u \in C^0(\Omega \setminus S)$. This immediately implies $|D^j u^{1+\alpha}|(\Omega) = |D^j u^{1+\alpha}|(S) = 0$ using [1] (3.90).

It will now be shown that R is open. Because $R_\delta = \Omega_\delta \setminus S_\delta$ and S_δ is again compact as the image of a compact set under projection, R_δ is relatively open in Ω_δ. We shall prove indirectly that $R_\delta \subset \operatorname{int} \Omega_\delta$. Assume that there was an $x_0 \in \partial \Omega_\delta \cap R_\delta$. Then there would exist only one $z_0 > \delta$ with $y_0 = (x_0, z_0) \in \partial U$. Furthermore there would be a sequence of points $x_j \in \Omega \setminus \Omega_\delta$ with

$$\lim_{j \to \infty} x_j = x_0 \text{ und } \limsup_{j \to \infty} u(x_j) \leq \delta.$$

However, this is a direct contradiction to the continuity of u in $x_0 \in R$ and $u(x_0) = z_0 > 0$. So we have in fact $R_\delta \subset \operatorname{int} \Omega_\delta$. Since R_δ is relatively open in Ω_δ, it must be open in Ω. Then as a union of open sets R must be open as well.

Inside this open set R we may apply the argument of [2] Theorem 14, which yields $u \in C^\omega(R)$. As a consequence we get $|D^c u^{1+\alpha}|(R) = 0$. Because $\dim_{\mathcal{H}}(S) \leq n - 6$ we already have $|D^c u^{1+\alpha}|(S) = 0$ and thus combined $|D^c u^{1+\alpha}|(\Omega_0) = 0$. Now [1] Proposition 3.92 says that $|D^c f|(\{\tilde{f} = 0\}) = 0$ for every $f \in BV$, where $\tilde{f}$ denotes the approximate limit of f. The set $\{\tilde{u} = 0\}$ is a superset of $\Omega \setminus \Omega_0$, while $\{u = 0\}$ is identical to $\{u^{1+\alpha} = 0\}$, so that $|D^c u^{1+\alpha}|(\Omega \setminus \Omega_0)$ vanishes as well and this finally implies $u^{1+\alpha} \in W^{1,1}(\Omega)$, which finishes the proof. ☐

The above proposition allows for a sensible selection of representatives of the sets $\{u > \epsilon\}$, which we may define as

$$\{u > \epsilon\} := \{x \in R : u(x) > \epsilon\}$$

for $\epsilon \geq 0$. In other words, we choose a representative of u given by continuity in $\Omega \setminus S$ and defined to be 0 in S. Then we have $R = \{u > 0\}$ and u is analytic where it is positive and fulfills the mean curvature equation (3). This will be the starting point for the following estimates.

First we need some control of the singular set S.

Lemma 1. *There is a sequence of open sets $U_j \supset S$ with finite perimeter,*

$$\lim_{j \to \infty} |U_j| = 0 \textit{ and } \lim_{j \to \infty} \mathcal{H}^{n-1}(\partial^* U_j) = 0.$$

Proof. We have $\mathcal{H}^{n-1}(S) = 0$, so for all $j \in \mathbb{N}$ there exists a $\delta = \delta(j) > 0$, such that $\mathcal{H}^{n-1}_{\delta(j)}(S) < \frac{1}{2^{j+1}}$, as well as a sequence of open balls $B_{\rho_{k,j}}(x_{k,j})$ with

$\bigcup_{k=1}^{\infty} B_{\rho_{k,j}}(x_{k,j}) \supset S,\ \rho_{k,j} < \frac{\delta(j)}{2}$ and

$$\begin{aligned}\sum_{k=1}^{\infty} \omega_{n-1}\rho_{k,j}^{n-1} &\leq \mathcal{H}^{n-1}_{\delta(j)}(S) + \frac{1}{2^{j+1}}\\ &\leq \frac{1}{2^j},\end{aligned}$$

where

$$\mathcal{H}^{n-1}_{\delta}(S) := \inf\left\{\sum_{k=1}^{\infty} \omega_{n-1}\left(\frac{\operatorname{diam}(B_k)}{2}\right)^{n-1} : \bigcup_{k=1}^{\infty} B_k \supset S,\ \operatorname{diam}(B_k) < \delta\right\},$$

so that $\mathcal{H}^{n-1}(S) = \lim_{\delta\to 0}\mathcal{H}^{n-1}_{\delta}(S)$.

We set $U_j := \bigcup_{k=1}^{\infty} B_{\rho_{k,j}}(x_{k,j}) \supset S$ and observe

$$\begin{aligned}\int |D\chi_{U_j}| &= \int |D\chi_{\bigcup_{k=1}^{\infty} B_{\rho_{k,j}}(x_{k,j})}|\\ &\leq \liminf_{N\to\infty}\int |D\chi_{\bigcup_{k=1}^{N} B_{\rho_{k,j}}(x_{k,j})}|\\ &\leq \liminf_{N\to\infty}\sum_{k=1}^{N}\int |D\chi_{B_{\rho_{k,j}}(x_{k,j})}|\\ &= \sum_{k=1}^{\infty}\int |D\chi_{B_{\rho_{k,j}}(x_{k,j})}|\\ &= n\omega_n \sum_{k=1}^{\infty}\rho_{k,j}^{n-1}\\ &= \frac{n\omega_n}{\omega_{n-1}}\sum_{k=1}^{\infty}\omega_{n-1}\rho_{k,j}^{n-1}\\ &\leq \frac{n\omega_n}{\omega_{n-1}}\frac{1}{2^j}\\ &\to 0,\ j\to\infty.\end{aligned}$$

The Sobolev inequality thus yields $|U_j| \to 0,\ j\to\infty$, as well. □

We begin with our estimates.

Lemma 2. *Let $\epsilon > 0$ be such that $\{u = \epsilon\}$ is a smooth hypersurface and*

$$\mathcal{H}^{n-1}(\partial\{u<\epsilon\}\cap\partial G) = \mathcal{H}^{n-1}(\partial\{u<\epsilon\}\cap\partial^* U_j) = 0\ \forall j\in\mathbb{N}. \tag{8}$$

Also let $G \subset\subset \Omega$ be an open set with smooth boundary. Then there holds

$$\begin{aligned}\int_{\{0<u<\epsilon\}\cap G\setminus U_j}\sqrt{1+|Du|^2}\,dx \leq{}& \frac{\alpha-1}{\alpha}\int_{\{0<u<\epsilon\}\cap G\setminus U_j}\frac{|Du|^2}{\sqrt{1+|Du|^2}}\,dx\\ &+\frac{\epsilon}{\alpha}\int_{\{u=\epsilon\}\cap G\cap \operatorname{ext}^* U_j}\frac{|Du|}{\sqrt{1+|Du|^2}}\,d\mathcal{H}^{n-1} + \frac{\epsilon}{\alpha}\mathcal{H}^{n-1}(\partial G\cap\{0<u<\epsilon\})\\ &+\frac{\epsilon}{\alpha}\mathcal{H}^{n-1}(\partial^* U_j),\end{aligned}$$

where the sets U_j are as in Lemma 1.

Remark. Because of Sard's theorem and [11] Proposition 2.16, Lemma 2 holds for almost all $\epsilon > 0$.

Proof. We start by rearranging the Euler equation (2) to get

$$\begin{aligned}\sqrt{1+|Du|^2} =& \frac{|Du|^2}{\sqrt{1+|Du|^2}} + \frac{u}{\alpha}\frac{\Delta u(1+|Du|^2) - Du\cdot D^2uDu}{(1+|Du|^2)^{\frac{3}{2}}} \\ =& \frac{\alpha-1}{\alpha}\frac{|Du|^2}{\sqrt{1+|Du|^2}} + \frac{1}{\alpha}\operatorname{div}\left(\frac{uDu}{\sqrt{1+|Du|^2}}\right),\end{aligned}$$

in $\{u > 0\}$, which we will integrate over the set $\{\delta < u < \epsilon\}\cap G\setminus U_j$, where we want to choose $\delta > 0$ in such a way that the level set $\{u = \delta\}$ is a smooth hypersurface and additionally $\mathcal{H}^{n-1}(\partial\{\delta < u < \epsilon\} \cap \partial G) = \mathcal{H}^{n-1}(\partial\{\delta < u < \epsilon\} \cap \partial^* U_j) = 0$ for all $j \in \mathbb{N}$. By the remark above this holds for almost all $0 < \delta < \epsilon$. On these level sets we then have $|Du| \neq 0$, so that their unit normals are given by $\pm\frac{Du}{|Du|}$. The outward unit normal of G will be designated ν_G, the one of U_j analogously ν_{U_j}. This yields

$$\begin{aligned}\int_{\{\delta<u<\epsilon\}\cap G\setminus U_j} \sqrt{1+|Du|^2}\,dx = \frac{\alpha-1}{\alpha}\int_{\{\delta<u<\epsilon\}\cap G\setminus U_j}\frac{|Du|^2}{\sqrt{1+|Du|^2}}\,dx \\ + \frac{1}{\alpha}\int_{\{\delta<u<\epsilon\}\cap G\setminus U_j}\operatorname{div}\left(\frac{uDu}{\sqrt{1+|Du|^2}}\right)dx.\end{aligned}$$

We may now partially integrate the second term on the right hand side to get

$$\begin{aligned}&\int\limits_{\{\delta<u<\epsilon\}\cap G\setminus U_j} \sqrt{1+|Du|^2}\,dx \\
&= \frac{\alpha-1}{\alpha}\int\limits_{\{\delta<u<\epsilon\}\cap G\setminus U_j}\frac{|Du|^2}{\sqrt{1+|Du|^2}}\,dx + \frac{1}{\alpha}\Bigg(-\int\limits_{\{u=\delta\}\cap G\cap \operatorname{ext}^* U_j}\frac{u|Du|}{\sqrt{1+|Du|^2}}\,d\mathcal{H}^{n-1} \\
&\quad+ \int\limits_{\{u=\epsilon\}\cap G\cap \operatorname{ext}^* U_j}\frac{u|Du|}{\sqrt{1+|Du|^2}}\,d\mathcal{H}^{n-1} + \int\limits_{\{\delta<u<\epsilon\}\cap \partial G\cap \operatorname{ext}^* U_j}\frac{uDu\cdot\nu_G}{\sqrt{1+|Du|^2}}\,d\mathcal{H}^{n-1} \\
&\quad- \int\limits_{\{\delta<u<\epsilon\}\cap G\cap \partial^* U_j}\frac{uDu\cdot\nu_{U_j}}{\sqrt{1+|Du|^2}}\,d\mathcal{H}^{n-1} + \int\limits_{\{\delta<u<\epsilon\}\cap \partial G\cap \partial^* U_j}\frac{uDu\cdot\nu_G}{\sqrt{1+|Du|^2}}\,d\mathcal{H}^{n-1}\Bigg) \\
&\le \frac{\alpha-1}{\alpha}\int\limits_{\{\delta<u<\epsilon\}\cap G\setminus U_j}\frac{|Du|^2}{\sqrt{1+|Du|^2}}\,dx + \frac{1}{\alpha}\Bigg(\int\limits_{\{u=\epsilon\}\cap G\cap \operatorname{ext}^* U_j}\frac{u|Du|}{\sqrt{1+|Du|^2}}\,d\mathcal{H}^{n-1} \\
&\quad+ \int\limits_{\{\delta<u<\epsilon\}\cap \partial G\cap \operatorname{ext}^* U_j}\frac{u|Du|}{\sqrt{1+|Du|^2}}\,d\mathcal{H}^{n-1} + \int\limits_{\{\delta<u<\epsilon\}\cap G\cap \partial^* U_j}\frac{u|Du|}{\sqrt{1+|Du|^2}} \\
&\quad+ \int\limits_{\{\delta<u<\epsilon\}\cap \partial G\cap \partial^* U_j}\frac{u|Du|}{\sqrt{1+|Du|^2}}\,d\mathcal{H}^{n-1}\Bigg).\end{aligned}$$

This implies

$$\int_{\{\delta<u<\epsilon\}\cap G\setminus U_j} \sqrt{1+|Du|^2}\,dx \le \frac{\alpha-1}{\alpha}\int_{\{\delta<u<\epsilon\}\cap G\setminus U_j} \frac{|Du|^2}{\sqrt{1+|Du|^2}}\,dx$$
$$+\frac{\epsilon}{\alpha}\int_{\{u=\epsilon\}\cap G\cap \operatorname{ext}^* U_j} \frac{|Du|}{\sqrt{1+|Du|^2}}\,d\mathcal{H}^{n-1} + \frac{\epsilon}{\alpha}\mathcal{H}^{n-1}(\partial G\cap\{0<u<\epsilon\})$$
$$+\frac{\epsilon}{\alpha}\mathcal{H}^{n-1}(\partial^* U_j)$$

and $\delta\to 0$ proves the claim. □

Lemma 3. *Under the assumptions of Lemma 2 the inequality*

$$\int_{\{0<u<\epsilon\}\cap G\setminus U_j} \frac{|Du|^2}{\sqrt{1+|Du|^2}}\,dx \le \epsilon\int_{\{u=\epsilon\}\cap G\cap \operatorname{ext}^* U_j} \frac{|Du|}{\sqrt{1+|Du|^2}}\,d\mathcal{H}^{n-1}$$
$$+\epsilon\mathcal{H}^{n-1}(\partial G\cap\{0<u<\epsilon\})+\epsilon\mathcal{H}^{n-1}(\partial^* U_j)$$

holds true.

Proof. We proceed as in the proof of Lemma 2 before, this time starting with the equation

$$u\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)=\frac{\alpha}{\sqrt{1+|Du|^2}},$$

which we integrate like in Lemma 2 to get

$$\begin{aligned}
&\int\limits_{\{\delta<u<\epsilon\}\cap G\setminus U_j} \frac{|Du|^2}{\sqrt{1+|Du|^2}}\,dx\\
&=-\int\limits_{\{\delta<u<\epsilon\}\cap G\setminus U_j} \frac{\alpha}{\sqrt{1+|Du|^2}}\,dx-\int\limits_{\{u=\delta\}\cap G\cap \operatorname{ext}^* U_j} \frac{|Du|u}{\sqrt{1+|Du|^2}}\,d\mathcal{H}^{n-1}\\
&\quad+\int\limits_{\{u=\epsilon\}\cap G\cap \operatorname{ext}^* U_j} \frac{|Du|u}{\sqrt{1+|Du|^2}}\,d\mathcal{H}^{n-1}+\int\limits_{\partial G\cap\{\delta<u<\epsilon\}\cap \operatorname{ext}^* U_j} \frac{uDu\cdot\nu_G}{\sqrt{1+|Du|^2}}\,d\mathcal{H}^{n-1}\\
&\quad-\int\limits_{\partial^* U_j\cap\{\delta<u<\epsilon\}\cap G} \frac{uDu\cdot\nu_{U_j}}{\sqrt{1+|Du|^2}}\,d\mathcal{H}^{n-1}+\int\limits_{\{\delta<u<\epsilon\}\cap\partial G\cap\partial^* U_j} \frac{uDu\cdot\nu_G}{\sqrt{1+|Du|^2}}\,d\mathcal{H}^{n-1}\\
&\le\int\limits_{\{u=\epsilon\}\cap G\cap \operatorname{ext}^* U_j} \frac{|Du|u}{\sqrt{1+|Du|^2}}\,d\mathcal{H}^{n-1}+\int\limits_{\partial G\cap\{\delta<u<\epsilon\}\cap \operatorname{ext}^* U_j} \frac{|Du|u}{\sqrt{1+|Du|^2}}\,d\mathcal{H}^{n-1}\\
&\quad+\int\limits_{\partial^* U_j\cap\{\delta<u<\epsilon\}\cap G} \frac{|Du|u}{\sqrt{1+|Du|^2}}\,d\mathcal{H}^{n-1}+\int\limits_{\{\delta<u<\epsilon\}\cap\partial G\cap\partial^* U_j} \frac{u|Du|}{\sqrt{1+|Du|^2}}\,d\mathcal{H}^{n-1}\\
&\le\epsilon\int\limits_{G\cap\{u=\epsilon\}\cap \operatorname{ext}^* U_j} \frac{|Du|}{\sqrt{1+|Du|^2}}\,d\mathcal{H}^{n-1}+\epsilon\mathcal{H}^{n-1}(\partial G\cap\{0<u<\epsilon\})\\
&\quad+\epsilon\mathcal{H}^{n-1}(\partial^* U_j).
\end{aligned}$$

Again, letting $\delta\to 0$ proves the lemma. □

Lemma 4. *Under the assumptions of Lemma 2 there holds*

$$\frac{\alpha\chi_{\{u>0\}}}{u\sqrt{1+|Du|^2}} \in L^1_{loc}(\Omega)$$

as well as the estimate

$$\int_{\{u=\epsilon\}\cap G\cap \mathrm{ext}^* U_j} \frac{|Du|}{\sqrt{1+|Du|^2}}\, d\mathcal{H}^{n-1} \leq \mathcal{H}^{n-1}(\partial G\cap\{u>\epsilon\}) \\ - \int_{\{u>\epsilon\}\cap G\setminus U_j} \frac{\alpha}{u\sqrt{1+|Du|^2}}\, dx + \mathcal{H}^{n-1}(\partial^* U_j).$$

Proof. This time we integrate the s.m.s.e. (3) directly over the set $\{u>\epsilon\}\cap G\setminus U_j$. Partial integration yields

$$\begin{aligned}
&\int_{G\cap\{u=\epsilon\}\cap \mathrm{ext}^* U_j} \frac{|Du|}{\sqrt{1+|Du|^2}}\, d\mathcal{H}^{n-1} \\
&= -\int_{\{u>\epsilon\}\cap G\setminus U_j} \frac{\alpha}{u\sqrt{1+|Du|^2}}\, dx + \int_{\partial G\cap\{u>\epsilon\}\cap \mathrm{ext}^* U_j} \frac{Du\cdot\nu_G}{\sqrt{1+|Du|^2}}\, d\mathcal{H}^{n-1} \\
&\quad - \int_{\partial^* U_j\cap\{u>\epsilon\}\cap G} \frac{Du\cdot\nu_{U_j}}{\sqrt{1+|Du|^2}}\, d\mathcal{H}^{n-1} + \int_{\{u>\epsilon\}\cap\partial G\cap\partial^* U_j} \frac{Du\cdot\nu_G}{\sqrt{1+|Du|^2}}\, d\mathcal{H}^{n-1} \\
&\leq \int_{\partial G\cap\{u>\epsilon\}\cap \mathrm{ext}^* U_j} \frac{|Du|}{\sqrt{1+|Du|^2}}\, d\mathcal{H}^{n-1} - \int_{\{u>\epsilon\}\cap G\setminus U_j} \frac{\alpha}{u\sqrt{1+|Du|^2}}\, dx \\
&\quad + \int_{\partial^* U_j\cap\{u>\epsilon\}\cap G} \frac{|Du|}{\sqrt{1+|Du|^2}}\, d\mathcal{H}^{n-1} + \int_{\{u>\epsilon\}\cap\partial G\cap\partial^* U_j} \frac{|Du|}{\sqrt{1+|Du|^2}}\, d\mathcal{H}^{n-1} \\
&\leq \mathcal{H}^{n-1}(\partial G\cap\{u>\epsilon\}) - \int_{\{u>\epsilon\}\cap G\setminus U_j} \frac{\alpha}{u\sqrt{1+|Du|^2}}\, dx + \mathcal{H}^{n-1}(\partial^* U_j),
\end{aligned}$$

which is the estimate to be proved. A rearranging of terms gives

$$\begin{aligned}
\int_{\{u>\epsilon\}\cap G\setminus U_j} \frac{\alpha}{u\sqrt{1+|Du|^2}}\, dx &\leq \mathcal{H}^{n-1}(\partial G\cap\{u>\epsilon\}) + \mathcal{H}^{n-1}(\partial^* U_j) \\
&\leq \mathcal{H}^{n-1}(\partial G) + \mathcal{H}^{n-1}(\partial^* U_j).
\end{aligned}$$

On the left hand side we may, by the boundedness of the integrand on the set $\{u>\epsilon\}$, let $j\to\infty$ and then, using the monotone convergence theorem, let $\epsilon\to 0$, which, because of the arbitrariness in the choice of G, proves the first part of the lemma. □

Proposition 2. *Let $u\in BV^{1+\alpha}(\Omega)\cap L^\infty(\Omega)$ be a bounded local minimizer of $\mathcal{F}$. Then $u\in W^{1,1}_{loc}(\Omega)$, $\{u=0\}$ is a set of locally finite perimeter in Ω, and we have*

$$\int_G |D\chi_{\{u=0\}}| \leq \mathcal{H}^{n-1}(\partial G\cap\{u>0\}) - \int_{\{u>0\}\cap G} \frac{\alpha}{u\sqrt{1+|Du|^2}}\, dx$$

for every open set $G\subset\subset\Omega$ with $\partial G\in C^1$. Also, if Ω is simply connected with $\partial\Omega\in C^2$, $u\in W^{1,1}(\Omega)$ and

$$\int_\Omega |D\chi_{\{u=0\}}| \leq \mathcal{H}^{n-1}(\partial\Omega) - \int_{\{u>0\}} \frac{\alpha}{u\sqrt{1+|Du|^2}}\, dx,$$

hold. In the latter case $\{u=0\}$ has finite perimeter in Ω.

Proof. For almost every $\epsilon > 0$ we may apply lemmata 2 to 4, which together yield the estimate

$$\frac{1}{\epsilon}\int_{\{0<u<\epsilon\}\cap G\setminus U_j}\sqrt{1+|Du|^2}\,dx \le \mathcal{H}^{n-1}(\partial G\cap\{u>0\}) - \int_{\{u>\epsilon\}\cap G\setminus U_j}\frac{\alpha}{u\sqrt{1+|Du|^2}}\,dx + 2\mathcal{H}^{n-1}(\partial^* U_j).2$$

Letting $j\to\infty$ gives

$$\frac{1}{\epsilon}\int_{\{0<u<\epsilon\}\cap G}\sqrt{1+|Du|^2}\,dx \le \mathcal{H}^{n-1}(\partial G\cap\{u>0\}) - \int_{\{u>\epsilon\}\cap G}\frac{\alpha}{u\sqrt{1+|Du|^2}}\,dx. \tag{9}$$

By choosing $\epsilon \ge ||u||_\infty$, we obtain $u\in W^{1,1}(\{u>0\}\cap G)$ for every $G\subset\subset\Omega$. Now choose a sequence $\epsilon_j\to 0$ such that the level sets $\{u=\epsilon_j\}$ are all smooth hypersurfaces and (8) holds. The definition

$$u_j(x) := \begin{cases} u(x), & \text{if } x\in\{u>\epsilon_j\}\setminus U_j \\ \epsilon_j, & \text{otherwise}\end{cases}$$

gives a sequence of functions $u_j\in BV(\Omega)$ with $u_j\to u\in L^1(\Omega)$. Since also

$$\begin{aligned}\int_G |Du_j| &\le \int_{G\cap\{u>\epsilon_j\}\setminus U_j}|Du|\,dx + ||u||_\infty\mathcal{H}^{n-1}(\partial^* U_j)\\ &\le \int_{G\cap\{u>0\}}|Du|\,dx + ||u||_\infty\mathcal{H}^{n-1}(\partial^* U_j)\\ &\le ||u||_\infty\left(\mathcal{H}^{n-1}(\partial G)+\mathcal{H}^{n-1}(\partial^* U_j)\right),\end{aligned}$$

the lower semicontinuity of the total variation with respect to convergence in L^1 implies $u\in BV(G)$. The argument used in the proof of Theorem 1 to show $u^{1+\alpha}\in W^{1,1}(\Omega)$ may be applied again here to achieve $u\in W^{1,1}(G)$ and thus also $u\in W^{1,1}_{loc}(\Omega)$, since $G\subset\subset\Omega$ is arbitrary.

Now we may estimate using the lower semicontinuity of perimeter, the coarea formula, (9), and the fact that u is weakly differentiable and thus $|Du|(\{u = 0\}) = 0$:

$$\begin{aligned}
&\liminf_{\epsilon\to 0} \int_G |D\chi_{\{u>\epsilon\}}| \\
&= \int_0^1 \liminf_{\epsilon\to 0} \int_G |D\chi_{\{u>t\epsilon\}}|\, dt \\
&\le \liminf_{\epsilon\to 0} \int_0^1 \int_G |D\chi_{\{u>t\epsilon\}}|\, dt \\
&= \liminf_{\epsilon\to 0} \frac{1}{\epsilon} \int_0^\epsilon \int_G |D\chi_{\{u>t\}}|\, dt \\
&\le \liminf_{\epsilon\to 0} \frac{1}{\epsilon} \int_{\{0<u<\epsilon\}\cap G} |Du|\, dx \\
&\le \liminf_{\epsilon\to 0} \frac{1}{\epsilon} \int_{\{0<u<\epsilon\}\cap G} \sqrt{1+|Du|^2}\, dx \\
&\le \mathcal{H}^{n-1}(\partial G \cap \{u>0\}) - \int_{\{u>0\}\cap G} \frac{\alpha}{u\sqrt{1+|Du|^2}}\, dx.
\end{aligned}$$

Since $\chi_{\{u>\epsilon_j\}} \to \chi_{\{u>0\}}$ in L^1, $\chi_{\{u>0\}} \in BV(G)$ follows and because $\chi_{\{u=0\}} = 1 - \chi_{\{u>0\}}$, the coincidence set $\{u = 0\}$ also has finite perimeter, whence the first assertion follows. To prove the second one, we choose $G = \Omega_\delta$ with $\delta > 0$, where Ω_δ denotes the set of all points $x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) > \delta$. As $\partial\Omega \in C^2$, for sufficiently small δ, $\Omega_\delta \in C^2$ and it follows that

$$\begin{aligned}
&\int_{\Omega_\delta} |D\chi_{\{u=0\}}| \\
\le&\mathcal{H}^{n-1}(\partial\Omega_\delta \cap \{u>0\}) - \int_{\{u>0\}\cap\Omega_\delta} \frac{\alpha}{u\sqrt{1+|Du|^2}}\, dx \\
\le&\mathcal{H}^{n-1}(\partial\Omega_\delta) - \int_{\{u>0\}\cap\Omega_\delta} \frac{\alpha}{u\sqrt{1+|Du|^2}}\, dx.
\end{aligned}$$

Letting $\delta \to 0$ thus completes the proof. □

Proof of Theorem 1. Theorem 1 is now simply a combination of results from Propositions 1 and 2. □

5. Proof of Theorem 2

Corollary 1. *Let $\{u < \epsilon\} \subset\subset \Omega$ for some $\epsilon > 0$. Then $\{u = 0\}$ has finite perimeter in Ω.*

Proof. Choose $\Omega \supset\supset G \supset \{u < \epsilon\}$ in Proposition 2. Then $\{u = 0\} \subset G$ and thus

$$\int_G |D\chi_{\{u=0\}}| = \int |D\chi_{\{u=0\}}| < \infty.$$

□

Proof of Theorem 2. If $G \subset\subset \Omega$ is an open set with $\partial G \in C^1$, then according to Proposition 2, there holds

$$\begin{aligned}
&\int_G |D\chi_{\{u=0\}}| \\
&\leq \int_{\{u>0\}} |D\chi_G| - \int_{\{u>0\}\cap G} \frac{\alpha}{u\sqrt{1+|Du|^2}}\,dx \\
&\leq \int_{\mathrm{ext}^*\{u=0\}} |D\chi_G| - \int_{\{u>0\}\cap G} \frac{\alpha}{u\sqrt{1+|Du|^2}}\,dx,
\end{aligned}$$

as $\{u>0\} = \mathrm{ext}\{u=0\} \subset \mathrm{ext}^*\{u=0\}$. We calculate

$$\begin{aligned}
&\int_{\{u>0\}\cap G} \frac{\alpha}{u\sqrt{1+|Du|^2}}\,dx \\
&\leq \int_{\mathrm{ext}^*\{u=0\}} |D\chi_G| - \int_G |D\chi_{\{u=0\}}|
\end{aligned}$$

Since $\partial G \in C^1$ and therefore $\partial^* G = \partial G$:

$$\begin{aligned}
&= \int_{\mathrm{ext}^*\{u=0\}} |D\chi_G| - \int_{\mathrm{int}^* G} |D\chi_{\{u=0\}}| \\
&= \mathcal{H}^{n-1}(\partial^* G \cap \mathrm{ext}^*\{u=0\}) - \mathcal{H}^{n-1}(\partial^*\{u=0\} \cap \mathrm{int}^* G) \\
&= \mathcal{H}^{n-1}(\partial^* G) - \mathcal{H}^{n-1}(\partial^* G \cap \mathrm{int}^*\{u=0\}) - \mathcal{H}^{n-1}(\partial^* G \cap \partial^*\{u=0\}) \\
&\quad - \mathcal{H}^{n-1}(\partial^*\{u=0\} \cap \mathrm{int}^* G) \\
&\leq \int |D\chi_G| - \int |D\chi_{G\cap\{u=0\}}|,
\end{aligned}$$

where we have used [11] Theorem 16.3 for the last inequality. It states that for two arbitrary sets A and B of finite perimeter,

$$\int |D\chi_{A\cap B}| = \mathcal{H}^{n-1}(\partial^* A \cap \mathrm{int}^* B) + \mathcal{H}^{n-1}(\partial^* B \cap \mathrm{int}^* A) + \mathcal{H}^{n-1}(\{\nu_A = \nu_B\}),$$

which implies the estimate used above.

If now $E \subset\subset \Omega$ is an arbitrary Caccioppoli set, then according to [11] Theorem 13.8 there exists a sequence of such smoothly bounded regions $G_j \subset\subset \Omega$ with $\chi_{G_j} \to \chi_E$ in L^1 and $\int |D\chi_{G_j}| \to \int |D\chi_E|$. For these we have

$$\begin{aligned}
&\int_{\{u>0\}\cap E} \frac{\alpha}{u\sqrt{1+|Du|^2}}\,dx \\
&\leq \liminf_{j\to\infty} \int_{\{u>0\}\cap G_j} \frac{\alpha}{u\sqrt{1+|Du|^2}}\,dx \\
&\leq \liminf_{j\to\infty} \left(\int |D\chi_{G_j}| - \int |D\chi_{G_j\cap\{u=0\}}| \right) \\
&\leq \lim_{j\to\infty} \int |D\chi_{G_j}| - \liminf_{j\to\infty} \int |D\chi_{G_j\cap\{u=0\}}| \\
&\leq \int |D\chi_E| - \int |D\chi_{E\cap\{u=0\}}| \\
&= \int_\Omega |D\chi_E| - \int_\Omega |D\chi_{E\cap\{u=0\}}|.
\end{aligned}$$

Here we assumed that also $\chi_{G_j} \to \chi_E$ pointwise a.e., which may be realized by choosing a suitable subsequence. Also we have applied Fatou's Lemma. To obtain the last estimate we used the lower semicontinuity of perimeter together with $\chi_{G_j \cap \{u=0\}} \to \chi_{E \cap \{u=0\}}$ in L^1. $\square$

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Fakultät für Mathematik, Universität Duisburg-Essen, 45127 Essen, Germany
E-mail address: tobias.tennstaedt@uni-due.de

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