

SCHRIFTENREIHE DER FAKULTÄT FÜR MATHEMATIK

The Maxwell Compactness Property  
for Bounded Weak Lipschitz Domains  
with Mixed Boundary Conditions in ND

by

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# The Maxwell Compactness Property for Bounded Weak Lipschitz Domains with Mixed Boundary Conditions in ND

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ABSTRACT. It is proved that the space of differential  $q$ -forms with weak exterior- and co-derivative, is compactly embedded into the space of square integrable  $q$ -forms. Mixed boundary conditions on weak Lipschitz domains are considered. Furthermore, canonical applications such as Maxwell estimates, Helmholtz decompositions and a static solution theory are proved.

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## 1. INTRODUCTION

The aim of this contribution is to prove a compact embedding, so called Weck's selection theorem [12, 13] or (generalized) Maxwell compactness property [12, 13, 10], of differential  $q$ -forms with weak exterior- and co-derivative into the space of square integrable  $q$ -forms, i.e.

$$\mathring{D}_{\Gamma_r}^q(\Omega) \cap \varepsilon^{-1} \mathring{\Delta}_{\Gamma_\nu}^q(\Omega) \hookrightarrow L^{2,q}(\Omega)$$

subject to mixed boundary conditions on bounded weak Lipschitz domains  $\Omega \subset \mathbb{R}^N$ . This generalises the results from [1], where bounded weak Lipschitz domains in the classical setting of  $\mathbb{R}^3$  were considered. The essential ingredient for its prove is Theorem 4.2. Similar results for strong Lipschitz domains can be found in [4, 2]. For a historical overview of the mathematical treatment of Weck's selection theorem (Maxwell compactness property) see [1, 6]. The central role of a compact embedding of this type can for example be seen in connection with Hilbert space complexes, where the embedding immediately provides

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closed ranges, solution theories by continuous inverses, Poincaré-type estimates, and access to Hodge-Helmholtz-type decompositions, Fredholm theory, div-curl-type lemmas, and a-posteriori error estimation see, [7, 8]. We elaborate on some of these applications in our Section 5. Finally we note that by the same arguments as in [10] our results extend to Riemannian manifolds.

## 2. NOTATION, PRELIMINARIES AND OUTLINE OF THE PROOF

Let  $\Omega \subset \mathbb{R}^N$  be a bounded weak Lipschitz domain. For a precise definition of weak Lipschitz domains, see Definitions 2.1 and 2.3. In short,  $\Omega$  is an  $N$ -dimensional  $C^{0,1}$ -submanifold of  $\mathbb{R}^N$  with boundary, i.e. a manifold with Lipschitz atlas. Let  $\Gamma := \partial\Omega$ , which is itself an  $(N-1)$ -dimensional Lipschitz-manifold without boundary, consist of two relatively open subsets  $\Gamma_\tau$  and  $\Gamma_\nu$  such that  $\bar{\Gamma}_\tau \cup \bar{\Gamma}_\nu = \Gamma$  and  $\Gamma_\tau \cap \Gamma_\nu = \emptyset$ . The separating set  $\bar{\Gamma}_\tau \cap \bar{\Gamma}_\nu$  will be assumed to be a, not necessarily connected,  $(N-2)$ -dimensional Lipschitz-submanifold of  $\Gamma$ . We will call  $(\Omega, \Gamma_\tau)$  a weak Lipschitz pair. The vector space  $\mathring{C}^{\infty,q}(\Omega)$  is defined as the subset of  $C^{\infty,q}(\Omega)$ , the set of smooth alternating differential forms of rank  $q$ , having compact support in  $\Omega$ . Together with the inner product

$$\langle E, H \rangle_{L^{2,q}(\Omega)} := \int_{\Omega} E \wedge \star H$$

it is an inner product space<sup>1</sup>. We may then define  $L^{2,q}(\Omega)$  as the completion of  $\mathring{C}^{\infty,q}(\Omega)$  with respect to the corresponding norm.  $L^{2,q}(\Omega)$  can be identified with those  $q$ -forms having  $L^2$ -coefficients with respect to any coordinate system. Using the weak version of Stokes' theorem

$$(1) \quad \langle dE, H \rangle_{L^{2,q+1}(\Omega)} = -\langle E, \delta H \rangle_{L^{2,q}(\Omega)}, \quad E \in \mathring{C}^{\infty,q}(\Omega), \quad H \in \mathring{C}^{\infty,q+1}(\Omega),$$

weak versions of the exterior derivative and co-derivative can be defined. Here  $d$  is the exterior derivative,  $\delta = (-1)^{N(q-1)} \star d \star$  the co-derivative and  $\star$  the Hodge-star-operator on  $\Omega$ . We thus introduce the Sobolev (Hilbert) spaces (equipped with their natural graph norms)

$$D^q(\Omega) := \{E \in L^{2,q}(\Omega) : dE \in L^{2,q+1}(\Omega)\}, \quad \Delta^q(\Omega) := \{E \in L^{2,q}(\Omega) : \delta E \in L^{2,q-1}(\Omega)\}$$

in the distributional sense. It holds

$$\star D^q(\Omega) = \Delta^{N-q}(\Omega), \quad \star \Delta^q(\Omega) = D^{N-q}(\Omega).$$

We further define the test forms

$$\mathring{C}_{\Gamma_\tau}^{\infty,q}(\Omega) := \{\varphi \in \mathring{C}^{\infty,q}(\mathbb{R}^N), \text{dist}(\text{supp } \varphi, \Gamma_\tau) > 0\}$$

and note that  $\mathring{C}_{\emptyset}^{\infty,q}(\Omega) = C^{\infty,q}(\bar{\Omega})$ . We now define boundary conditions. First let

$$(2) \quad \mathring{D}_{\Gamma_\tau}^q(\Omega) := \overline{\mathring{C}_{\Gamma_\tau}^{\infty,q}(\Omega)}^{D^q(\Omega)}, \quad \mathring{\Delta}_{\Gamma_\nu}^q(\Omega) := \overline{\mathring{C}_{\Gamma_\nu}^{\infty,q}(\Omega)}^{\Delta^q(\Omega)}$$

as closures of test forms. For the full boundary case  $\Gamma_\tau = \Gamma$  (resp.  $\Gamma_\nu = \Gamma$ ) we set

$$\mathring{D}^q(\Omega) := \mathring{D}_{\Gamma_\tau}^q(\Omega), \quad \mathring{\Delta}^q(\Omega) := \mathring{\Delta}_{\Gamma_\nu}^q(\Omega).$$

Furthermore, we introduce the weak spaces

$$(3) \quad \begin{aligned} \mathring{D}_{\Gamma_\tau}^q(\Omega) &:= \{E \in D^q(\Omega) : \langle E, \delta \varphi \rangle_{L^{2,q}(\Omega)} = -\langle dE, \varphi \rangle_{L^{2,q+1}(\Omega)} \text{ for all } \varphi \in \mathring{C}_{\Gamma_\tau}^{\infty,q+1}(\Omega)\}, \\ \mathring{\Delta}_{\Gamma_\nu}^q(\Omega) &:= \{H \in \Delta^q(\Omega) : \langle H, d\varphi \rangle_{L^{2,q}(\Omega)} = -\langle \delta H, \varphi \rangle_{L^{2,q-1}(\Omega)} \text{ for all } \varphi \in \mathring{C}_{\Gamma_\tau}^{\infty,q-1}(\Omega)\}, \end{aligned}$$

and again for  $\Gamma_\tau = \Gamma$  (resp.  $\Gamma_\nu = \Gamma$ ) we set

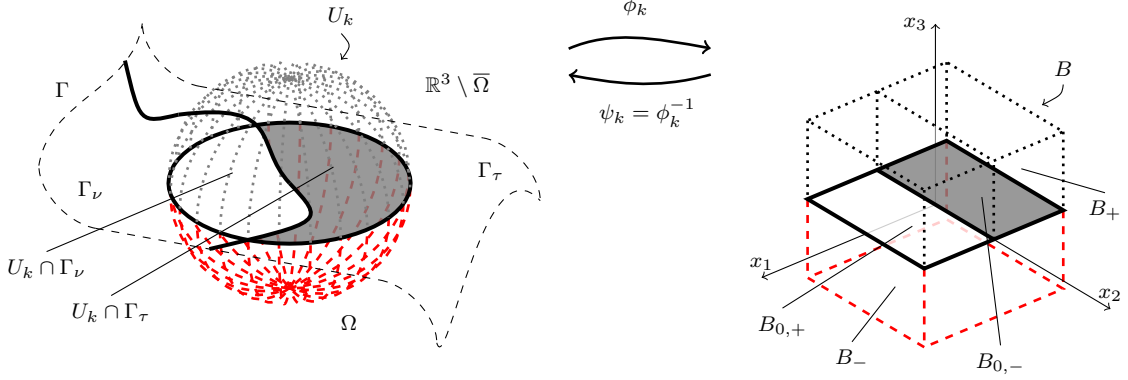
$$\mathring{D}^q(\Omega) := \mathring{D}_{\Gamma_\tau}^q(\Omega), \quad \mathring{\Delta}^q(\Omega) := \mathring{\Delta}_{\Gamma_\nu}^q(\Omega).$$

We note that in definitions (1) and (2) the smooth test forms can by mollification be replaced by their respective Lipschitz continuous counterpart, e.g.  $\mathring{C}_{\Gamma_\tau}^{\infty,q}(\Omega)$  can be replaced by  $\mathring{C}_{\Gamma_\tau}^{0,1,q}(\Omega)$ . Similarly, in definition (3) the smooth test forms can by completion be replaced by their respective closures, i.e.  $\mathring{C}_{\Gamma_\nu}^{\infty,q+1}(\Omega)$  and  $\mathring{C}_{\Gamma_\tau}^{\infty,q-1}(\Omega)$  can be replaced by  $\mathring{\Delta}_{\Gamma_\nu}^{q+1}(\Omega)$  and  $\mathring{D}_{\Gamma_\tau}^{q-1}(\Omega)$ , respectively. In (2) and (3) homogeneous tangential and normal traces on  $\Gamma_\tau$ , respectively  $\Gamma_\nu$ , are generalised. Clearly

$$\mathring{D}_{\Gamma_\tau}^q(\Omega) \subset \mathring{D}_{\Gamma_\tau}^q(\Omega), \quad \mathring{\Delta}_{\Gamma_\nu}^q(\Omega) \subset \mathring{\Delta}_{\Gamma_\nu}^q(\Omega)$$

and it will later be shown that in fact equality holds under our regularity assumption on the boundary. In case of full boundary conditions the equality even holds without any assumptions on the regularity of

<sup>1</sup>For simplicity we work in a real Hilbert space setting.

FIGURE 1. Mappings  $\phi_k$  and  $\psi_k$  between a ball  $U_k$  and the cube  $B$ .

the boundary, as can be seen by a short functional analytic argument, see [1], but which is unavailable for the mixed boundary case. We define the closed subspaces

$$\mathbf{D}_0^q(\Omega) := \{E \in \mathbf{D}^q(\Omega) : dE = 0\}, \quad \Delta_0^q(\Omega) := \{E \in \Delta^q(\Omega) : \delta E = 0\}$$

as well as  $\mathring{\mathbf{D}}_{\Gamma_\tau, 0}^q(\Omega) := \mathring{\mathbf{D}}_{\Gamma_\tau}^q(\Omega) \cap \mathbf{D}_0^q(\Omega)$  and  $\mathring{\Delta}_{\Gamma_\nu, 0}^q(\Omega) := \mathring{\Delta}_{\Gamma_\nu}^q(\Omega) \cap \Delta_0^q(\Omega)$ . Analogously for the weak spaces

$$\mathring{\mathbf{D}}_{\Gamma_\tau, 0}^q(\Omega) := \mathring{\mathbf{D}}_{\Gamma_\tau}^q(\Omega) \cap \mathbf{D}_0^q(\Omega), \quad \mathring{\Delta}_{\Gamma_\nu, 0}^q(\Omega) := \mathring{\Delta}_{\Gamma_\nu}^q(\Omega) \cap \Delta_0^q(\Omega).$$

Note that by switching  $\Gamma_\tau$  and  $\Gamma_\nu$  we can define the respective boundary conditions on the other part of the boundary as well. Furthermore, let  $\varepsilon$  be a bounded, symmetric, uniformly positive definite transformation on  $L^{2,q}$ -forms. Transformations of this type will from now on be called admissible.

**2.1. Lipschitz domains.** Let  $\Omega \subset \mathbb{R}^N$  be a bounded domain with boundary  $\Gamma := \partial\Omega$ . We introduce the setting we will be working in. Define (cf. Figure 2)

$$B := (-1, 1)^N \subset \mathbb{R}^N, \quad B_\pm := \{x \in B : \pm x_N > 0\}, \quad B_0 := \{x \in B : x_N = 0\}, \\ B_{0,\pm} := \{x \in B_0 : \pm x_1 > 0\}, \quad B_{0,0} := \{x \in B_0 : x_1 = 0\}.$$

**Definition 2.1.**  $\Omega$  is called weak Lipschitz, if the boundary  $\Gamma$  is a Lipschitz submanifold, i.e., if there is a finite open covering  $U_1, \dots, U_K \subset \mathbb{R}^N$  of  $\Gamma$  and vector fields  $\phi_k : U_k \rightarrow B$ , such that for  $k = 1, \dots, K$

- (i)  $\phi_k \in C^{0,1}(U_k, B)$  is bijective and  $\psi_k := \phi_k^{-1} \in C^{0,1}(B, U_k)$ ,
- (ii)  $\phi_k(U_k \cap \Omega) = B_-$

hold.

**Remark 2.2.** For  $k = 1, \dots, K$  we have  $\phi_k(U_k \setminus \bar{\Omega}) = B_+$  and  $\phi_k(U_k \cap \Gamma) = B_0$ .

**Definition 2.3.** Let  $\Omega$  be weak Lipschitz. A relatively open subset  $\Gamma_\tau$  of  $\Gamma$  is called weak Lipschitz, if  $\Gamma_\tau$  is a Lipschitz submanifold of  $\Gamma$ , i.e., there is an open covering  $U_1, \dots, U_K \subset \mathbb{R}^N$  of  $\Gamma$  and vector fields  $\phi_k := U_k \rightarrow B$ , such that for  $k = 1, \dots, K$  and in addition to (i), (ii) in Definition 2.1 one of

- (iii)  $U_k \cap \Gamma_\tau = \emptyset$ ,
- (iii')  $U_k \cap \Gamma_\tau = U_k \cap \Gamma \Rightarrow \phi_k(U_k \cap \Gamma_\tau) = B_0$ ,
- (iii'')  $\emptyset \neq U_k \cap \Gamma_\tau \neq U_k \cap \Gamma \Rightarrow \phi_k(U_k \cap \Gamma_\tau) = B_{0,-}$

holds. We define  $\Gamma_\nu := \Gamma \setminus \bar{\Gamma}_\tau$  to be the relatively open complement of  $\Gamma_\tau$ .

**Definition 2.4.** A pair  $(\Omega, \Gamma_\tau)$  conforming to Definitions 2.1 and 2.3 will be called weak Lipschitz.

If  $(\Omega, \Gamma_\tau)$  is weak Lipschitz so is  $(\Omega, \Gamma_\nu)$ .

**Remark 2.5.** For the cases (iii), (iii') and (iii'') in Definition 2.3 we further have

- (iii)  $U_k \cap \Gamma_\tau = \emptyset \Rightarrow U_k \cap \Gamma_\nu = U_k \cap \Gamma \Rightarrow \phi_k(U_k \cap \Gamma_\nu) = B_0$ ,
- (iii')  $U_k \cap \Gamma_\tau = U_k \cap \Gamma \Rightarrow U_k \cap \Gamma_\nu = \emptyset$ ,
- (iii'')  $\emptyset \neq U_k \cap \Gamma_\tau \neq U_k \cap \Gamma \Rightarrow \emptyset \neq U_k \cap \Gamma_\nu \neq U_k \cap \Gamma \Rightarrow \phi_k(U_k \cap \Gamma_\nu) = B_{0,+}$  and  $\phi_k(U_k \cap \bar{\Gamma}_\tau \cap \bar{\Gamma}_\nu) = B_{0,0}$ .

In the literature a bounded domain  $\Omega \subset \mathbb{R}^N$  is called (strong) Lipschitz, if there are an open covering  $U_1, \dots, U_K \subset \mathbb{R}^N$  and rigid body motions  $R_k = A_k + a_k$ ,  $A_k$  orthogonal,  $a_k \in \mathbb{R}^N$ ,  $k = 1, \dots, K$ , such that with  $\xi_k \in C^{0,1}(I^{N-1}, I)$ ,  $k = 1, \dots, K$ , and  $I = (-1, 1)$

$$R_k(U_k \cap \Omega) = \{x \in B : x_N < \xi_k(x')\}, \quad x' = (x_1, x_2, \dots, x_{N-1}),$$

holds. Then  $R_k(U_k \cap \Gamma) = \{x \in B : x_N = \xi_k(x')\}$ . A relatively open subset  $\Gamma_\tau \subset \Gamma$  is called (strong) Lipschitz, if with  $\zeta_k \in C^{0,1}(I^{N-2}, I)$

$$\emptyset \neq U_k \cap \Gamma_\tau \neq U_k \cap \Gamma \quad \Rightarrow \quad R_k(U_k \cap \Gamma_\tau) = \{x \in B : x_N = \xi_k(x'), \quad x_1 < \zeta_k(x_2, \dots, x_{N-1})\}$$

holds. With this  $R_k(U_k \setminus \bar{\Omega}) = \{x \in B : x_N > \xi_k(x')\}$  and for  $\emptyset \neq U_k \cap \Gamma_\tau \neq U_k \cap \Gamma$

$$\begin{aligned} R_k(U_k \cap \Gamma_\nu) &= \{x \in B : x_N = \xi_k(x'), \quad x_1 > \zeta_k(x_2, \dots, x_{N-1})\}, \\ R_k(U_k \cap \bar{\Gamma}_\tau \cap \bar{\Gamma}_\nu) &= \{x \in B : x_N = \xi_k(x'), \quad x_1 = \zeta_k(x_2, \dots, x_{N-1})\}. \end{aligned}$$

Clearly it holds

- $\Omega$  strong Lipschitz  $\Rightarrow \Omega$  weak Lipschitz,
- $\Omega$  strong Lipschitz and  $\Gamma_\tau$  strong Lipschitz  $\Rightarrow (\Omega, \Gamma_\tau)$  weak Lipschitz pair.

For later purposes we introduce special notations for the half-cube domain

$$(4) \quad \Xi := B_-, \quad \gamma := \partial \Xi$$

and its relatively open boundary parts  $\gamma_\tau$  and  $\gamma_\nu := \gamma \setminus \bar{\gamma}_\tau$ . We will only consider the cases

$$(5) \quad \gamma_\nu = \emptyset, \quad \gamma_\nu = B_0, \quad \gamma_\nu = B_{0,+}$$

and we note that  $\Xi$  and  $\gamma_\tau$  are strong Lipschitz.

**2.2. Outline of the proof.** Let  $(\Omega, \Gamma_\tau)$  be a weak Lipschitz pair for a bounded domain  $\Omega \subset \mathbb{R}^N$ .

- As a first step, we observe  $\mathring{H}_{\Gamma_\tau}^{1,q}(\Omega) = \mathring{H}_{\Gamma_\tau}^{1,q}(\Omega)$ , i.e., for the  $H^{1,q}$ -spaces the strong and weak definitions of the boundary conditions coincide.
- In the second and essential step, we construct various  $H^{1,q}$ -potentials on simple domains, mainly for the half-cube  $\Xi$  from (4) with the special boundary constellations (5), i.e.,

$$\mathring{D}_{\Gamma_\nu,0}^q(\Xi) = \mathring{D}_{\gamma_\nu,0}^q(\Xi) = d \mathring{H}_{\gamma_\nu}^{1,q-1}(\Xi), \quad \mathring{\Delta}_{\Gamma_\nu,0}^q(\Xi) = \mathring{\Delta}_{\Gamma_\nu,0}^q(\Xi) = \delta \mathring{H}_{\gamma_\nu}^{1,q+1}(\Xi).$$

Potentials of this type are called regular potentials.

- In the third step it is shown that the strong and weak definitions of the boundary conditions coincide on the half-cube  $\Xi$  from (4) with the special boundary constellation (5), i.e.,

$$(6) \quad \mathring{D}_{\Gamma_\nu}^q(\Xi) = \mathring{D}_{\Gamma_\nu}^q(\Xi), \quad \mathring{\Delta}_{\Gamma_\nu}^q(\Xi) = \mathring{\Delta}_{\Gamma_\nu}^q(\Xi).$$

- The fourth step proves the compact embedding on the half-cube  $\Xi$  from (4) with the special boundary constellations (5), i.e.,

$$(7) \quad \mathring{D}_{\Gamma_\tau}^q(\Xi) \cap \varepsilon^{-1} \mathring{\Delta}_{\Gamma_\nu}^q(\Xi) \hookrightarrow L^{2,q}(\Xi)$$

is compact.

- In the fifth step, (6) is established for weak Lipschitz domains, i.e.

$$\mathring{D}_{\Gamma_\tau}^q(\Omega) = \mathring{D}_{\Gamma_\tau}^q(\Omega), \quad \mathring{\Delta}_{\Gamma_\nu}^q(\Omega) = \mathring{\Delta}_{\Gamma_\nu}^q(\Omega).$$

- In the last step, we finally prove the compact embedding (7) for weak Lipschitz pairs, i.e.,

$$\mathring{D}_{\Gamma_\tau}^q(\Omega) \cap \varepsilon^{-1} \mathring{\Delta}_{\Gamma_\nu}^q(\Omega) \hookrightarrow L^{2,q}(\Omega)$$

is compact.

## 3. REGULAR POTENTIALS

In this section the compact embedding is proved on the half-cube  $\Xi \subset \mathbb{R}^N$ . This will be achieved by constructing  $\mathbf{H}^1$ -potentials for  $d$ -free and  $\delta$ -free  $L^{2,q}$ -forms, which will enable us to use Rellich's selection theorem. The special domain  $\Xi$ , together with the global identity chart, is an  $N$ -dimensional manifold. Hence  $q$ -forms  $E \in L^{2,q}(\Xi)$  can be represented in cartesian coordinates by their components  $E_I$ , i.e. (using summation convention)  $E = E_I dx^I$ . Here we use the ordered multi index notation  $dx^I = dx^{i_1} \wedge \dots \wedge dx^{i_q}$  for  $I = (i_1, \dots, i_q) \in \{1, \dots, N\}^q$ . The inner product for  $E, H \in L^{2,q}(\Xi)$  is, using their representations  $E = E_I dx^I$  and  $H = H_J dx^J$ , given by

$$\langle E, H \rangle_{L^{2,q}(\Xi)} = \int_{\Xi} E \wedge \star H = \sum_I \int_{\Xi} E_I H_I = \sum_I \langle E_I, H_I \rangle_{L^2(\Xi)} = \langle \vec{E}, \vec{H} \rangle_{L^2(\Xi)}$$

where we introduce the vector proxy notation

$$\vec{E} = [E_I]_I \in L^2(\Xi; \mathbb{R}^{N_q}), \quad N_q := \binom{N}{q}.$$

We can now define the Sobolev space  $\mathbf{H}^{k,q}(\Xi)$  as the subset of  $L^{2,q}(\Xi)$  having each component  $E_I$  in  $\mathbf{H}^k$ . In these cases, we have for  $|\alpha| \leq k$

$$\partial^\alpha E = \sum_I \partial^\alpha E_I dx^I \quad \text{and} \quad \langle E, H \rangle_{\mathbf{H}^{k,q}(\Xi)} := \sum_{0 \leq |\alpha| \leq k} \langle \partial^\alpha E, \partial^\alpha H \rangle_{L^{2,q}(\Xi)}$$

and we use the vector proxy notation also for the gradient, i.e.

$$\nabla \vec{E} = [\partial_n E_I]_{n,I} = [\dots \nabla E_I \dots]_I \in L^2(\Xi; \mathbb{R}^{N \times N_q}).$$

Hence, for  $E, H \in \mathbf{H}^{1,q}(\Xi)$

$$\begin{aligned} \langle E, H \rangle_{\mathbf{H}^{1,q}(\Xi)} &= \langle E, H \rangle_{L^{2,q}(\Xi)} + \sum_{n=1}^N \langle \partial_n E, \partial_n H \rangle_{L^{2,q}(\Xi)} = \sum_I \left( \int_{\Xi} E_I H_I + \sum_n \int_{\Xi} \partial_n E_I \partial_n H_I \right) \\ &= \sum_I (\langle E_I, H_I \rangle_{L^2(\Xi)} + \langle \nabla E_I, \nabla H_I \rangle_{L^2(\Xi)}) = \langle \vec{E}, \vec{H} \rangle_{L^2(\Xi)} + \langle \nabla \vec{E}, \nabla \vec{H} \rangle_{L^2(\Xi)} = \langle \vec{E}, \vec{H} \rangle_{\mathbf{H}^1(\Xi)}. \end{aligned}$$

Boundary conditions for  $\mathbf{H}^{1,q}$ -forms can again be defined strongly and weakly, i.e., by closure

$$\mathbf{H}_{\Gamma_\tau}^{1,q}(\Xi) := \overline{\mathbf{C}_{\gamma_\tau}^{\infty,q}(\Xi)}^{\mathbf{H}^{1,q}}$$

and by partial integration

$$\mathring{\mathbf{H}}_{\Gamma_\tau}^{1,q}(\Xi) := \{E \in \mathbf{H}^{1,q}(\Xi) : \langle \vec{E}, \operatorname{div} \vec{\phi} \rangle_{L^2(\Xi)} = -\langle \nabla \vec{E}, \vec{\phi} \rangle_{L^2(\Xi)} \text{ for all } \phi \in \mathring{\mathbf{C}}_{\Gamma_\nu}^{\infty,q}(\Xi)\}.$$

We also introduce the following spaces

$$\begin{aligned} \mathbf{D}^{k,q}(\Xi) &:= \left\{ E \in \mathbf{H}^{k,q}(\Xi) : dE \in \mathbf{H}^{k,q+1}(\Xi) \right\}, \\ \Delta^{k,q}(\Xi) &:= \left\{ E \in \mathbf{H}^{k,q}(\Xi) : \delta E \in \mathbf{H}^{k,q-1}(\Xi) \right\}. \end{aligned}$$

One of the main tools in the following arguments is a universal extension operator for the Sobolev spaces  $\mathbf{D}^{k,q}$  given in [3], which is based on the universal extension operator for standard Sobolev spaces  $\mathbf{H}^k$  introduced by E.M. Stein in [11]. "Universality" in this context means that the operator, which is given by a single formula, is able to extend all orders of Sobolev spaces. More precisely the following theorem, which is taken from [3, Theorem 3.6], holds

**Theorem 3.1.** *Let  $\Omega \subset \mathbb{R}^N$  be a bounded strong Lipschitz domain,  $k \in \mathbb{N}_0$  and  $0 \leq q \leq N$ . Then there exists a universal extension operator with the following properties:*

$$\mathcal{E} : \mathbf{D}^{k,q}(\Omega) \rightarrow \mathbf{D}^{k,q}(\mathbb{R}^N)$$

satisfying

- (i)  $\mathcal{E}E = E$  a.e. in  $\Omega$ .

(ii) *The extension operator  $\mathcal{E}$  is continuous, i.e.*

$$\exists c > 0 \quad \forall E \in \mathbf{D}^{k,q}(\Omega) \quad |\mathcal{E}E|_{\mathbf{D}^{k,q}(\Omega)} \leq c |E|_{\mathbf{D}^{k,q}(\mathbb{R}^N)}.$$

*Note that  $c$  depends on  $\Omega, N, k$ , and  $q$ , but not on  $E$ .*

We start out with a density result for  $\mathbf{H}^{1,q}$ -forms, i.e., the strong and weak definitions of the boundary conditions coincide for  $\mathbf{H}^{1,q}$ -forms. This is an immediate consequence of the corresponding scalar result, whose proof can be found in [4, Lemma 2, Lemma 3] and with a simplified proof in [1, Lemma 3.1].

**Lemma 3.2.** *Let  $\Omega \subset \mathbb{R}^N$  be a bounded domain and  $(\Omega, \Gamma_\tau)$  be a weak Lipschitz pair as well as*

$$\mathring{\mathbf{H}}_{\Gamma_\tau}^{1,q}(\Omega) := \{u \in \mathbf{H}^{1,q}(\Omega) : u|_{\Gamma_\tau} = 0\}.$$

*Then  $\mathring{\mathbf{H}}_{\Gamma_\tau}^{1,q}(\Omega) = \mathring{\mathbf{H}}_{\Gamma_\tau}^{1,q}(\Omega) = \mathring{\mathbf{H}}_{\Gamma_\tau}^{1,q}(\Omega)$ .*

**3.1.  $\mathbf{H}^{1,q}$ -potentials without boundary conditions.** The next two lemmas ensure the existence of  $\mathbf{H}^{1,q}$ -potentials without boundary conditions.

**Lemma 3.3.** *Let  $\Omega \subset \mathbb{R}^N$  be a bounded strong Lipschitz domain. Then there exists a continuous linear operator*

$$\mathcal{T}_d : \mathbf{L}^{2,q}(\Omega) \rightarrow \mathbf{H}^{1,q-1}(\mathbb{R}^N) \cap \Delta_0^{q-1}(\mathbb{R}^N),$$

*such that for all  $E \in \mathbf{D}_0^q(\Omega) \cap \mathcal{H}_N^q(\Omega)^\perp$*

$$d(\mathcal{T}_d E) = E \text{ in } \Omega.$$

*Epecially  $\mathbf{D}_0^q(\Omega) = d \mathbf{H}^{1,q-1}(\Omega) = d(\mathbf{H}^{1,q-1}(\Omega) \cap \Delta_0^{q-1}(\Omega))$  and the ‘regular’ potential depends continuously on the data. Particularly, these are closed subspaces of  $\mathbf{L}^{2,q}(\Omega)$  and  $\mathcal{T}_d$  is a right inverse to  $d$ .*

Here  $\mathcal{H}_N^q(\Omega) := \mathbf{D}_0^q(\Omega) \cap \mathring{\Delta}_0^q(\Omega)$  are the harmonic Neumann forms and  $\cdot^\perp$  denotes orthogonality with respect to the  $\mathbf{L}^{2,q}(\Omega)$  scalar product.

*Proof.* Suppose  $E \in \mathbf{D}_0^q(\Omega) \cap \mathcal{H}_N^q(\Omega)^\perp(\Omega)$ . As  $\Omega$  is bounded and strong Lipschitz, we have by Helmholtz decomposition and with closed subspaces (see [9, Lemma 1, Lemma 3 and Korollar 3.2])

$$\begin{aligned} \mathbf{L}^{2,q}(\Omega) &= d \mathbf{D}^{q-1}(\Omega) \oplus \mathring{\Delta}_0^q(\Omega) = \mathbf{D}_0^q(\Omega) \oplus \delta \mathring{\Delta}^{q+1}(\Omega) \\ (8) \quad &= d \mathbf{D}^{q-1}(\Omega) \oplus \mathcal{H}_N^q(\Omega) \oplus \delta \mathring{\Delta}^{q+1}(\Omega), \end{aligned}$$

where  $\oplus$  denotes the orthogonal sum in  $\mathbf{L}^{2,q}(\Omega)$ . Hence there exists an  $H \in \mathbf{D}^{q-1}(\Omega)$  with  $dH = E$  in  $\Omega$ . Let  $\pi_\Omega$  be the Helmholtz projector onto  $\delta \mathring{\Delta}^q(\Omega)$ . Then we have  $\pi_\Omega H \in \mathbf{D}^{q-1}(\Omega) \cap \delta \mathring{\Delta}^q(\Omega)$  with  $d\pi_\Omega H = dH = E$  and by the Friedrichs-Poincaré-type estimate, see [9, Lemma 2]

$$|\pi_\Omega H|_{\mathbf{L}^{2,q-1}(\Omega)} \leq c |d\pi_\Omega H|_{\mathbf{L}^{2,q}(\Omega)} = c |E|_{\mathbf{L}^{2,q}(\Omega)}.$$

Let  $\mathcal{E} : \mathbf{D}^{q-1}(\Omega) \rightarrow \mathbf{D}^{q-1}(\mathbb{R}^N)$  be the Stein extension operator from [3], i.e.,  $\mathcal{E}(\pi_\Omega H) \in \mathbf{D}^{q-1}(\mathbb{R}^N)$  with compact support. Projecting again, now onto  $\Delta_0^{q-1}(\mathbb{R}^N)$ , we obtain a form

$$\pi_{\mathbb{R}^N} \mathcal{E}(\pi_\Omega H) \in \mathbf{D}^{q-1}(\mathbb{R}^N) \cap \Delta_0^{q-1}(\mathbb{R}^N).$$

Using regularity in the whole space, see [5], we conclude

$$\pi_{\mathbb{R}^N} \mathcal{E}(\pi_\Omega H) \in \mathbf{H}^{1,q-1}(\mathbb{R}^N) \cap \Delta_0^{q-1}(\mathbb{R}^N)$$

and  $d\pi_{\mathbb{R}^N} \mathcal{E}(\pi_\Omega H) = d\mathcal{E}(\pi_\Omega H) = d\pi_\Omega H = dH = E$  in  $\Omega$ .  $\square$

By Hodge- $\star$ -duality we get a corresponding result for the  $\delta$ -operator.

**Lemma 3.4.** *Let  $\Omega \subset \mathbb{R}^N$  be a bounded strong Lipschitz domain. Then there exists a continuous linear operator*

$$\mathcal{T}_\delta : \mathbf{L}^{2,q}(\Omega) \rightarrow \mathbf{H}^{1,q+1}(\mathbb{R}^N) \cap \mathbf{D}_0^{q+1}(\mathbb{R}^N),$$

*such that for all  $H \in \Delta_0^q(\Omega) \cap \mathcal{H}_D(\Omega)^\perp$*

$$\delta(\mathcal{T}_\delta H) = H \text{ in } \Omega.$$

*Epecially  $\Delta_0^q(\Omega) = \delta \mathbf{H}^{1,q+1}(\Omega) = \delta(\mathbf{H}^{1,q+1}(\Omega) \cap \mathbf{D}_0^{q+1}(\Omega))$  and the ‘regular’ potential depends continuously on the data. In particular these are closed subspaces of  $\mathbf{L}^{2,q}(\Omega)$  and  $\mathcal{T}_\delta$  is a right inverse to  $\delta$ .*

Here  $\mathcal{H}_D^q(\Omega) := \mathring{\mathbf{D}}_0^q(\Omega) \cap \Delta_0^q(\Omega)$  are the harmonic Dirichlet forms.



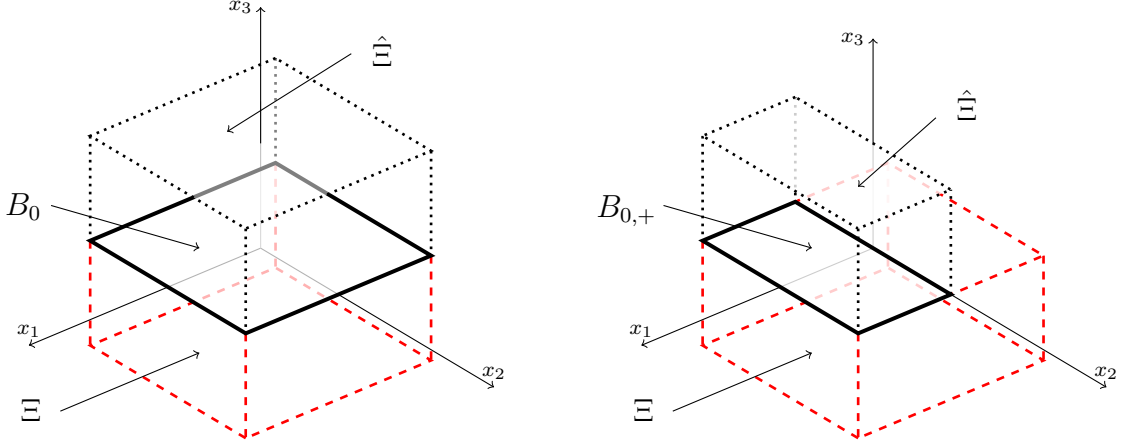


FIGURE 2. The half-cube  $\Xi = B_-$ , extended by  $\hat{\Xi}$  to the polygonal domain  $\tilde{\Xi}$ , and the rectangles  $\gamma_\nu = B_0$  and  $\gamma_\nu = B_{0,+}$ .

**3.2.  $H^{1,q}$ -potentials with boundary conditions on the half cube.** Now we start constructing  $H^{1,q}$ -potentials on  $\Xi$  with boundary conditions. Let us recall our special setting on the half-cube

$$\Xi = B_- \quad \text{and} \quad \gamma_\nu = \emptyset, \quad \gamma_\nu = B_0 \quad \text{or} \quad \gamma_\nu = B_{0,+}.$$

Furthermore, cf. Figure 2, we extend  $\Xi$  over  $\gamma_\nu$  by

$$\tilde{\Xi} = \text{int}(\Xi \cup \hat{\Xi}), \quad \hat{\Xi} := \begin{cases} \{x \in B : x_N > 0\} = B_+ & , \text{ if } \gamma_\nu = B_0, \\ \{x \in B : x_N > 0, x_1 > 0\} = \{x \in B_+ : x_1 > 0\} =: B_{+,+} & , \text{ if } \gamma_\nu = B_{0,+}. \end{cases}$$

**Theorem 3.5.** *There exists a continuous linear operator*

$$\mathcal{S}_d : \mathring{\mathbf{D}}_{\gamma_\nu,0}^q(\Xi) \rightarrow H^{1,q-1}(\mathbb{R}^3) \cap \mathring{H}_{\gamma_\nu}^{1,q-1}(\Xi),$$

such that for all  $H \in \mathring{\mathbf{D}}_{\gamma_\nu,0}^q(\Xi)$

$$d(\mathcal{S}_d H) = H \quad \text{in } \Xi.$$

*Epecially  $\mathring{\mathbf{D}}_{\gamma_\nu,0}^q(\Xi) = \mathring{\mathbf{D}}_{\gamma_\nu,0}^q(\Xi) = d \mathring{H}_{\gamma_\nu}^{1,q-1}(\Xi) = d \mathring{D}_{\gamma_\nu}^{q-1}(\Xi) = d \mathring{\mathbf{D}}_{\gamma_\nu}^{q-1}$  and the ‘regular’  $\mathring{H}_{\gamma_\nu}^{1,q-1}(\Xi)$ -potential depends continuously on the data. In particular these are closed subspaces of  $L^{2,q}(\Xi)$  and  $\mathcal{S}_d$  is a right inverse to  $d$ .*

*Proof.* The case  $\gamma_\nu = \emptyset$  is done in Lemma 3.3. Hence let  $\gamma_\nu = B_0$  or  $\gamma_\nu = B_{0,+}$ . Suppose  $H \in \mathring{\mathbf{D}}_{\gamma_\nu,0}^q(\Xi)$  and define  $\tilde{H} \in L^{2,q}(\tilde{\Xi})$  by

$$(9) \quad \tilde{H} := \begin{cases} H & \text{in } \Xi, \\ 0 & \text{in } \hat{\Xi}. \end{cases}$$

It follows  $d\tilde{H} = 0$  in  $\tilde{\Xi}$ , i.e.  $\tilde{H} \in \mathring{D}_0^q(\tilde{\Xi})$ . Because  $\tilde{\Xi}$  is topologically trivial, Lemma 3.3 yields  $E = \mathcal{T}_d \tilde{H} \in H^{1,q-1}(\mathbb{R}^N) \cap \mathring{D}_0^{q-1}(\mathbb{R}^N)$  with  $dE = \tilde{H}$  in  $\tilde{\Xi}$ . In particular  $E \in H^{1,q-1}(\hat{\Xi})$  and  $dE = 0$  in  $\hat{\Xi}$ . Using Lemma 3.3 again, we obtain  $F \in H^{1,q-2}(\hat{\Xi})$  with

$$dF = E \quad \text{in } \hat{\Xi}.$$

Since  $E \in H^{1,q-1}(\hat{\Xi})$  we have  $F \in \mathring{D}^{1,q-2}(\hat{\Xi})$ . Let  $\mathcal{E} : \mathring{D}^{1,q-2}(\hat{\Xi}) \rightarrow \mathring{D}^{1,q-2}(\mathbb{R}^N)$  again be the Stein extension operator. Then

$$\mathcal{S}_d : \begin{array}{ccc} \mathring{\mathbf{D}}_{\gamma_\nu,0}^q(\Xi) & \longrightarrow & H^{1,q-1}(\mathbb{R}^N) \\ H & \longmapsto & E - d(\mathcal{E}F) \end{array}$$

is linear and continuous. Since  $\mathcal{S}_d H = 0$  in  $\hat{\Xi}$ , we have  $\mathcal{S}_d H|_{\gamma_\nu} = 0$ , which means  $\mathcal{S}_d H \in \mathring{H}_{\gamma_\nu}^{1,q-1}(\Xi)$ .

Hence  $\mathcal{S}_d H \in \mathring{H}_{\gamma_\nu}^{1,q-1}(\Xi) \subset \mathring{\mathbf{D}}_{\gamma_\nu}^{q-1}(\Xi) \subset \mathring{\mathbf{D}}_{\gamma_\nu}^{q-1}(\Xi)$  by Lemma 3.2. Moreover

$$d(\mathcal{S}_d H) = H \quad \text{in } \Xi,$$

and we immediately see

$$d\mathring{H}_{\gamma_\nu}^{1,q-1}(\Xi) \subset d\mathring{D}_{\gamma_\nu}^{q-1}(\Xi) \subset \mathring{D}_{\gamma_\nu,0}^q(\Xi) \subset \mathring{\mathbf{D}}_{\gamma_\nu,0}^q(\Xi) \subset d\mathring{H}_{\gamma_\nu}^{1,q-1}(\Xi),$$

completing the proof.  $\square$

Again by Hodge- $\star$ -duality, the following theorem follows.

**Theorem 3.6.** *There exists a continuous linear operator*

$$\mathcal{S}_\delta : \mathring{\mathbf{D}}_{\gamma_\nu,0}^q(\Xi) \rightarrow \mathbf{H}^{1,q+1}(\mathbb{R}^N) \cap \mathring{H}_{\gamma_\nu}^{1,q+1}(\Xi),$$

such that for all  $H \in \mathring{\mathbf{D}}_{\gamma_\nu,0}^q(\Xi)$

$$\delta(\mathcal{S}_\delta H) = H \quad \text{in } \Xi.$$

Especially  $\mathring{\mathbf{D}}_{\gamma_\nu,0}^q(\Xi) = \mathring{\mathbf{D}}_{\gamma_\nu,0}^q(\Xi) = \delta\mathring{H}_{\gamma_\nu}^{1,q+1}(\Xi) = \delta\mathring{\mathbf{D}}_{\gamma_\nu}^{q+1}(\Xi) = \delta\mathring{\mathbf{A}}_{\gamma_\nu}^{q+1}$  and the ‘regular’  $\mathring{H}_{\gamma_\nu}^{1,q+1}(\Xi)$ -potential depends continuously on the data. In particular these are closed subspaces of  $\mathbf{L}^{2,q}(\Xi)$  and  $\mathcal{S}_\delta$  is a right inverse to  $\delta$ .

**Remark 3.7.** *Inspection of the above proof shows that the latter theorem holds for more general domains. Let  $\Omega \subset \mathbb{R}^N$  be a bounded strong Lipschitz domain, such that  $\mathbb{R}^N \setminus \bar{\Omega}$  is connected, and let  $\Gamma_\nu = \bigcup_{k=1}^K \Gamma_{\nu,k}$ ,  $K \in \mathbb{N}$ , with disjoint, relatively open and simply connected strong Lipschitz surface patches  $\Gamma_{\nu,k} \subset \Gamma$ , where  $\text{dist}(\Gamma_{\nu,k}, \Gamma_{\nu,\ell}) > 0$  for all  $1 \leq k \neq \ell \leq K$ . Now extend  $\Omega$  over  $\Gamma_{\nu,k}$  by  $\hat{\Omega}_k$ , let  $\tilde{\Omega}$  denote the interior of  $\bar{\Omega} \cup \bar{\tilde{\Omega}}_1 \cup \dots \cup \bar{\tilde{\Omega}}_K$  and define  $\tilde{H}$  like in (9). Then  $\tilde{H} \in \mathring{D}_0^q(\tilde{\Omega})$ . Lemma 3.3 yields  $\mathcal{T}_d \tilde{H} \in \mathbf{H}^{1,q-1}(\mathbb{R}^N) \cap \mathring{\Delta}_0^{q-1}(\mathbb{R}^N)$  with  $d(\mathcal{T}_d \tilde{H}) = \tilde{H}$  in  $\tilde{\Omega}$ . Again  $d(\mathcal{T}_d \tilde{H}) = 0$  in  $\hat{\Omega}_k$  for  $k = 1, \dots, K$ . Continuing analogously and since the  $\hat{\Omega}_k$  are simply connected, there exist unique potentials  $F_1, \dots, F_K \in \mathbf{H}^{1,q-2}(\hat{\Omega}_k)$  with  $\mathcal{T}_d \tilde{H} = dF_k$  in  $\hat{\Omega}_k$ . As before  $F_k \in \mathbf{D}^{1,q-2}(\hat{\Omega}_k)$ . Let  $\mathcal{E}_k : \mathbf{D}^{1,q-2}(\hat{\Omega}_k) \rightarrow \mathbf{D}^{1,q-2}(\mathbb{R}^N)$ ,  $k = 1, \dots, K$ , be Stein extension operators. By cutting off appropriately it can be arranged that  $\text{supp}(\mathcal{E}_k \varphi_k) \cap \bar{\tilde{\Omega}}_\ell = \emptyset$  for all  $1 \leq k \neq \ell \leq K$ . We define*

$$\mathcal{S}_d H := \mathcal{T}_d \tilde{H} - \sum_{k=1}^K d(\mathcal{E}_k F_k) \in \mathbf{H}^{1,q-1}(\mathbb{R}^N).$$

Again from  $\mathcal{S}_d H = 0$  in  $\hat{\Omega}_k$ ,  $k = 1, \dots, K$ ,  $\mathcal{S}_d H|_{\Gamma_\nu} = 0$  follows, which means  $\mathcal{S}_d H \in \mathring{H}_{\Gamma_\nu}^{1,q-1}(\Omega)$  and therefore

$$\mathcal{S}_d H \in \mathring{H}_{\Gamma_\nu}^{1,q-1}(\Omega) \subset \mathring{D}_{\Gamma_\nu}^{q-1}(\Omega) \subset \mathring{\mathbf{D}}_{\Gamma_\nu}^{q-1}(\Omega).$$

Moreover  $d(\mathcal{S}_d H) = H$  in  $\Omega$ , as  $d(\mathcal{S}_d H) = d(\mathcal{T}_d \tilde{H}) = \tilde{H}$  even in  $\tilde{\Omega}$ .

**3.3. Weak equals strong for the half-cube in terms of boundary conditions.** Now the two main density results immediately follow. We note that this has already been proved for the  $\mathbf{H}^{1,q}(\Omega)$ -spaces in Lemma 3.2, i.e.,  $\mathring{\mathbf{H}}_{\Gamma_\tau}^{1,q}(\Omega) = \mathring{H}_{\Gamma_\tau}^{1,q}(\Omega)$ .

**Theorem 3.8.**  $\mathring{D}_{\gamma_\nu}^q(\Xi) = \mathring{D}_{\gamma_\nu}^q(\Xi)$  and  $\mathring{\mathbf{D}}_{\gamma_\nu}^q(\Xi) = \mathring{\mathbf{D}}_{\gamma_\nu}^q(\Xi)$ .

*Proof.* Suppose  $E \in \mathring{\mathbf{D}}_{\gamma_\nu}^q(\Xi)$  and thus  $dE \in \mathring{\mathbf{D}}_{\gamma_\nu,0}^{q+1}(\Xi)$ . By Theorem 3.5 there exists  $H = \mathcal{S}_d dE \in \mathring{H}_{\gamma_\nu}^{1,q}(\Xi)$  with  $dH = dE$ . By Theorem 3.5 we get  $E - H \in \mathring{\mathbf{D}}_{\gamma_\nu,0}^q(\Xi) = \mathring{D}_{\gamma_\nu,0}^q(\Xi)$  and hence  $E \in \mathring{D}_{\gamma_\nu}^q(\Xi)$ .  $\square$

#### 4. THE COMPACT EMBEDDING

**4.1. Compact embedding on the half-cube.** First we show the main result on the half-cube  $\Xi = B_-$  with the special boundary patch

$$\gamma_\nu = \emptyset, \quad \gamma_\nu = B_0 \quad \text{or} \quad \gamma_\nu = B_{0,+}$$

from the latter section. To this end we consider the densely defined and closed unbounded linear operator

$$d := d_\tau^{q-1} : \mathring{D}_{\gamma_\tau}^{q-1}(\Xi) \subset \mathbf{L}^{2,q-1}(\Xi) \rightarrow \mathbf{L}_\varepsilon^{2,q}(\Xi)$$

and its adjoint

$$-\delta^* := -\delta_\nu^q = (d_\tau^{q-1})^* : \varepsilon^{-1} \mathring{\mathbf{D}}_{\gamma_\nu}^q(\Xi) \subset \mathbf{L}_\varepsilon^{2,q}(\Xi) \rightarrow \mathbf{L}^{2,q-1}(\Xi).$$

Note that by Theorem 3.8 we have  $\mathring{\mathbf{D}}_{\gamma_\nu}^q(\Xi) = \mathring{\mathbf{D}}_{\gamma_\nu}^q(\Xi)$ . Here,  $\mathbf{L}_\varepsilon^{2,q}(\Xi)$  denotes  $\mathbf{L}^{2,q}(\Xi)$  equipped with the inner product  $\langle \cdot, \cdot \rangle_{\mathbf{L}_\varepsilon^{2,q}(\Xi)} = \langle \varepsilon \cdot, \cdot \rangle_{\mathbf{L}^{2,q}(\Xi)}$ . Let  $\oplus_\varepsilon$  denote the orthogonal sum with respect to the  $\mathbf{L}_\varepsilon^{2,q}$ -scalar product. The projection theorem yields

**Lemma 4.1.**  $\mathbf{L}^{2,q}(\Xi) = \mathbf{d} \mathring{\mathbf{D}}_{\gamma_\tau}^{q-1}(\Xi) \oplus_\varepsilon \varepsilon^{-1} \mathring{\mathbf{\Delta}}_{\gamma_\nu,0}^q(\Xi) = \mathbf{d} \mathring{\mathbf{H}}_{\gamma_\tau}^{1,q-1}(\Xi) \oplus_\varepsilon \varepsilon^{-1} \delta \mathring{\mathbf{H}}_{\gamma_\nu}^{1,q+1}(\Xi)$

*Proof.* By the projection theorem

$$\mathbf{L}^{2,q}(\Xi) = \overline{\mathbf{d} \mathring{\mathbf{D}}_{\gamma_\tau}^{q-1}(\Xi)} \oplus_\varepsilon \varepsilon^{-1} \mathring{\mathbf{\Delta}}_{\gamma_\nu,0}^q(\Xi).$$

It holds  $\overline{\mathbf{d} \mathring{\mathbf{D}}_{\gamma_\tau}^{q-1}(\Xi)} = \mathbf{d} \mathring{\mathbf{D}}_{\gamma_\tau}^{q-1}(\Xi) = \mathbf{d} \mathring{\mathbf{H}}_{\gamma_\tau}^{1,q-1}(\Xi)$  by Theorem 3.5 and  $\mathring{\mathbf{\Delta}}_{\gamma_\nu,0}^q(\Xi) = \mathring{\mathbf{\Delta}}_{\gamma_\nu,0}^q(\Xi) = \delta \mathring{\mathbf{H}}_{\gamma_\nu}^{1,q+1}(\Xi)$  by Theorem 3.6.  $\square$

**Theorem 4.2.** *The embedding  $\mathring{\mathbf{D}}_{\gamma_\tau}^q(\Xi) \cap \varepsilon^{-1} \mathring{\mathbf{\Delta}}_{\gamma_\nu}^q(\Xi) \hookrightarrow \mathbf{L}^{2,q}(\Xi)$  is compact.*

*Proof.* Let  $(H_n)_{n \in \mathbb{N}}$  be a bounded sequence in  $\mathring{\mathbf{D}}_{\gamma_\tau}^q(\Xi) \cap \varepsilon^{-1} \mathring{\mathbf{\Delta}}_{\gamma_\nu}^q(\Xi)$ . By Lemma 4.1 we can decompose

$$H_n = H_n^{\mathbf{d}} + H_n^\delta = \mathbf{d} E_n^{\mathbf{d}} + \varepsilon^{-1} \delta E_n^\delta \in (\mathbf{d} \mathring{\mathbf{H}}_{\gamma_\tau}^{1,q-1}(\Xi) \cap \varepsilon^{-1} \mathring{\mathbf{\Delta}}_{\gamma_\nu}^q(\Xi)) \oplus_\varepsilon (\varepsilon^{-1} \delta \mathring{\mathbf{H}}_{\gamma_\nu}^{1,q+1}(\Xi) \cap \mathring{\mathbf{D}}_{\gamma_\tau}^q(\Xi)),$$

with  $\mathbf{d} H_n^\delta = \mathbf{d} H_n$  and  $\delta \varepsilon H_n^{\mathbf{d}} = \delta \varepsilon H_n$ . Furthermore, we can estimate

$$\begin{aligned} |E_n^{\mathbf{d}}|_{\mathbf{H}^{1,q-1}(\Xi)} &\leq c |H_n^{\mathbf{d}}|_{\mathbf{L}^{2,q}(\Xi)} \leq c |H_n|_{\mathbf{L}^{2,q}(\Xi)}, \\ |E_n^\delta|_{\mathbf{H}^{1,q+1}(\Xi)} &\leq c |H_n^\delta|_{\mathbf{L}^{2,q}(\Xi)} \leq c |H_n|_{\mathbf{L}^{2,q}(\Xi)}. \end{aligned}$$

By Rellich's selection theorem  $(E_n^{\mathbf{d}})$  and  $(E_n^\delta)$  w.l.o.g. converge in  $\mathbf{L}^{2,q-1}(\Xi)$  and  $\mathbf{L}^{2,q+1}(\Xi)$  respectively and

$$\begin{aligned} |H_n^{\mathbf{d}} - H_m^{\mathbf{d}}|_{\mathbf{L}_\varepsilon^{2,q}(\Xi)}^2 &= \langle H_n^{\mathbf{d}} - H_m^{\mathbf{d}}, \mathbf{d}(E_n^{\mathbf{d}} - E_m^{\mathbf{d}}) \rangle_{\mathbf{L}_\varepsilon^{2,q}(\Xi)} \\ &= -\langle \delta \varepsilon (H_n^{\mathbf{d}} - H_m^{\mathbf{d}}), E_n^{\mathbf{d}} - E_m^{\mathbf{d}} \rangle_{\mathbf{L}^{2,q-1}(\Xi)} \leq c |E_n^{\mathbf{d}} - E_m^{\mathbf{d}}|_{\mathbf{L}^{2,q-1}(\Xi)}. \end{aligned}$$

Thus  $(H_n^{\mathbf{d}})$  converges in  $\mathbf{L}^{2,q}(\Xi)$  and an analogous computation shows the convergence of  $(H_n^\delta)$ . Altogether  $(H_n)$  converges in  $\mathbf{L}^{2,q}(\Xi)$ .  $\square$

**Remark 4.3.** *The use of Helmholtz decompositions and regular potentials in the proof of Theorem 4.2 demonstrates the main idea behind an elegant proof of a compact embedding. This general idea carries over to proofs of compact embeddings related to other kinds of Hilbert complexes as well.*

**4.2. The compact embedding for weak Lipschitz domains.** The aim of this section is to transfer Theorem 4.2 to arbitrary weak Lipschitz pairs  $(\Omega, \Gamma_\tau)$ . To this end we will employ a technical lemma, whose proof is sketched in [10, Section 3] and [14, Remark 2]. We give a detailed proof in the appendix. Let us consider the following situation: Let  $\Theta, \tilde{\Theta}$  be two domains in  $\mathbb{R}^N$  with boundaries  $\Upsilon := \partial\Theta$ ,  $\tilde{\Upsilon} := \partial\tilde{\Theta}$  and  $\Upsilon_0 \subset \Upsilon$  relatively open. Moreover, let

$$\phi : \Theta \rightarrow \tilde{\Theta}, \quad \psi := \phi^{-1} : \tilde{\Theta} \rightarrow \Theta$$

be Lipschitz diffeomorphisms, this is,  $\phi \in C^{0,1}(\Theta, \tilde{\Theta})$  and  $\psi = \phi^{-1} \in C^{0,1}(\tilde{\Theta}, \Theta)$ . Then  $\tilde{\Theta} = \phi(\Theta)$ ,  $\tilde{\Upsilon} = \phi(\Upsilon)$  and we define  $\tilde{\Upsilon}_0 := \phi(\Upsilon_0)$ .

**Lemma 4.4.** *Let  $E \in \mathring{\mathbf{D}}_{\Upsilon_0}^q(\Theta)$  resp.  $E \in \mathring{\mathbf{D}}_{\tilde{\Upsilon}_0}^q(\tilde{\Theta})$  and  $H \in \varepsilon^{-1} \mathring{\mathbf{\Delta}}_{\Upsilon_0}^q(\Theta)$  resp.  $H \in \varepsilon^{-1} \mathring{\mathbf{\Delta}}_{\tilde{\Upsilon}_0}^q(\tilde{\Theta})$ . Then*

$$\begin{aligned} \psi^* E &\in \mathring{\mathbf{D}}_{\tilde{\Upsilon}_0}^q(\tilde{\Theta}) \text{ resp. } \mathring{\mathbf{D}}_{\Upsilon_0}^q(\Theta) & \text{ and } & \quad \mathbf{d} \psi^* E = \psi^* \mathbf{d} E, \\ \psi^* H &\in \mu^{-1} \mathring{\mathbf{\Delta}}_{\tilde{\Upsilon}_0}^q(\tilde{\Theta}) \text{ resp. } \mu^{-1} \mathring{\mathbf{\Delta}}_{\Upsilon_0}^q(\Theta) & \text{ and } & \quad \delta \mu \psi^* H = \pm \star \mathbf{d} \psi^* \star \varepsilon H = \pm \star \psi^* \star \delta \varepsilon H, \end{aligned}$$

where  $\mu := (-1)^{qN-1} \star \psi^* \star \varepsilon \phi^*$  is an admissible transformation. Moreover, there exists  $c > 0$ , independent of  $E$  and  $H$ , such that

$$|\psi^* E|_{\mathbf{D}^q(\tilde{\Theta})} \leq c |E|_{\mathbf{D}^q(\Theta)}, \quad |\psi^* H|_{\mu^{-1} \mathring{\mathbf{\Delta}}^q(\tilde{\Theta})} \leq c |H|_{\varepsilon^{-1} \mathring{\mathbf{\Delta}}^q(\Theta)}.$$

From now on we make the following

**General Assumption:** Let  $(\Omega, \Gamma_\tau)$  be a weak Lipschitz pair as in Definitions 2.1 and 2.3. In particular,  $\Omega$  is bounded.

We adjust Lemma 4.4 to our situation: Let  $U_1, \dots, U_K$  be an open covering of  $\Gamma$  according to Definitions 2.1 and 2.3 and set  $U_0 := \Omega$ . Therefore  $U_0, \dots, U_K$  is an open covering of  $\bar{\Omega}$ . Moreover let  $\chi_k \in \mathring{C}^\infty(U_k)$ ,  $k \in \{0, \dots, K\}$ , be a partition of unity subordinate to the open covering  $U_0, \dots, U_K$ . Now suppose  $k \in \{1, \dots, K\}$ . We define

$$\Omega_k := U_k \cap \Omega, \quad \Gamma_k := U_k \cap \Gamma, \quad \Gamma_{\tau,k} := U_k \cap \Gamma_\tau, \quad \Gamma_{\nu,k} := U_k \cap \Gamma_\nu,$$

$$\begin{aligned}\hat{\Gamma}_k &:= \partial\Omega_k, & \Sigma_k &:= \hat{\Gamma}_k \setminus \Gamma, & \hat{\Gamma}_{\tau,k} &:= \text{int}(\Gamma_{\tau,k} \cup \bar{\Sigma}_k), & \hat{\Gamma}_{\nu,k} &:= \text{int}(\Gamma_{\nu,k} \cup \bar{\Sigma}_k), \\ & & \sigma &:= \gamma \setminus \bar{B}_0, & \hat{\gamma}_\tau &:= \text{int}(\gamma_\tau \cup \bar{\sigma}), & \hat{\gamma}_\nu &:= \text{int}(\gamma_\nu \cup \bar{\sigma}).\end{aligned}$$

Lemma 4.4 will from now on be used with

$$\Theta := \Omega_k, \quad \tilde{\Theta} := \Xi, \quad \phi := \phi_k : \Omega_k \rightarrow \Xi, \quad \psi := \psi_k : \Xi \rightarrow \Omega_k$$

and with one of the following cases

$$\Upsilon_0 := \Gamma_{\tau,k}, \quad \Upsilon_0 := \hat{\Gamma}_{\tau,k}, \quad \Upsilon_0 := \Gamma_{\nu,k} \quad \text{or} \quad \Upsilon_0 := \hat{\Gamma}_{\nu,k}.$$

Then  $\Upsilon = \hat{\Gamma}_k$  and  $\tilde{\Upsilon} = \phi_k(\hat{\Gamma}_k) = \gamma$  as well as (depending on the respective case)

$$\begin{aligned}\tilde{\Upsilon}_0 = \phi_k(\Gamma_{\tau,k}) &= \gamma_\tau, & \tilde{\Upsilon}_0 = \phi_k(\hat{\Gamma}_{\tau,k}) &= \hat{\gamma}_\tau, & \gamma_\tau &\in \{\emptyset, B_0, B_{0,-}\}, & \gamma_\nu &= \gamma \setminus \bar{\gamma}_\tau, \\ \tilde{\Upsilon}_0 = \phi_k(\Gamma_{\nu,k}) &= \gamma_\nu, & \tilde{\Upsilon}_0 = \phi_k(\hat{\Gamma}_{\nu,k}) &= \hat{\gamma}_\nu, & \gamma_\nu &\in \{\emptyset, B_0, B_{0,+}\}, & \gamma_\tau &= \gamma \setminus \bar{\gamma}_\nu.\end{aligned}$$

**Remark 4.5.** *Theorems 3.5, 3.6, Remark 3.7, as well as Theorems 3.8, 4.2 hold for  $\gamma_\nu = B_{0,-}$  without any (substantial) modification as well.*

It is straightforward to show

**Lemma 4.6.** *Let  $k \in \{1, \dots, K\}$ . For  $E \in \mathring{\mathbf{D}}_{\Gamma_\tau}^q(\Omega)$  and  $H \in \mathring{\mathbf{\Delta}}_{\Gamma_\nu}^q(\Omega)$  we have*

$$E \in \mathring{\mathbf{D}}_{\Gamma_{\tau,k}}^q(\Omega_k), \quad \chi_k E \in \mathring{\mathbf{D}}_{\hat{\Gamma}_{\tau,k}}^q(\Omega_k), \quad H \in \mathring{\mathbf{\Delta}}_{\Gamma_{\nu,k}}^q(\Omega_k), \quad \chi_k H \in \mathring{\mathbf{\Delta}}_{\hat{\Gamma}_{\nu,k}}^q(\Omega_k).$$

**Theorem 4.7.**  $\mathring{\mathbf{D}}_{\Gamma_\tau}^q(\Omega) = \mathring{\mathbf{D}}_{\Gamma_\tau}^q(\Omega)$  and  $\mathring{\mathbf{\Delta}}_{\Gamma_\nu}^q(\Omega) = \mathring{\mathbf{\Delta}}_{\Gamma_\nu}^q(\Omega)$ .

*Proof.* Suppose  $E \in \mathring{\mathbf{D}}_{\Gamma_\tau}^q(\Omega)$ . Then  $\chi_0 E \in \mathring{\mathbf{D}}^q(\Omega) \subset \mathring{\mathbf{D}}_{\Gamma_\tau}^q(\Omega)$  by mollification. Let  $k \in \{1, \dots, K\}$ . Then  $E \in \mathring{\mathbf{D}}_{\Gamma_{\tau,k}}^q(\Omega_k)$  by Lemma 4.6. Lemma 4.4, Theorem 3.8 (with  $\gamma_\nu := \gamma_\tau$ ) and Remark 4.5 yield

$$\psi_k^* E \in \mathring{\mathbf{D}}_{\gamma_\tau}^q(\Xi) = \mathring{\mathbf{D}}_{\gamma_\tau}^q(\Xi), \quad \gamma_\tau = \phi_k(\Gamma_{\tau,k}) \in \{\emptyset, B_0, B_{0,-}\}.$$

Then  $E = \phi_k^* \psi_k^* E \in \mathring{\mathbf{D}}_{\Gamma_{\tau,k}}^q(\Omega_k)$  and thus  $\chi_k E \in \mathring{\mathbf{D}}_{\hat{\Gamma}_{\tau,k}}^q(\Omega_k) \subset \mathring{\mathbf{D}}_{\Gamma_\tau}^q(\Omega)$ .

Hence  $E = \sum_k \chi_k E \in \mathring{\mathbf{D}}_{\Gamma_\tau}^q(\Omega)$ .  $\mathring{\mathbf{\Delta}}_{\Gamma_\nu}^q(\Omega) = \mathring{\mathbf{\Delta}}_{\Gamma_\nu}^q(\Omega)$  follows analogously or by Hodge- $\star$ -duality.  $\square$

**Remark 4.8.** *By Theorem 4.7, Lemma 4.6 also holds for the spaces  $\mathring{\mathbf{D}}_{\Gamma_\tau}^q(\Omega)$  and  $\mathring{\mathbf{\Delta}}_{\Gamma_\nu}^q(\Omega)$ . More precisely, for  $E \in \mathring{\mathbf{D}}_{\Gamma_\tau}^q(\Omega)$  and  $H \in \mathring{\mathbf{\Delta}}_{\Gamma_\nu}^q(\Omega)$  we have for  $k \in \{1, \dots, K\}$*

$$E \in \mathring{\mathbf{D}}_{\Gamma_{\tau,k}}^q(\Omega_k), \quad \chi_k E \in \mathring{\mathbf{D}}_{\hat{\Gamma}_{\tau,k}}^q(\Omega_k), \quad H \in \mathring{\mathbf{\Delta}}_{\Gamma_{\nu,k}}^q(\Omega_k), \quad \chi_k H \in \mathring{\mathbf{\Delta}}_{\hat{\Gamma}_{\nu,k}}^q(\Omega_k).$$

Now the compact embedding for weak Lipschitz pairs  $(\Omega, \Gamma_\tau)$  can be proved.

**Theorem 4.9.** *Let  $\varepsilon \in \mathbf{L}^\infty(\Omega)$  be an admissible transformation on  $q$ -forms. Then the embedding*

$$\mathring{\mathbf{D}}_{\Gamma_\tau}^q(\Omega) \cap \varepsilon^{-1} \mathring{\mathbf{\Delta}}_{\Gamma_\nu}^q(\Omega) \hookrightarrow \mathbf{L}^{2,q}(\Omega)$$

*is compact.*

*Proof.* Suppose  $(E_n)$  is a bounded sequence in  $\mathring{\mathbf{D}}_{\Gamma_\tau}^q(\Omega) \cap \varepsilon^{-1} \mathring{\mathbf{\Delta}}_{\Gamma_\nu}^q(\Omega)$ . Then by mollification

$$E_{0,n} := \chi_0 E_n \in \mathring{\mathbf{D}}^q(\Omega) \cap \varepsilon^{-1} \mathring{\mathbf{\Delta}}^q(\Omega),$$

$E_{0,n}$  even has compact support in  $\Omega$ , and by classical results  $(E_{0,n})$ , see [12, 13, 10], contains an  $\mathbf{L}^{2,q}(\Omega)$ -converging subsequence, again denoted by  $(E_{0,n})$ . Hence  $E_{0,n} \rightarrow E_0$  in  $\mathbf{L}^{2,q}(\Omega)$  with some  $E_0 \in \mathbf{L}^{2,q}(\Omega)$ . Let  $k \in \{1, \dots, K\}$ . By Lemma 4.6 and Remark 4.8

$$E_{k,n} := \chi_k E_n \in \mathring{\mathbf{D}}_{\hat{\Gamma}_{\tau,k}}^q(\Omega_k), \quad \varepsilon E_{k,n} \in \mathring{\mathbf{\Delta}}_{\hat{\Gamma}_{\nu,k}}^q(\Omega_k)$$

and the sequence  $(E_{k,n})$  is bounded in  $\mathring{\mathbf{D}}_{\hat{\Gamma}_{\tau,k}}^q(\Omega_k) \cap \varepsilon^{-1} \mathring{\mathbf{\Delta}}_{\hat{\Gamma}_{\nu,k}}^q(\Omega_k)$  by the product rule. By Lemma 4.4

$$|\psi_k^* E_{k,n}|_{\mathbf{D}^q(\Xi)} \leq c |E_{k,n}|_{\mathbf{D}^q(\Omega_k)},$$

showing that  $(\psi_k^* E_{k,n})$  is bounded in  $\mathring{\mathbf{D}}_{\hat{\gamma}_\tau}^q(\Xi)$ . Analogously,  $(\psi_k^* E_{k,n})$  is bounded in  $\mu_k^{-1} \mathring{\mathbf{\Delta}}_{\hat{\gamma}_\nu}^q(\Xi)$  with the admissible transformation  $\mu_k := (-1)^{qN-1} \star \psi_k^* \star \varepsilon \phi_k^*$ . Thus  $(\psi_k^* E_{k,n})$  is bounded in

$$\mathring{\mathbf{D}}_{\hat{\gamma}_\tau}^q(\Xi) \cap \mu_k^{-1} \mathring{\mathbf{\Delta}}_{\hat{\gamma}_\nu}^q(\Xi) \subset \mathring{\mathbf{D}}_{\hat{\gamma}_\tau}^q(\Xi) \cap \mu_k^{-1} \mathring{\mathbf{\Delta}}_{\hat{\gamma}_\nu}^q(\Xi), \quad \gamma_\nu \in \{\emptyset, B_0, B_{0,+}\}, \quad \hat{\gamma}_\tau = \gamma \setminus \bar{\gamma}_\nu.$$

Therefore, w.l.o.g.  $\psi_k^* E_{k,n} \xrightarrow{n \rightarrow \infty} \hat{E}_k$  in  $L^{2,q}(\Xi)$  with some  $\hat{E}_k \in L^{2,q}(\Xi)$  by Theorem 4.2. Now

$$E_{k,n} \in L^{2,q}(\Omega_k), \quad E_k := \phi_k^* \hat{E}_k \in L^2(\Omega_k)$$

and Lemma 4.4 yields

$$|E_{k,n} - E_k|_{L^{2,q}(\Omega_k)} \leq c \left| \psi_k^* E_{k,n} - \hat{E}_k \right|_{L^{2,q}(\Xi)}.$$

Hence  $E_{k,n} \xrightarrow{n \rightarrow \infty} E_k$  in  $L^{2,q}(\Omega_k)$  and  $E_{k,n} \xrightarrow{n \rightarrow \infty} E_k$  in  $L^{2,q}(\Omega)$  for their extensions by zero to  $\Omega$ . Finally  $E_n = \sum_k \chi_k E_n = \sum_k E_{k,n} \xrightarrow{n \rightarrow \infty} \sum_k E_k$  in  $L^{2,q}(\Omega)$ .  $\square$

## 5. APPLICATIONS

From now on let  $\Omega \subset \mathbb{R}^N$  be a bounded domain and let  $(\Omega, \Gamma_\tau)$  be a weak Lipschitz pair as well as  $\varepsilon : L^{2,q}(\Omega) \rightarrow L^{2,q}(\Omega)$  be admissible. This sections' results immediately follow in the framework of a general functional analytic toolbox, see [7, 8].

**5.1. The Maxwell estimate.** A first consequence of the compact embedding Theorem 4.9, i.e.,

$$\mathring{D}_{\Gamma_\tau}^q(\Omega) \cap \varepsilon^{-1} \mathring{\Delta}_{\Gamma_\nu}^q(\Omega) \hookrightarrow L^{2,q}(\Omega)$$

is that the space of so-called 'Dirichlet-Neumann forms'

$$\mathcal{H}_\varepsilon^q(\Omega) := \mathring{D}_{\Gamma_\tau,0}^q(\Omega) \cap \varepsilon^{-1} \mathring{\Delta}_{\Gamma_\nu,0}^q(\Omega)$$

is finite dimensional because the unit ball in  $\mathcal{H}_\varepsilon^q(\Omega)$  is compact. By a standard indirect argument Theorem 4.9 immediately implies the so-called Maxwell estimate:

**Theorem 5.1.** *There exists a constant  $c_m > 0$ , such that for all  $E \in \mathring{D}_{\Gamma_\tau}^q(\Omega) \cap \varepsilon^{-1} \mathring{\Delta}_{\Gamma_\nu}^q(\Omega) \cap \mathcal{H}_\varepsilon^q(\Omega)^{\perp_\varepsilon}$*

$$|E|_{L_\varepsilon^{2,q}(\Omega)} \leq c_m \left( |\mathrm{d}E|_{L^{2,q+1}(\Omega)}^2 + |\delta(\varepsilon E)|_{L^{2,q-1}(\Omega)}^2 \right)^{1/2}.$$

Here we denote by  $\cdot^{\perp_\varepsilon}$  orthogonality with respect to  $L_\varepsilon^{2,q}(\Omega)$ -scalar product.

**5.2. Helmholtz decompositions.** Applying the projection theorem to the densely defined and closed unbounded linear operators

$$\mathrm{d} := \mathrm{d}_\tau^{q-1} : \mathring{D}_{\Gamma_\tau}^{q-1}(\Omega) \subset L^{2,q-1}(\Omega) \rightarrow L_\varepsilon^{2,q}(\Omega)$$

with adjoint, see Theorem 4.7,

$$-\delta^* := -\delta_\nu^q = (\mathrm{d}_\tau^{q-1})^* : \varepsilon^{-1} \mathring{\Delta}_{\Gamma_\nu}^q(\Omega) \subset L_\varepsilon^{2,q}(\Omega) \rightarrow L^{2,q-1}(\Omega)$$

and

$$\varepsilon^{-1} \delta := \varepsilon^{-1} \delta_\nu^{q+1} : \varepsilon^{-1} \mathring{\Delta}_{\Gamma_\nu}^{q+1}(\Omega) \subset L^{2,q+1}(\Omega) \rightarrow L_\varepsilon^{2,q}(\Omega)$$

with adjoint

$$-\mathrm{d} := -\mathrm{d}_\tau^q = (\varepsilon^{-1} \delta_\nu^{q+1})^* : \mathring{D}_{\Gamma_\tau}^q(\Omega) \subset L_\varepsilon^{2,q}(\Omega) \rightarrow L^{2,q+1}(\Omega)$$

we obtain

$$L^{2,q}(\Omega) = \overline{\mathrm{d} \mathring{D}_{\Gamma_\tau}^{q-1}(\Omega)} \oplus_\varepsilon \varepsilon^{-1} \mathring{\Delta}_{\Gamma_\nu,0}^q(\Omega)$$

and

$$L^{2,q}(\Omega) = \mathring{D}_{\Gamma_\tau,0}^q(\Omega) \oplus_\varepsilon \overline{\varepsilon^{-1} \delta \mathring{\Delta}_{\Gamma_\nu}^{q+1}(\Omega)}.$$

Hence  $\mathring{D}_{\Gamma_\tau,0}^q(\Omega) = \overline{\mathrm{d} \mathring{D}_{\Gamma_\tau}^{q-1}(\Omega)} \oplus_\varepsilon \mathcal{H}_\varepsilon^q(\Omega)$ , where  $\mathcal{H}_\varepsilon^q(\Omega) = \mathring{D}_{\Gamma_\tau,0}^q(\Omega) \cap \varepsilon^{-1} \mathring{\Delta}_{\Gamma_\nu,0}^q(\Omega)$ . Altogether

$$L^{2,q}(\Omega) = \overline{\mathrm{d} \mathring{D}_{\Gamma_\tau}^{q-1}(\Omega)} \oplus_\varepsilon \mathcal{H}_\varepsilon^q(\Omega) \oplus_\varepsilon \overline{\varepsilon^{-1} \delta \mathring{\Delta}_{\Gamma_\nu}^{q+1}(\Omega)}.$$

We arrive at

**Theorem 5.2.** *The following orthogonal decompositions hold:*

$$\begin{aligned} L^{2,q}(\Omega) &= \mathrm{d} \mathring{D}_{\Gamma_\tau}^{q-1}(\Omega) \oplus_\varepsilon \varepsilon^{-1} \mathring{\Delta}_{\Gamma_\nu,0}^q(\Omega) = \mathring{D}_{\Gamma_\tau,0}^q(\Omega) \oplus_\varepsilon \varepsilon^{-1} \delta \mathring{\Delta}_{\Gamma_\nu}^{q+1}(\Omega) \\ &= \mathrm{d} \mathring{D}_{\Gamma_\tau}^{q-1}(\Omega) \oplus_\varepsilon \mathcal{H}_\varepsilon^q(\Omega) \oplus_\varepsilon \varepsilon^{-1} \delta \mathring{\Delta}_{\Gamma_\nu}^{q+1}(\Omega). \end{aligned}$$

Furthermore

$$\begin{aligned} \mathrm{d} \mathring{D}_{\Gamma_\tau}^q(\Omega) &= \mathrm{d} \left( \mathring{D}_{\Gamma_\tau}^q(\Omega) \cap \varepsilon^{-1} \delta \mathring{\Delta}_{\Gamma_\nu}^{q+1}(\Omega) \right) = \mathrm{d} \left( \mathring{D}_{\Gamma_\tau}^q(\Omega) \cap \varepsilon^{-1} \mathring{\Delta}_{\Gamma_\nu,0}^q(\Omega) \cap \mathcal{H}_\varepsilon^q(\Omega)^{\perp_\varepsilon} \right), \\ \delta \mathring{\Delta}_{\Gamma_\nu}^q(\Omega) &= \delta \left( \mathring{\Delta}_{\Gamma_\nu}^q(\Omega) \cap \varepsilon \mathrm{d} \mathring{D}_{\Gamma_\tau}^{q-1}(\Omega) \right) = \delta \left( \mathring{\Delta}_{\Gamma_\nu}^q(\Omega) \cap \varepsilon \left( \mathring{D}_{\Gamma_\tau,0}^q(\Omega) \cap \mathcal{H}_\varepsilon^q(\Omega)^{\perp_\varepsilon} \right) \right) \end{aligned}$$

and

$$d\mathring{D}_{\Gamma_\tau}^{q-1}(\Omega) = \mathring{D}_{\Gamma_\tau,0}^q(\Omega) \cap \mathcal{H}_\varepsilon^q(\Omega)^{\perp\varepsilon}, \quad \delta\mathring{\Delta}_{\Gamma_\nu}^{q+1}(\Omega) = \mathring{\Delta}_{\Gamma_\nu,0}^q(\Omega) \cap \mathcal{H}_\varepsilon^q(\Omega)^\perp.$$

The ranges  $d\mathring{D}_{\Gamma_\tau}^{q-1}(\Omega)$  and  $\delta\mathring{\Delta}_{\Gamma_\nu}^{q+1}(\Omega)$  are closed subspaces of  $\mathbb{L}^{2,q}(\Omega)$ . Moreover, the  $d$ - resp.  $\delta$ -potentials are uniquely determined in  $\mathring{D}_{\Gamma_\tau}^q(\Omega) \cap \varepsilon^{-1}\mathring{\Delta}_{\Gamma_\nu,0}^q(\Omega) \cap \mathcal{H}_\varepsilon^q(\Omega)^{\perp\varepsilon}$  and  $\mathring{\Delta}_{\Gamma_\nu}^q(\Omega) \cap \varepsilon(\mathring{D}_{\Gamma_\tau,0}^q(\Omega) \cap \mathcal{H}_\varepsilon^q(\Omega)^{\perp\varepsilon})$ , respectively, and depend continuously on their respective images.

*Proof.* Theorem 5.1 implies closedness of the ranges and continuity of the potentials.  $\square$

**5.3. Static solution theory.** As a further application we turn to the boundary value problem of generalized electro- and magnetostatics with mixed boundary values: Let  $F \in \mathbb{L}^{2,q+1}(\Omega)$ ,  $G \in \mathbb{L}^{2,q-1}(\Omega)$ ,  $E_\tau \in \mathbb{D}^q(\Omega)$ ,  $E_\nu \in \varepsilon^{-1}\Delta^q(\Omega)$  and  $\varepsilon$  be admissible. Find  $E \in \mathbb{D}^q(\Omega) \cap \varepsilon^{-1}\Delta^q(\Omega)$  with

$$(10) \quad \begin{aligned} dE &= F, \\ \delta\varepsilon E &= G, \\ E - E_\tau &\in \mathring{D}_{\Gamma_\tau}^q(\Omega), \\ \varepsilon(E - E_\nu) &\in \mathring{\Delta}_{\Gamma_\nu}^q(\Omega). \end{aligned}$$

For uniqueness, we require the additional conditions

$$(11) \quad \langle \varepsilon E, D_\ell \rangle_{\mathbb{L}^{2,q}(\Omega)} = \alpha_\ell \in \mathbb{R}, \quad \ell = 1, \dots, d,$$

where  $d$  is the dimension and  $\{D_\ell\}$  an  $\varepsilon$ -orthonormal basis of  $\mathcal{H}_\varepsilon^q(\Omega)$ . The boundary values on  $\Gamma_\tau$  and  $\Gamma_\nu$ , respectively, are realized by the given forms  $E_\tau$  and  $E_\nu$ , respectively.

**Theorem 5.3.** (10) admits a solution, if and only if

$$E_\tau \in \mathbb{D}^q(\Omega), \quad E_\nu \in \varepsilon^{-1}\Delta^q(\Omega),$$

and

$$(12) \quad F - dE_\tau \perp \mathring{\Delta}_{\Gamma_\nu,0}^{q+1}(\Omega), \quad G - \delta\varepsilon E_\nu \perp \mathring{D}_{\Gamma_\tau,0}^{q-1}(\Omega).$$

The solution  $E \in \mathbb{D}^q(\Omega) \cap \varepsilon^{-1}\Delta^q(\Omega)$  can be chosen in a way such that condition (11) with  $\alpha \in \mathbb{R}^d$  is fulfilled, which then uniquely determines the solution. Furthermore the solution depends linearly and continuously on the data.

Note that (12) is equivalent to  $F - dE_\tau \in d\mathring{D}_{\Gamma_\tau}^q(\Omega)$  and  $G - \delta\varepsilon E_\nu \in \delta\mathring{\Delta}_{\Gamma_\nu}^q(\Omega)$ . For homogeneous boundary data, i.e.,  $E_\tau = E_\nu = 0$ , the theorem immediately follows from a functional analytic toolbox, which even states a sharper result: The linear static Maxwell-operator

$$M \quad : \quad \begin{array}{ccc} \mathring{D}_{\Gamma_\tau}^q(\Omega) \cap \varepsilon^{-1}\mathring{\Delta}_{\Gamma_\nu}^q(\Omega) & \longrightarrow & d\mathring{D}_{\Gamma_\tau}^q(\Omega) \times \delta\mathring{\Delta}_{\Gamma_\nu}^q(\Omega) \times \mathbb{R}^d \\ E & \longmapsto & (dE, \delta\varepsilon E, (\langle \varepsilon E, D_\ell \rangle_{\mathbb{L}^{2,q}(\Omega)})_{\ell=1}^d) \end{array}$$

is a topological isomorphism. Its inverse  $M^{-1}$  maps not only continuously onto  $\mathring{D}_{\Gamma_\tau}^q(\Omega) \cap \varepsilon^{-1}\mathring{\Delta}_{\Gamma_\nu}^q(\Omega)$ , but also compactly into  $\mathbb{L}^{2,q}(\Omega)$  by Theorem 4.9. For homogeneous kernel data, i.e., for

$$M_0 \quad : \quad \begin{array}{ccc} \mathring{D}_{\Gamma_\tau}^q(\Omega) \cap \varepsilon^{-1}\mathring{\Delta}_{\Gamma_\nu}^q(\Omega) \cap \mathcal{H}_\varepsilon^q(\Omega)^{\perp\varepsilon} & \longrightarrow & d\mathring{D}_{\Gamma_\tau}^q(\Omega) \times \delta\mathring{\Delta}_{\Gamma_\nu}^q(\Omega) \\ E & \longmapsto & (dE, \delta\varepsilon E) \end{array}$$

we have  $\|M_0^{-1}\| \leq (c_m^2 + 1)^{1/2}$ .

#### APPENDIX A. PROOF OF LEMMA 4.4

We start out by proving the assertions for the exterior derivative.

**A.1. Without Boundary Conditions.** Let  $E = \sum_I E_I dx^I \in \mathbb{D}^q(\Theta)$ . We have to prove  $\psi^*E \in \mathbb{D}^q(\tilde{\Theta})$  with  $d\psi^*E = \psi^*dE$ .

(i) Let us first assume  $\Phi \in \mathbb{C}^{0,1,q}(\Theta)$ , i.e.,  $\Phi_I \in \mathbb{C}^{0,1}(\Theta)$  for all  $I$ . In the following we denote by  $\tilde{\cdot}$  the composition with  $\psi$ . We have

$$\begin{aligned} d\psi_j &= \sum_i \partial_i \psi_j dx^i, & \psi^*\Phi &= \sum_I \tilde{\Phi}_I \psi^* dx^i = \sum_I \tilde{\Phi}_I (d\psi_{i_1}) \wedge \cdots \wedge (d\psi_{i_q}), \\ d\Phi &= \sum_{I,j} \partial_j \Phi_I (dx_j) \wedge (dx^I). \end{aligned}$$

By Rademacher's theorem we know that  $\tilde{\Phi}_I = \Phi_I \circ \psi$  and  $\psi_j$  belong to  $C^{0,1}(\tilde{\Theta}) \subset H^1(\tilde{\Theta})$  and that the chainrule holds, i.e.,  $\partial_i \tilde{\Phi}_I = \sum_j \tilde{\partial}_j \tilde{\Phi}_I \partial_i \psi_j$ . As  $\psi_j \in H^1(\tilde{\Theta})$  we get  $d\psi_j \in D_0^1(\tilde{\Theta})$  by

$$\langle d\psi_j, \delta\varphi \rangle_{L^{2,1}(\tilde{\Theta})} = -\langle \psi_j, \delta\delta\varphi \rangle_{L^{2,0}(\tilde{\Theta})} = 0$$

for all  $\varphi \in \mathring{C}^{\infty,2}(\tilde{\Theta})$ . Thus by definition we see

$$\begin{aligned} d\psi^*\Phi &= \sum_I (d\tilde{\Phi}_I) \wedge (d\psi_{i_1}) \wedge \cdots \wedge (d\psi_{i_q}) = \sum_{I,i} \partial_i \tilde{\Phi}_I (dx^i) \wedge (d\psi_{i_1}) \wedge \cdots \wedge (d\psi_{i_q}) \\ &= \sum_{I,i,j} \tilde{\partial}_j \tilde{\Phi}_I \partial_i \psi_j (dx^i) \wedge (d\psi_{i_1}) \wedge \cdots \wedge (d\psi_{i_q}) = \sum_{I,j} \tilde{\partial}_j \tilde{\Phi}_I (d\psi_j) \wedge (d\psi_{i_1}) \wedge \cdots \wedge (d\psi_{i_q}). \end{aligned}$$

On the other hand it holds

$$\psi^* d\Phi = \sum_{I,j} \tilde{\partial}_j \tilde{\Phi}_I (\psi^* dx_j) \wedge (\psi^* dx^I) = \sum_{I,j} \tilde{\partial}_j \tilde{\Phi}_I (d\psi_j) \wedge (d\psi_{i_1}) \wedge \cdots \wedge (d\psi_{i_q}).$$

Therefore,  $\psi^*\Phi \in D^q(\tilde{\Theta})$  and  $d\psi^*\Phi = \psi^* d\Phi$ .

- (ii) For general  $E \in D^q(\Theta)$  we pick  $\Phi \in \mathring{C}^{\infty,q+1}(\tilde{\Theta})$ . Note  $\text{supp } \Phi \subset \subset \tilde{\Theta} = \phi(\Theta)$ . Replacing  $\psi$  by  $\phi$  in (i) we have  $\phi^* \star \Phi \in D^{N-q-1}(\Theta)$  with  $d\phi^* \star \Phi = \phi^* d\star\Phi$  and since  $\phi^* \star \Phi = \sum_I (\star\tilde{\Phi}_I) \phi^* dx^I$  holds,  $\text{supp } \phi^* \star \Phi \subset \subset \Theta$ . By standard mollification we obtain a sequence  $(\Psi_n) \subset \mathring{C}^{\infty,N-q-1}(\Theta)$  with  $\Psi_n \rightarrow \phi^* \star \Phi$  in  $D^{N-q-1}(\Theta)$ . Furthermore  $\star\Psi_n \in \mathring{C}^{\infty,q+1}(\Theta)$ . Then

$$\begin{aligned} \langle \psi^* E, \delta\Phi \rangle_{L^{2,q}(\tilde{\Theta})} &= \int_{\tilde{\Theta}} \psi^* E \wedge \star\delta\Phi = \pm \int_{\tilde{\Theta}} \psi^* E \wedge \psi^* \phi^* d\star\Phi = \pm \int_{\tilde{\Theta}} \psi^* (E \wedge \phi^* d\star\Phi) \\ &= \pm \int_{\Theta} E \wedge \phi^* d\star\Phi = \pm \int_{\Theta} E \wedge d\phi^* \star\Phi \leftarrow \pm \int_{\Theta} E \wedge d\Psi_n \\ &= \pm \int_{\Theta} E \wedge \star\star d\star\Psi_n = \pm \langle E, \delta\star\Psi_n \rangle_{L^{2,q}(\Theta)} \\ &= \pm \langle dE, \star\Psi_n \rangle_{L^{2,q+1}(\Theta)} \rightarrow \pm \langle dE, \star\phi^* \star\Phi \rangle_{L^{2,q+1}(\Theta)} = \pm \int_{\Theta} dE \wedge \phi^* \star\Phi \\ &= \pm \int_{\tilde{\Theta}} \psi^* (dE \wedge \phi^* \star\Phi) = \pm \int_{\tilde{\Theta}} (\psi^* dE) \wedge \star\Phi = -\langle \psi^* dE, \Phi \rangle_{L^{2,q+1}(\tilde{\Theta})} \end{aligned}$$

and hence  $\psi^* E \in D^q(\tilde{\Theta})$  with  $d\psi^* E = \psi^* dE$ .

- (iii) Let  $E \in D^q(\Theta)$ . By (ii) we know  $\psi^* E \in D^q(\tilde{\Theta})$  with  $d\psi^* E = \psi^* dE$ . Hence

$$\begin{aligned} |\psi^* E|_{L^{2,q}(\tilde{\Theta})}^2 &= \int_{\tilde{\Theta}} \psi^* E \wedge \star\psi^* E = \int_{\Theta} \phi^* \psi^* E \wedge \phi^* \star\psi^* E \\ &= \pm \int_{\Theta} E \wedge \star(\star\phi^* \star\psi^*) E \leq c |E|_{L^{2,q}(\Theta)}^2, \end{aligned}$$

and

$$|d\psi^* E|_{L^{2,q+1}(\tilde{\Theta})} = |\psi^* dE|_{L^{2,q+1}(\tilde{\Theta})} \leq c |dE|_{L^{2,q+1}(\Theta)}.$$

**A.2. With Strong Boundary Condition.** Let  $E \in \mathring{D}_{\Upsilon_0}^q(\Theta)$  and  $(E_n) \subset \mathring{C}_{\Upsilon_0}^{\infty,q}(\Theta)$  with  $E_n \rightarrow E$  in  $D^q(\Theta)$ . By Appendix A.1 (ii) we know  $\psi^* E, \psi^* E_n \in D^q(\tilde{\Theta})$  with  $d\psi^* E_n = \psi^* dE_n$  as well as  $d\psi^* E = \psi^* dE$ . Furthermore,  $\psi^* E_n$  has compact support away from  $\tilde{\Upsilon}_0$ . By standard mollification we see  $\psi^* E_n \in \mathring{D}_{\tilde{\Upsilon}_0}^q(\tilde{\Theta})$ . Moreover, by A.1 (iii)  $\psi^* E_n \rightarrow \psi^* E$  in  $D^q(\tilde{\Theta})$ . Therefore  $\psi^* E \in \mathring{D}_{\tilde{\Upsilon}_0}^q(\tilde{\Theta})$  with  $d\psi^* E = \psi^* dE$ .

**A.3. With Weak Boundary Condition.** Let  $E \in \mathring{D}_{\Upsilon_0}^q(\Theta)$  and  $\Phi \in \mathring{C}_{\tilde{\Upsilon}_1}^{\infty,q+1}(\tilde{\Theta})$ , where  $\Upsilon_1 = \Upsilon \setminus \tilde{\Upsilon}_0$ . By Appendix A.1 (ii) we again know  $\psi^* E \in D^q(\tilde{\Theta})$  with  $d\psi^* E = \psi^* dE$ . Moreover by Appendix A.2  $\phi^* \star \Phi \in \mathring{D}_{\Upsilon_1}^{N-q-1}(\Theta)$  and hence  $\star\phi^* \star \Phi \in \mathring{\Delta}_{\Upsilon_1}^{q+1}(\Theta)$ . We repeat the calculation from Appendix A.1 (ii) to arrive at

$$\begin{aligned} \langle \psi^* E, \delta\Phi \rangle_{L^{2,q}(\tilde{\Theta})} &= \int_{\tilde{\Theta}} \psi^* E \wedge \star\delta\Phi = \pm \langle E, \star\phi^* d\star\Phi \rangle_{L^{2,q}(\Theta)} \\ &= \pm \langle E, \star d\phi^* \star\Phi \rangle_{L^{2,q}(\Theta)} = \pm \langle E, \delta\star\phi^* \star\Phi \rangle_{L^{2,q}(\Theta)} \\ &= \pm \langle dE, \star\phi^* \star\Phi \rangle_{L^{2,q+1}(\Theta)} = -\langle \psi^* dE, \Phi \rangle_{L^{2,q+1}(\tilde{\Theta})} = -\langle d\psi^* E, \Phi \rangle_{L^{2,q+1}(\tilde{\Theta})} \end{aligned}$$

and therefore  $\psi^*E \in \mathring{\mathbf{D}}_{\tilde{\Gamma}_0}^q(\tilde{\Theta})$ .

**A.4. Assertions for the co-derivative.** It holds by Appendix A.1 (ii)

$$\varepsilon H \in \Delta^q(\Theta) \Leftrightarrow \star \varepsilon H \in \mathbf{D}^{N-q}(\Theta) \Leftrightarrow \psi^* \star \varepsilon \phi^* \psi^* H \in \mathbf{D}^{N-q}(\tilde{\Theta}) \Leftrightarrow \mu \psi^* H \in \Delta^q(\tilde{\Theta}).$$

Moreover, using Appendix A.1 (iii)  $\mu$  is admissible since for all  $H \in \mathbf{L}^{2,q}(\tilde{\Theta})$

$$\begin{aligned} \langle \mu H, H \rangle_{\mathbf{L}^{2,q}(\tilde{\Theta})} &= \pm \langle \star \psi^* \star \varepsilon \phi^* H, H \rangle_{\mathbf{L}^{2,q}(\tilde{\Theta})} = \pm \langle \psi^* \star \varepsilon \phi^* H, \star H \rangle_{\mathbf{L}^{2,N-q}(\tilde{\Theta})} \\ &= \pm \int_{\tilde{\Theta}} \psi^* \star \varepsilon \phi^* H \wedge H = \pm \int_{\Theta} \star \varepsilon \phi^* H \wedge \star \phi^* H \\ &= \pm \langle \varepsilon \phi^* H, \phi^* H \rangle_{\mathbf{L}^{2,q}(\Theta)} \geq c |\phi^* H|_{\mathbf{L}^{2,q}(\Theta)}^2 \geq c |H|_{\mathbf{L}^{2,q}(\tilde{\Theta})}^2. \end{aligned}$$

Furthermore

$$\delta \mu \psi^* H = \pm \star d \psi^* \star \varepsilon H = \pm \star \psi^* \star \delta \varepsilon H.$$

The remaining assertions now follow by Appendix A.1 - A.3 and Hodge- $\star$ -duality.

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