Recursive constructions for s-resolvable t-designs

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Abstract

In this paper we investigate simple t-designs having s-resolutions for $t \geq 3$ and $1 \leq s < t$. The study focuses particularly on recursive construction methods for these designs. One of the results, viewed as the main theorem, presents a general and effective method for finding s-resolvable t-designs, and it also yields statements about large sets of s-designs as by-products. As an example, we show the construction of a 3-resolvable infinite family of simple 4-designs with parameters 4-$(2^n + 2, 7, 70(2^n - 2)/3)$, $\gcd(n, 6) = 1$, $n \geq 5$.

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1 Introduction

In this paper we are interested in non-trivial simple t-designs which are s-resolvable with $s > 1$. Little attention has been given to the topic yet. Most known results on resolvable t-designs concern the case $s = 1$. In particular, the $(1, 1)$-resolvable case, i.e. the blocks of the design can be partitioned into classes where each class consists of blocks which partition the points. By constrast, when the investigated designs are the trivial t-designs, we deal with the question of partitioning the complete k-$(v, k, 1)$ design into s-designs which are widely known as large sets. In fact, there are a great deal of results concerning large sets, for example [1, 11, 13, 14, 15, 16, 17, 18, 19, 21, 22, 28]. The great interest in large sets might be due to the celebrated result of Teirlinck about the existence of non-trivial simple t-designs for arbitrary large t, whose proof involves large sets, see [21, 22]. The subject of the paper inherently relates to a recent article of the author [27] in which t-designs having s-resolutions are the basis of a recursive method for constructing non-trivial simple t-designs. However, since very little is currently known about s-resolutions for non-trivial t-designs, the applications of the method as shown in [27] have been necessarily restricted to the trivial ingredient designs with large sets only. Moreover, to a certain extent the method in [27] is based on a recursive construction of simple t-designs in [26], also called the basic construction. The aim of the present paper primarily concerns recursion methods for constructions of t-designs having s-resolutions. Among others, we show as the main theorem that the basic construction indeed gives a general construction for s-resolvable t-designs. Actually, it may be viewed as an effective and useful tool for finding designs with resolutions. Interestingly, we obtain statements about large sets as by-products of the main result. To illustrate the main theorem, we show the construction of a 3-resolvable infinite family of simple 4-designs with parameters 4-$(2^n+2, 7, 70(2^n-2)/3)$, $\gcd(n,6)=1$, $n \geq 5$. The paper is organized as follows. Section 2 recalls some basic facts about t-designs, and presents the definition and elementary facts about s-resolutions. Section 3 deals with a brief account of known infinite families of s-resolvable t-designs for $s \geq 2$. Section 4 includes the basic construction in [26] and presents a proof of the main theorem for constructing s-resolvable t-designs; the section is closed with statements about large sets, a by-product of the main result. Section 5 examines further recursive constructions for s-resolvable t-designs. Section 6 displays an application of the main theorem. The paper closes in Section 7.

2 Preliminaries

We recall a few basic definitions. A t-design, with parameters denoted by t-(v,k,λ), is a pair $(X, \mathfrak{B})$, where X is a v-set of *points* and $\mathfrak{B}$ is a collection of k-subsets of X, called *blocks*, having the property that every t-subset of X is a subset of exactly λ blocks in $\mathfrak{B}$. The parameter λ is called the *index* of the design. A t-design is called *simple* if no two blocks are identical, i.e. no block of $\mathcal{B}$ is repeated; otherwise, it is called *non-simple* (i.e. $\mathfrak{B}$ is a multiset). It can be shown by simple counting that a t-(v,k,λ) design is an s-(v,k,λ_s) design for $0 \leq s \leq t$, where $\lambda_s = \lambda\binom{v-s}{t-s}/\binom{k-s}{t-s}$. Since λ_s is an integer, necessary conditions for the parameters of a t-design are that $\binom{k-s}{t-s}$ divides $\lambda\binom{v-s}{t-s}$, for $0 \leq s \leq t$. For given t, v and k, we denote by $\lambda_{\min}(t,k,v)$, or $\lambda_{\min}$ for short, the smallest positive integer such that these conditions are satisfied for all $0 \leq s \leq t$. By complementing each block in X of a t-(v,k,λ) design, we obtain a t-$(v, v-k, \lambda^*)$ design, called the *complementary design*, with $\lambda^* = \lambda\binom{v-k}{t}/\binom{k}{t}$, hence we shall assume that $k \leq v/2$. If $\mathfrak{B}$ is a collection of all k-subsets of X, then $(X,\mathfrak{B})$ is a t-$(v,k,\lambda_{\max})$ design with $\lambda_{\max} = \binom{v-t}{k-t}$, and is called the *complete* design or the *trivial* design. Thus the value $\lambda_{\max}$ is the maximum λ of a simple t-(v,k,λ) design. From a given t-(v,k,λ) design $(X,\mathfrak{B})$ we obtain a t-$(v,k,\binom{v-t}{k-t}-\lambda)$ design $(X,\mathfrak{B}^*)$, called the *supplementary design* of $(X,\mathfrak{B})$, where $\mathfrak{B}^*$ is the set of all k-subsets of X which are not a block in $\mathfrak{B}$. Let $x \in X$. The $(t-1)$-$(v-1,k-1,\lambda)$ design, called the *derived design* at the point x, is the design with the point set $X \setminus \{x\}$ and the block set are all blocks $B \in \mathfrak{B}$ containing x with x removed. Finally, the $(t-1)$-$(v-1, k, \frac{v-k}{k-t+1}\lambda)$ design, called the *residual design* at the point x, with the point set $X \setminus \{x\}$ and the block set are the blocks of $\mathfrak{B}$ not containing x.

We refer the reader to [4, 10] for more information about designs.

Definition 2.1 *A t-(v,k,λ)-design $(X,\mathfrak{B})$ is said to be s-resolvable, or to have an s-resolution, with $0 < s < t$, if its block set $\mathfrak{B}$ can be partitioned into $N \geq 2$ classes*

$\mathfrak{A}_1, \ldots, \mathfrak{A}_N$ *such that each* $(X, \mathfrak{A}_i)$ *is an* s-(v, k, δ) *design for* $i = 1, \ldots, N$. *Each* $\mathfrak{A}_i$ *is called an* s*-resolution class or simply a resolution class. The set of* N *classes is called an* s*-resolution of the design.*

To display the full data of s-(v, k, δ) designs in the resolution we also write (s, δ)-resolution. Thus a $(1, 1)$-resolution of a t-design, for example, is a partition of its blocks into classes of 1-$(v, k, 1)$ designs; each class consists of v/k mutually disjoint blocks which partition the point set and are usually called a parallel class of the design.

If the complete design k-$(v, k, 1)$ is s-resolvable with N resolution classes, where each class is an s-(v, k, δ) design, then we say that there exists a *large set* of size N of s-designs, and it is denoted by $\mathrm{LS}[N](s, k, v)$, or by $\mathrm{LS}_\delta(s, k, v)$ to emphasize the value δ.

Following are some simple lemmas about basic properties of s-resolutions for t-designs. We omit the proofs.

Lemma 2.1 *If a* t-(v, k, λ) *design is* s*-resolvable with* N *resolution classes, then* N *divides* $\frac{\binom{v}{t}}{\binom{k}{t}}\lambda$.

Lemma 2.2 *If a* t-(v, k, λ) *design is* s*-resolvable with* N *resolution classes, then*

1. *its* $(t-1)$-$(v-1, k-1, \lambda)$ *derived design and* $(t-1)$-$(v-1, k, \frac{v-k}{k-t+1}\lambda)$ *residual design are* $(s-1)$*-resolvable with* N *resolution classes;*

2. *its* t-$(v, v-k, \frac{\binom{v-k}{t}}{\binom{k}{t}}\lambda)$ *complementary design is* s*-resolvable with* N *resolution classes.*

The next lemma is trivial but useful.

Lemma 2.3 *Suppose that* $(X, \mathfrak{B}_1)$ *and* $(X, \mathfrak{B}_2)$ *are two* s*-resolvable* t*-designs with the same block size having* N *resolution classes each such that* $\mathfrak{B}_1 \cap \mathfrak{B}_2 = \emptyset$. *Then* $(X, \mathfrak{B}_1 \cup \mathfrak{B}_2)$ *is an* s*-resolvable* t*-design with* N *resolution classes.*

Proof. Let $C_1, \ldots, C_N$ (resp. $D_1, \ldots, D_N$) be the resolution classes of $(X, \mathfrak{B}_1)$ (resp. $(X, \mathfrak{B}_2)$). Then $C_1 \cup D_1, \ldots, C_N \cup D_N$ are the resolution classes of $(X, \mathfrak{B}_1 \cup \mathfrak{B}_2)$. □

3 Some known s-resolvable infinite families of simple t-designs

In this section we briefly address some infinite families of t-designs which are shown to have s-resolutions with $s \geq 2$. Let G be an s-homogeneous permutation group on a set X (i.e. G is a permutation group of the set X such that for any two s-subsets S_1 and S_2 of X, there exists a $g \in G$ such that $S_1^g = S_2$). It is well-known that for any k-subset $K \subseteq X$, $k \geq s$, the G-orbit of K will form the block set of an s-design defined on the point set X. More precisely, let $\mathfrak{B}$ denote the G-orbit of K and let $|X| = v$. Then $(X, \mathfrak{B})$ is an s-(v, k, δ) design admitting G as an automorphism group with

$$\delta = \binom{k}{s} |G_S|/|G_K|,$$

where S and K are an s-subset and a k-subset of X and G_S and G_K are the (setwise) stabilizer of S and K in G, respectively.

The construction of t-(v, k, λ) designs using the action of an s-homogeneous group G on a v-set X, where $t > s \geq 2$, yields the commonly known examples of s-resolvable t-designs. Actually, the obtained t-design $(X, \mathfrak{B})$ has the block set $\mathfrak{B}$ as a disjoint union of G-orbits of k-subsets of X. If $\mathfrak{B}$ contains more than one G-orbit and all these G-orbits have the same size, then $(X, \mathfrak{B})$ is s-resolvable, and each G-orbit is a resolution class.

First of all [3] Baker shows that the Steiner quadruple system 3-$(4^m, 4, 1)$ constructed from an even dimensional affine space over the field of two elements is 2-resolvable, having 2-$(4^m, 4, 1)$ designs as resolution classes. Also, Teirlinck shows for example that there exist 2-resolvable 3-$(2p^n + 2, 4, 1)$ designs with $p \in \{7, 31, 127\}$, with 2-$(2p^n + 2, 4, 1)$ designs as resolution classes, for any positive integer n, [23].

In the following we focus on the case of the projective general linear group $G = \mathrm{PGL}(2, q)$, $q = 2^n$, acting on the projective line $X = \mathrm{GF}(q) \cup \{\infty\}$. In fact, G acts sharply 3-transitively on X, and thus $|G| = (q + 1)q(q - 1)$ (in particular, G is 3-homogeneous and the stabilizer G_S is isomorphic to the symmetric group S_3, for any 3-subset S of X).

We consider some known infinite families of simple 4-designs with $k = 5, 6, 8, 9$.

- $k = 5$. The first example is the family of simple 4-$(q+1, 5, 5)$ designs constructed by Alltop [2], for every $q = 2^n$, $n \geq 5$, n odd. Each 4-design $(X, \mathfrak{B})$ in the family has block set $\mathfrak{B}$ as a disjoint union of $(q - 2)/6$ G-orbits of 5-subsets B of X with stabilizer $G_B \cong E_4$, the elementary 2-group of order 4. It follows that, each G-orbit of B is a 3-$(q + 1, 5, 15)$ design. Thus $(X, \mathfrak{B})$ is 3-resolvable and has $N = (q - 2)/6$ resolution classes.

We have checked that all the $4 - (q + 1, k, \lambda)$ designs for $k = 6, 8, 9$, constructed by Bierbrauer in [6, 7, 8] are 3-resolvable. Following are some examples.

- $k = 6$. The family of 4-designs $(X, \mathfrak{B})$ with parameters 4-$(2^n + 1, 6, 10)$, $\gcd(n, 6) = 1$, and $n \geq 5$, in [6], whose blocks are the disjoint union of $(q-2)/6$ G-orbits of 6-subsets B of X with stabilizer $G_B \cong S_3$ in G. Each G-orbit of B is then a 3-$(2^n + 1, 6, 20)$ design. And $(X, \mathcal{B})$ is 3-resolvable with $N = (q-2)/6$ resolution classes.

  In [8] it is shown that there is a family of 4-designs $(X, \mathfrak{B}_1)$ with parameters 4-$(2^n + 1, 6, 60)$ whose blocks are the disjoint union of $(q - 2)/6$ G-orbits of 6-subsets B of X with stabilizer $G_B = 1$. Each of these G-orbits is a 3-$(2^n + 1, 6, 120)$ design. So $(X, \mathfrak{B}_1)$ is 3-resolvable with $N = (q - 2)/6$ resolution classes. Moreover, it holds that $\mathfrak{B} \cap \mathfrak{B}_1 = \emptyset$. By Lemma 2.3, $\mathfrak{B} \cup \mathfrak{B}_1$ forms a 3-resolvable 4-design with parameters 4-$(2^n+1, 6, 70)$. The number of resolution classes remains unchanged with $N = (q-2)/6$ and each class is a 3-$(2^n+1, 6, 140)$ design.

- $k = 9$. The family of 4-designs $(X, \mathcal{B})$ with parameters 4-$(2^n+1, 9, 84)$, $\gcd(n, 6) = 1$, and $n \geq 5$, in [7], whose blocks are the disjoint union of $(q-2)/6$ G-orbits of 9-subsets B of X with stabilizer $G_B \cong S_3$ in G. Each G-orbit of B forms a 3-$(2^n+1, 9, 84)$ design. And again $(X, \mathcal{B})$ is 3-resolvable with $N = (q-2)/6$ resolution classes.

- $k = 8$. The family of 4-designs $(X, \mathcal{B})$ with parameters 4-$(2^n+1, 8, 35)$, $\gcd(n, 6) = 1$, and $n \geq 5$, in [8], whose blocks are the disjoint union of $(q-2)/6$ G-orbits of 8-subsets B of X with stabilizer $G_B \cong E_8$, the elementary abelian 2-group of order 8, in G. Each G-orbit of B forms a 3-$(2^n+1, 8, 42)$ design. $(X, \mathcal{B})$ is 3-resolvable with $N = (q-2)/6$ resolution classes.

Similarly, when the projective special linear groups $\mathrm{PSL}(2, q)$ are used to construct t-designs, $t \geq 4$, it is likely that it would yield s-resolvable t-designs with $s = 2$ or 3. As examples, it is shown in [12] that there are 3-resolvable 4-$(48, 5, \lambda)$ designs for $\lambda = 8, 12, 16, 20$, constructed with $\mathrm{PSL}(2, 47)$. In [5] Betten, Laue, Molodtsov and Wassermann have constructed 5-$(84, 6, 1)$ designs whose blocks are unions of long block orbits of $\mathrm{PSL}(2, 83)$, (i.e. orbits of length $|\mathrm{PSL}(2, 83)|$) and showed that there are 3 non-isomorphic such 5-$(84, 6, 1)$ designs. Thus these designs are 3-resolvable with 18 resolution classes and each class is a 3-$(84, 6, 60)$ design.

4 The main theorem on construction of s-resolvable t-designs

We include a summary of the basic construction as described in [26] in the following theorem. This is necessary for the proof of the main theorem.

Theorem 4.1 (Basic construction) *Let v, k, t be integers with $v > k > t \geq 2$. Let X be a v-set and let $X = X_1 \cup X_2$ be a partition of X with $|X_1| = v_1$ and $|X_2| = v_2$. Let $D_i = (X_1, \mathfrak{B}^{(i)})$ be the complete i-$(v_1, i, 1)$ design for $i = 0, \ldots, t$ and let $D_i = (X_1, \mathfrak{B}^{(i)})$ be a simple t-$(v_1, i, \lambda_t^{(i)})$ design for $i = t+1, \ldots, k$. Similarly, let $\bar{D}_i = (X_2, \bar{\mathfrak{B}}^{(i)})$ be the complete i-$(v_2, i, 1)$ design for $i = 0, \ldots, t$, and let $\bar{D}_i = (X_2, \bar{\mathfrak{B}}^{(i)})$ be a simple t-$(v_2, i, \bar{\lambda}_t^{(i)})$ design for $i = t+1, \ldots, k$. Define*

$$\mathfrak{B} = \mathfrak{B}_{(0,k)} \times [u_0] \cup \mathfrak{B}_{(1,k-1)} \times [u_1] \cup \cdots \cup \mathfrak{B}_{(k-1,1)} \times [u_{k-1}] \cup \mathfrak{B}_{(k,0)} \times [u_k],$$

where

$$\mathfrak{B}_{(i,k-i)} = \{B = B_i \cup \bar{B}_{k-i} \mid B_i \in \mathfrak{B}^{(i)}, \bar{B}_{k-i} \in \bar{\mathfrak{B}}^{(k-i)}\}.$$

Assume that

$$L_{0,t} = L_{1,t-1} = L_{2,t-2} = \cdots = L_{t,0} := \Lambda, \tag{1}$$

for a positive integer Λ, *where*

$$L_{r,t-r} \quad = \quad \sum_{i=0}^{k} u_i.\lambda_r^{(i)}.\bar{\lambda}_{t-r}^{(k-i)}, \tag{2}$$

$r = 0, \ldots, t,$ *and* $u_i \in \{0,1\}$, *for* $i = 0, \ldots, k$. *Then* $(X, \mathfrak{B})$ *is a simple* t-(v,k,Λ) *design.*

Some explanations of the symbols in the theorem need to be included.

- Two degenerate cases for designs occur when either $k = t = 0$ or $v = k$.
  - The case $k = t = 0$ gives an "empty" design, denoted by $\emptyset$; however note that the number of blocks of the empty design is 1.
  - The case $v = k$ gives a degenerate k-design having just one block consisting of all v points.
- The notation $X \times [u]$, where X is a finite set and $u \in \{0,1\}$, has the following meaning. $X \times [0]$ is the empty set $\emptyset$, and $X \times [1] = X$. In particular, $\mathfrak{B}_{(i,k-i)} \times [u_i]$ indicates that either it is an empty set $\emptyset$ (when $u_i = 0$) or the set $\mathfrak{B}_{(i,k-i)}$ itself (when $u_i = 1$). The case $u_i = 0$ means that the pair $(D_i, \bar{D}_{k-i})$ is not involved in the construction.
- Any t-subset T of X is denoted by $T_{(r,t-r)}$ where $|T \cap X_1| = r$ and $|T \cap X_2| = t-r$, for $r = 0, \ldots, t$. And $L_{r,t-r}$ is the number of blocks in $\mathfrak{B}$ containing $T_{(r,t-r)}$.

We should mention that Theorem 1 in [25] is a special case of the basic construction, in particular, the easiest case with $v_1 = 1$ and $v_2 = v$ has widely been used to generate new designs from two specific known designs.

We are now in a position to prove the main theorem.

Theorem 4.2 *Let* s t, k, v *be positive integers with* $v > k > t > s \geq 1$. *Let* N *be a fixed integer. Assume that, for* $i = 0, 1, \ldots, k$, *each pair* $(\mathfrak{B}^{(i)}, \bar{\mathfrak{B}}^{(k-i)})$ *of the basic construction has the property that either* $\mathfrak{B}^{(i)}$ *or* $\bar{\mathfrak{B}}^{(k-i)}$ *has an* s*-resolution with* N *resolution classes. Then any* t-(v,k,Λ) *resulting design of the basic construction is* s*-resolvable with* N *resolution classes.*

Proof. The main idea of the proof is that if each pair $(\mathfrak{B}^{(i)}, \bar{\mathfrak{B}}^{(k-i)})$ of the basic construction has the required property, then we can contruct an s-resolution with N classes for any resulting design. We keep the notation used in the basic construction. Assume that $\mathfrak{B}^{(i)}$ is s-resolvable with N resolution classes. Then we write the t-$(v_1, i, \lambda_t^{(i)})$ design $(X_1, \mathfrak{B}^{(i)})$ as the union of N mutually disjoint s-$(v_1, i, \delta_s^{(i)})$ designs

$$(X_1, \mathfrak{C}_1^{(i)}), (X_1, \mathfrak{C}_2^{(i)}), \ldots, (X_1, \mathfrak{C}_N^{(i)})$$

where $\delta_s^{(i)} = \frac{\lambda_s^{(i)}}{N} = \frac{\binom{v_1-s}{t-s}}{\binom{i-s}{t-s}} \lambda_t^{(i)}/N$. Similarly, if $\bar{\mathfrak{B}}^{(k-i)}$ is s-resolvable with N resolution classes, then we write the t-$(v_2, k-i, \bar{\lambda}_t^{(k-i)})$ design $(X_2, \bar{\mathfrak{B}}^{(k-i)})$ as the union of N mutually disjoint s-$(v_2, k-i, \bar{\delta}_s^{(k-i)})$ designs

$$(X_2, \bar{\mathfrak{C}}_1^{(k-i)}), (X_2, \bar{\mathfrak{C}}_2^{(k-i)}), \ldots, (X_2, \bar{\mathfrak{C}}_N^{(k-i)})$$

with $\bar{\delta}_s^{(k-i)} = \frac{\bar{\lambda}_s^{(k-i)}}{N} = \frac{\binom{v_2-s}{t-s}}{\binom{k-i-s}{t-s}} \bar{\lambda}_t^{(k-i)}/N$.

The basic construction states that $(X, \mathfrak{B})$ is a t-(v, k, Λ) design, when

$$L_{0,t} = L_{1,t-1} = \cdots = L_{t-1,1} = L_{t,0} = \Lambda$$

where

$$\mathfrak{B} = \mathfrak{B}_{(0,k)} \times [u_0] \cup \mathfrak{B}_{(1,k-1)} \times [u_1] \cup \cdots \cup \mathfrak{B}_{(k-1,1)} \times [u_{k-1}] \cup \mathfrak{B}_{(k,0)} \times [u_k],$$

and

$$\mathfrak{B}_{(i,k-i)} = \{B = B_i \cup \bar{B}_{k-i} \mid B_i \in \mathfrak{B}^{(i)}, \bar{B}_{k-i} \in \bar{\mathfrak{B}}^{(k-i)}\}.$$

$$L_{r,t-r} = \sum_{i=0}^{k} u_i . \lambda_r^{(i)} . \bar{\lambda}_{t-r}^{(k-i)},$$

$r = 0, \ldots, t$, and $u_i \in \{0, 1\}$.

If $\mathfrak{B}^{(j)}$ is s-resolvable with N classes we write $\mathfrak{B}_{(j,k-j)} = (\mathfrak{B}^{(j)} \cup \bar{\mathfrak{B}}^{(k-j)})$ as

$$\mathfrak{B}_{(j,k-j)} = (\mathfrak{C}_1^{(j)} \cup \bar{\mathfrak{B}}^{(k-j)}) \cup (\mathfrak{C}_2^{(j)} \cup \bar{\mathfrak{B}}^{(k-j)}) \cup \cdots \cup (\mathfrak{C}_N^{(j)} \cup \bar{\mathfrak{B}}^{(k-j)}),$$

whereas if $\bar{\mathfrak{B}}^{(k-j)}$ is s-resolvable we write

$$\mathfrak{B}_{(j,k-j)} = (\mathfrak{B}^{(j)} \cup \bar{\mathfrak{C}}_1^{(k-j)}) \cup (\mathfrak{B}^{(j)} \cup \bar{\mathfrak{C}}_2^{(k-j)}) \cup \cdots \cup (\mathfrak{B}^{(j)} \cup \bar{\mathfrak{C}}_N^{(k-j)}).$$

Define

$$J := \{j \mid j = 0, \ldots, k \ : \ \mathfrak{B}^{(j)} \text{ is } s\text{-resolvable}\}$$

$$\bar{J} := \{j \mid j = 0, \ldots, k \ : \ \bar{\mathfrak{B}}^{(j)} \text{ is } s\text{-resolvable}\}$$

For $i = 1, \ldots, N$ define

$$\mathfrak{B}_i := \bigcup_{j \in J} (\mathfrak{C}_i^{(j)} \cup \bar{\mathfrak{B}}^{(k-j)}) \times [u_j] \ \cup \ \bigcup_{j \in \bar{J}} (\mathfrak{B}^{(k-j)} \cup \bar{\mathfrak{C}}_i^{(j)}) \times [u_{k-j}].$$

Since the pairs $(j, k-j)$, $j \in J$ and $(k-j, j)$, $j \in \bar{J}$ cover all $k+1$ pairs $(j, k-j)$ exactly once, for $j = 0, \ldots, k$, we have

$$\mathfrak{B} = \mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \cdots \cup \mathfrak{B}_N.$$

We claim that $(X, \mathfrak{B}_i)$ is an s-(v, k, δ_s) design with $\delta_s = \frac{\Lambda_s}{N} = \frac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}} \Lambda/N$, for $i = 1, \ldots, N$.

In fact, if $(X,\mathfrak{B})$ is a t-(v,k,Λ) design, the equalities

$$L_{0,t}=L_{1,t-1}=\cdots=L_{t-1,1}=L_{t,0}=\Lambda$$

are satisfied. In particular, $(X,\mathfrak{B})$ is an s-design with parameters s-(v,k,Λ_s)=s-$(v,k,\frac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}}\Lambda)$. In other words, the equalities

$$L_{0,s}=L_{1,s-1}=\cdots=L_{s-1,1}=L_{s,0}=\Lambda_s$$

are satisfied, where

$$L_{r,s-r}=\sum_{i=0}^{k}u_i.\lambda_r^{(i)}.\bar{\lambda}_{s-r}^{(k-i)},$$

$r=0,\ldots,s$. From the assumption that either $\mathfrak{B}^{(j)}$ or $\bar{\mathfrak{B}}^{(k-j)}$ is s-resolvable for each pair $(\mathfrak{B}^{(j)},\bar{\mathfrak{B}}^{(k-j)})$ we may write $L_{r,s-r}$ as follows

$$\begin{aligned}L_{r,s-r} &= \sum_{j\in J}u_j\delta_r^{(j)}.N.\bar{\lambda}_{s-r}^{(k-j)} + \sum_{j\in\bar{J}}u_{k-j}\lambda_r^{(k-j)}.\bar{\delta}_{s-r}^{(j)}.N\\ &= N.(\sum_{j\in J}u_j\delta_r^{(j)}.\bar{\lambda}_{s-r}^{(k-j)} + \sum_{j\in\bar{J}}u_{k-j}\lambda_r^{(k-j)}.\bar{\delta}_{s-r}^{(j)})\end{aligned}$$

Set

$$L^*_{r,s-r}:=\sum_{j\in J}u_j\delta_r^{(j)}.\bar{\lambda}_{s-r}^{(k-j)} + \sum_{j\in\bar{J}}u_{k-j}\lambda_r^{(k-j)}.\bar{\delta}_{s-r}^{(j)}.$$

Then we have

$$L^*_{0,s}=L^*_{1,s-1}=\cdots=L^*_{s-1,1}=L^*_{s,0}=\Lambda_s/N$$

which proves the claim. Finally, since $\mathfrak{B}_1,\ldots,\mathfrak{B}_N$ are mutually disjoint, the t-design $(X,\mathfrak{B})$ is the union of N pairwise disjoint s-designs $(X,\mathfrak{B}_i)$. This completes the proof of the theorem. □

The following corollary presents simple and useful cases of Theorem 4.2.

Corollary 4.3

1. *Assume that there exist s-resolvable t-designs with parameters t-$(v,k-1,\lambda_t^{(k-1)})$ and t-$(v,k,\lambda_t^{(k)})$ having the same number of resolution classes, say N, such that*
$$\lambda_{t-1}^{(k-1)}-\lambda_t^{(k-1)}=\lambda_t^{(k)}.$$
Then there exists an s-resolvable t-$(v+1,k,\lambda_{t-1}^{(k-1)})$ design with N resolution classes.

2. *Assume that there exists an s-resolvable t-$(2k+1,k,\lambda_t)$ design with N resolution classes. Then there exists an s-resolvable t-$(2k+2,k+1,\lambda_t\frac{2k+2-t}{k+1-t})$ design with N resolution classes.*

Proof. (1.) is a special case of Theorem 4.2 with $|X_1| = 1$, $|X_2| = v$. (2.) is a special case of (1.) with $v = 2k + 1$. □

We now consider some consequences of the main theorem for large sets. This is the case when all ingredient designs are the complete designs and s-resolutions are replaced by large sets. Following are some statements about large sets as by-products of Theorem 4.2.

Corollary 4.4 *If there exist an* $LS[N](t, k, v)$ *and an* $LS[N](t, k+1, v)$*, then there exists an* $LS[N](t, k+1, v+1)$.

Proof. Take $|X_1| = 1$, $|X_2| = v$ and use the basic construction with complete designs having t-large sets. □

Corollary 4.5 *Assume that there exist* $LS[N](t, i, v)$ *for* $t < i \leq k$*. Then the following hold.*

(i) For any given $1 \leq j < k - t$ *there exists an* $LS[N](t, t+1+h, v+j)$ *for* $j \leq h < k - t$.

(ii) If $k \geq 2t + 1$*, then there also exists an* $LS[N](t, k, 2v)$

Proof. (i) Applying Theorem 4.2 recursively by starting with $|X_1| = 1$, $|X_2| = v$.

(ii) Applying Theorem 4.2 with $|X_1| = |X_2| = v$. □

Corollary 4.6 *If there exists an* $LS[N](t, k, 2k+1)$*, then there exists an* $LS[N](t, k+1, 2k+2)$.

Proof. Applying Theorem 4.2 with $|X_1| = 1$ and $|X_2| = 2k+1$ by taking into account Lemma 2.2. The corollary may also be viewed as Corollary 4.3 (2.) for large sets. □

We hope that more involved consequences of the main theorem for large sets could be found.

5 Further recursive constructions of s-resolvable t-designs

In this section we examine two known constructions for t-designs in the papers [24, 25] and show that they both can be extended to constructions of s-resolvable t-designs.

Theorem 5.1 *Let* $(X, \mathfrak{B})$ *be an* s*-resolvable* t-(v, k, λ) *design with* N *resolution classes having the property that* $|B_1 \cap B_2| < k - k'$ *for any two distinct blocks* $B_1, B_2 \in \mathfrak{B}$*, where* $k' > 0$ *is a fixed integer. If there exists a* t-$(v-k, k', \lambda')$ *design, then there exists an* s*-resolvable* t-$(v, k+k', \Lambda)$ *design with* N *resolution classes where*

$$\Lambda = \lambda \frac{1}{\binom{v-t}{k-t}} \sum_{i=0}^{t} \binom{t}{i} \binom{v-t}{k-t+i} \lambda'_i.$$

Proof. First note that if the assumption of s-resolvability would be dropped, the theorem would become Theorem 5 in [25], in which the construction of t-$(v, k+k', \Lambda)$ design is as follows. For each block $B \in \mathfrak{B}$ let $(X \setminus B, \mathfrak{B}')$ be a t-$(v-k, k', \lambda')$ design on the point set $(X \setminus B)$. Define

$$\mathfrak{B}^*_B := \{B \cup B' \mid B' \in \mathfrak{B}'\}$$

and

$$\mathfrak{B}^* := \bigcup_{B \in \mathfrak{B}} \mathfrak{B}^*_B.$$

Then $(X, \mathfrak{B}^*)$ is the constructed t-$(v, k+k', \Lambda)$ design.

Now assume that $(X, \mathfrak{B})$ is s-resolvable with N resolution classes $\mathfrak{C}_1, \ldots, \mathfrak{C}_N$, i.e. $(X, \mathfrak{C}_i)$ is an s-(v, k, δ) design, $i = 1, \ldots, N$. Define

$$\mathfrak{B}^*_{\mathfrak{C}_i} = \bigcup_{B \in \mathfrak{C}_i} \mathfrak{B}^*_B.$$

Then $(X, \mathfrak{B}^*_{\mathfrak{C}_i})$ is an s-$(v, k+k', \Delta_s)$ design with

$$\Delta_s = \delta_s \frac{1}{\binom{v-s}{k-s}} \sum_{i=0}^{s} \binom{s}{i} \binom{v-s}{k-s+i} \lambda'_i,$$

where $\lambda'_s = \lambda' \frac{\binom{v-k-s}{t-s}}{\binom{k'-s}{t-s}}$. Since $\mathfrak{B}^* = \mathfrak{B}^*_{\mathfrak{C}_1} \cup \cdots \cup \mathfrak{B}^*_{\mathfrak{C}_N}$ and $\mathfrak{B}^*_{\mathfrak{C}_i}$'s are pairwise disjoint, the constructed design $(X, \mathfrak{B}^*)$ is thus s-resolvable. $\square$

Remark 5.1 Remark that if $t > k'$ in Theorem 5.1, then each trivial i-$(v-k, i, 1)$ design for $i < t$ can be viewed as a t-$(v-k, i, \lambda')$ design with $\lambda' = 0$. In particular, Theorem 5.1 remains valid for $t > k'$.

Theorem 5.2 *Assume that there exists an s-resolvable t-(v, k, λ) design with N resolution classes such that $v\lambda_0(\lambda_0 - \lambda_1) < \binom{v}{k}$. Then there exists an s-resolvable t-$(v+1, k, (v+1-t)\lambda)$ design with N resolution classes.*

Proof. Again if the assumption of s-resolvability would be removed, the theorem would become Theorem A in [24], where the existence of the t-$(v+1, k, (v+1-t)\lambda)$ design is shown as the union of $v+1$ mutually disjoint designs. Let $(Y, \mathfrak{D})$ denote the t-(v, k, λ) design. Let $X = \{x_1, \ldots, x_{v+1}\}$ be a $(v+1)$-set. Let $X_i = X \setminus \{x_i\}$ and $(X_i, \mathfrak{B}_i)$ be a copy of $(Y, \mathfrak{D})$ defined on X_i. If the condition of the theorem is satisfied, then there are $v+1$ mutually disjoint $\mathfrak{B}_i$. And the block set of the constructed t-$(v+1, k, (v+1-t)\lambda)$ design $(X, \mathfrak{B})$ is defined by $\mathfrak{B} = \bigcup_{i=1}^{v+1} \mathfrak{B}_i$.

Now by the assumption, $(Y, \mathfrak{D})$ is s-resolvable with N resolution classes. As $(X_i, \mathfrak{B}_i)$ is a copy of $(Y, \mathfrak{D})$, we denote $\mathfrak{C}^{(i)}_1, \ldots, \mathfrak{C}^{(i)}_N$ the N resolution classes of $(X_i, \mathfrak{B}_i)$ i.e. $(X_i, \mathfrak{C}^{(i)}_j)$ is an s-(v, k, δ) design, $j = 1, \ldots, N$ and $\mathfrak{B}_i = \mathfrak{C}^{(i)}_1 \cup \cdots \cup \mathfrak{C}^{(i)}_N$.

Define

$$\mathfrak{C}_j = \mathfrak{C}^{(1)}_j \cup \cdots \cup \mathfrak{C}^{(v+1)}_j,$$

for $j = 1, \ldots, N$. Then $\mathfrak{C}_j \cap \mathfrak{C}_{j'}$, for $j \neq j'$ and we have

$$\mathfrak{B} = \mathfrak{C}_1 \cup \cdots \cup \mathfrak{C}_N.$$

We show that $(X, \mathfrak{C}_j)$ is a simple s-$(v+1, k, \Delta)$ design with $\Delta = (v+1-s)\delta$. Let $S = \{x_{j_1}, \ldots, x_{j_s}\} \subseteq X$ be an s-set. Then S appears in δ blocks of $(X_i, \mathfrak{B}_i)$ for $i \neq j_1, \ldots, j_s$. Thus S appears in $(v+1-s)\delta$ blocks of $(X, \mathfrak{C}_j)$, i.e. $(X, \mathfrak{C}_j)$ is an s-$(v+1, k, (v+1-s)\delta)$ design. Hence $(X, \mathfrak{B})$ is s-resolvable with N resolution classes. □

It is worth mentioning that for the sake of simplicity we have presented Theorem 5.2 as it is. Actually, Theorem 5.2 can be stated in a more general form by using the result of Magliveras and Plambeck in [20].

5.1 An application: A 3-resolvable infinite family of 4-designs with parameters 4-$(2^n + 2, 7, 70(2^n - 2)/3)$

As an application of Theorem 4.2, we construct 3-resolvable 4-designs with parameters 4-$(2^n + 2, 7, 70(2^n - 3)/3)$. In [9] it is shown that the 4-$(2^n + 1, 6, 10)$ design $(X, \mathfrak{B})$, with $\gcd(n, 6) = 1$, $n \geq 5$, constructed in [6] has the property that any two distinct blocks intersect in at most 4 points. Applying Theorem 5.1 to $(X, \mathfrak{B})$ by using the trivial 1-$(2^n - 5, 1, 1)$ design as the required t-$(v - k, k', \lambda')$ design, see Remark 5.1, gives a 4-$(2^n + 1, 7, 70(2^n - 5)/3)$ design. Now, since the 4-$(2^n + 1, 6, 10)$ design is 3-resolvable with $N = (2^n - 2)/6$ resolution classes, it follows by Theorem 5.1 that the 4-$(2^n + 1, 7, 70(2^n - 5)/3)$ design is 3-resolvable with the same number of resolutions.

Further, as shown in the previous section, there exists a 3-resolvable 4-$(2^n + 1, 6, 70)$ design, $\gcd(n, 6) = 1$ for $n \geq 5$, with $N = (2^n - 2)/6$ resolution classes. We apply Theorem 4.2 or Corollary 4.3 to these two families as ingredient designs for $|X_1| = 1$ and $|X_2| = 2^n + 1$. More precisely, since $|X_1| = 1$, we only have two expressions $L_{0,4}$ and $L_{1,3}$, where

$$\begin{aligned}
L_{0,4} &= \bar{\lambda}_4^{(7)} + \bar{\lambda}_4^{(6)} = 70(2^n - 5)/3 + 70 = 70(2^n - 2)/3, \\
L_{1,3} &= \bar{\lambda}_3^{(6)} = 70(2^n - 2)/3.
\end{aligned}$$

Thus the requirement $L_{0,4} = L_{1,3}$ is satisfied and we obtain the following theorem.

Theorem 5.3 *For any integer* $n \geq 5$ *with* $\gcd(n, 6) = 1$ *there exists a 3-resolvable* 4-$(2^n + 2, 7, 70(2^n - 2)/3)$ *design with* $N = (2^n - 2)/6$ *resolution classes.*

6 Conclusion

The primary aim of the work is an investigation of recursive constructions for t-designs having s-resolutions, emphasizing the cases $s > 1$. A brief account of known infinite families of s-resolvable t-designs was included, whose existence is essentially based on the construction of t-designs with an s-homogeneous group. The main theorem provides a general and effective tool for constructing designs having resolutions and particularly yields statements about large sets as by-products. By virtue of the existence of general recursive constructions for t-designs using resolutions of ingredient designs, we hope that this topic would attract more attention in the future as it was the case with large sets in the past.

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