SCHRIFTENREIHE DER FAKULTÄT FÜR MATHEMATIK

A Short Note on a Weighted Friedrichs Inequality

by

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A SHORT NOTE ON A WEIGHTED FRIEDRICHS INEQUALITY

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ABSTRACT. In this short note we derive an upper bound for the constant $c_\alpha > 0$ in the weighted Friedrichs type inequality

$$\forall \varphi \in \mathring{\mathsf{H}}^1(\Omega) \qquad |\varphi|_{\mathsf{L}^2(\Omega)} \le c_\alpha \sqrt{\langle \alpha \nabla \varphi, \nabla \varphi \rangle_{\mathsf{L}^2(\Omega)}},$$

where $\Omega \in \mathbb{R}^d, d \ge 1$ is a bounded domain, and α a bounded, self-adjoint, and uniformly positive definite matrix valued function. The contents of this note follow in a straightforward manner from well known results. In particular, for a constant diagonal matrix α we obtain the bound

$$c_\alpha \le \left(\pi \sqrt{\frac{\alpha_1}{l_1^2} + \cdots + \frac{\alpha_d}{l_d^2}} \right)^{-1},$$

where l_i are the side lengths of a d-interval encompassing Ω, and α_i are the diagonal entries of α. We apply the derived upper bound in a posteriori error estimation for an elliptic problem.

1. INTRODUCTION

We denote by $x := (x_1, \dots, x_d)$ the Euclidean coordinates in $\mathbb{R}^d, d \ge 1$, and by $\Omega \subset \mathbb{R}^d$ a bounded domain. The calculations performed in this note are invariant with respect to translations of the domain, so without loss of generality we assume Ω is contained in the open d-interval

$$I := \prod_{i=1}^d (0, l_i), \qquad 0 < l_i < \infty.$$

We denote by $\langle \cdot, \cdot \rangle_{\mathsf{L}^2(\Omega)}$ and $|\cdot|_{\mathsf{L}^2(\Omega)}$ the inner product and norm for scalar- or vector-valued functions in $\mathsf{L}^2(\Omega)$. We introduce the subindex notation $\langle \cdot, \cdot \rangle_{\mathsf{L}^2(\Omega), \rho} := \langle \rho \cdot, \cdot \rangle_{\mathsf{L}^2(\Omega)}$, which induces $|\cdot|_{\mathsf{L}^2(\Omega), \rho}$, where ρ belongs to the space of essentially bounded functions $\mathsf{L}^\infty(\Omega)$. If ρ is self-adjoint and uniformly positive definite, they become an inner product and a norm in $\mathsf{L}^2(\Omega)$, respectively. We define the usual Sobolev spaces

$$\mathsf{H}^1(\Omega) := \{\varphi \in \mathsf{L}^2(\Omega) \mid \nabla \varphi \in \mathsf{L}^2(\Omega)\}, \qquad \mathsf{D}(\Omega) := \{\phi \in \mathsf{L}^2(\Omega) \mid \operatorname{div} \phi \in \mathsf{L}^2(\Omega)\},$$

which are Hilbert spaces equipped with the graph norms $|\cdot|_{\mathsf{H}^1(\Omega)}$ and $|\cdot|_{\mathsf{D}(\Omega)}$, respectively. The space of functions belonging to $\mathsf{H}^1(\Omega)$, and vanishing on $\partial\Omega$, is defined as the H^1-closure of smooth test functions with compact supports in Ω, i.e.,

$$\mathring{\mathsf{H}}^1(\Omega) := \overline{\mathring{\mathsf{C}}^\infty(\Omega)}^{\mathsf{H}^1(\Omega)}.$$

For these functions the Friedrichs inequality

$$\forall \varphi \in \mathring{\mathsf{H}}^1(\Omega) \qquad |\varphi|_{\mathsf{L}^2(\Omega)} \le c |\nabla \varphi|_{\mathsf{L}^2(\Omega)}$$

holds, where $c > 0$ is called the Friedrichs constant. Note that c is assumed to be the best possible, i.e., smallest possible constant for which the Friedrichs inequality holds. A commonly

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utilized upper bound for c is [6]

$$(1.1) \qquad c \leq \left(\pi \sqrt{\frac{1}{l_1^2} + \cdots + \frac{1}{l_d^2}} \right)^{-1}.$$

This short note is dedicated to finding upper bounds for the constant $c_\alpha > 0$ in the weighted Friedrichs type inequality

$$(1.2) \qquad \forall \varphi \in \mathring{\mathsf{H}}^1(\Omega) \qquad |\varphi|_{\mathsf{L}^2(\Omega)} \leq c_\alpha |\nabla \varphi|_{\mathsf{L}^2(\Omega),\alpha}$$

for bounded Ω. Here $\alpha \in \mathsf{L}^\infty(\Omega)$ is a self-adjoint uniformly positive definite matrix valued function, i.e., it satisfies

$$(1.3) \qquad \forall \xi \in \mathbb{R}^d \qquad \alpha(x)\xi \cdot \xi \geq \underline{\alpha}|\xi|^2, \qquad \underline{\alpha} > 0 \qquad \text{(f.a.a. } x \in \Omega\text{)}.$$

Estimates for c_α can be calculated by using estimates for c, since obviously

$$|\varphi|_{\mathsf{L}^2(\Omega)} \leq c|\nabla \varphi|_{\mathsf{L}^2(\Omega)} \leq \frac{c}{\sqrt{\underline{\alpha}}}|\nabla \varphi|_{\mathsf{L}^2(\Omega),\alpha}$$

holds. Note that since the first estimation step is done using the Friedrichs inequality, and contains the full gradient on the right hand side, it is inevitable that the final estimation step involves a division with the smallest eigenvalue of α. Estimating further by using (1.1) we obtain the estimate

$$(1.4) \qquad c_\alpha \leq \left(\pi \sqrt{\left(\frac{1}{l_1^2} + \cdots + \frac{1}{l_d^2}\right) \underline{\alpha}} \right)^{-1},$$

which blows up as $\underline{\alpha}$ approaches zero.

Having computable upper bounds of Friedrichs and Poincaré type constants related to both weighted and non-weighted variants of corresponding inequalities is important in a posteriori error estimation. Error upper bounds typically contain these constants, and are especially important for functional type error estimates, where *guaranteed* upper bounds of the exact error are desired. In this short note we omit a literature overview of a posteriori error estimation, and instead refer the reader to the books [2, 5, 7, 9, 12].

Some references with upper bounds of Friedrichs and Poincaré type constants are the book [6] and the paper [8] (see also [3]). We also cite the interesting survey article [4]. Some more recent work on the subject include [11], where a weighted Friedrichs inequality similar to (1.2) is considered. The author calculates numerically two-sided bounds of a Friedrichs type constant in weighted norms. This approach allows for mixed boundary conditions. In [10] Friedrichs and Poincaré inequalities in non-weighted norms with mixed boundary conditions are considered. This approach involves decomposing the domain into smaller subdomains for which Friedrichs and Poincaré constants are known. The resulting upper bounds depend on the decomposition.

In this note we show that in the case of full homogeneous Dirichlet boundary conditions, there is a simple way to obtain an upper bound of c_α with better properties than (1.4). The upper bound, derived in Section 2, follows from well known results. In Section 3 we demonstrate the benefit of using this upper bound by a numerical example where we perform a posteriori error estimation of an elliptic problem.

2. Weighted Friedrichs Type Inequality

The calculations of this section are based on the well known one-dimensional inequality

$$(2.1) \qquad \forall \varphi \in \mathring{\mathsf{H}}^1((0,l)) \qquad \int_0^l |\varphi(y)|^2 \, \mathrm{d}y \leq \frac{l^2}{\pi^2} \int_0^l |\varphi'(y)|^2 \, \mathrm{d}y,$$

where $0 < l < \infty$. Using this inequality one can proof a Friedrichs type inequality involving only one partial derivative, and by an additional estimation step obtain an inequality involving

the full gradient. In the case of bounded domains, this would result in the estimate (1.1). However, we will need the intermediate result involving only one partial derivative. Note that since we want to control all partial derivatives separately, we cannot rotate the domain.

Lemma 2.1. *Let Ω be bounded, and $i \in \{1, \ldots, d\}$. Then we have the estimate*

$$\forall \varphi \in \overset{\circ}{\mathsf{H}}{}^1(\Omega) \qquad |\varphi|_{\mathsf{L}^2(\Omega)} \leq \frac{l_i}{\pi} |\partial_i \varphi|_{\mathsf{L}^2(\Omega)}.$$

Proof. Consider first the real valued case and $i = 1$. For any $\varphi \in \overset{\circ}{\mathsf{C}}{}^\infty(\Omega)$ its zero-extension $\hat{\varphi} : I \to \mathbb{R}$ belongs to $\overset{\circ}{\mathsf{C}}{}^\infty(I)$. For any $\tilde{x} := (x_2, \ldots, x_d)$ belonging to $\tilde{I} := (0, l_2) \times \cdots \times (0, l_d)$, the function $\hat{\varphi}(x_1, \tilde{x})$ is a real valued function of one variable vanishing at the endpoints of the interval $(0, l_1)$, so by (2.1) we have

$$\int_0^{l_1} |\hat{\varphi}(x_1, \tilde{x})|^2 \, \mathrm{d}x_1 \leq \frac{l_1^2}{\pi^2} \int_0^{l_1} |\partial_1 \hat{\varphi}(x_1, \tilde{x})|^2 \, \mathrm{d}x_1.$$

By integrating the above with respect to $\tilde{x}$ in $\tilde{I}$, we obtain

$$|\hat{\varphi}|_{\mathsf{L}^2(I)}^2 \leq \frac{l_1^2}{\pi^2} |\partial_1 \hat{\varphi}|_{\mathsf{L}^2(I)}^2 \qquad \Rightarrow \qquad |\varphi|_{\mathsf{L}^2(\Omega)}^2 \leq \frac{l_1^2}{\pi^2} |\partial_1 \varphi|_{\mathsf{L}^2(\Omega)}^2,$$

since the norms are nonzero only in Ω. By density the above holds for any $\varphi \in \overset{\circ}{\mathsf{H}}{}^1(\Omega)$. By an identical procedure the assertion follows for $i \in \{2, \ldots, d\}$ for real valued functions. Having established the assertion for real valued functions, it is clear that it holds for complex valued functions as well. $\qquad \square$

We now consider the constant c_α in the inequality (1.2). We assume that $\alpha \in \mathsf{L}^\infty(\Omega)$ is a self-adjoint diagonal matrix

$$(2.2) \qquad \alpha := \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_d \end{pmatrix}$$

satisfying uniform positive definiteness (1.3), which in this case is equivalent to

$$(2.3) \qquad \forall i \in \{1, \ldots, d\} \qquad \alpha_i(x) \geq \underline{\alpha_i} > 0 \qquad (\text{f.a.a. } x \in \Omega).$$

Now $\underline{\alpha} = \min\{\underline{\alpha_1}, \ldots, \underline{\alpha_d}\}$.

Theorem 2.2. *Let Ω be bounded and $\alpha \in \mathsf{L}^\infty(\Omega)$ be a self-adjoint diagonal matrix defined by (2.2)–(2.3). Then we have the estimate*

$$c_\alpha \leq \left(\pi \sqrt{\frac{\underline{\alpha_1}}{l_1^2} + \cdots + \frac{\underline{\alpha_d}}{l_d^2}} \right)^{-1}.$$

Proof. Let $\varphi \in \overset{\circ}{\mathsf{H}}{}^1(\Omega)$. Since α is diagonal, the weighted norm can be written as

$$|\nabla \varphi|_{\mathsf{L}^2(\Omega), \alpha}^2 = |\partial_1 \varphi|_{\mathsf{L}^2(\Omega), \alpha_1}^2 + \cdots + |\partial_d \varphi|_{\mathsf{L}^2(\Omega), \alpha_d}^2.$$

Lemma 2.1 gives

$$|\varphi|_{\mathsf{L}^2(\Omega)}^2 \leq \frac{l_i^2}{\pi^2} |\partial_i \varphi|_{\mathsf{L}^2(\Omega)}^2 \leq \frac{l_i^2}{\pi^2 \underline{\alpha_i}} |\partial_i \varphi|_{\mathsf{L}^2(\Omega), \alpha_i}^2$$

for any $i \in \{1, \ldots, d\}$. By multiplying the above by $\underline{\alpha_i}/l_i^2$ and summing up the d inequalities, we obtain

$$\left(\frac{\underline{\alpha_1}}{l_1^2} + \cdots + \frac{\underline{\alpha_d}}{l_d^2} \right) |\varphi|_{\mathsf{L}^2(\Omega)}^2 \leq \frac{1}{\pi^2} |\nabla \varphi|_{\mathsf{L}^2(\Omega), \alpha}^2,$$

which implies the assertion. $\qquad \square$

Remark 2.3.

(i) *Theorem 2.2 with $\alpha = \mathrm{id}$ results in the estimate* (1.1).
(ii) *It is easy to see that the upper bound of Theorem 2.2 is always smaller or equal to the upper bound* (1.4).
(iii) *The above procedure furnishes upper bounds of c_α even when the diagonal matrix α is not uniformly positive definite (see Appendix A).*
(iv) *An upper bound similar to Theorem 2.2 can be obtained provided that the homogeneous Dirichlet boundary conditions hold in at least one direction. I.e., for the unit square in $\mathbb{R}^2$, it is enough that one pair of opposing boundaries have the boundary condition; if this pair is the one parallel to the x_2-axis, then the boundary condition is in the direction of the x_1-axis, and we have $c_\alpha \leq (\pi\sqrt{\underline{\alpha}_1/l_1^2})^{-1}$. However, since the opposing boundary parts must have a "straight line of sight" to each other, the possible domains are quite limited. More variety in domains is achieved, if one uses (instead of (2.1))*

$$\int_0^l |\varphi(y)|^2 \, \mathrm{d}y \leq \frac{l^2}{2} \int_0^l |\varphi'(y)|^2 \, \mathrm{d}y,$$

which holds for all functions $\varphi \in \mathsf{H}^1((0,l))$ vanishing either on the beginning or the end of the interval. In this way only one of the opposing boundaries need to have the boundary condition. For this inequality see, e.g., [1, p. 158].

Under certain conditions non-diagonal α can be handled as well. For readability we consider only the three dimensional case. For any self-adjoint $\alpha(x) = \{\alpha_{ij}(x)\}_{i,j=1}^3$ from $\mathsf{L}^\infty(\Omega)$ we define

$$(2.4) \qquad \tilde{\alpha} := \begin{pmatrix} \tilde{\alpha}_1 & 0 & 0 \\ 0 & \tilde{\alpha}_2 & 0 \\ 0 & 0 & \tilde{\alpha}_3 \end{pmatrix}, \qquad \begin{aligned} \tilde{\alpha}_1 &:= \alpha_{11} - (|\Re\alpha_{12}| + |\Re\alpha_{13}|), \\ \tilde{\alpha}_2 &:= \alpha_{22} - (|\Re\alpha_{12}| + |\Re\alpha_{23}|), \\ \tilde{\alpha}_3 &:= \alpha_{33} - (|\Re\alpha_{13}| + |\Re\alpha_{23}|). \end{aligned}$$

It is easy to verify that $\tilde{\alpha}$ is self-adjoint, and that

$$(2.5) \qquad \forall \phi \in \mathsf{L}^2(\Omega) \qquad |\phi|_{\mathsf{L}^2(\Omega),\tilde{\alpha}} \leq |\phi|_{\mathsf{L}^2(\Omega),\alpha}$$

holds. If $\tilde{\alpha}$ is also uniformly positive definite, i.e., it satisfies

$$\forall i \in \{1,2,3\} \qquad \tilde{\alpha}_i(x) \geq \underline{\tilde{\alpha}}_i > 0 \qquad (\text{f.a.a. } x \in \Omega),$$

we can directly use Theorem 2.2 to obtain an estimate of c_α.

Theorem 2.4. *Let $\Omega \in \mathbb{R}^3$ be bounded and $\alpha \in \mathsf{L}^\infty(\Omega)$ be a self-adjoint matrix valued function for which $\tilde{\alpha}$ is uniformly positive definite. Then we have the estimate*

$$c_\alpha \leq \left(\pi\sqrt{\frac{\underline{\tilde{\alpha}}_1}{l_1^2} + \frac{\underline{\tilde{\alpha}}_2}{l_2^2} + \frac{\underline{\tilde{\alpha}}_3}{l_3^2}}\right)^{-1}.$$

Proof. Let $\varphi \in \mathring{\mathsf{H}}^1(\Omega)$. Theorem 2.2 gives

$$|\varphi|_{\mathsf{L}^2(\Omega)} \leq \left(\pi\sqrt{\frac{\underline{\tilde{\alpha}}_1}{l_1^2} + \frac{\underline{\tilde{\alpha}}_2}{l_2^2} + \frac{\underline{\tilde{\alpha}}_3}{l_3^2}}\right)^{-1} |\nabla\varphi|_{\mathsf{L}^2(\Omega),\tilde{\alpha}},$$

and with (2.5) we have the assertion. $\qquad\square$

Remark 2.5. *For $\tilde{\alpha}$ to be uniformly positive definite would require that the off-diagonal entries of α be comparatively small compared to its diagonal entries. However, now Remark 2.3 (iii) holds with respect to $\tilde{\alpha}$. In particular, for an upper bound of c_α it is enough that $\tilde{\alpha}$ is positive semi-definite such that one of the diagonal entries of $\tilde{\alpha}$ is uniformly positive definite.*

We demonstrate the derived results in the real valued setting.

Example 1 (Diagonal matrix α). Let $\Omega \subset (0,1)^2$ and α be the uniformly positive definite constant matrix

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}, \qquad \delta > 0.$$

The estimate (1.4) gives the upper bound

$$(2.6) \qquad c_\alpha \leq \left(\pi \sqrt{2 \min\{1, \delta\}} \right)^{-1},$$

and Theorem 2.2 gives

$$(2.7) \qquad c_\alpha \leq \left(\pi \sqrt{1 + \delta} \right)^{-1}.$$

It is easy to see that the latter does not blow up as δ becomes smaller. Table 1 shows the values of the bounds with different δ.

TABLE 1. Example 1: Values of the upper bounds (2.6) and (2.7) with different δ.

δ	10^{-6}	10^{-4}	10^{-2}	1	10^2	10^4	10^6
(2.6)	225.07908	22.50791	2.25079	0.22508	0.22508	0.22508	0.22508
(2.7)	0.31831	0.31829	0.31673	0.22508	0.03167	0.00318	0.00032

Example 2 (Solution theory for a reaction-diffusion problem)**.** Consider the following reaction-diffusion problem: find $u \in \mathring{\mathsf{H}}^1(\Omega)$ satisfying

$$- \operatorname{div} \alpha \nabla u + \rho\, u = f,$$

where $f \in \mathsf{L}^2(\Omega)$, $\rho \in \mathsf{L}^\infty(\Omega)$, and $\alpha \in \mathsf{L}^\infty(\Omega)$ is a symmetric uniformly positive definite matrix valued function. The variational formulation of this problem reads as

$$\forall \varphi \in \mathring{\mathsf{H}}^1(\Omega) \qquad \langle \nabla u, \nabla \varphi \rangle_{\mathsf{L}^2(\Omega),\alpha} + \langle u, \varphi \rangle_{\mathsf{L}^2(\Omega),\rho} = \langle f, \varphi \rangle_{\mathsf{L}^2(\Omega)}.$$

By setting $\varphi = u$ in the bilinear form on the left hand side, we obtain

$$\langle \nabla u, \nabla u \rangle_{\mathsf{L}^2(\Omega),\alpha} + \langle u, u \rangle_{\mathsf{L}^2(\Omega),\rho} = (1 - \epsilon) |\nabla u|_{\mathsf{L}^2(\Omega),\alpha}^2 + \epsilon |\nabla u|_{\mathsf{L}^2(\Omega),\alpha}^2 + |u|_{\mathsf{L}^2(\Omega),\rho}^2$$

$$\geq (1 - \epsilon) \underline{\alpha} |\nabla u|_{\mathsf{L}^2(\Omega)}^2 + \left(\frac{\epsilon}{c_\alpha^2} + \rho \right) |u|_{\mathsf{L}^2(\Omega)}^2,$$

where $0 < \epsilon < 1$. We observe that this form is coercive provided that

$$\left(\frac{\epsilon}{c_\alpha^2} + \rho \right) > 0$$

holds, and under this condition a unique solution exists in $\mathring{\mathsf{H}}^1(\Omega)$ by the Riez representation theorem. Let $\Omega \subset (0,1)^2$, $\rho \in \mathbb{R}$, and

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 100 \end{pmatrix}.$$

Using (1.4) to estimate c_α (see Example 1), we see that for existence and uniqueness of a solution, the necessary condition is $\rho > -\epsilon\, 2\pi^2$, but using Theorem 2.2 the necessary condition becomes $\rho > -\epsilon\, 101\pi^2$, allowing for a larger range of admissible ρ.

Example 3 (Non-diagonal matrix α). Let $\Omega \subset (0,1)^3$, and

$$\alpha = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 300 & 1 \\ 1 & 1 & 3 \end{pmatrix}, \qquad \tilde{\alpha} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 298 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\tilde{\alpha}$ is calculated according to (2.4). Now $\underline{\alpha} = 2$, and (1.4) gives the bound $c_\alpha \le \left(\pi\sqrt{6}\right)^{-1}$. On the other hand, Theorem 2.4 gives the upper bound $c_\alpha \le \left(\pi\sqrt{300}\right)^{-1}$, which is sharper.

3. Application to A Posteriori Error Estimation

As stated in the introduction, the motivation for deriving computable upper bounds for the constant c_α is that it is essential in a posteriori error estimation for numerical approximations of elliptic partial differential equations. As an example we consider the diffusion problem in a bounded domain Ω with homogeneous Dirichlet boundary conditions on the whole boundary: find $u \in \mathring{\mathsf{H}}^1(\Omega)$ satisfying

$$- \operatorname{div} \alpha\nabla u = f,$$

where $f \in \mathsf{L}^2(\Omega)$ and $\alpha \in \mathsf{L}^\infty(\Omega)$ is a symmetric uniformly positive definite matrix valued function. The variational formulation for this problem reads as

$$(3.1) \qquad \forall\varphi \in \mathring{\mathsf{H}}^1(\Omega) \qquad \langle\nabla u, \nabla\varphi\rangle_{\mathsf{L}^2(\Omega),\alpha} = \langle f, \varphi\rangle_{\mathsf{L}^2(\Omega)}.$$

Since (1.2) is satisfied, a unique solution $u \in \mathring{\mathsf{H}}^1(\Omega)$ exists by the Riesz representation theorem. By setting $\varphi = u$ in (3.1) we obtain

$$|\nabla u|^2_{\mathsf{L}^2(\Omega),\alpha} = \langle f, u\rangle_{\mathsf{L}^2(\Omega)} \le |f|_{\mathsf{L}^2(\Omega)}|u|_{\mathsf{L}^2(\Omega)} \le c_\alpha|f|_{\mathsf{L}^2(\Omega)}|\nabla u|_{\mathsf{L}^2(\Omega),\alpha},$$

and we see that the solution depends continuously on the given right hand side:

$$|\nabla u|_{\mathsf{L}^2(\Omega),\alpha} \le c_\alpha|f|_{\mathsf{L}^2(\Omega)}.$$

We now present the functional type a posteriori error upper bound, which can be found in, e.g., the books [5, 7, 9].

Theorem 3.1. *Let $\tilde{u} \in \mathring{\mathsf{H}}^1(\Omega)$ be an arbitrary approximation of u, and $\tilde{c}_\alpha$ be any approximation of c_α from above. Then we have the estimate*

$$\forall y \in \mathsf{D}(\Omega) \qquad |\nabla(u - \tilde{u})|_{\mathsf{L}^2(\Omega),\alpha} \le \tilde{c}_\alpha|f + \operatorname{div} y|_{\mathsf{L}^2(\Omega)} + |y - \alpha\nabla\tilde{u}|_{\mathsf{L}^2(\Omega),\alpha^{-1}} := M(\tilde{c}_\alpha, \tilde{u}, y).$$

Proof. We begin by subtracting the term $\langle\nabla\tilde{u}, \nabla\varphi\rangle_{\mathsf{L}^2(\Omega),\alpha}$ from both sides of (3.1) and obtain

$$\begin{aligned}
\langle\nabla(u - \tilde{u}), \nabla\varphi\rangle_{\mathsf{L}^2(\Omega),\alpha} &= \langle f, \varphi\rangle_{\mathsf{L}^2(\Omega)} - \langle\nabla\tilde{u}, \nabla\varphi\rangle_{\mathsf{L}^2(\Omega),\alpha} \\
&= \langle f + \operatorname{div} y, \varphi\rangle_{\mathsf{L}^2(\Omega)} + \langle y - \alpha\nabla\tilde{u}, \nabla\varphi\rangle_{\mathsf{L}^2(\Omega)} \\
&\le |f + \operatorname{div} y|_{\mathsf{L}^2(\Omega)}|\varphi|_{\mathsf{L}^2(\Omega)} + |y - \alpha\nabla\tilde{u}|_{\mathsf{L}^2(\Omega),\alpha^{-1}}|\nabla\varphi|_{\mathsf{L}^2(\Omega),\alpha} \\
&\le \left(c_\alpha|f + \operatorname{div} y|_{\mathsf{L}^2(\Omega)} + |y - \alpha\nabla\tilde{u}|_{\mathsf{L}^2(\Omega),\alpha^{-1}}\right)|\nabla\varphi|_{\mathsf{L}^2(\Omega),\alpha},
\end{aligned}$$

where we have used $\langle\operatorname{div} y, \varphi\rangle_{\mathsf{L}^2(\Omega)} + \langle y, \nabla\varphi\rangle_{\mathsf{L}^2(\Omega)} = 0$. Setting $\varphi = u - \tilde{u}$ finishes the proof. $\square$

Remark 3.2. *By using the upper bound $c_\alpha \le c/\sqrt{\underline{\alpha}}$ for the value of $\tilde{c}_\alpha$ we obtain the most commonly used form of this functional type a posteriori error upper bound for the diffusion problem.*

Note that the above estimate is sharp, i.e., theoretically there is no gap between the exact error and the estimate. This is seen by setting $y = \alpha\nabla u \in \mathsf{D}(\Omega)$. The first term of the error functional M vanishes, and it becomes apparent that sharpness does not depend on c_α. However, obtaining good error bounds requires not only choosing y close to the exact flux $\alpha\nabla u$, but also having good upper bounds for the unknown constant c_α. Especially in the case when $-\operatorname{div} y$ is not close to f, a large over-estimation of the constant c_α will lead to a large over-estimation of the error, as we will now demonstrate.

Example 4 (Error estimation with Raviart-Thomas averaging). We solve the diffusion problem (3.1) in in the L-shaped domain $\Omega = (0,1)^2 \setminus [(1/2,1) \times (0,1/2)]$ with

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 10^{-4} \end{pmatrix}, \qquad f = 1.$$

We use linear nodal finite elements to solve (3.1), and denote the approximation by $\tilde{u}$. The free function y in the functional M is obtained by averaging $\alpha \nabla \tilde{u}$ to the edges of the mesh resulting in a function from the linear Raviart-Thomas finite element space, which is a subspace of $\mathsf{D}(\Omega)$. We denote this averaging operator by G_{RT}. Using (1.4) to estimate the value of c_α (see Example 1), we have the estimate

$$(3.2) \qquad |\nabla(u - \tilde{u})|_{\mathsf{L}^2(\Omega),\alpha} \le M(22.50791, \tilde{u}, G_{\mathrm{RT}}(\alpha \nabla \tilde{u}))$$

and by using Theorem 2.2 we obtain the estimate

$$(3.3) \qquad |\nabla(u - \tilde{u})|_{\mathsf{L}^2(\Omega),\alpha} \le M(0.31829, \tilde{u}, G_{\mathrm{RT}}(\alpha \nabla \tilde{u})).$$

Since $-\operatorname{div} G_{\mathrm{RT}}(\alpha \nabla \tilde{u})$ is only a rough approximation of f, the quality of the latter estimate is better, as is seen from Table 2.

TABLE 2. Example 4: Values of the upper bounds (3.2) and (3.3) with different meshes.

#elements	384	1536	6144	24576	98304
(3.2)	18.4444	17.1419	16.1891	14.9832	13.2664
(3.3)	1.5563	0.9166	0.5705	0.3809	0.2695

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Appendix A. Positive semi-definite matrices α

We shortly demonstrate that upper bounds for c_α can be obtained even if the self-adjoint $\alpha \in \mathsf{L}^\infty(\Omega)$ is only positive semi-definite: Let $\Omega \subset (-1, 1)^3$ and α be the constant matrix

$$\alpha = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The smallest eigenvalue is $\underline{\alpha} = 0$, so the estimate (1.4) cannot be used. However, as stated in Remark 2.3 (iii), we can still obtain upper bounds for c_α; we can apply Lemma 2.1 with $i = 3$ to obtain

$$|\varphi|_{\mathsf{L}^2(\Omega)} \leq \frac{2}{\pi} |\partial_3 \varphi|_{\mathsf{L}^2(\Omega)} = \frac{2}{\pi} |\nabla \varphi|_{\mathsf{L}^2(\Omega), \alpha} \qquad \Rightarrow \qquad c_\alpha \leq \frac{2}{\pi}.$$

The quantity $|\nabla \cdot|_{\mathsf{L}^2(\Omega), \alpha}$ is now not a norm, however.

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