An extending theorem for s-resolvable t-designs

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Abstract

An extending theorem for s-resolvable t-designs is presented, which may be viewed as an extension of Qiu-rong Wu's result. The theorem yields recursive constructions for s-resolvable t-designs, and mutually disjoint t-designs. For example, it can be shown that if there exists a large set $LS[29](4,5,33)$, then there exists a family of 3-resolvable 4-$(5+29m, 6, \frac{5}{2}m(1+29m))$ designs for $m \geq 1$, with 5 resolution classes. Moreover, for any given integer $h \geq 1$, there exist $(5 \cdot 2^h - 5)$ mutually disjoint simple 3-$(3+m(5 \cdot 2^h - 3), 4, m)$ designs for all $m \geq 1$. In addition, we give a brief account of t-designs derived from the result of Wu.

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1 Introduction

In [11] Teirlinck presented a recursive construction method for large sets $LS[n](t, t+1, v)$, and also mentioned (on page 351) that the result was implicit in his earlier paper [10], but it is by no means obvious. In [4] Khosrosvshahi and Ajoodani-Namini gave a theorem for extending t-designs for $k = t+1$, whose application to large sets provided the same result as that of Teirlinck. Shortly after, Qiu-rong Wu [14] generalized the construction for any $k \geq t+1$ and obtained a striking result on extending t-designs and large sets. Based on Wu's result, Kramer, Magliveras and O'Brien [5] proved among others the existence of large sets $LS[3](4, 6, 9m+5)$ for any $m \geq 1$. And Kreher [7] showed the existence of $LS[2](6, 8, 16m+23)$ for all $m \geq 0$. In the present paper we are interested in simple s-resolvable t-designs and we will prove an extending theorem for s-resolvable t-designs along the lines of the extending theorem of Wu. It should be mentioned that a general method for constructing t-designs was presented in [12], in which s-resolvable t-designs play a crucial role, and the first investigation of these designs was recently given in [13].

We recall a few basic definitions. A t-design, denoted by t-(v, k, λ), is a pair $(X, \mathcal{B})$, where X is a v-set of *points* and $\mathcal{B}$ is a collection of k-subsets of X, called *blocks*, such that every t-subset of X is a subset of exactly λ blocks of $\mathcal{B}$. A t-design is called *simple* if no two blocks are identical, otherwise, it is called *non-simple*. All designs in this paper are simple designs. For any fixed subset Y of X with $|Y| = u \leq t$, define $\mathcal{B}_Y = \{B \setminus Y : Y \subset B \in \mathcal{B}\}$. Then $(X \setminus Y, \mathcal{B}_Y)$ is a $(t-u)$-$(v-u, k-u, \lambda)$ design, called a *derived design* of $(X, \mathcal{B})$. It is well-known that a t-(v, k, λ) design is also an s-(v, k, λ_s) design for $0 \leq s \leq t$, where $\lambda_s = \lambda\binom{v-s}{t-s}/\binom{k-s}{t-s}$. If $\mathcal{B}$ is the set of all k-subsets of X, then $(X, \mathcal{B})$ is a t-$(v, k, \binom{v-t}{k-t})$ design, called the *complete* design or the *trivial* design. A t-(v, k, λ) design $(X, \mathcal{B})$ is said to be s-resolvable, for $0 < s < t$, if its block set $\mathcal{B}$ can be partitioned into $N \geq 2$ classes $\mathcal{A}_1, \ldots, \mathcal{A}_N$ such that each $(X, \mathcal{A}_i)$ is an s-(v, j, δ) design for $i = 1, \ldots, N$. Each $\mathcal{A}_i$ is called an s-resolution class or simply a resolution class and the set of N classes is called an s-resolution of $(X, \mathcal{B})$. If the complete k-$(v, k, 1)$ design is t-resolvable with N resolution classes, where each class is a t-(v, k, λ) design, then we say that there exists a *large set* of size N of t-designs denoted by $LS[N](t, k, v)$ or by $LS_\lambda(t, k, v)$ to emphasize the value λ. Moreover, if there is an $LS[N](t, k, v)$, then there is an $LS[N](t-u, k-u, v-u)$, for $u \leq t$.

For more information about s-resolvable t-designs with $1 < s < t$ we refer the reader to [12, 13]. It should be remarked that s-resolvable t-designs have been used in the construction of t-designs [12].

2 The main theorem of Wu and a proposition of Teirlinck

We begin by recalling the main extending theorem for t-designs of Qiu-rong Wu in [14] and a proposition of Teirlinck about large sets in [11].

Theorem 2.1 (Wu) *Suppose that there exist*

(i) simple t-(v_1, k, λ_1) and t-(v_2, k, λ_2) designs D_1 and D_2 such that $\frac{\lambda_1}{\binom{v_1-t}{k-t}} = \frac{\lambda_2}{\binom{v_2-t}{k-t}} = z$;

(ii) $LS[n](k-2, k-1, v_1-1)$ and $LS[n](k-2, k-1, v_2-1)$, where n is an integer such that zn is an integer.

Then there exists a simple t-$(v_1+v_2-k+1, k, \lambda)$ design D_3 with $\lambda = z\binom{v_1+v_2-k+1-t}{k-t}$.

Corollary 2.2 (Wu) *Suppose that there exist a simple t-(v, k, λ) design with $z = \frac{\lambda}{\binom{v-t}{k-t}}$ and a large set $LS[n](k-2, k-1, v-1)$, where n is an integer such that zn is an integer. Then there exists a simple t-$(v+m(v-k+1), k, z\binom{v-t+m(v-k+1)}{k-t})$ design, for any $m > 0$.*

Proposition 2.3 (Teirlinck) *If there exists a large set $LS[n](t, t+1, v)$, then there exists a large set $LS[n](t, t+1, v+m(v-t))$ for any positive integer m.*

Note that Proposition 2.3 is Proposition 9 given in [11], where Teirlinck also mentioned that it was implicitly in [10].

3 An Extending Theorem for s-resovable t-designs

Theorem 3.1 *Let D_1 and D_2 be simple t-(v_1, k, λ_1) and t-(v_2, k, λ_2) designs respectively such that $\frac{\lambda_1}{\binom{v_1-t}{k-t}} = \frac{\lambda_2}{\binom{v_2-t}{k-t}} = z$. Suppose that*

(i) D_1 and D_2 are both s-resolvable with N resolution classes and $z = \frac{Nu}{n}$, where u, n are positive integers;

(ii) there exist $LS[n](k-2, k-1, v_1-1)$ and $LS[n](k-2, k-1, v_2-1)$.

Then there exists a simple s-resolvable t-$(v_1+v_2-k+1, k, \lambda)$ design D_3 with N resolution classes, where $\lambda = z\binom{v_1+v_2-k+1-t}{k-t}$.

The following simple lemma is needed for the proof of Theorem 3.1.

Lemma 3.2 *Let $(X, \mathcal{D})$ be a t-(v, k, λ) design such that $z = \frac{\lambda}{\binom{v-t}{k-t}} = \frac{Nu}{n}$, where N, u, n are positive integers. Suppose that $(X, \mathcal{D})$ is s-resolvable with N resolution classes, where each class is an s-(v, k, δ_s) design. Then $z' = \frac{\delta_s}{\binom{v-s}{k-s}} = \frac{u}{n}$.*

Proof. First note that $\lambda_s = \lambda\frac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}}$. Since $(X, \mathcal{D})$ is a disjoint union of N designs with parameters s-(v, k, δ_s), we have $\lambda_s = N\delta_3$. Thus

$$\delta_s = \frac{\lambda_s}{N} = \frac{u}{n}\binom{v-t}{k-t}\frac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}}.$$

This simplifies to

$$\delta_s = \frac{u}{n}\binom{v-s}{k-s}.$$

Hence $z' = \frac{\delta_3}{\binom{v-s}{k-s}} = \frac{u}{n}$. □

Proof. (of Theorem 3.1)

The proof consists of two parts. Part 1 is the construction of D_3. Part 2 is the proof of s-resolvability of D_3. Part 1 is the proof of Theorem 2.1, given by Qui-rong Wu in [14]. In order to follow the proof in Part 2 we need to describe the construction of D_3 in Part 1.

Part 1: Construction of D_3.

Let $X = \{1, 2, \ldots, v_1+v_2-k+1\}$. Define $X_j = \{1, 2, \ldots, v_1-j\}$, $j = 0, 1, \ldots, k-1$, and $Y_j = \{v_1+2-j, v_1+3-j, \ldots, v_1+v_2-k+1\}$, $j = 1, 2, \ldots, k$.

Note that $X_j \cup Y_j = X \setminus \{v_1+1-j\}$, for $0 < j < k$.

Partition $P_k(X)$ into $(k+1)$ classes: $C_0, \dots, C_k$, as follows. $C_0 = P_k(X_0)$, $C_k = P_k(Y_k)$, and for $0 < j < k$, $C_j = \{A \cup A',\ A \in P_{k-j}(X_j),\ A' \in P_j(Y_j)\}$.

Let $P_t(X) = \{T_1, T_2, \dots, T_{\binom{v_1+v_2-k+1}{t}}\}$ be the set of all t-subsets of X, and let n_{ij} denote the number of blocks $B \in C_j$ containing T_i. Then

$$\sum_{j=0}^{k} n_{ij} \quad = \quad \binom{v_1+v_2-k+1-t}{k-t} \tag{1}$$

The main idea is to *construct a collection $\mathcal{B}_j$ of k-subsets of X from C_j such that any t-subset T_i of X is contained in zn_{ij} blocks in $\mathcal{B}_j$ for $j = 0, \dots, k$.* Thus by equation (1), $(X, \bigcup_{j=0}^{k} \mathcal{B}_j)$ is a t-$(v_1+v_2-k+1, k, z\binom{v_1+v_2-k+1-t}{k-t})$ design.

The description of $\mathcal{B}_j$ is as follows. Consider two cases (a): $j = 0, k$ and (b): $0 < j < k$.

Case (a): $j = 0, k$.
$\mathcal{B}_0$ is a collection of k-subsets of X_0 such that $(X_0, \mathcal{B}_0)$ is a copy of the t-(v_1, k, λ_1) design D_1. $\mathcal{B}_k$ is a collection of k-subsets of Y_k such that $(Y_k, \mathcal{B}_k)$ is a copy of the t-(v_2, k, λ_2) design D_2.

Then it is clear that if $T_i \subset X_0$, then $zn_{i,0} = \lambda_1$. Thus T_i is contained in $zn_{i,0}$ blocks of $\mathcal{B}_0$. Similarly, if $T_i \subset Y_k$, then T_i is contained in $zn_{i,k}$ blocks of $\mathcal{B}_k$.

Case (b): $0 < j < k$.
The construction of $\mathcal{B}_j$, $0 < j < k$, is based on the large sets $LS[n](k-2, k-1, v_1-1)$ and $LS[n](k-2, k-1, v_2-1)$ and their derived large sets. First, consider $X_1 = \{1, 2, \dots, v_1-1\}$. Let $(X_1, \mathcal{A}_{1,1}), (X_1, \mathcal{A}_{2,1}), \dots, (X_1, \mathcal{A}_{n,1})$ be a large set of $(k-2)$-$(v_1-1, k-1, \frac{v_1-k-1}{n})$ designs. Now $X_1 \setminus X_j = \{v_1-j+1, v_1-j+2, \dots, v_1-1\}$. Deleting the points $v_1-j+1, v_1-j+2, \dots, v_1-1$ gives the corresponding derived designs $(X_j, \mathcal{A}_{1,j}), (X_j, \mathcal{A}_{2,j}), \dots, (X_j, \mathcal{A}_{n,j})$ which form a large set of $(k-1-j)$-$(v_1-j, k-j, \frac{v_1-k-1}{n})$ designs.

Similarly, consider $Y_{k-1} = \{v_1-k+3, v_1-k+4, \dots, v_1-k+v_2+1\}$. Let $(Y_{k-1}, \mathcal{A}'_{1,k-1}), (Y_{k-1}, \mathcal{A}'_{2,k-1}), \dots, (Y_{k-1}, \mathcal{A}'_{n,k-1})$ be a large set of $(k-2)$-$(v_2-1, k-1, \frac{v_2-k-1}{n})$ designs. By deleting the points $v_1+3-k, v_1+4-k, \dots, v_1+1-j$ we obtain the corresponding derived designs $(Y_j, \mathcal{A}'_{1,j}), (Y_j, \mathcal{A}'_{2,j}), \dots, (Y_j, \mathcal{A}'_{n,j})$ which form a large set of $(j-1)$-$(v_2-k+j, j, \frac{v_2-k-1}{n})$ designs.

Let σ be any given permutation on $\{1, 2, \dots, n\}$, define a subset $C_{(j,\sigma)}$ of C_j as follows.

$$C_{(j,\sigma)} = \bigcup_{i=1}^{n} \mathcal{A}_{i,j} \uplus \mathcal{A}'_{\sigma(i),j},$$

where $\mathcal{A}_{i,j} \uplus \mathcal{A}'_{\sigma(i),j} = \{A \cup A' : A \in \mathcal{A}_{i,j},\ A' \in \mathcal{A}'_{\sigma(i),j}\}$. Then it is shown that T_i is contained in n_{ij}/n blocks in $C_{(j,\sigma)}$ for every i. Finally, let $m = zn = Nu$ and let $\sigma_1, \sigma_2, \dots, \sigma_m$ be m permutations on $\{1, 2, \dots, n\}$ with $\sigma_i = (12\cdots n)^i$, $i = 1, 2 \dots, m$. Define

$$\mathcal{B}_j = \bigcup_{i=1}^{m} C_{(j,\sigma_i)}.$$

Then T_i is contained in $m(n_{i,j}/n) = zn_{i,j}$ blocks of $\mathcal{B}_j$. Cases (a) and (b) together show that $(X, \bigcup_{j=0}^{k} \mathcal{B}_j)$ is the desired design D_3.

Part 2: Resolvability of D_3.

Let $(X_0, \mathcal{B}_{0,1}), (X_0, \mathcal{B}_{0,2}), \ldots, (X_0, \mathcal{B}_{0,N})$ be an s-resolution of $D_1 = (X_0, \mathcal{B}_0)$, with $\mathcal{B}_0 = \bigcup_{i=1}^{N} \mathcal{B}_{0,i}$ and each $(X_0, \mathcal{B}_{0,i})$ is an s-(v_1, k, δ_1) design. Let D_1' denote the s-(v_1, k, δ_1) design.

Similarly, let $(Y_k, \mathcal{B}_{k,1}), (Y_k, \mathcal{B}_{k,2}), \ldots, (Y_k, \mathcal{B}_{k,N})$ be an s-resolution of $D_2 = (Y_k, \mathcal{B}_k)$ with $\mathcal{B}_k = \bigcup_{i=1}^{N} \mathcal{B}_{k,i}$ and each $(Y_k, \mathcal{B}_{k,i})$ is an s-(v_2, k, δ_2) design. Let D_2' denote the s-(v_2, k, δ_2) design.

By Lemma 3.2 we have $z' = \frac{\delta_1}{\binom{v_1-s}{k-s}} = \frac{\delta_2}{\binom{v_2-s}{k-s}} = \frac{u}{n}$.

Let $P = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$ be the set of $m = zn = Nu$ permutations on $\{1, 2, \ldots, n\}$ with $\sigma_i = (12\cdots n)^i$, $i = 1, \ldots, m$. Let $P = P_1 \cup P_2 \cup \cdots \cup P_N$ be a partition of P with $|P_i| = u$ for $i = 1, \ldots, N$.

In Part 1, for $1 < j < k$ we have

$$\begin{aligned} \mathcal{B}_j &= \bigcup_{i=1}^{m} C_{(j,\sigma_i)} \\ &= \bigcup_{i=1}^{N} \bigcup_{h \in P_i} C_{(j,\sigma_h)} \\ &= \bigcup_{i=1}^{N} \mathcal{B}_{j,i}, \end{aligned}$$

where $\mathcal{B}_{j,i} := \bigcup_{h \in P_i} C_{(j,\sigma_h)}$. Here $(X, \bigcup_{j=0}^{k} \mathcal{B}_j)$ is the constructed t-$(v_1 + v_2 - k + 1, k, z\binom{v_1+v_2-k+1-t}{k-t})$ design D_3.

Define $D_3^{(i)} := (X, \bigcup_{j=0}^{k} \mathcal{B}_{j,i})$ for $i = 1, \ldots, N$. Then $D_3^{(i)}$ is the design constructed from the pair $D_1' = (X, \mathcal{B}_{0,i})$ and $D_2' = (X, \mathcal{B}_{k,i})$, and thus $D_3^{(i)}$ has parameters s-$(v_1 + v_2 - k + 1, k, z'\binom{v_1+v_2-k+1-s}{k-s})$. Since $D_3^{(1)}, \ldots, D_3^{(N)}$ are pairwise disjoint, they form an s-resolution of D_3. The proof is complete. □

The next corollary providing a statement about large sets is an immediate consequence of Theorem 3.1.

Corollary 3.3 *Suppose there are large sets $LS[n](t, k, v_1)$, $LS[n](t, k, v_2)$, $LS[n](k-2, k-1, v_1-1)$ and $LS[n](k-2, k-1, v_2-1)$. Then there exists a large set $LS[n](t, k, v_1+v_2-k+1)$.*

Proof. Here D_1 and D_2 are the complete k-$(v_1, k, 1)$ and k-$(v_2, k, 1)$ designs having both a t-resolution with n resolution classes. The constructed design D_3 is the complete k-$(v_1 + v_2 - k + 1, k, 1)$ design having again a t-resolution with n classes. □

Remark that Corollary 3.3 is Theorem 2 in [14], which is the main theorem for large sets of [14].

Corollary 3.4 *Suppose that there exists an s-resolvable t-(v,k,λ) design with N resolution classes such that $z = \frac{\lambda}{\binom{v-t}{k-t}} = \frac{Nu}{n}$, where u, n are positive integers. If there exists an $LS[n](k-2,k-1,v-1)$, then there exists an s-resolvable t-$(v+m(v-k+1),k,z\binom{v-t+m(v-k+1)}{k-t})$ design with N resolution classes for any $m > 0$.*

Proof. From the existence of an $LS[n](k-2,k-1,v-1)$ by assumption, it follows that there exists an $LS[n](k-2,k-1,v-1+m(v-k+1))$ for any $m \geq 0$, by Proposition 2.3. Consider two starting steps of a recursion using Theorem 3.1. For $m = 0$, take $D_2 = D_1$, where D_1 is an s-resolvable t-(v,k,λ) design. Applying Theorem 3.1 gives an s-resolvable t-$(v+v-k+1,k,z\binom{v+v-k+1-t}{k-t})$ design. For $m = 1$, take D_2 as an s-resolvable t-$(v+v-k+1,k,z\binom{v+v-k+1-t}{k-t})$ design. Large sets $LS[n](k-2,k-1,v-1)$ and $LS[n](k-2,k-1,v+(v-k))$ exist for $m = 0, 1$. Thus Theorem 3.1 gives an s-resolvable t-$(v+2(v-k+1),k,z\binom{v+2(v-k+1)-t}{k-t})$ design. Hence, using Theorem 3.1 recursively will complete the proof. □

A simple form of Corollary 3.4 for large sets is as follows.

Corollary 3.5 *Suppose that there exist large sets $LS[n](t,k,v)$ and $LS[n](k-2,k-1,v-1)$. Then there exist large sets $LS[n](t,k,v+m(v-k+1))$ for all $m \geq 0$.*

An immediate consequence of Theorem 3.1 for mutually disjoint t-designs can be expressed as follows.

Corollary 3.6 *Let D_1 and D_2 be the union of N mutually disjoint t-(v_1,k,λ_1) and t-(v_2,k,λ_2) designs respectively such that $\frac{N\lambda_1}{\binom{v_1-t}{k-t}} = \frac{N\lambda_2}{\binom{v_2-t}{k-t}} = \frac{Nu}{n}$, where u, n are positive integers. Suppose that there exist $LS[n](k-2,k-1,v_1-1)$ and $LS[n](k-2,k-1,v_2-1)$. Then there exist N mutually disjoint t-(v_1+v_2-k+1,k,λ) designs with $\lambda = \frac{u}{n}\binom{v_1+v_2-k+1-t}{k-t}$.*

In this context Corollary 3.4 becomes

Corollary 3.7 *Suppose that there exist N mutually disjoint t-(v,k,λ) designs such that $z = \frac{N\lambda}{\binom{v-t}{k-t}} = \frac{Nu}{n}$, where u, n are positive integers. If there exists an $LS[n](k-2,k-1,v-1)$, then there exist N mutually disjoint t-$(v+m(v-k+1),k,\frac{u}{n}\binom{v-t+m(v-k+1)}{k-t})$ designs for any $m > 0$.*

4 Applications

First of all, we show the existence of simple 3-resolvable 4-$(34,6,75)$ and 4-$(35,7,31\cdot 25)$ designs with $N = 5$ resolution classes. Consider 3-resolvable 4-$(33,5,5)$ and 4-$(33,6,70)$ designs, both having $N = 5$ resolution classes. The former is constructed by Alltop and the latter by Bierbrauer, see [13]. Next, employ Corollary 4.3 of [13] which states that if there exist s-resolvable t-designs with parameters t-$(v,k-1,\lambda_t^{(k-1)})$ and

t-$(v, k, \lambda_t^{(k)})$ having the same number of resolution classes, such that $\lambda_{t-1}^{(k-1)} - \lambda_t^{(k-1)} = \lambda_t^{(k)}$, then there exists an s-resolvable t-$(v+1, k, \lambda_{t-1}^{(k-1)})$ design. It is clear that the condition of the corollary is satisfied for the 4-$(33, 5, 5)$ and 4-$(33, 6, 70)$ designs, thus we obtain a 4-$(34, 6, 75)$ design.

Now consider a 3-resolvable 4-$(34, 7, 700)$ design with $N = 5$ resolution classes in Theorem 6.1 of [13]. Again, applying Corollary 4.3 of [13] to the 4-$(34, 6, 75)$ and 4-$(34, 7, 700)$ designs will give a 3-resolvable 4-$(35, 7, 31 \cdot 25)$ design.

We record this result in the following proposition.

Proposition 4.1 *There exist simple* 3*-resolvable* 4*-designs with* $N = 5$ *resolution classes having parameters* 4-$(34, 6, 75)$ *and* 4-$(35, 7, 31 \cdot 25)$*.*

Now, by using these 4-$(33, 5, 5)$, 4-$(34, 6, 75)$, and 4-$(35, 7, 31 \cdot 25)$ designs for Corollary 3.4, we may state the following.

Proposition 4.2

1. *If there exists an* $LS[29](3, 4, 32)$*, then there exists a* 3*-resolvable* 4-$(4 + 29m, 5, 5m)$ *design for any* $m \geq 1$*.*
2. *If there exists an* $LS[29](4, 5, 33)$*, then there exists a* 3*-resolvable* 4-$(5+29m, 6, \frac{5}{2}m(1+29m))$ *design for any* $m \geq 1$*.*
3. *If an* $LS[29](5, 6, 34)$ *exists, then there exists a* 3*-resolvable* 4-$(6+29m, 7, \frac{5}{3}m\binom{2+29m}{2})$ *design for any* $m \geq 1$*.*

The existence of any infinite family of 3-resolvale 4-designs in Proposition 4.2 thus reduces to the existence of a single large set. Hence the following problem is a great challenge.

Open problem 4.1 *Does there exist any of the following large sets* $LS[29](3, 4, 32)$*,* $LS[29](4, 5, 33)$*,* $LS[29](5, 6, 34)$*?*

Note that $LS[29](4, 5, 33)$, $LS[29](3, 4, 32)$ and $LS[29](2, 3, 31)$ are the derived large sets of $LS[29](5, 6, 34)$. Among these large sets, only $LS[29](2, 3, 31)$ is known to exist.

A derived design of the 4-$(33, 5, 5)$ design above is a 2-resolvable 3-$(32, 4, 5)$ design. Since an $LS[29](2, 3, 31)$ exists, we obtain the following result by Corollary 3.4.

Theorem 4.3 *There exists a* 2*-resolvable* 3-$(3 + 29m, 4, 5m)$ *design with* $N = 5$ *resolution classes for any* $m \geq 1$*.*

We now show an interesting example of mutually disjoint 3-designs by using Corollary 3.7. In [3] Etzion and Hartman show that for $v = 5 \cdot 2^h$, $h \geq 1$, there exist $5 \cdot 2^h - 5$ mutually disjoint 3-$(5 \cdot 2^h, 4, 1)$ Steiner quadruple systems. However, the existence of a large set of 3-$(5 \cdot 2^h, 4, 1)$ designs remains an open problem for $h \geq 2$. For $h = 1$, i.e., $v = 10$, Kramer and Mesner show in [6] that the maximal number of mutually disjoint 3-$(10, 4, 1)$ designs is 5. In other words, there is no large set of 3-$(10, 4, 1)$ designs.

Since there are $N = 5 \cdot 2^h - 5$ mutually disjoint 3-$(5 \cdot 2^h, 4, 1)$ designs for any given $h \geq 1$, we have $z = \frac{N\lambda}{\binom{v-t}{k-t}} = \frac{Nu}{n} = \frac{5 \cdot 2^h - 5}{5 \cdot 2^h - 3}$. In addition, since $LS[5 \cdot 2^h - 3](2, 3, 5 \cdot 2^h - 1)$ exists, Corollary 3.7 yields $5 \cdot 2^h - 5$ mutually disjoint 3-designs with parameters 3-$(3 + m(5 \cdot 2^n - 3), 4, m)$ for all $m \geq 1$. Thus we have

Theorem 4.4 *For any given integer $h \geq 1$, there exist $N = 5 \cdot 2^h - 5$ mutually disjoint simple* 3-$(3 + m(5 \cdot 2^h - 3), 4, m)$ *designs for all $m \geq 1$.*

5 Some series of t-designs from Wu's result

Closer inspection of the literature reveals that works related to the result of Wu have focused on large sets rather than on finding t-designs. Here we include a short account of simple t-designs for $t = 4, 5$ concerning the latter case.

1. There exist simple 4-$(18, 5, h2)$ designs for $h = 1, 2, 3$ with $z := \frac{\lambda}{\binom{v-t}{k-t}} = \frac{h}{7}$. Further there is a $LS[7](3, 4, 17)$ [2]. Using Corollary 2.2 we obtain a 4-$(4 + 14m, 5, h2m)$ design for every $m \geq 1$.

2. There exist simple 5-$(33, 6, h4)$ designs for $h = 1, 2, 3$ with $z := \frac{\lambda}{\binom{v-t}{k-t}} = \frac{h}{7}$. Further there is a $LS[7](4, 5, 32)$ [8]. From Corollary 2.2 we obtain a 5-$(5 + 28m, 6, h4m)$ design for every $m \geq 1$.

   Using the following result of Teirlinck [9]: an $LS_{\lambda_{\min}}(3, 4, v)$ exists if $v \equiv 0 \bmod 3$, we can derive more infinite classes of simple 4-designs from Corollary 2.2. Here are two examples.

3. There exist simple 4-$(31, 5, h3)$ designs for $h = 1, 2, 3, 4$, as derived designs of 5-$(32, 6, h3)$ designs [1], with $z := \frac{\lambda}{\binom{v-t}{k-t}} = \frac{h}{9}$. Since there exists a $LS_{\lambda_{\min}}(3, 4, 30) = LS[9](3, 4, 30)$, Corollary 2.2 gives a 4-$(4 + 27m, 5, h3m)$ design for any $m \geq 1$.

4. There exist simple 4-$(37, 5, h3)$ designs for $h = 3, 4$ with $z := \frac{\lambda}{\binom{v-t}{k-t}} = \frac{h}{11}$. Since there is a $LS_{\lambda_{\min}}(3, 4, 36) = LS[11](3, 4, 36)$ we have a 4-$(4+33m, 5, h3m)$ design for any $m \geq 1$.

In summary, we obtain the following.

Theorem 5.1 *There exist the following simple infinite series of t-designs with parameters:*

1. 4-$(4 + 14m, 5, h2m)$, $h = 1, 2, 3$, $m \geq 1$.
2. 5-$(5 + 28m, 6, h4m)$, $h = 1, 2, 3$, $m \geq 1$.
3. 4-$(4 + 27m, 5, h3m)$, $h = 1, 2, 3, 4$, $m \geq 1$.
4. 4-$(4 + 33m, 5, h3m)$, $h = 3, 4$, $m \geq 1$.

In general, using Corollary 2.2 and Teirlinck's result [9], we can prove the following.

Theorem 5.2 *Let v be a positive integer with $v \equiv 1 \bmod 3$. Suppose that there exists a simple 4-$(v, 5, \lambda)$ design. Then, there exists a simple 4-$(4+m(v-4), 5, \lambda m)$ design for any $m \geq 1$.*

Proof. Let $\lambda_{\min}$ denote the smallest possible value for which a 3-$(v-1, 4, \lambda_{\min})$ design exists. From the assumption there is a 3-$(v-1, 4, \lambda)$ design. Thus $\lambda = h\lambda_{\min}$. Since $v-1 \equiv 0 \bmod 3$, there is an $LS_{\lambda_{\min}}(3, 4, v-1) = LS[N](3, 4, v-1)$, where $N = \frac{v-4}{\lambda_{\min}}$. Now for the 4-$(v, 5, \lambda)$ design we have $z := \frac{\lambda}{\binom{v-t}{k-t}} = \frac{\lambda}{v-4} = \frac{h}{N}$. Hence Corollary 2.2 gives a 4-$(4+m(v-4), 5, \lambda m)$ design for any $m \geq 1$. □

Furthermore, there exist large sets $LS_{\lambda_{\min}}(4, 5, 20u+4)$ if $\gcd(u, 30) = 1$, and $LS_{60}(4, 5, 60u+4)$ if $\gcd(u, 60) = 1$ or 2 [11]. Similarly, we obtain the following result by using these large sets.

Theorem 5.3 1. *If there exists a simple 5-$(v, 6, \lambda)$ design for $v = 20u+5$ and $\gcd(u, 30) = 1$, then there exists a simple 5-$(5+m(v-5), 6, \lambda m)$ design for any $m \geq 1$.*

2. *If there exists a simple 5-$(v, 6, \lambda)$ design for $v = 60u+5$ and $\gcd(u, 60) = 1$ or 2, then there exists a simple 5-$(5+m(v-5), 6, \lambda m)$ design for any $m \geq 1$.*

6 Conclusion

The main result of the paper presents an extending theorem for s-resolvable t-designs along the lines of the extending theorem for t-designs and large sets of Qiu-rong Wu. A particular feature of the method is that it will produce an infinite series of t-designs having s-resolutions on the basis of a single pair of an appropriate s-resolvable t-design and a specific large set. Another consequence of the result is a recursive construction for mutually disjoint t-designs.

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