

A new look at an old construction: constructing (simple) 3-designs from resolvable 2-designs

Douglas R. Stinson*

David R. Cheriton School of Computer Science
University of Waterloo
Waterloo, Ontario, N2L 3G1, Canada

Colleen M. Swanson

Computer Science & Engineering Division
University of Michigan
Ann Arbor, MI 48109, USA

Tran van Trung

Institut für Experimentelle Mathematik
Universität Duisburg-Essen
Ellernstrasse 29, 45326 Essen, Germany

September 23, 2013

Abstract

In 1963, Shrikhande and Raghavarao [5] published a recursive construction for designs that starts with a resolvable design (the “master design”) and then uses a second design (“the indexing design”) to take certain unions of blocks in each parallel class of the master design. Several variations of this construction have been studied by different authors. We revisit this construction, concentrating on the case where the master design is a resolvable BIBD and the indexing design is a 3-design. We show that this construction yields a 3-design under certain circumstances. The resulting 3-designs have block size $k = v/2$ and they are resolvable. We also construct some previously unknown simple designs by this method.

1 Introduction

We begin with a few definitions. Suppose that v, b, r and k are positive integers such that $2 \leq k < v$. A (v, b, r, k) -incomplete block design (or (v, b, r, k) -IBD) is a pair (X, \mathcal{B}) , where X is a set of v points and \mathcal{B} is a collection (i.e., a multiset) of b subsets of X (called *blocks*), such that

1. every block contains exactly k points, and
2. every point is contained in exactly r blocks.

It follows that $bk = vr$ in a (v, b, r, k) -IBD. Here are a few definitions relating to additional properties that IBDs might satisfy.

- Suppose $k \mid v$. A *parallel class* in a (v, b, r, k) -IBD (X, \mathcal{B}) is a set of v/k disjoint blocks. (X, \mathcal{B}) is *resolvable* if \mathcal{B} can be partitioned into r parallel classes.

*D. Stinson’s research is supported by NSERC discovery grant 203114-11

- A (v, b, r, k) -IBD (X, \mathcal{B}) is *simple* if \mathcal{B} does not contain any repeated blocks, i.e., \mathcal{B} is a set.
- A (v, b, r, k) -IBD (X, \mathcal{B}) is *trivial* if \mathcal{B} consists of all $\binom{v}{k}$ possible k -subsets of points.

A (v, b, r, k, λ) -balanced incomplete block design (or a (v, b, r, k, λ) -BIBD) is a (v, b, r, k) -IBD in which every pair of points occurs in exactly λ blocks. It is necessarily the case that $\lambda = r(k-1)/(v-1)$. Sometimes the notation (v, k, λ) -BIBD is used, since the parameters r and b can be computed from v, k and λ (specifically, $r = \lambda(v-1)/(k-1)$ and $b = vr/k$).

Suppose that λ, t, k , and v are positive integers such that $t \leq k < v$. A t - (v, k, λ) -design is a pair (X, \mathcal{B}) , where X is a set of v points and \mathcal{B} is a collection (i.e., a multiset) of k -subsets of X (called *blocks*), such that every subset of t points occurs in exactly λ blocks. It is well-known that a subset of $j \leq t-1$ points occurs in λ_j blocks in \mathcal{B} , where

$$\lambda_j = \frac{\lambda \binom{v-j}{t-j}}{\binom{k-j}{t-j}}. \quad (1)$$

If we define $b = \lambda_0$ and $r = \lambda_1$, then a t - (v, k, λ) -design is a (v, b, r, k) -IBD. A 2- (v, k, λ) -design is the same thing as a (v, k, λ) -BIBD. Also, a 1- (v, k, r) -design is equivalent to a $(v, vr/k, r, k)$ -IBD. A resolvable 2- $(v, 2, 1)$ -design is equivalent to a *one-factorization* of the complete graph K_v . The parallel classes in this design are called *one-factors*.

1.1 Related Work

There are some examples of related work, which we mention now. There is an unpublished construction due to Lonz and Vanstone of 3- $(v, 4, 3)$ -designs from resolvable 2- $(v, 2, 1)$ -designs (see [3]). They take all possible unions of two blocks from a parallel class of a 2- $(v, 2, 1)$ -design; this immediately yields a 3- $(v, 4, 3)$ -design. In [4], this method was extended to show the existence of simple 3- $(v, 4, 3)$ -designs. Recently, this approach was generalized by Jimbo, Kunihara, Laue and Sawa in [2]. The master design is a resolvable 3- (v, k, λ) -design and the indexing design is a 3-design on v/k points. The construction [2] can also be applied when the master design is a one-factorization (this is a trivial 2-design that can be viewed as a degenerate 3-design with $\lambda = 0$).

We employ the same construction as in [2], except the master design is a resolvable 2-design instead of a 3-design. We still end up with a 3-design provided that the other ingredient in the construction is a 3-design on v/k points with block size $v/(2k)$. All of our constructions, as well as the other constructions mentioned above, can be considered to be applications of the Shrikhande-Raghavarao construction.

1.2 Organization of the Paper

In Section 2, we present the main construction and discuss when this construction will yield (resolvable) 3-designs. Section 3 presents conditions that guarantee that the main construction will yield a nontrivial design. In Section 4, we investigate the simplicity of the constructed designs. We give some conditions that will guarantee that the constructed designs are simple. We also look at several specific constructed 3-designs and show by computer that they are simple even though we cannot prove theoretically that they are. Simple 3-designs for these parameters were not previously known to exist. Finally, Section 5 is a brief conclusion.

2 The Construction

We begin by describing a construction of Shrikhande and Raghavarao published in 1963 [5].

Construction 1. *We need two ingredients for the construction:*

1. *Let (X, \mathcal{B}) be a resolvable (v, b, r, k) -IBD and let Π_1, \dots, Π_r denote the parallel classes in the resolution of (X, \mathcal{B}) . There are $w = v/k$ blocks in each parallel class. Let the blocks in Π_i be named B_i^j , $1 \leq j \leq w$. We will call (X, \mathcal{B}) the **master design**.*
2. *Suppose (Y, \mathcal{C}) is a (w, b', r', k') -IBD, where $Y = \{1, \dots, w\}$. We will call (Y, \mathcal{C}) the **indexing design**.*

Now, for each i , $1 \leq i \leq r$, and for each $C \in \mathcal{C}$, define

$$D_{i,C} = \bigcup_{j \in C} B_i^j.$$

That is, for every block C of the indexing design and for every parallel class Π_i of the master design, we construct a block $D_{i,C}$ by taking the union of the blocks in Π_i indexed by C . Finally, let

$$\mathcal{D} = \{D_{i,C} : 1 \leq i \leq r, C \in \mathcal{C}\}.$$

(X, \mathcal{D}) is the **constructed design**.

We will study the design (X, \mathcal{D}) obtained from Construction 1. The following easily proven result was first shown in [5].

Theorem 2.1. *Suppose that (X, \mathcal{B}) is a resolvable (v, b, r, k) -IBD, and suppose (Y, \mathcal{C}) is a (w, b', r', k') -IBD, where $w = v/k$. Let (X, \mathcal{D}) be defined as in Construction 1. Then the constructed design (X, \mathcal{D}) is a (v, b'', r'', k'') -IBD, where*

$$\begin{aligned} b'' &= rb', \\ r'' &= rr', \text{ and} \\ k'' &= kk'. \end{aligned}$$

If the two input designs are BIBDs, then so is the constructed design. The following result is also from [5].

Theorem 2.2. *Suppose that (X, \mathcal{B}) is a resolvable (v, k, λ) -BIBD, and suppose (Y, \mathcal{C}) is a (w, k', λ'_2) -BIBD, where $w = v/k$. Let (X, \mathcal{D}) be defined as in Construction 1. Then the constructed design (X, \mathcal{D}) is a $(v, b'', r'', k'', \lambda'')$ -BIBD, where b'', r'' and k'' are the same as in Theorem 2.1, and*

$$\lambda'' = \lambda r' + (r - \lambda) \lambda'_2.$$

We are interested in determining conditions which guarantee that the constructed design is a 3-design. For this analysis, we will assume that the indexing design (Y, \mathcal{C}) is a 3- (w, k', λ') -design, which requires $k' \geq 3$.

Let $x, y, z \in X$ be three distinct points and suppose these points occur in α blocks of the master design (X, \mathcal{B}) . There are $3(\lambda - \alpha)$ parallel classes in which exactly two points of $\{x, y, z\}$ occur in a block, and $r - \alpha - 3(\lambda - \alpha)$ parallel classes in which x, y , and z occur in different blocks.

For a set of three points $x, y, z \in X$, let $\lambda_{x,y,z}$ denote the number of blocks in the constructed design \mathcal{D} that contain x, y and z . The following lemma is obtained by simple counting.

Lemma 2.3. *Suppose that (X, \mathcal{B}) is a resolvable (v, k, λ) -BIBD and (Y, \mathcal{C}) is a 3 - (w, k', λ') -design where $w = v/k$. Let (X, \mathcal{D}) be defined as in Construction 1. Suppose that three points x, y, z occur in exactly α blocks in \mathcal{B} . Then,*

$$\lambda_{x,y,z} = \alpha r' + 3(\lambda - \alpha) \lambda'_2 + (r - \alpha - 3(\lambda - \alpha)) \lambda'. \quad (2)$$

Theorem 2.4. *Suppose that (X, \mathcal{B}) is a resolvable (v, k, λ) -BIBD and (Y, \mathcal{C}) is a 3 - (w, k', λ') -design where $w = v/k$. Let (X, \mathcal{D}) be defined as in Construction 1. Then (X, \mathcal{D}) is a 3 -design if and only if one of the following conditions is satisfied:*

1. (X, \mathcal{B}) is a 3 -design,
2. $k = 2$, or
3. $k' = v/(2k)$.

Proof. Since $k' > 2$, if we substitute (1) into (2), we see that

$$\lambda_{x,y,z} = \lambda' \left(3\lambda \frac{w-2}{k'-2} + r - 3\lambda \right) + \lambda' \alpha \left(\frac{(w-1)(w-2)}{(k'-1)(k'-2)} - 3 \frac{w-2}{k'-2} + 2 \right). \quad (3)$$

Therefore $\lambda_{x,y,z}$ is of the form $c_1 + \alpha c_2$, for constants c_1 and c_2 . Clearly, the design (X, \mathcal{D}) is a 3 -design if and only if α is a constant or $c_2 = 0$. There are three possible cases that can occur:

1. If $\alpha > 0$ is a constant, this implies that the master design (X, \mathcal{B}) is a 3 -design. In this situation (X, \mathcal{D}) will be a 3 -design for any choice of $k' > 2$.
2. If $\alpha = 0$, then it must be the case that $k = 2$ (here, the master design can be thought of as a “degenerate” 3 -design). Again, (X, \mathcal{D}) will be a 3 -design for any choice of $k' > 2$.
3. We now use (3) to derive the conditions that yield $c_2 = 0$. Because $k' > 2$, we have

$$\begin{aligned} & \frac{(w-1)(w-2)}{(k'-1)(k'-2)} - 3 \frac{w-2}{k'-2} + 2 = 0 \\ \iff & (w-1)(w-2) - 3(w-2)(k'-1) + 2(k'-1)(k'-2) = 0 \\ \iff & w^2 - 3wk' + 2(k')^2 = 0 \\ \iff & (w-k')(w-2k') = 0. \end{aligned}$$

That is, $c_2 = 0$ when $k' = v/k$ or $k' = v/(2k)$. Note that if $k' = v/k$, then (X, \mathcal{D}) is not a 3 -design since each block in \mathcal{B} would consist of all the points of X . So (X, \mathcal{D}) is a 3 -design if and only if $k' = v/(2k)$, in which case $2k \mid v$. □

The first case in Theorem 2.4, where the master design is a 3 -design, is the case that was studied in [2]. The second case, where $k = 2$, was also studied in [2]. We will study the third case in more detail. First, we record the designs that are constructed in this case.

Corollary 2.5. *Suppose the following designs exist:*

1. a resolvable (v, b, r, k, λ) -BIBD, and

2. a $3-(w, w/2, \lambda')$ design, where $w = v/k > 4$ is even.

Then there exists a $3-(v, v/2, \mu)$ design, where

$$\mu = \lambda' \left(\frac{3\lambda w}{w-4} + r \right) \quad (4)$$

Proof. The value of μ can be obtained from (3), where the coefficient of α is equal to 0. We have

$$\mu = \lambda' \left(3\lambda \frac{w-2}{\frac{w}{2}-2} + r - 3\lambda \right) = \lambda' \left(\frac{3\lambda w}{w-4} + r \right).$$

□

Theorem 2.4 requires that $k' > 2$. It is also useful to consider the variant where $k' = 2$; in this case, we can take (Y, \mathcal{C}) to be a (trivial) $2-(w, 2, 1)$ -design. As in Lemma 2.3, suppose that three points x, y, z occur in exactly α blocks in \mathcal{B} . From (2) with $\lambda' = 0$, we have that

$$\lambda_{x,y,z} = \alpha(w-1) + 3(\lambda - \alpha) = \alpha(w-4) + 3\lambda. \quad (5)$$

If α is not constant, it is clear that $\lambda_{x,y,z}$ will be constant if and only if $w = 4$. In this case, the indexing design is a $2-(4, 2, 1)$ -design, which is a one-factorization of K_4 . Therefore, we have the following variant of Corollary 2.5.

Corollary 2.6. *Suppose there is a resolvable (v, b, r, k, λ) -BIBD, where $v/k = 4$. Then there is a $3-(v, v/2, 3\lambda)$ -design.*

2.1 Resolvability of the Constructed Designs

In this section, we consider when the constructed design will be resolvable. This question is easy to answer; we state the following simple result without proof.

Lemma 2.7. *The constructed design (X, \mathcal{D}) in Construction 1 is resolvable whenever the indexing design (Y, \mathcal{C}) is resolvable.*

However, if we look at Corollaries 2.5 and 2.6, we can proceed in a slightly different manner. Both of these corollaries yield 3-designs with block size $v/2$. A result of Tran [6, Theorem 2.7] shows that a resolvable $3-(v, v/2, \mu)$ -design exists whenever a $3-(v, v/2, \mu)$ -design exists. Thus we have the following extensions of Corollaries 2.5 and 2.6.

Corollary 2.8. *Suppose the following designs exist:*

1. a resolvable (v, b, r, k, λ) -BIBD, and
2. a $3-(w, w/2, \lambda')$ design, where $w = v/k > 4$ is even.

Then there exists a resolvable $3-(v, v/2, \mu)$ design, where

$$\mu = \lambda' \left(\frac{3\lambda w}{w-4} + r \right).$$

Corollary 2.9. *Suppose there is a resolvable (v, b, r, k, λ) -BIBD where $v/k = 4$. Then there is a resolvable $3-(v, v/2, 3\lambda)$ -design.*

3 Nontriviality of the Constructed Designs

It is easy to see that Construction 1 will often produce nontrivial designs. Here is a small example to illustrate.

Example 3.1. Suppose we start with the trivial $(18, 816, 136, 3)$ -IBD consisting of all the 3-subsets of an 18-set. This design is resolvable into $r = 136$ parallel classes by Baranyi's Theorem. Apply Construction 1 where the indexing design is a $(6, 20, 10, 3)$ -IBD consisting of all the 3-subsets of a 6-set. The constructed design contains 2720 blocks. However, a trivial IBD with 18 points and block size 9 contains $\binom{18}{9} = 48620$ blocks. Therefore the constructed design is nontrivial. \square

More generally, we will show that the constructed design will be nontrivial whenever the master design and indexing design are both *simple* designs. This is because a trivial design with block size $v/2$ contains $\binom{v}{v/2}$ blocks, and the number of blocks in the constructed design will turn out to be smaller than that. Suppose $2k$ divides v and $v/k > 4$. We apply Construction 1 with the master design being a simple design. The number of blocks in the master design is at most $\binom{v}{k}$ and therefore $r \leq \binom{v-1}{k-1}$. Since the indexing design is simple, it follows that the number of blocks in the constructed design is at most

$$r \binom{\frac{v}{k}}{\frac{v}{2k}} \leq \binom{v-1}{k-1} \binom{\frac{v}{k}}{\frac{v}{2k}}.$$

If we can prove the inequality

$$\binom{v-1}{k-1} \binom{\frac{v}{k}}{\frac{v}{2k}} < \binom{v}{\frac{v}{2}}, \quad (6)$$

then we will have shown that the constructed design is nontrivial. To do this, we will make use of the well-known inequalities

$$\binom{v-1}{k-1} < \left(\frac{v}{k}\right)^{k-1} \quad (7)$$

and

$$\frac{4^n}{2\sqrt{n}} < \binom{2n}{n} < \frac{4^n}{\sqrt{\pi n}}. \quad (8)$$

From (7) and (8), it is clear that (6) will hold provided that we can prove that the following inequality holds:

$$\left(\frac{v}{k}\right)^{k-1} \frac{4^{v/2k}}{\sqrt{\pi(v/2k)}} < \frac{4^{v/2}}{2\sqrt{v/2}}. \quad (9)$$

After some simplification, it is easy to see that (9) is equivalent to

$$\left(\frac{v}{k}\right)^{k-1} < 2^{v(k-1)/k} \sqrt{\frac{\pi}{4k}}. \quad (10)$$

The inequality (10) can be rewritten as

$$\left(\frac{2^{v/k}}{\frac{v}{k}}\right)^{k-1} > \sqrt{\frac{4k}{\pi}}. \quad (11)$$

Since $v/k \geq 4$, the left side of (11) is at least 4^{k-1} . Since $k \geq 2$, it is easy to see that $4^{k-1} \geq 2k$. Finally it is obvious that $2k > \sqrt{\frac{4k}{\pi}}$. Therefore (11) holds, and we have proven the following.

Theorem 3.1. *Suppose that $2k$ divides v and $v/k \geq 4$. Suppose we apply Construction 1 where the master and indexing designs are both simple designs. Then the constructed design is nontrivial.*

4 Simple Designs

In certain situations, Corollaries 2.5 and 2.6 yield simple designs. This is the theme we pursue in this section.

Suppose (X, \mathcal{B}) is a resolvable (v, b, r, k) -IBD and let $w = v/k$. Suppose the assumed resolution of (X, \mathcal{B}) consists of r parallel classes Π_1, \dots, Π_r , where $\Pi_i = \{B_i^1, \dots, B_i^w\}$ for $1 \leq i \leq r$. Let $1 \leq \alpha \leq w - 1$ be an integer. We say that two parallel classes Π_i and Π_j satisfy the α -*partial replacement property* (or *PRP*) if there exist two parallel classes Π_i^* and Π_j^* of blocks in \mathcal{B} such that

$$\Pi_i^* \cup \Pi_j^* = \Pi_i \cup \Pi_j \quad \text{and} \quad |\Pi_i^* \cap \Pi_i| = \alpha. \quad (12)$$

We say that the resolution of (X, \mathcal{B}) is α -*PRP-free* if there do not exist two parallel classes in $\{\Pi_1, \dots, \Pi_r\}$ that satisfy α -PRP. Further, we say that the resolution of (X, \mathcal{B}) is *PRP-free* if there do not exist two parallel classes in $\{\Pi_1, \dots, \Pi_r\}$ that satisfy α -PRP for any positive integer $\alpha < w$.

As an example of a resolvable BIBD that is *not* PRP-free, consider a one-factorization of K_{4n} that contains a sub-one-factorization of K_{2n} . Each one-factor of a K_{4n} has $2n$ edges and each one-factor of a K_{2n} contains n edges. There are $2n - 1$ one-factors of the K_{4n} that contain a one-factor of the K_{2n} . The n edges in any one-factor of K_{2n} may be swapped with the n edges in any other one-factor of K_{2n} . Therefore, the two corresponding one-factors in the one-factorization of K_{4n} satisfy n -PRP.

Lemma 4.1. *If a (v, b, r, k) -IBD has a unique resolution, then it is PRP-free.*

Proof. Let $w = v/k$. Suppose there are two parallel classes Π_i and Π_j that satisfy α -PRP for some α , where $1 \leq \alpha \leq w - 1$. Let Π_i^* and Π_j^* be the two parallel classes that satisfy (12). If we replace Π_i and Π_j by Π_i^* and Π_j^* , then we obtain a second resolution of the given design. \square

Theorem 4.2. *Suppose we apply Construction 1 with the master design having a resolution that is k' -PRP-free, where the indexing design has block size k' . Then the constructed design is simple. Furthermore, in the case where the indexing design is trivial, the constructed design is simple if and only if the resolution of the master design is k' -PRP-free.*

Proof. Suppose that (X, \mathcal{D}) is not simple; then $D_{i,C} = D_{i',C'}$ where $(i, C) \neq (i', C')$. If $i = i'$ and $D_{i,C} = D_{i',C'}$, then $C = C'$. Therefore, we can assume $i \neq i'$. If we define

$$\Pi_i^* = \{B_i^j : j \in C\} \cup \{B_{i'}^j : j \notin C'\}$$

and

$$\Pi_{i'}^* = \{B_{i'}^j : j \in C'\} \cup \{B_i^j : j \notin C\},$$

then we see that the parallel classes Π_i and $\Pi_{i'}$ in (X, \mathcal{B}) satisfy k' -PRP.

Now, assume that the indexing design is trivial and there are two parallel classes Π_i and $\Pi_{i'}$ in the resolution of (X, \mathcal{B}) that satisfy k' -PRP. So there exist Π_i^* and $\Pi_{i'}^*$ such that

$$\Pi_i^* \cup \Pi_{i'}^* = \Pi_i \cup \Pi_{i'} \quad \text{and} \quad |\Pi_i^* \cap \Pi_i| = k'.$$

Let $C = \{j : B_i^j \in \Pi_i^* \cap \Pi_i\}$ and $C' = \{j : B_{i'}^j \in \Pi_{i'}^* \cap \Pi_{i'}\}$. Note that C and C' are both blocks in the indexing design, because the indexing design is trivial. Then it is easy to see that $D_{i,C} = D_{i',C'}$, so the constructed design is not simple. \square

We give a well-known class of BIBDs having unique resolutions.

Lemma 4.3. *For all prime powers q and for all integers $m \geq 2$, there exists a PRP-free resolvable $(q^m, q^{m-1}, (q^{m-1} - 1)/(q - 1)$ -BIBD.*

Proof. The hyperplanes of the m -dimensional affine geometry $\text{AG}(m, q)$ over \mathbb{F}_q yield a resolvable $(q^m, q^{m-1}, (q^{m-1} - 1)/(q - 1)$ -BIBD where each parallel class consists of q blocks. Furthermore, any two blocks from different parallel classes intersect in exactly q^{m-2} points. From this fact, it is easily seen that this design has a unique resolution. Hence, from Lemma 4.1, the resolution is PRP-free. \square

Remark. *Any affine resolvable BIBD has a unique resolution and therefore is PRP-free.*

Corollary 4.4. *Suppose $q = 2^n > 4$ and there exists a 3 -($q, q/2, \lambda'$) design. Then there exists a simple 3 -(q^m, q^{m-1}, μ) design, where*

$$\mu = \lambda' \left(\frac{q^m - 4}{q - 4} \right).$$

Proof. We apply Theorem 4.2, starting with the resolvable $(q^m, q^{m-1}, (q^{m-1} - 1)/(q - 1)$ -BIBD from Lemma 4.3. This design has $r = (q^{m-1} - 1)/(q - 1)$ and $w = q$. Corollary 2.5 establishes that the constructed design will be a 3 -(q^m, q^{m-1}, μ)-design, where

$$\begin{aligned} \mu &= \lambda' \left(\frac{3\lambda w}{w - 4} + r \right) \\ &= \lambda' \left(\frac{3(q^{m-1} - 1)}{q - 1} \times \frac{q}{q - 4} + \frac{q^m - 1}{q - 1} \right) \\ &= \lambda' \left(\frac{3(q^m - q) + (q^m - 1)(q - 4)}{(q - 1)(q - 4)} \right) \\ &= \lambda' \left(\frac{q^{m+1} - q^m - 4q + 4}{(q - 1)(q - 4)} \right) \\ &= \lambda' \left(\frac{q^m - 4}{q - 4} \right). \end{aligned}$$

Finally, Theorem 4.2 shows that the constructed design is simple. \square

As an example, suppose we take $q = 8$ and $m = 2$. Here, the master design is the affine plane of order 8, i.e., a resolvable $(64, 72, 9, 8, 1)$ -BIBD. If the indexing design is a 3 -($8, 4, 1$)-design, then the constructed 3 -($64, 32, 15$)-design is simple. If the indexing design is the trivial 3 -($8, 4, 5$)-design, then the constructed 3 -($64, 32, 75$)-design is again simple.

It also turns out that many 3 -designs constructed from Corollary 2.8 are simple, even if we cannot prove theoretically that they are. In fact, we have checked several designs by computer and verified computationally that they are simple 3 -designs. These designs all arise from trivial indexing designs. They are listed in Table 1 and details are provided in the rest of this section.

Table 1: Simple 3-designs constructed from Corollaries 2.5 and 2.6

master design	w	constructed design
2-(24, 4, 3)	6	3-(24, 12, 50)
2-(24, 6, 5)	4	3-(24, 12, 15)
2-(24, 3, 2)	8	3-(24, 12, 175)
2-(28, 7, 6)	4	3-(28, 14, 18)
2-(30, 5, 4)	6	3-(30, 15, 65)
2-(30, 3, 2)	10	3-(30, 15, 819)
2-(32, 8, 7)	4	3-(32, 16, 21)
2-(36, 9, 8)	4	3-(36, 18, 24)

4.1 A Simple 3-(24, 12, 15)-design

We construct a simple 3-(24, 12, 15)-design on point set $X = \{0, 1, \dots, 22, \infty\}$. The cyclic group \mathbb{Z}_{23} acts on the design. \mathbb{Z}_{23} permutes the points $\{0, 1, \dots, 22\}$ (cyclically) and fixes the point ∞ .

We start with a resolvable (24, 6, 5)-BIBD having a cyclic automorphism of order 23 (see [1, p. 408]). The four base blocks of the BIBD form a parallel class:

$$\{\infty, 0, 1, 7, 15, 20\}, \{2, 3, 4, 5, 10, 14\}, \{6, 11, 13, 17, 19, 22\}, \{8, 9, 12, 16, 18, 21\}.$$

Here are the six base blocks of the constructed 3-(24, 12, 15) design:

$$\begin{aligned} B_1 &= \{\infty, 0, 1, 7, 15, 20, 2, 3, 4, 5, 10, 14\} \\ B_2 &= \{\infty, 0, 1, 7, 15, 20, 6, 11, 13, 17, 19, 22\} \\ B_3 &= \{\infty, 0, 1, 7, 15, 20, 8, 9, 12, 16, 18, 21\} \\ B_4 &= \{2, 3, 4, 5, 10, 14, 6, 11, 13, 17, 19, 22\} \\ B_5 &= \{2, 3, 4, 5, 10, 14, 8, 9, 12, 16, 18, 21\} \\ B_6 &= \{6, 11, 13, 17, 19, 22, 8, 9, 12, 16, 18, 21\}. \end{aligned}$$

The other blocks are obtained by developing the base blocks through \mathbb{Z}_{23} .

The next line gives data about intersection numbers of the design:

$$69, 0, 46, 0, 506, 2208, 3864, 2208, 506, 0, 46, 0, 0.$$

There are 13 numbers corresponding to 13 possible intersection numbers, namely $0, 1, 2, \dots, 12$. For example, 69 pairs of blocks have zero intersection; they form the parallel classes of the design. The second value equals 0, which says that there are no pairs of blocks intersecting in 1 point. The third value equals 46 pairs, which says that there are 46 pairs of blocks intersecting in 2 points, etc. The data show that any two blocks of the design intersect in at most 10 points, so the design is simple.

4.2 A Simple 3-(28, 14, 18)-design

We construct a simple 3-(28, 14, 18)-design on point set $X = \{0, 1, \dots, 26, \infty\}$. The cyclic group \mathbb{Z}_{27} acts on the design. \mathbb{Z}_{27} permutes the points $\{0, 1, \dots, 26\}$ (cyclically) and fixes the point ∞ .

We start with a resolvable $(28, 7, 6)$ -BIBD having a cyclic automorphism of order 27 (see [1, p. 408]). The four base blocks of the BIBD form a parallel class:

$$\{\infty, 0, 5, 14, 15, 24, 25\}, \{1, 10, 17, 20, 22, 23, 26\}, \{2, 6, 8, 9, 13, 19, 21\}, \{3, 4, 7, 11, 12, 16, 18\}.$$

Here are the six base blocks of the constructed 3 -(28, 14, 18) design:

$$\begin{aligned} B_1 &= \{\infty, 0, 5, 14, 15, 24, 25, 1, 10, 17, 20, 22, 23, 26\} \\ B_2 &= \{\infty, 0, 5, 14, 15, 24, 25, 2, 6, 8, 9, 13, 19, 21\} \\ B_3 &= \{\infty, 0, 5, 14, 15, 24, 25, 3, 4, 7, 11, 12, 16, 18\} \\ B_4 &= \{1, 10, 17, 20, 22, 23, 26, 2, 6, 8, 9, 13, 19, 21\} \\ B_5 &= \{1, 10, 17, 20, 22, 23, 26, 3, 4, 7, 11, 12, 16, 18\} \\ B_6 &= \{2, 6, 8, 9, 13, 19, 21, 3, 4, 7, 11, 12, 16, 18\}. \end{aligned}$$

The other blocks are obtained by developing the base blocks through \mathbb{Z}_{27} .

The next line gives the intersection numbers of the design:

$$81, 0, 0, 0, 54, 1080, 3132, 4428, 3132, 1080, 54, 0, 0, 0, 0.$$

This data shows that the design is simple.

4.3 A Simple 3 -(30, 15, 65)-design

We construct a simple 3 -(30, 15, 65)-design on point set $X = \{0, 1, \dots, 28, \infty\}$. The cyclic group \mathbb{Z}_{29} acts on the design. \mathbb{Z}_{29} permutes the points $\{0, 1, \dots, 28\}$ (cyclically) and fixes the point ∞ .

We start with a resolvable $(30, 5, 4)$ -BIBD having a cyclic automorphism of order 29 (see [1, p. 408]). The six base blocks of the BIBD form a parallel class:

$$\{\infty, 4, 8, 17, 18\}, \{0, 1, 7, 24, 27\}, \{2, 12, 15, 16, 23\}, \\ \{9, 11, 13, 21, 26\}, \{3, 14, 20, 22, 25\}, \{5, 6, 10, 19, 28\}.$$

Here are the 20 base blocks of the constructed 3 -(30, 15, 65) design:

$$\begin{aligned} B_1 &= \{\infty, 4, 8, 17, 18, 0, 1, 7, 24, 27, 2, 12, 15, 16, 23\} \\ B_2 &= \{\infty, 4, 8, 17, 18, 0, 1, 7, 24, 27, 9, 11, 13, 21, 26\} \\ B_3 &= \{\infty, 4, 8, 17, 18, 0, 1, 7, 24, 27, 3, 14, 20, 22, 25\} \\ B_4 &= \{\infty, 4, 8, 17, 18, 0, 1, 7, 24, 27, 5, 6, 10, 19, 28\} \\ B_5 &= \{\infty, 4, 8, 17, 18, 2, 12, 15, 16, 23, 9, 11, 13, 21, 26\} \\ B_6 &= \{\infty, 4, 8, 17, 18, 2, 12, 15, 16, 23, 3, 14, 20, 22, 25\} \\ B_7 &= \{\infty, 4, 8, 17, 18, 2, 12, 15, 16, 23, 5, 6, 10, 19, 28\} \\ B_8 &= \{\infty, 4, 8, 17, 18, 9, 11, 13, 21, 26, 3, 14, 20, 22, 25\} \\ B_9 &= \{\infty, 4, 8, 17, 18, 9, 11, 13, 21, 26, 5, 6, 10, 19, 28\} \\ B_{10} &= \{\infty, 4, 8, 17, 18, 3, 14, 20, 22, 25, 5, 6, 10, 19, 28\} \\ B_{11} &= \{0, 1, 7, 24, 27, 2, 12, 15, 16, 23, 9, 11, 13, 21, 26\} \\ B_{12} &= \{0, 1, 7, 24, 27, 2, 12, 15, 16, 23, 3, 14, 20, 22, 25\} \\ B_{13} &= \{0, 1, 7, 24, 27, 2, 12, 15, 16, 23, 5, 6, 10, 19, 28\} \\ B_{14} &= \{0, 1, 7, 24, 27, 9, 11, 13, 21, 26, 3, 14, 20, 22, 25\} \end{aligned}$$

$$\begin{aligned}
B_{15} &= \{0, 1, 7, 24, 27, 9, 11, 13, 21, 26, 5, 6, 10, 19, 28\} \\
B_{16} &= \{0, 1, 7, 24, 27, 3, 14, 20, 22, 25, 5, 6, 10, 19, 28\} \\
B_{17} &= \{2, 12, 15, 16, 23, 9, 11, 13, 21, 26, 3, 14, 20, 22, 25\} \\
B_{18} &= \{2, 12, 15, 16, 23, 9, 11, 13, 21, 26, 5, 6, 10, 19, 28\} \\
B_{19} &= \{2, 12, 15, 16, 23, 3, 14, 20, 22, 25, 5, 6, 10, 19, 28\} \\
B_{20} &= \{9, 11, 13, 21, 26, 3, 14, 20, 22, 25, 5, 6, 10, 19, 28\}.
\end{aligned}$$

The other blocks are obtained by developing the base blocks through \mathbb{Z}_{29} .

The next line gives the intersection numbers of the design:

$$290, 0, 0, 58, 1044, 10382, 24940, 47386, 47386, 24940, 10382, 1044, 58, 0, 0, 0.$$

This data shows that the design is simple.

4.4 A Simple 3-(24, 12, 50)-design

A 3-(24, 12, 50)-design constructed from a resolvable (24, 4, 3)-BIBD having a cyclic automorphism of order 23 (see [1, p. 407]). The six base blocks of the BIBD form a parallel class:

$$\{\infty, 0, 7, 10\}, \{1, 8, 12, 22\}, \{2, 5, 6, 11\}, \{3, 9, 14, 18\}, \{4, 16, 17, 19\}, \{13, 15, 20, 21\}.$$

The intersection numbers of the 3-(24, 12, 50)-design (from 0 to 12 respectively) are

$$230, 0, 0, 1242, 9982, 23414, 36064, 23414, 9982, 1242, 0, 0, 0.$$

4.5 A Simple 3-(24, 12, 175)-design

A 3-(24, 12, 175)-design constructed from a resolvable (24, 3, 2)-BIBD having a cyclic automorphism of order 23 (see [1, p. 407]). The eight base blocks of the BIBD form a parallel class:

$$\{\infty, 16, 20\}, \{0, 7, 21\}, \{1, 3, 11\}, \{4, 5, 18\}, \{6, 12, 17\}, \{2, 10, 13\}, \{8, 9, 14\}, \{15, 19, 22\}.$$

The intersection numbers of the 3-(24, 12, 175)-design (from 0 to 12 respectively) are:

$$805, 46, 1380, 30314, 99958, 289432, 452180, 289432, 99958, 30314, 1380, 46, 0.$$

4.6 A Simple 3-(30, 15, 819)-design

A 3-(30, 15, 819)-design constructed from a resolvable (30, 3, 2)-BIBD having a cyclic automorphism of order 29 (see [1, p. 407]). The ten base blocks of the BIBD form a parallel class:

$$\begin{aligned}
&\{\infty, 2, 22\}, \{0, 1, 19\}, \{6, 8, 27\}, \{9, 15, 16\}, \{7, 10, 14\}, \\
&\{11, 25, 28\}, \{12, 18, 23\}, \{13, 21, 26\}, \{4, 20, 24\}, \{3, 5, 17\}.
\end{aligned}$$

The intersection numbers of the 3-(30, 15, 819) are

$$\begin{aligned}
&3654, 0, 928, 115594, 242730, 1356272, 4482530, 7150008, \\
&7150008, 4482530, 1356272, 242730, 115594, 928, 0, 0.
\end{aligned}$$

4.7 A Simple 3-(32, 16, 21)-design

A 3-(32, 16, 21)-design constructed from a resolvable (32, 8, 7)-BIBD having a cyclic automorphism of order 31 (see [1, p. 408]). The four base blocks of the BIBD form a parallel class:

$$\{\infty, 0, 1, 5, 16, 18, 25, 28\}, \{3, 9, 17, 19, 20, 21, 24, 26\}, \\ \{2, 6, 7, 12, 14, 15, 23, 27\}, \{4, 8, 10, 11, 13, 22, 29, 30\}.$$

The intersection numbers of the 3-(32, 16, 21)-design are

$$93, 0, 0, 0, 0, 372, 930, 4836, 4836, 4836, 930, 372, 0, 0, 0, 0, 0.$$

4.8 A Simple 3-(36, 18, 24)-design

A 3-(36, 18, 24)-design constructed from a resolvable (36, 9, 8)-BIBD having a cyclic automorphism of order 35 (see [1, p. 408]). The four base blocks of the BIBD form a parallel class:

$$\{\infty, 1, 10, 11, 13, 15, 25, 26, 29\}, \{0, 9, 16, 17, 18, 19, 22, 24, 30\}, \\ \{3, 4, 6, 12, 21, 23, 27, 31, 34\}, \{2, 5, 7, 8, 14, 20, 28, 32, 33\}.$$

The intersection numbers of the 3-(36, 18, 24)-design are

$$105, 0, 0, 0, 0, 70, 280, 2100, 4970, 7000, 4970, 2100, 280, 70, 0, 0, 0, 0, 0.$$

5 Conclusion

There are probably many other possible applications of the constructions in this paper to obtain (new) simple 3-designs. It would be very nice to find additional easily checked conditions that would guarantee that a constructed 3-design is simple.

References

- [1] C.J. Colbourn and J.H. Dinitz. *Handbook of Combinatorial Designs, Second Edition*, Chapman & Hall/ CRC, 2006.
- [2] M. Jimbo, Y. Kuniyama, R. Laue, and Masanori Sawa. Unifying some known infinite families of combinatorial 3-designs. *Journal of Combinatorial Theory A*, **118** (2011), 1072–1085.
- [3] D. Jungnickel and S.A. Vanstone. On resolvable designs $S_3(3, 4, v)$. *Journal of Combinatorial Theory A*, **43** (1986), 334–337.
- [4] K.T. Phelps, D.R. Stinson, and S.A. Vanstone. The existence of simple $S_3(3, 4, v)$. *Discrete Mathematics* **77** (1989), 255–258.
- [5] S.S. Shrikhande and D. Raghavarao. A method of construction of incomplete block designs, *Sankhyā A* **25** (1963) 399–402.
- [6] Tran van Trung. Recursive constructions for 3-designs and resolvable 3-designs, *Journal of Statistical Planning and Inference* **95** (2001), 341–358.