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Preprint 2020-02

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Abstract

This work concerns the study of asymptotic behavior of coupled systems of $p(x)$-laplacian differential inclusions. We obtain that the generalized semiflow generated by the coupled system has a global attractor, we prove continuity of solutions with respect to initial conditions and triple parameters and we prove upper semicontinuity of a family of global attractors for Reaction-Diffusion systems with spatially variable exponents when the exponents go to constants greater than 2 in the topology of $L^\infty(\Omega)$ and the diffusion coefficients go to infinity.

Keywords: Reaction-Diffusion coupled systems, variable exponents, attractors, upper semicontinuity, large diffusion.

AMS 2010 Subject Classification: Primary: 35K55, 35K92; Secondary: 35A16, 35B40, 35B41.

1 Introduction

PDEs for which the flow is essentially determined by an ordinary differential equation have been studied by many researchers, see for example [2, 8, 9, 13, 18, 19, 20, 27, 28, 32]. In [30, 31, 32] the authors investigated in which way the parameter $p(x)$ affects the dynamic of problems involving the $p(x)$-Laplacian.

In this work we consider the following nonlinear coupled system

$$\begin{cases} \frac{\partial u_s}{\partial t} - \operatorname{div}(D_s|\nabla u_s|^{p_s(x)-2}\nabla u_s) + |u_s|^{p_s(x)-2}u_s \in F(u_s,v_s) \quad t>0 \\ \frac{\partial v_s}{\partial t} - \operatorname{div}(D_s|\nabla v_s|^{q_s(x)-2}\nabla v_s) + |v_s|^{q_s(x)-2}v_s \in G(u_s,v_s) \quad t>0 \\ u_s(0)=u_{0s},\ v_s(0)=v_{0s} \end{cases}, \tag{1}$$

where $u_{0s}, v_{0s} \in H := L^2(\Omega)$, $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a smooth bounded domain, $D_s \in [1,\infty)$, $p_s(\cdot), q_s(\cdot) \in C(\bar{\Omega})$, $p_s^- := \min_{x\in\overline{\Omega}} p_s(x) \geq p$, $q_s^- := \min_{x\in\overline{\Omega}} q_s(x) \geq q$, $p_s^+ := \max_{x\in\overline{\Omega}} p_s(x) \leq L$, $q_s^+ := \max_{x\in\overline{\Omega}} q_s(x) \leq L$, for all $s \in \mathbb{N}$. We assume that $p_s(\cdot) \to p$,

$q_s(\cdot) \to q$ in $L^\infty(\Omega)$ and $D_s \to \infty$ as $s \to \infty$, where $L, p, q > 2$ are positive constants. $F, G : L^2(\Omega) \times L^2(\Omega) \to P(L^2(\Omega))$ are bounded, upper semicontinuous and positively sublinear multivalued operators.

In this work, to study global attractors for the system (1) which we do not have guaranty of uniqueness of solution, we follows the general frameworks [5, 24, 25].

The paper is organized as follows. In Section 2 we remaind some definitions and we present properties on the operator and guarantee the existence of global solutions and global attractors. In Section 3 we obtain uniform estimates for solutions of (1). In Section 4 we prove that the solutions $\{u_s\}$ of the PDE (1) go to the solution u of the limit problem (18) which is an ODE system, and, after that, we obtain the upper semicontinuity of the global attractors for the problem (1).

2 Existence of global solutions and global attractor

Let us first remaind some definitions. Consider the system

$$(P) \begin{cases} u_t + Au \in F(u, v) \\ v_t + Bv \in G(u, v) \\ u(0) = u_0, v(0) = v_0. \end{cases}$$

where A and B are monotone operators of subdifferential type defined in a real Hilbert space H.

Definition 1 *[25] A strong solution [weak solution] of (P) is a pair (u, v) satisfying: $u, v \in C([0, T]; H)$ for which there exists $f, g \in L^1(0, T; H)$, $f(t) \in F(u(t), v(t))$, $g(t) \in G(u(t), v(t))$ a.e. in $(0, T)$, and such that (u, v) is a strong solution [weak solution] (see Definition 3.1 and Theorem 3.4 in [7]) over $(0, T)$ to the system (P_1) below:*

$$(P_1) \begin{cases} u_t + Au = f \\ v_t + Bv = g \\ u(0) = u_0, v(0) = v_0 \end{cases}$$

Definition 2 *[3, 4, 25] Let U be a topological space. A mapping $G : U \to P(H)$ is called upper semicontinuous at $u \in U$, if*

(i) $G(u)$ is nonempty, bounded, closed and convex.

(ii) For each open subset D in H satisfying $G(u) \subset D$, there exists a neighborhood V of u, such that $G(v) \subset D$, for each $v \in V$.

If G is upper semicontinuous at each $u \in U$, then it is called upper semicontinuous on U.

Definition 3 *[3, 4, 25] Let M be a Lebesgue measurable subset in $\mathbb{R}^q$, $q \geq 1$. By a selection of $E : M \to P(H)$ we mean a function $f : M \to H$ such that $f(y) \in E(y)$ a.e. $y \in M$, and we denote by SelE the set*

$$SelE \doteq \{f,\ f : M \to H \text{ is a measurable selection of } E\}.$$

In order to get global solutions we impose suitable conditions on terms F and G.

Definition 4 *[25] The pair (F,G) of maps $F,G : H\times H \to P(H)$, which takes bounded subsets of $H\times H$ into bounded subsets of H, is called positively sublinear if there exist $a>0$, $b>0$, $c>0$ and $m_0>0$ such that for each $(u,v)\in H\times H$ with $\|u\|_H > m_0$ or $\|v\|_H > m_0$ for which either there exists $f_0 \in F(u,v)$ satisfying $\langle u, f_0\rangle > 0$ or there exists $g_0 \in G(u,v)$ with $\langle v, g_0\rangle > 0$, we have both*

$$\|f\|_H \leq a\|u\|_H + b\|v\|_H + c \ \text{ and } \ \|g\|_H \leq a\|u\|_H + b\|v\|_H + c$$

for each $f\in F(u,v)$ and each $g\in G(u,v)$.

Now, let us remaind the definitions of the Lebesgue and Sobolev spaces with variable exponents. Considering $p\in L^{\infty}_{+}(\Omega) := \{q \in L^{\infty}(\Omega):\ \text{ess inf}\, q \geq 1\}$, then

$$L^{p(\cdot)}(\Omega) := \left\{ u : \Omega \to \mathbb{R};\, u \ \text{ is measurable, } \int_\Omega |u(x)|^{p(x)}dx < \infty \right\}$$

is a Banach space with the norm

$$\|u\|_{p(x)} := \inf\left\{ \lambda > 0; \rho\Big(\frac{u}{\lambda}\Big) \leq 1 \right\},$$

where $\rho(u) := \int_\Omega |u(x)|^{p(x)}dx$. The following inequality will be used later

$$\min\{\|u\|^{p^-}_{p(x)}, \|u\|^{p^+}_{p(x)}\} \leq \int_\Omega |u(x)|^{p(x)}dx \leq \max\{\|u\|^{p^-}_{p(x)}, \|u\|^{p^+}_{p(x)}\} \tag{2}$$

Furthermore,

$$W^{1,p(\cdot)}(\Omega) := \left\{ u \in L^{p(\cdot)}(\Omega); |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}$$

which is a Banach space with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} := \|\nabla u\|_{p(x)} + \|u\|_{p(x)}.$$

The authors in [33] proved that the operator

$$A^s u := -\text{div}(D_s|\nabla u|^{p_s(x)-2}\nabla u) + |u|^{p_s(x)-2}u$$

is the realization of the operator $A_1^s : X_s \to X_s^*,\ X_s := W^{1,p_s(\cdot)}(\Omega)$,

$$A_1^s u(v) := \int_\Omega D_s |\nabla u(x)|^{p_s(x)-2}\nabla u(x)\cdot \nabla v(x) dx + \int_\Omega |u(x)|^{p_s(x)-2}u(x)v(x)dx,$$

i.e., $A^s(u) = A_1^s u,\ if\ \ u \in \mathcal{D}(A^s) := \{u \in X_s;\ A_1^s u \in H\}$ and it is a maximal monotone operator in H. Besides, A^s generates a compact semigroup and is the subdifferential of the proper, convex and lower semicontinuous function $\varphi_{p_s(x)} : H \to \mathbb{R}\cup\{+\infty\}$ defined by

$$\varphi_{p_s(x)}(u) := \begin{cases} \Big[\displaystyle\int_\Omega \frac{D_s}{p_s(x)}|\nabla u|^{p_s(x)}dx + \int_\Omega \frac{1}{p_s(x)}|u|^{p_s(x)}dx\Big], & \text{if } u \in X_s,\\ +\infty, & \text{otherwise.}\end{cases} \tag{3}$$

Moreover, they proved that the system (1) has a strong global solution (u_s, v_s).

Using the following elementary assertion we can obtain estimates on the operator only considering two cases.

Proposition 5 *[1] Let λ, μ be arbitrary nonnegative numbers. For every positive α, β, $\alpha \geq \beta$,*

$$\lambda^\alpha + \mu^\beta \geq \frac{1}{2^\alpha} \begin{cases} (\lambda+\mu)^\alpha & \text{if } \lambda+\mu < 1, \\ (\lambda+\mu)^\beta & \text{if } \lambda+\mu \geq 1. \end{cases}$$

Then it is easy to show that for every $u \in X_s$

$$\langle A^s u, u \rangle_{X_s^*, X_s} \geq \frac{1}{2^{p_s^+}} \begin{cases} \|u\|_{X_s}^{p_s^+} & \text{if } \|u\|_{X_s} < 1, \\ \|u\|_{X_s}^{p_s^-} & \text{if } \|u\|_{X_s} \geq 1. \end{cases} \tag{4}$$

From now on, we will denote $X_s := W^{1,p_s(\cdot)}(\Omega)$, $Y_s := W^{1,q_s(\cdot)}(\Omega)$, $X := W^{1,p}(\Omega)$ and $Y := W^{1,q}(\Omega)$.

It is a known result that $X_s, Y_s \subset H$ with continuous and dense embeddings (see [15, 28]). Moreover, it is easy to see that

$$\|u_s\|_H \leq 4(|\Omega|+1)^2 \|u_s\|_{X_s}, \tag{5}$$

for all $u_s \in X_s$ and for all $s \in \mathbb{N}$.

By [25] we get:

- If $D_s(u_0, v_0)$ is the set of all solutions of (1) with initial data (u_0, v_0) then

  $$\mathbb{G}_s := \bigcup_{(u_0,v_0) \in H \times H} D_s(u_0, v_0)$$

  is a generalized semiflow in $H \times H$ which is called the generalized semiflow associated with (1).

- If $\mathbb{G}_s$ is eventually bounded then $\mathbb{G}_s$ is asymptotically compact.

Therefore, according to Theorem 9 in [24] (see also Remark 6 in [24]), in order to assure the existence of a compact invariant B-attractor for (1), it is enough to guarantee that the generalized semiflow $\mathbb{G}_s$ defined by (1) is B-dissipative and so, eventually bounded and φ-dissipative.

In this work we denote $A^s(w) := -\text{div}(D_s|\nabla w|^{p_s(x)-2}\nabla w) + |w|^{p_s(x)-2}w$, and analogously $B^s(w) := -\text{div}(D_s|\nabla w|^{q_s(x)-2}\nabla w) + |w|^{q_s(x)-2}w$; S^s the semigroup generated by A^s and T_s the multivalued semigroup defined by $\mathbb{G}_s$.

Theorem 6 *Let $F, G : H \times H \to P(H)$ bounded, upper semicontinuous and positively sublinear operators. There exists a bounded set B_s in $H \times H$ and $t_0 > 0$ such that for any $\varphi_s \in \mathbb{G}_s$, $\varphi_s(t) \in B_s$, $\forall\ t \geq t_0$. Then, in particular, the generalized semiflow $\mathbb{G}_s$ defined by (1) is B-dissipative.*

Proof. Let $\varphi_s = (u_s, v_s) \in \mathbb{G}_s$ a solution of (1). Then there exists a pair $(f_s, g_s) \in \text{Sel } F(u_s, v_s) \times \text{Sel } G(u_s, v_s)$, $f_s, g_s \in L^1(0,T;H)$ for each $T > 0$ such that u_s, v_s satisfy the problem

$$\begin{cases} \frac{\partial u_s}{\partial t} + A^s(u_s) = f_s & \text{in} \quad (0,T) \times \Omega, \\ \frac{\partial v_s}{\partial t} + B^s(v_s) = g_s & \text{in} \quad (0,T) \times \Omega, \\ u_s(0,x) = u_{0s}(x),\ v_s(0,x) = v_{0s}(x) & \text{in} \quad \Omega. \end{cases} \tag{6}$$

Multiplying the first equation by u_s we obtain

$$\Big\langle \frac{\partial u_s(t)}{\partial t}, u_s(t) \Big\rangle_H + \Big\langle A^s(u_s(t)), u_s(t) \Big\rangle_H = \langle f_s(t), u_s(t) \rangle_H.$$

Let $I := (0,T)$, $I_{1s} := \{t \in I : \|u_s(t)\|_{X_s} < 1\}$ and $I_{2s} := \{t \in I : \|u_s(t)\|_{X_s} \geq 1\}$. Then by (4)

$$\frac{1}{2}\frac{d}{dt}\|u_s(t)\|_H^2 + \frac{1}{2^{p_s^+}}\|u_s(t)\|_{X_s}^{p_s^+} \leq \langle f_s(t), u_s(t)\rangle_H \quad \text{if} \quad t \in I_{1s},$$

and

$$\frac{1}{2}\frac{d}{dt}\|u_s(t)\|_H^2 + \frac{1}{2^{p_s^+}}\|u_s(t)\|_{X_s}^{p_s^-} \leq \langle f_s(t), u_s(t)\rangle_H \quad \text{if} \quad t \in I_{2s}.$$

Thus,

$$\frac{1}{2}\frac{d}{dt}\|u_s(t)\|_H^2 \leq \begin{cases} -\frac{\sigma}{\alpha^{p_s^+}}\|u_s(t)\|_H^{p_s^+} + \langle f_s(t), u_s(t)\rangle_H & \text{if} \quad t \in I_{1s}, \\ -\frac{\sigma}{\alpha^{p_s^-}}\|u_s(t)\|_H^{p_s^-} + \langle f_s(t), u_s(t)\rangle_H & \text{if} \quad t \in I_{2s}. \end{cases} \tag{7}$$

where $\alpha := 4(|\Omega|+1)^2$ and $\sigma := \frac{1}{2^L}$.

In an analogous way, multiplying the second equation in (6) by v_s we obtain

$$\frac{1}{2}\frac{d}{dt}\|v_s(t)\|_H^2 \leq \begin{cases} -\frac{\sigma}{\alpha^{q_s^+}}\|v_s(t)\|_H^{q_s^+} + \langle g_s(t), v_s(t)\rangle_H \, \text{if} \, t \in \tilde{I_{1s}} \\ -\frac{\sigma}{\alpha^{q_s^-}}\|v_s(t)\|_H^{q_s^-} + \langle g_s(t), v_s(t)\rangle_H \, \text{if} \, t \in \tilde{I_{2s}} \end{cases},$$

where $\tilde{I_{1s}} := \{t \in I : \|v_s(t)\|_{Y_s} < 1\}$, $\tilde{I_{2s}} := \{t \in I : \|v_s(t)\|_{Y_s} \geq 1\}$.

Now, let $r_s := \frac{p_s^+}{p_s^-} > 1$ and r_s' such that $\frac{1}{r_s} + \frac{1}{r_s'} = 1$ then by Young's inequality

$$\|u_s(t)\|_H^{p_s^-} \leq \frac{1}{r_s'} + \frac{1}{r_s}\|u_s(t)\|_H^{p_s^+},$$

and so

$$-\frac{\sigma}{\alpha^{p_s^+}}\|u_s(t)\|_H^{p_s^+} \leq r_s\Big(-\frac{\sigma}{\alpha^{p_s^+}}\|u_s(t)\|_H^{p_s^-} + \frac{\sigma}{\alpha^{p_s^+} r_s'}\Big). \tag{8}$$

Using (8) in (7) we get

$$\frac{1}{2}\frac{d}{dt}\|u_s(t)\|_H^2 \leq -C_2\|u_s(t)\|_H^{p_s^-} + \langle f_s(t), u_s(t)\rangle_H + C_1 \quad \forall\, t \in I = (0,T),$$

where $C_2 := \frac{1}{(2\alpha)^L}$ and $C_1 := \frac{L\sigma}{p\alpha^p}$.

In a analogous way, taking $\tilde{r}_s := \frac{q_s^+}{q_s^-} > 1$ and $\tilde{r}_s'$ such that $\frac{1}{\tilde{r}_s} + \frac{1}{\tilde{r}_s'} = 1$ we get

$$\frac{1}{2}\frac{d}{dt}\|v_s(t)\|_H^2 \leq -\tilde{C}_2\|v_s(t)\|_H^{q_s^-} + \langle g_s(t), v_s(t)\rangle_H + \tilde{C}_1 \quad \forall\, t \in I = (0,T),$$

where $\tilde{C}_2 = C_2 = \frac{1}{(2\alpha)^L}$ and $\tilde{C}_1 := \frac{L\sigma}{q\alpha^q}$.

Thus, we obtain

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\|u_s(t)\|_H^2 \leq -C_2\|u_s(t)\|_H^{p_s^-} + \langle f_s(t), u_s(t)\rangle_H + C_1 \\ \frac{1}{2}\frac{d}{dt}\|v_s(t)\|_H^2 \leq -\tilde{C}_2\|v_s(t)\|_H^{q_s^-} + \langle g_s(t), v_s(t)\rangle_H + \tilde{C}_1 \end{cases} \tag{9}$$

where $C_2, \tilde{C}_2, C_1, \tilde{C}_1$ are positive real numbers depending on $|\Omega|, L, p, q$.

We can suppose, without losing generality that $p_s^- \geq q_s^-$. If $p_s^- = q_s^-$ we obtain a similar expression as (9) with q_s^- on the place of p_s^-. If $p_s^- > q_s^-$, taking $\theta_s := \frac{p_s^-}{q_s^-} > 1$, θ_s' such that $\frac{1}{\theta_s} + \frac{1}{\theta_s'} = 1$ and $\epsilon > 0$ we have

$$\|u_s(t)\|_H^{q_s^-} = \frac{\epsilon}{\epsilon}\|u_s(t)\|_H^{q_s^-} \leq \frac{1}{\theta_s' \epsilon^{\theta_s'}} + \frac{1}{\theta_s}\epsilon^{\theta_s}\|u_s(t)\|_H^{p_s^-}$$

and then

$$-C_2\|u_s(t)\|_H^{p_s^-} \leq \frac{\theta_s}{\epsilon^{\theta_s}}\Big[\frac{C_2}{\theta_s' \epsilon^{\theta_s'}} - C_2\|u_s(t)\|_H^{q_s^-}\Big].$$

So, we have that

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\|u_s(t)\|_H^2 \leq -\frac{C_2\theta_s}{\epsilon^{\theta_s}}\|u_s(t)\|_H^{q_s^-} + \langle f_s(t), u_s(t)\rangle_H + C_1 + \frac{\theta_s C_2}{\theta_s' \epsilon^{\theta_s}\epsilon^{\theta_s'}} \\ \frac{1}{2}\frac{d}{dt}\|v_s(t)\|_H^2 \leq -\tilde{C}_2\|v_s(t)\|_H^{q_s^-} + \langle g_s(t), v_s(t)\rangle_H + \tilde{C}_1 \end{cases} \tag{10}$$

Now, we use that (F, G) is positively sublinear (see Definition 4) to estimate $\langle f_s(t), u_s(t)\rangle_H$ and $\langle g_s(t), v_s(t)\rangle_H$. To do this, we have to consider the following three steps:

1. If $\|u_s(t)\|_H \leq m_0$ and $\|v_s(t)\|_H \leq m_0$ then as F and G take bounded subsets of $H \times H$ into bounded subsets of H there exists $C > 0$ such that

$$\langle f_s(t), u_s(t)\rangle_H \leq \|f_s(t)\|_H\|u_s(t)\|_H \leq Cm_0$$

and

$$\langle g_s(t), v_s(t)\rangle_H \leq \|g_s(t)\|\|_s v(t)\| \leq Cm_0.$$

2. If $\|u_s(t)\|_H > m_0$ or $\|v_s(t)\|_H > m_0$ and $\langle f_0, u_s(t)\rangle \leq 0$ and $\langle g_0, v_s(t)\rangle \leq 0$ $\forall$ $f_0 \in F(u_s(t), v_s(t))$ and $\forall$ $g_0 \in G(u_s(t), v_s(t))$ then $\langle f_s(t), u_s(t)\rangle_H \leq 0$ and $\langle g_s(t), v_s(t)\rangle_H \leq 0$.

3. If $\|u_s(t)\|_H > m_0$ or $\|v_s(t)\|_H > m_0$ and $\langle f_0, u_s(t)\rangle > 0$ or $\langle g_0, v_s(t)\rangle > 0$ for some $f_0 \in F(u_s(t), v_s(t))$ or for some $g_0 \in G(u_s(t), v_s(t))$ then, for $\epsilon > 0$, $\kappa_s := \frac{q_s^-}{2} > 1$ and $\nu_s := \frac{q_s^-}{(q_s^-)'} > 1$, we get

$$\begin{aligned}
\langle f_s(t), u_s(t)\rangle &\leq \|f_s(t)\|_H\|u_s(t)\|_H \\
&\leq \frac{\epsilon}{\epsilon}a\|u_s(t)\|_H^2 + \frac{\epsilon}{\epsilon}b\|u_s(t)\|_H\|v_s(t)\|_H + \frac{\epsilon}{\epsilon}c\|u_s(t)\|_H \\
&\leq \frac{1}{\kappa_s'}\Big(\frac{a}{\epsilon}\Big)^{\kappa_s'} + \frac{1}{\kappa_s}\epsilon^{\kappa_s}\|u_s(t)\|_H^{q_s^-} + \frac{1}{(q_s^-)'}\Big(\frac{b}{\epsilon}\Big)^{(q_s^-)'}\|v_s(t)\|_H^{(q_s^-)'} \\
&\quad + \frac{1}{q_s^-}\epsilon^{q_s^-}\|u_s(t)\|_H^{q_s^-} + \frac{1}{(q_s^-)'}\Big(\frac{c}{\epsilon}\Big)^{(q_s^-)'} + \frac{1}{q_s^-}\epsilon^{q_s^-}\|u_s(t)\|_H^{q_s^-} \\
&= \Big(\frac{2}{q_s^-}\epsilon^{\frac{q_s^-}{2}} + \frac{2}{q_s^-}\epsilon^{q_s^-}\Big)\|u_s(t)\|_H^{q_s^-} + \frac{\epsilon}{\epsilon}\frac{1}{(q_s^-)'}\Big(\frac{b}{\epsilon}\Big)^{(q_s^-)'}\|v_s(t)\|_H^{(q_s^-)'} \\
&\quad + \Big(\frac{1}{\kappa_s'}\Big(\frac{a}{\epsilon}\Big)^{\kappa_s'} + \frac{1}{(q_s^-)'}\Big(\frac{c}{\epsilon}\Big)^{(q_s^-)'}\Big) \\
&\leq \Big(\frac{2}{q_s^-}\epsilon^{\frac{q_s^-}{2}} + \frac{2}{q_s^-}\epsilon^{q_s^-}\Big)\|u_s(t)\|_H^{q_s^-} + \frac{\epsilon^{\nu_s}}{\nu_s}\|v_s(t)\|_H^{q_s^-} \\
&\quad + \Big[\frac{1}{\nu_s'}\Big(\frac{1}{\epsilon}\frac{1}{(q_s^-)'}\Big(\frac{b}{\epsilon}\Big)^{(q_s^-)'}\Big)^{\nu_s'} + \frac{1}{\kappa_s'}\Big(\frac{a}{\epsilon}\Big)^{\kappa_s'} + \frac{1}{(q_s^-)'}\Big(\frac{c}{\epsilon}\Big)^{(q_s^-)'}\Big]
\end{aligned}$$

and on an analogous way

$$\langle g_s(t), v_s(t)\rangle \le \Big(\frac{2}{q_s^-}\epsilon^{\frac{q_s^-}{2}} + \frac{2}{q_s^-}\epsilon^{q_s^-}\Big)\|v_s(t)\|_H^{q_s^-} + \frac{\epsilon^{\nu_s}}{\nu_s}\|u_s(t)\|_H^{q_s^-}$$
$$+\Big[\frac{1}{\nu_s'}\Big(\frac{1}{\epsilon}\frac{1}{(q_s^-)'}\Big(\frac{a}{\epsilon}\Big)^{(q_s^-)'}\Big)^{\nu_s'} + \frac{1}{\kappa_s'}\Big(\frac{b}{\epsilon}\Big)^{\kappa_s'} + \frac{1}{(q_s^-)'}\Big(\frac{c}{\epsilon}\Big)^{(q_s^-)'}\Big].$$

Therefore, joining up 1.), 2.) and 3.) we get

$$\langle f_s(t), u_s(t)\rangle \le \Big(\frac{2}{q_s^-}\epsilon^{\frac{q_s^-}{2}} + \frac{2}{q_s^-}\epsilon^{q_s^-}\Big)\|u_s(t)\|_H^{q_s^-} + \frac{\epsilon^{\nu_s}}{\nu_s}\|v_s(t)\|_H^{q_s^-} + m_0 C$$
$$+\Big[\frac{1}{\nu_s'}\Big(\frac{1}{\epsilon}\frac{1}{(q_s^-)'}\Big(\frac{b}{\epsilon}\Big)^{(q_s^-)'}\Big)^{\nu_s'} + \frac{1}{\kappa_s'}\Big(\frac{a}{\epsilon}\Big)^{\kappa_s'} + \frac{1}{(q_s^-)'}\Big(\frac{c}{\epsilon}\Big)^{(q_s^-)'}\Big] \tag{11}$$

and

$$\langle g_s(t), v_s(t)\rangle \le \Big(\frac{2}{q_s^-}\epsilon^{\frac{q_s^-}{2}} + \frac{2}{q_s^-}\epsilon^{q_s^-}\Big)\|v_s(t)\|_H^{q_s^-} + \frac{\epsilon^{\nu_s}}{\nu_s}\|u_s(t)\|_H^{q_s^-} + m_0 C$$
$$+\Big[\frac{1}{\nu_s'}\Big(\frac{1}{\epsilon}\frac{1}{(q_s^-)'}\Big(\frac{a}{\epsilon}\Big)^{(q_s^-)'}\Big)^{\nu_s'} + \frac{1}{\kappa_s'}\Big(\frac{b}{\epsilon}\Big)^{\kappa_s'} + \frac{1}{(q_s^-)'}\Big(\frac{c}{\epsilon}\Big)^{(q_s^-)'}\Big]. \tag{12}$$

Using (11) and (12) in (10) we get

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\|u_s(t)\|_H^2 \le \Big(-\frac{C_2\theta_s}{\epsilon^{\theta_s}} + \frac{2}{q_s^-}\epsilon^{\frac{q_s^-}{2}} + \frac{2}{q_s^-}\epsilon^{q_s^-}\Big)\|u_s(t)\|_H^{q_s^-} + \frac{\epsilon^{\nu_s}}{\nu_s}\|v_s(t)\|_H^{q_s^-} + C_3(\epsilon,s) \\ \frac{1}{2}\frac{d}{dt}\|v_s(t)\|_H^2 \le \Big(-\tilde{C}_2 + \frac{2}{q_s^-}\epsilon^{\frac{q_s^-}{2}} + \frac{2}{q_s^-}\epsilon^{q_s^-}\Big)\|v_s(t)\|_H^{q_s^-} + \frac{\epsilon^{\nu_s}}{\nu_s}\|u_s(t)\|_H^{q_s^-} + C_4(\epsilon,s) \end{cases}$$

where

$$C_3(\epsilon,s) = m_0 C + \Big[\frac{1}{\nu_s'}\Big(\frac{1}{\epsilon}\frac{1}{(q_s^-)'}\Big(\frac{b}{\epsilon}\Big)^{(q_s^-)'}\Big)^{\nu_s'} + \frac{1}{\kappa_s'}\Big(\frac{a}{\epsilon}\Big)^{\kappa_s'} + \frac{1}{(q_s^-)'}\Big(\frac{c}{\epsilon}\Big)^{(q_s^-)'}\Big] + C_1 + \frac{\theta_s C_2}{\theta_s'\epsilon^{\theta_s}\epsilon^{\theta_s'}}$$

and

$$C_4(\epsilon,s) = m_0 C + \Big[\frac{1}{\nu_s'}\Big(\frac{1}{\epsilon}\frac{1}{(q_s^-)'}\Big(\frac{a}{\epsilon}\Big)^{(q_s^-)'}\Big)^{\nu_s'} + \frac{1}{\kappa_s'}\Big(\frac{b}{\epsilon}\Big)^{\kappa_s'} + \frac{1}{(q_s^-)'}\Big(\frac{c}{\epsilon}\Big)^{(q_s^-)'}\Big] + \tilde{C}_1.$$

Thus, adding the last two inequalities we obtain

$$\frac{1}{2}\frac{d}{dt}\Big(\|u_s(t)\|_H^2 + \|v_s(t)\|_H^2\Big) \le \Big(-\frac{C_2\theta_s}{\epsilon^{\theta_s}} + \frac{2}{q_s^-}\epsilon^{\frac{q_s^-}{2}} + \frac{2}{q_s^-}\epsilon^{q_s^-} + \frac{\epsilon^{\nu_s}}{\nu_s}\Big)\|u_s(t)\|_H^{q_s^-}$$
$$+\Big(-\tilde{C}_2 + \frac{2}{q_s^-}\epsilon^{\frac{q_s^-}{2}} + \frac{2}{q_s^-}\epsilon^{q_s^-} + \frac{\epsilon^{\nu_s}}{\nu_s}\Big)\|v_s(t)\|_H^{q_s^-} + C_3(\epsilon,s) + C_4(\epsilon,s).$$

As $\epsilon > 0$ is arbitrary, take ϵ_0 sufficiently small such that

$$\frac{2}{q_s^-}\epsilon_0^{\frac{q_s^-}{2}} + \frac{2}{q_s^-}\epsilon_0^{q_s^-} + \frac{\epsilon_0^{\nu_s}}{\nu_s} < \frac{\tilde{C}_2}{2} \quad \text{and} \quad \frac{C_2\theta_s}{\epsilon_0^{\theta_s}} \ge \tilde{C}_2.$$

Then
$$\frac{1}{2}\frac{d}{dt}\Big(\|u_s(t)\|_H^2 + \|v_s(t)\|_H^2\Big) \le -C_5\Big(\|u(t)\|_H^{q_s^-} + \|v_s(t)\|_H^{q_s^-}\Big) + C_6(s)$$
where $C_5 := \frac{\tilde{C}_2}{2} > 0$ and $C_6(s) = C_3(\epsilon_0, s) + C_4(\epsilon_0, s) > 0$.

Thus,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(\|u_s(t)\|_H^2 + \|v_s(t)\|_H^2\Big) &\le -C_5\Big(\|u_s(t)\|_H^{2\frac{q_s^-}{2}} + \|v_s(t)\|_H^{2\frac{q_s^-}{2}}\Big) + C_6(s)\\ &\le -\frac{C_5}{2^{\frac{q_s^-}{2}}}\Big(\|u_s(t)\|_H^2 + \|v_s(t)\|_H^2\Big)^{\frac{q_s^-}{2}} + C_6(s).\end{aligned}$$

Therefore, the function $y_s(t) := \|u_s(t)\|_H^2 + \|v_s(t)\|_H^2$ satisfies the inequality

$$y_s'(t) \le -\frac{2C_5}{2^{\frac{q_s^-}{2}}} y_s(t)^{\frac{q_s^-}{2}} + 2C_6(s), \qquad t > 0.$$

From Lemma 5.1 in [34] we obtain

$$y_s(t) \le \left(\frac{2C_6(s)}{\frac{2C_5}{2^{q_s^-/2}}}\right)^{2/q_s^-} + \left[\frac{2C_5}{2^{q_s^-/2}}\left(\frac{q_s^-}{2} - 1\right)t\right]^{\frac{-1}{\left(\frac{q_s^-}{2}-1\right)}}.$$

So, considering $r_s := \left(\frac{2C_6(s)2^{q_s^-/2}}{2C_5}\right)^{2/q_s^-} + \left[\frac{2C_5}{2^{q_s^-/2}}\left(\frac{q_s^-}{2} - 1\right)\right]^{\frac{-1}{\left(\frac{q_s^-}{2}-1\right)}}$ and $t_0 = 1$ we have $\|u_s(t)\|_H^2 + \|v_s(t)\|_H^2 \le r_s$, for all $t \ge t_0$. ■

We conclude that $\mathbb{G}_s$ has a compact invariant global B-attractor $\mathcal{A}_s$. The global B-attractor $\mathcal{A}_s$ is unique and given by

$$\mathcal{A}_s = \bigcup_{B \in B(H\times H)} \omega_s(B) = \omega_{sB}(H \times H).$$

Futhermore $\mathcal{A}_s$ is the maximal compact invariant subset of $H \times H$, and is minimal among all closed global B-attractors. We also have that $\mathcal{A}_s$ is the union of all complete bounded orbits in $H \times H$ (see Theorem 15 in [24]).

3 Uniform estimates

In this section we prove uniform estimates in $H \times H$ and $X_s \times Y_s$ on the solutions of (1).

Lemma 7 *If (u_s, v_s) is a solution of (1), then there exist positive numbers r_0 and a constant $t_0 > 0$ such that $\|(u_s(t), v_s(t))\|_{H\times H} \le r_0$, for each $t \ge t_0$ and $s \in \mathbb{N}$.*

Proof. The same arguments employed in the proof of Theorem 6 can also be applied here, but now use from the beginning the hypothesis $p_s^- \geq p$, $q_s^- \geq q$, $p_s^+ \leq L$, $q_s^+ \leq L$, for all $s \in \mathbb{N}$ and we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_s(t)\|_H^2 \leq -C_2\|u_s(t)\|_H^p + \langle f_s(t), u_s(t)\rangle_H + C_1 \quad \forall\, t \in I = (0,T),$$

and

$$\frac{1}{2}\frac{d}{dt}\|v_s(t)\|_H^2 \leq -\tilde{C}_2\|v_s(t)\|_H^q + \langle g_s(t), v_s(t)\rangle_H + \tilde{C}_1 \quad \forall\, t \in I = (0,T),$$

Thus, we obtain

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\|u_s(t)\|_H^2 \leq -C_2\|u_s(t)\|_H^p + \langle f_s(t), u_s(t)\rangle_H + C_1 \\ \frac{1}{2}\frac{d}{dt}\|v_s(t)\|_H^2 \leq -\tilde{C}_2\|v_s(t)\|_H^q + \langle g_s(t), v_s(t)\rangle_H + \tilde{C}_1 \end{cases}$$

where $C_2, \tilde{C}_2, C_1, \tilde{C}_1$ are positive real numbers depending on $|\Omega|, L, p, q$. Now, repeating the procedure with $\theta := p/q$, $\kappa := q/2$, $\nu := q/q'$ we obtain the result. ∎

Remark 8 *The constants r_0 and t_0 in Lemma 7 are independents of the initial values and can be chosen uniformly on $s \in \mathbb{N}$.*

Corollary 9 *There exists a bounded set B_0 in $H \times H$ such that $\mathcal{A}_s \subset B_0$, for all $s \in \mathbb{N}$.*

Lemma 10 *If (u_s, v_s) is a solution of (1), then there exists a positive number $K = K(u_{0s}, v_{0s}, t_0)$ such that $\|(u_s(t), v_s(t))\|_{H\times H} \leq K$, $\forall t \in [0, t_0]$. If the initial values are all in a bounded set of $H\times H$, then K is uniform on s and we have that $\|(u_s(t), v_s(t))\|_{H\times H} \leq K$, for each s and for each $t \in [0, t_0]$. In this case we can consider $t_0 = 0$ in Lemma 7.*

Proof. As (u_s, v_s) is a solution of (1) there exists a pair $(f_s, g_s) \in$ Sel $F(u_s, v_s) \times$ Sel $G(u_s, v_s)$, $f_s, g_s \in L^1(0,T;H)$ such that u_s, v_s satisfy the problem

$$\begin{cases} \frac{\partial u_s}{\partial t} + A^s(u_s) = f_s \quad \text{in} \quad (0,T)\times\Omega, \\ \frac{\partial v_s}{\partial t} + B^s(v_s) = g_s \quad \text{in} \quad (0,T)\times\Omega, \\ u_s(0,x) = u_{0s}(x),\ v_s(0,x) = v_{0s}(x) \quad \text{in} \quad \Omega. \end{cases} \tag{13}$$

Then, multiplying the first equation on (13) by $u_s(t)$ and the second one by $v_s(t)$, summing up and using that $\langle A^s(u_s(t)), u_s(t)\rangle \geq 0$ and $\langle B^s(v_s(t)), v_s(t)\rangle \geq 0$ it follows that

$$\frac{1}{2}\frac{d}{dt}\left(\|u_s(t)\|_H^2 + \|v_s(t)\|_H^2\right) \leq \langle f_s(t)), u_s(t)\rangle + \langle g_s(t)), v_s(t)\rangle.$$

Now, we use that (F, G) is positively sublinear to estimate $\langle f_s(t)), u_s(t)\rangle$ and $\langle g_s(t)), v_s(t)\rangle$ and we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|u_s(t)\|_H^2 + \|v_s(t)\|_H^2\right) \leq C_1\left(\|u_s(t)\|_H^2 + \|v_s(t)\|_H^2\right) + C_2. \tag{14}$$

where C_1 is a positive real number depending on a, b, c and C_2 is a positive real number depending on m_0. Integrating (14) from 0 to $t \leq t_0$ we obtain

$$\|u_s(t)\|_H^2 + \|v_s(t)\|_H^2 \leq \left(\|u_{0s}\|_H^2 + \|v_{0s}\|_H^2\right) + \int_0^t 2C_1\left(\|u_s(\tau)\|_H^2 + \|v_s(\tau)\|_H^2\right) d\tau + 2C_2 t_0.$$

By Gronwall-Bellman inequality

$$\|u_s(t)\|_H^2 + \|v_s(t)\|_H^2 \leq \left(\|u_{0s}\|_H^2 + \|v_{0s}\|_H^2 + 2C_2t_0\right) e^{2C_1t_0}, \quad \forall t \in [0, t_0],$$

and the assertion on the lemma follows. ■

Lemma 11 *If $\varphi_s := (u_s, v_s) \in \mathbb{G}_s$, then there exist positive constants $K > 0$ and $t_1 > t_0$, independents of s, such that*

$$\|\varphi_s(t)\|_{X_s \times Y_s} = \|u_s(t)\|_{X_s} + \|v_s(t)\|_{Y_s} < K$$

for every $t \geq t_1$ and $s \in \mathbb{N}$, where t_0 is the positive constant in Lemma 7.

Proof. Take $t_1 > t_0$. As (u_s, v_s) is a solution of (1) there exists a pair $(f_s, g_s) \in \operatorname{Sel} F(u_s, v_s) \times \operatorname{Sel} G(u_s, v_s)$, $f_s, g_s \in L^1(0, T; H)$ such that u_s, v_s satisfy the problem

$$\begin{cases} \frac{\partial u_s}{\partial t} + A^s(u_s) = f_s & \text{in} \quad (0,T) \times \Omega, \\ \frac{\partial v_s}{\partial t} + B^s(v_s) = g_s & \text{in} \quad (0,T) \times \Omega. \end{cases}$$

Considering $\varphi_{p_s(x)}$ as in (3) we obtain

$$\begin{aligned} \frac{d}{dt}\varphi_{p_s(x)}(u_s(t)) &= \left\langle \partial\varphi_{p_s(x)}(u_s(t)), \frac{\partial u_s}{\partial t}(t) \right\rangle \\ &= \left\langle f_s(t) - \frac{\partial u_s}{\partial t}(t), \frac{\partial u_s}{\partial t}(t) - f_s(t) + f_s(t) \right\rangle \\ &= -\left\| f_s(t) - \frac{\partial u_s}{\partial t}(t) \right\|_H^2 + \left\langle f_s(t) - \frac{\partial u_s}{\partial t}(t), f_s(t) \right\rangle \end{aligned}$$

for a.e. t in $(0, T)$. Therefore,

$$\frac{d}{dt}\varphi_{p_s(x)}(u_s(t)) + \frac{1}{2}\left\| f_s(t) - \frac{\partial u_s}{\partial t}(t) \right\|_H^2 \leq \frac{1}{2}\|f_s(t)\|_H^2.$$

Now by using Lemma 7 and the fact that F and G are bounded, there exists a positive constant C_0 such that $\|f_s(t)\|_H \leq C_0$ for all $t \geq t_0$ and $s \in \mathbb{N}$. In particular,

$$\frac{d}{dt}\varphi_{p_s(x)}(u_s(t)) \leq \frac{1}{2}\|f_s(t)\|_H^2 \leq \frac{1}{2}C_0^2, \ \forall\, t \geq t_0, \ s \in \mathbb{N}. \tag{15}$$

By definition of subdifferential we have the following inequality

$$\varphi_{p_s(x)}(u_s(t)) \leq \langle \partial\varphi_{p_s(x)}(u_s(t)), u_s(t) \rangle.$$

Thus

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_s(t)\|_H^2 + \varphi_{p_s(x)}(u_s(t)) &\leq \left\langle \frac{\partial u_s}{\partial t}(t), u_s(t) \right\rangle + \langle \partial\varphi_{p_s(x)}(u_s(t)), u_s(t) \rangle \\ &= \langle f_s(t), u_s(t) \rangle \leq \|f_s(t)\|_H \|u_s(t)\|_H \leq C_0 r_0 \end{aligned} \tag{16}$$

for all $t \geq t_0$ and $s \in \mathbb{N}$. Let $t \geq t_0$ and $r := t_1 - t_0 > 0$. Integrating (16) from t to $t+r$ we obtain

$$\int_t^{t+r} \varphi_{p_s(x)}(u_s(\tau))d\tau \leq \frac{1}{2}\|u_s(t)\|_H^2 + C_0 r_0 r \leq \frac{1}{2}r_0^2 + C_0 r_0 r =: A \tag{17}$$

for all $s \in \mathbb{N}$. From (15), (17) and the Uniform Gronwall Lemma (see [34]), we obtain

$$\varphi_{p_s(x)}(u_s(t)) \leq \frac{A}{r} + \frac{1}{2}C_0^2 r =: \tilde{r_1},$$

for all $t \geq t_1$ and $s \in \mathbb{N}$. Using (3) we obtain $\|u_s(t)\|_{X_s} \leq K_1$ for all $t \geq t_1$ and $s \in \mathbb{N}$ for a positive constant K_1. In a similar way, we conclude $\|v_s(t)\|_{Y_s} \leq K_2$ for all $t \geq t_1$ and $s \in \mathbb{N}$ for a positive constant K_2 and the assertion on the lemma follows. ■

Corollary 12 *a) There exists a bounded set B_1^s in $X_s \times Y_s$ such that $\mathcal{A}_s \subset B_1^s$.*
b) Let (u_s, v_s) be a solution of problem (1). Given $t_1 > 0$ there exists a positive constant r_2, independent of s, such that

$$\|u_s(t)\|_X + \|v_s(t)\|_Y < r_2,$$

for all $t \geq t_1$ and $s \in \mathbb{N}$.
c) $\mathcal{A} := \overline{\bigcup_{s\in\mathbb{N}} \mathcal{A}_s}$ is a compact subset of $H \times H$.

Lemma 13 *If $(u_s, v_s) \in \mathbb{G}_s$ and there exists $C > 0$ such that $\|u_{0s}\|_{X_s} + \|v_{0s}\|_{Y_s} \leq C$ for all $s \in \mathbb{N}$, then we have that there exists a positive constant $\tilde{K}$ such that*

$$\|(u_s(t), v_s(t)\|_{X_s \times Y_s} \leq \tilde{K} \quad \forall s \in \mathbb{N} \text{ and } \forall t \in [0, t_1].$$

In this case we can consider $t_1 = 0$ in Lemma 11.

Proof. Given $t_1 > 0$, if (u_s, v_s) is a solution of (1) then multiplying the first equation by $\frac{\partial u_s}{\partial t}(t)$ we have that

$$\left\|\frac{\partial u_s}{\partial t}(t)\right\|_H^2 + \left\langle A^s(u_s(t)), \frac{\partial u_s}{\partial t}(t)\right\rangle = \left\langle f_s(t), \frac{\partial u_s}{\partial t}(t)\right\rangle.$$

As $\langle A^s(u_s(t)), \frac{\partial u_s}{\partial t}(t)\rangle = \frac{d}{dt}\varphi_{p_s(x)}(u_s(t))$ we obtain

$$\frac{1}{2}\left\|\frac{\partial u_s}{\partial t}(t)\right\|_H^2 + \frac{d}{dt}\varphi_{p_s(x)}(u_s(t)) \leq \frac{1}{2}\|f_s(t)\|_H^2$$

and then

$$\frac{d}{dt}\varphi_{p_s(x)}(u_s(t)) \leq \frac{1}{2}\|f_s(t)\|_H^2.$$

Using Lemma 10 and the fact that F is bounded we conclude

$$\frac{d}{dt}\varphi_{p_s(x)}(u_s(t)) \leq C_1, \qquad \text{for all } t \in [0, t_1], s \in \mathbb{N},$$

where $C_1 > 0$ is a constant. Therefore, integrating the equation above from 0 to τ, for $\tau \leq t_1$, we obtain

$$\varphi_{p_s(x)}(u_s(\tau)) \leq \varphi_{p_s(x)}(u_{0s}) + C_1 t_1, \qquad \text{for all } \tau \in [0, t_1], s \in \mathbb{N}.$$

In a similar way, we obtain

$$\varphi_{q_s(x)}(v_s(\tau)) \leq \varphi_{q_s(x)}(v_{0s}) + C_2 t_1, \qquad \text{for all } \tau \in [0, t_1], s \in \mathbb{N}.$$

where $C_2 > 0$ is a constant and the result follows. ■

Corollary 14 *Let (u_s, v_s) be a solution of (1) with initial value u_{0s}, v_{0s}. If there is $C > 0$ such that $\|u_{0s}\|_{X_s} + \|v_{0s}\|_{Y_s} \leq C$ for all $s \in \mathbb{N}$, then given $t_1 > 0$ there exists a positive constant $\widetilde{R_1}$ such that*

$$\|u_s(t)\|_X + \|v_s(t)\|_Y \leq \widetilde{R_1},$$

for all $t \in [0, t_1]$ and $s \in \mathbb{N}$.

4 The limit problem and convergence properties

Our objective in this section is to prove that the limit problem of problem (1) as D_s increases to infinity and $p_s(\cdot) \to p > 2$, $q_s(\cdot) \to q > 2$ in $L^\infty(\Omega)$ as $s \to \infty$ is described by an ordinary differential system. Firstly we observe that the gradients of the solutions of problem (1) converge in norm to zero as $s \to \infty$, which allows us to guess the limit problem

$$\begin{cases} \dot{u} + \phi_p(u) \in \widetilde{F}(u, v) \\ \dot{v} + \phi_q(v) \in \widetilde{G}(u, v) \\ u(0) = u_0, v(0) = v_0, \end{cases} \tag{18}$$

where $\phi_p(w) := |w|^{p-2}w$, $\widetilde{F} := F_{|\mathbb{R}\times\mathbb{R}}, \widetilde{G} := G_{|\mathbb{R}\times\mathbb{R}} : \mathbb{R} \times \mathbb{R} \to P(\mathbb{R})$ if we identify $\mathbb{R}$ with the constant functions which are in H, since Ω is a bounded set.

The proof of the next result follows the ideas of [27], but some adjustments are necessary for this variable exponent case. To obtain the limit equation we prove firstly the

Theorem 15 *If (u_s, v_s) is a solution of (1), then for each $t > t_1$, the sequences of real numbers $\{\|\nabla u_s(t)\|_H\}_{s\in\mathbb{N}}$ and $\{\|\nabla v_s(t)\|_H\}_{s\in\mathbb{N}}$ both possess subsequences $\{\|\nabla u_{s_j}(t)\|_H\}$ and $\{\|\nabla v_{s_j}(t)\|_H\}$ that converges to zero as $j \to +\infty$, where t_1 is the positive constant in Lemma 11.*

Proof. Let $T > 0$ and $t \in (t_1, T)$. Let (u_s, v_s) be a solution of the problem (1). Therefore, there are $f_s, g_s \in L^1(0, T; H)$, with $f_s(t) \in F(u_s(t), v_s(t))$, $g_s(t) \in G(u_s(t), v_s(t))$ *a.e.* in $(0, T)$, and such that (u_s, v_s) is a solution of the system:

$$\begin{cases} \dfrac{\partial u_s}{\partial t} + A^s u_s = f_s & \text{in } (0, T) \\ \dfrac{\partial v_s}{\partial t} + B^s = g_s & \text{in } (0, T) \\ u_s(0) = u_{0s}, v_s(0) = v_{0s} \end{cases} \tag{19}$$

Doing the inner product of the first equation of (19) with $u_s(\tau)$, it comes that

$$\frac{1}{2}\frac{d}{dt}\|u_s(\tau)\|_H^2 + D_s\int_\Omega |\nabla u_s(\tau)|^{p_s(x)}dx + \int_\Omega |u_s(\tau)|^{p_s(x)}dx = \langle f_s(\tau), u_s(\tau)\rangle. \tag{20}$$

Analogously, we have that

$$\frac{1}{2}\frac{d}{dt}\|v_s(\tau)\|_H^2 + D_s\int_\Omega |\nabla v_s(\tau)|^{q_s(x)}dx + \int_\Omega |v_s(\tau)|^{q_s(x)}dx = \langle g_s(\tau), v_s(\tau)\rangle. \tag{21}$$

Now by using Lemma 7 and the fact that F and G are bounded, there exists a positive constant C_0 such that $\|f_s(\tau)\|_H \le C_0$ and $\|g_s(\tau)\|_H \le C_0$ for all $\tau \ge t_0$ and $s \in \mathbb{N}$. Then

$$\langle f_s(\tau), u_s(\tau)\rangle \le \|f_s(t)\|_H\|u_s(\tau)\|_H \le C_0 r_0$$

and

$$\langle g_s(\tau), v_s(\tau)\rangle \le \|g_s(t)\|_H\|v_s(\tau)\|_H \le C_0 r_0.$$

Then, adding the equations (20) and (21), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|u_s(\tau)\|_H^2 + \|v_s(\tau)\|_H^2\Big) + D_s\int_\Omega |\nabla u_s(\tau)|^{p_s(x)}dx + D_s\int_\Omega |\nabla v_s(\tau)|^{q_s(x)}dx\\ &+\int_\Omega |u_s(\tau)|^{p_s(x)}dx + \int_\Omega |v_s(\tau)|^{q_s(x)}dx \le C_3,\ a.e. \text{ in } (t_1, T).\end{aligned}$$

As $\int_\Omega |u_s(\tau)|^{p_s(x)}dx + \int_\Omega |v_s(\tau)|^{q_s(x)}dx \ge 0$, we have in particular that

$$\frac{1}{2}\frac{d}{dt}\Big(\|u_s(\tau)\|_H^2 + \|v_s(\tau)\|_H^2\Big) + D_s\int_\Omega |\nabla u_s(\tau)|^{p_s(x)}dx + D_s\int_\Omega |\nabla v_s(\tau)|^{q_s(x)}dx \le C_3, \tag{22}$$

$a.e.$ in (t_1, T). Integrating the inequality (22) from t_1 to T, we obtain

$$\begin{aligned}\frac{1}{2}\Big(\|u_s(T)\|_H^2 + \|v_s(T)\|_H^2\Big) &+ D_s\int_{t_1}^T\int_\Omega |\nabla u_s(\tau)|^{p_s(x)}dxd\tau\\ &+ D_s\int_{t_1}^T\int_\Omega |\nabla v_s(\tau)|^{q_s(x)}dxd\tau\\ &\le \int_{t_1}^T C_3 d\tau + \frac{1}{2}\Big(\|u_s(t_1)\|_H^2 + \|v_s(t_1)\|_H^2\Big)\\ &\le C_3 T + r_0^2 := k(T).\end{aligned}$$

In particular

$$D_s\int_{t_1}^T\int_\Omega |\nabla u_s(\tau)|^{p_s(x)}dxd\tau \le k(T)$$

and

$$D_s\int_{t_1}^T\int_\Omega |\nabla v_s(\tau)|^{q_s(x)}dxd\tau \le k(T),$$

that implies

$$\int_{t_1}^{T}\int_{\Omega}|\nabla u_s(\tau)|^{p_s(x)}dxd\tau \leq \frac{1}{D_s}k(T) \to 0 \text{ as } s \to +\infty.$$

Therefore there exists a subsequence s_j such that

$$\int_{\Omega}|\nabla u_{s_j}(\tau)|^{p_{s_j}(x)}dx \to 0 \text{ as } j \to +\infty,\ \tau - a.e. \text{ in } (t_1, T),$$

and so there exists a subset $J \subset (t_1, T)$ with Lebesgue measure $m((t_1, T)/J) = 0$ such that

$$\int_{\Omega}|\nabla u_{s_j}(\tau)|^{p_{s_j}(x)}dx \to 0 \ \text{ as } j \to +\infty,\ \forall\, \tau \in J.$$

Given $t \in (t_1, T)$ we claim that there is at least one $\nu \in J$ with $\nu < t$, on the contrary we would have $(t_1, t) \cap J = \emptyset$, so $m((t_1, T)/J) > 0$ which is a contradiction. Now pick one $\nu \in J$ with $t_1 < \nu < t$ and let $h = t - \nu$. Let $\varepsilon > 0$ and $j_0 = j_0(\varepsilon) > 0$ be such that if $j > j_0$ then

$$\int_{\Omega}|\nabla u_{s_j}(\nu)|^{p_{s_j}(x)}dx < \frac{\varepsilon}{L}.$$

We have that

$$\frac{d}{d\tau}\varphi_{p_{s_j}(x)}(u_{s_j}(\nu+\tau)) = \left\langle \partial\varphi_{p_{s_j}(x)}(u_{s_j}(\nu+\tau)), \frac{d}{d\tau}u_{s_j}(\nu+\tau) \right\rangle,\ \ a.e. \text{ in } (0, T).$$

Therefore

$$\begin{aligned}
&\int_{\Omega}\frac{D_{s_j}}{p_{s_j}(x)}|\nabla u_{s_j}(\nu+h)|^{p_{s_j}(x)}dx + \int_{\Omega}\frac{1}{p_{s_j}(x)}|u_{s_j}(\nu+h)|^{p_{s_j}(x)}dx\\
&-\int_{\Omega}\frac{D_{s_j}}{p_{s_j}(x)}|\nabla u_{s_j}(\nu)|^{p_{s_j}(x)}dx - \int_{\Omega}\frac{1}{p_{s_j}(x)}|u_{s_j}(\nu)|^{p_{s_j}(x)}dx\\
&= \varphi_{p_{s_j}(x)}(u_{s_j}(\nu+h)) - \varphi_{p_{s_j}(x)}(u_{s_j}(\nu))\\
&= \int_0^h \frac{d}{d\tau}\varphi_{p_{s_j}(x)}(u_{s_j}(\nu+\tau))\, d\tau\\
&= \int_0^h \left\langle \partial\varphi_{p_{s_j}(x)}(u_{s_j}(\nu+\tau)), \frac{d}{d\tau}u_{s_j}(\nu+\tau) \right\rangle\, d\tau\\
&= \int_0^h \left\langle f_{s_j}(\nu+\tau) - \frac{d}{d\tau}u_{s_j}(\nu+\tau), \frac{d}{d\tau}u_{s_j}(\nu+\tau) \right\rangle\, d\tau\\
&= \int_0^h \left\langle f_{s_j}(\nu+\tau), \frac{d}{d\tau}u_{s_j}(\nu+\tau) \right\rangle\, d\tau - \int_0^h \left\langle \frac{d}{d\tau}u_{s_j}(\nu+\tau), \frac{d}{d\tau}u_{s_j}(\nu+\tau) \right\rangle\, d\tau\\
&\leq \frac{1}{2}\int_0^h \|f_{s_j}(\nu+\tau)\|_H^2\, d\tau - \frac{1}{2}\int_0^h \left\|\frac{d}{d\tau}u_{s_j}(\nu+\tau)\right\|_H^2\, d\tau\\
&\leq \frac{1}{2}\int_0^h \|f_{s_j}(\nu+\tau)\|_H^2\, d\tau \leq \frac{1}{2}\int_0^h C_0^2 d\tau = \frac{1}{2}C_0^2 h.
\end{aligned}$$

Thus,

$$\int_\Omega \frac{D_{s_j}}{p_{s_j}(x)} |\nabla u_{s_j}(\nu+h)|^{p_{s_j}(x)} dx \leq \int_\Omega \frac{D_{s_j}}{p_{s_j}(x)} |\nabla u_{s_j}(\nu)|^{p_{s_j}(x)} dx + \int_\Omega \frac{1}{p_{s_j}(x)} |u_{s_j}(\nu)|^{p_{s_j}(x)} dx + \frac{1}{2} C_0^2 h.$$

Then,

$$\int_\Omega |\nabla u_{s_j}(\nu+h)|^{p_{s_j}(x)} dx \leq \frac{L}{2} \int_\Omega |\nabla u_{s_j}(\nu)|^{p_{s_j}(x)} dx + \frac{L}{2D_{s_j}} \int_\Omega |u_{s_j}(\nu)|^{p_{s_j}(x)} dx + \frac{LC_0^2 h}{2D_{s_j}}.$$

So, using (2) and Lemma 11

$$\int_\Omega |\nabla u_{s_j}(\nu+h)|^{p_{s_j}(x)} dx \leq \frac{L}{2} \int_\Omega |\nabla u_{s_j}(\nu)|^{p_{s_j}(x)} dx + \frac{L}{2D_{s_j}} K^L + \frac{LC_0^2 |T-t_1|}{2D_{s_j}},$$

where K is the positive constant which appear in the Lemma 11.

Thus, choose $j_1 = j_1(\varepsilon)$ sufficiently large such that

$$\frac{L}{2D_{s_j}} K^L + \frac{LC_0^2 |T-t_1|}{2D_{s_j}} < \varepsilon/2,$$

whenever $j > j_1$ and, moreover, we consider $j_2 = j_2(\varepsilon) = \max\{j_0, j_1\}$. For $j > j_2$ we have

$$\begin{aligned} \int_\Omega |\nabla u_{s_j}(t)|^{p_{s_j}(x)} dx &= \int_\Omega |\nabla u_{s_j}(\nu + t - \nu)|^{p_{s_j}(x)} dx \\ &\leq \frac{L}{2} \int_\Omega |\nabla u_{s_j}(\nu)|^{p_{s_j}(x)} dx + \frac{L}{2D_{s_j}} K^L + \frac{LC_0^2 |T-t_1|}{2D_{s_j}} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, for $j > j_2$

$$\min\{\|\nabla u_{s_j}(t)\|_{p_{s_j}(x)}^{p_{s_j}^-}, \|\nabla u_{s_j}(t)\|_{p_{s_j}(x)}^{p_{s_j}^+}\} \leq \int_\Omega |\nabla u_{s_j}(t)|^{p_{s_j}(x)} dx < \varepsilon.$$

As $p_{s_j}(x) > 2$, $\|\nabla u_{s_j}(t)\|_H \leq 2(|\Omega|+1)\|\nabla u_{s_j}(t)\|_{p_{s_j}(x)}$ we obtain

$$\| \nabla u_{s_j}(t) \|_H \to 0 \ \text{ as } j \to +\infty.$$

Analogously we conclude that $\| \nabla v_{s_j}(t) \|_H \to 0$ as $j \to +\infty$. ■

Proposition 16 *If (u_s, v_s) is a solution of problem (1) in $(0, t_1)$, then for each $t \in [0, t_1]$, the sequences of real numbers $\{\|\nabla u_s(t)\|_p\}_{s\in\mathbb{N}}$ and $\{\|\nabla v_s(t)\|_q\}_{s\in\mathbb{N}}$ remain limited as $s \to +\infty$ whenever the initial values will be such that $\|u_{0s}\|_{X_s} + \|v_{0s}\|_{Y_s} \leq C$ for all $s \in \mathbb{N}$. If the initial data are equal to a same constant, i. e., if $(u_s(0), v_s(0)) = (u_0, v_0) \in \mathbb{R}\times\mathbb{R}$, $\forall\, s \in \mathbb{N}$, then for each $t \in [0, t_1]$, the sequences of real numbers $\{\|\nabla u_s(t)\|_p\}_{s\in\mathbb{N}}$ and $\{\|\nabla v_s(t)\|_q\}_{s\in\mathbb{N}}$ converges to zero as $s \to +\infty$, respectively.*

Proof. In fact, let (u_s, v_s) be a solution of problem (1) in $(0, t_1)$. Therefore, there are $f_s, g_s \in L^1(0, t_1; H)$, with $f_s(t) \in F(u_s(t), v_s(t))$, $g_s(t) \in G(u_s(t), v_s(t))$ *a.e.* in $(0, t_1)$, and such that (u_s, v_s) is a solution of the system:

$$\begin{cases} \dfrac{\partial u_s}{\partial t} + A^s u_s = f_s & \text{in } (0, t_1) \\ \dfrac{\partial v_s}{\partial t} + B^s v_s = g_s & \text{in } (0, t_1) \\ u_s(0) = u_{0s}, v_s(0) = v_{0s} \end{cases}$$

Doing the inner product of the first equation with $\frac{\partial u_s(t)}{\partial t}$, we obtain

$$\begin{aligned} \left\| \frac{\partial u_s(t)}{\partial t} \right\|_H^2 + \frac{d}{dt}\varphi_{p_s(x)}(u_s(t)) &= \left\langle f_s(t), \frac{\partial u_s(t)}{\partial t} \right\rangle \\ &\leq \|f_s(t)\|_H \left\| \frac{\partial u_s(t)}{\partial t} \right\|_H \\ &\leq \frac{1}{2}\|f_s(t)\|_H^2 + \frac{1}{2}\left\| \frac{\partial u_s(t)}{\partial t} \right\|_H^2 . \end{aligned}$$

In particular,

$$\frac{d}{dt}\varphi_{p_s(x)}(u_s(t)) \leq \frac{1}{2}\|f_s(t)\|_H^2. \tag{23}$$

Since $\|u_{0s}\|_{X_s} + \|v_{0s}\|_{Y_s} \leq C$ for all $s \in \mathbb{N}$, using (5) we have that the initial values are in a bounded set of $H \times H$. Using Lemma 10 and the fact that F and G take bounded sets of $H \times H$ in bounded sets of H, it follows that there exists a positive constant C such that $\|f_s(t)\|_H^2 \leq C$, $\forall\, t \in [0, t_1]$ and $s \in \mathbb{N}$. Computing the integral of 0 to τ, $\tau \in [0, t_1]$ in (23), we obtain

$$\varphi_{p_s(x)}(u_s(\tau)) \leq \varphi_{p_s(x)}(u_{0s}) + \frac{1}{2}Ct_1,$$

$\forall\, \tau \in [0, t_1]$ and $s \in \mathbb{N}$. Therefore,

$$\begin{aligned} &D_s \int_\Omega \frac{1}{p_s(x)}|\nabla u_s(\tau)|^{p_s(x)}dx + \int_\Omega \frac{1}{p_s(x)}|u_s(\tau)|^{p_s(x)}dx \\ &\leq D_s \int_\Omega \frac{1}{p_s(x)}|\nabla u_{0s}|^{p_s(x)}dx + \int_\Omega \frac{1}{p_s(x)}|u_{0s}|^{p_s(x)}dx + \frac{1}{2}Ct_1, \quad \forall\, \tau \in [0, t_1] \text{ and } \forall\, s \in \mathbb{N}. \end{aligned}$$

Then

$$\int_\Omega |\nabla u_s(\tau)|^{p_s(x)}dx \leq \frac{L}{2}\int_\Omega |\nabla u_{0s}|^{p_s(x)}dx + \frac{L}{2D_s}\left(\int_\Omega |u_{0s}|^{p_s(x)}dx + Ct_1\right),$$

for all $\tau \in [0, t_1]$ and $s \in \mathbb{N}$. Analogously we prove that

$$\int_\Omega |\nabla v_s(\tau)|^{q_s(x)}dx \leq \frac{L}{2}\int_\Omega |\nabla v_{0s}|^{q_s(x)}dx + \frac{L}{2D_s}\left(\int_\Omega |v_{0s}|^{q_s(x)}dx + Ct_1\right).$$

Since $p_s(x) \geq p$, $q_s(x) \geq q$ and $\|u_{0s}\|_{X_s} + \|v_{0s}\|_{Y_s} \leq C$ for all $s \in \mathbb{N}$ the result follows using (2). ■

The Lemma 15 confirms that the equation (18) is a good candidate for the limit problem.

Lemma 17 *[26] The problem* (18) *has a global solution.*

Theorem 18 *[26] The problem* (18) *defines a generalized semiflow* $\mathbb{G}^\infty$ *which has a global B-attractor* $\mathcal{A}^\infty$.

The next result guarantees that (18) is in fact the limit problem for (1), as $s \to \infty$.

Theorem 19 *Let* (u_s, v_s) *be a solution of the problem* (1). *Suppose that the initial values* $(u_s(0), v_s(0)) = (u_{0s}, v_{0s}) \to (u_0, v_0) \in \mathbb{R} \times \mathbb{R}$ *in the topology of* $H \times H$ *as* $s \to +\infty$. *Then there exist a solution* (u, v) *of the problem* (18) *satisfying* $(u(0), v(0)) = (u_0, v_0)$ *and a subsequence* $\{(u_{s_j}, v_{s_j})\}_j$ *of* $\{(u_s, v_s)\}_s$ *such that, for each* $T > 0$, $u_{s_j} \to u$, $v_{s_j} \to v$ *in* $C([0,T]; H)$ *as* $j \to +\infty$.

Proof. Let be $T > 0$ fixed arbitrarily large. Let (u_s, v_s) be a solution of the problem (1) with $(u_s(0), v_s(0)) = (u_{0s}, v_{0s}) \to (u_0, v_0) \in \mathbb{R} \times \mathbb{R}$ in $H \times H$ as $s \to +\infty$. Therefore, there are $f_s, g_s \in L^1(0,T; H)$, with

$$f_s(t) \in F(u_s(t), v_s(t)),\ g_s(t) \in G(u_s(t), v_s(t)) \ \ a.e. \text{ in } (0,T),$$

and such that (u_s, v_s) is a solution of the system (P_s^1) below:

$$(P_s^1) \begin{cases} \dfrac{\partial u_s}{\partial t} + A^s u_s = f_s & \text{in } (0,T) \\ \dfrac{\partial v_s}{\partial t} + B^s v_s = g_s & \text{in } (0,T) \\ u_s(0) = u_{0s}, v_s(0) = v_{0s} \end{cases}$$

We denote $u_s(\cdot) := I(u_{0s}) f_s(\cdot)$ and $v_s(\cdot) := I(v_{0s}) g_s(\cdot)$ and also denote by $z_s(\cdot) := I(u_0) f_s(\cdot)$ and $w_s(\cdot) := I(v_0) g_s(\cdot)$ being the solutions of the problems

$$(P_{f_s,u_0}) \begin{cases} \dfrac{\partial z_s}{\partial t} + A^s z_s = f_s \\ z_s(0) = u_0 \end{cases}$$

and

$$(P_{g_s,v_0}) \begin{cases} \dfrac{\partial w_s}{\partial t} + B^s w_s = g_s \\ w_s(0) = v_0, \end{cases}$$

respectively.

Doing the inner product of the first equation in (P_s^1) with u_s and computing the integral of 0 to t, $t \leq T$, we obtain

$$\frac{1}{2} \parallel u_s(t) \parallel_H^2 \leq \frac{1}{2} \parallel u_{0s} \parallel_H^2 + \int_0^t \langle f_s(\tau), u_s(\tau) \rangle d\tau.$$

As $\{u_{0s}\}$ is a convergent sequence we have that there exists a positive constant R such that $\| u_{0s} \|_H^2 \leq R^2$. Thus,

$$\frac{1}{2} \| u_s(t) \|_H^2 \leq \frac{1}{2} R^2 + \int_0^t \langle f_s(\tau), u_s(\tau) \rangle d\tau.$$

Using the hypothesis that the couple (F, G) is positively sublinear and the Gronwall's inequality we obtain that there are positive constants α, β, γ and C such that

$$\| u_s(t) \|_H \leq C + \gamma T + \textstyle\int_0^t [\alpha \| u_s(\tau) \|_H + \beta \| v_s(\tau) \|_H] d\tau.$$

So, there is a positive constant M independent of $t \in [0, T]$ such that

$$\| u_s(t) \|_H \leq M + \int_0^t [\alpha \| u_s(\tau) \|_H + \beta \| v_s(\tau) \|_H] d\tau.$$

Analogously, there exists a positive constant $\widetilde{M}$ independent of $t \in [0, T]$ such that

$$\| v_s(t) \|_H \leq \widetilde{M} + \int_0^t [\beta \| u_s(\tau) \|_H + \alpha \| v_s(\tau) \|_H] d\tau.$$

Adding this two inequalities and denoting by $N \doteq M + \widetilde{M}$ and $\rho \doteq \alpha + \beta$ we have

$$\| u_s(t) \|_H + \| v_s(t) \|_H \leq N + \rho \int_0^t [\| u_s(\tau) \|_H + \| v_s(\tau) \|_H] d\tau$$

and so it follows by the Gronwall-Bellman's inequality that

$$\| u_s(t) \|_H + \| v_s(t) \|_H \leq N e^{\rho T},$$

for all $t \in [0, T]$ and for all $s \in \mathbb{N}$.

As F and G takes bounded sets of $H \times H$ in bounded sets of H, it follows by the inequality above that there exists $D > 0$ such that

$$\| f_s(t) \|_H \leq D \quad \text{and} \quad \| g_s(t) \|_H \leq D, \quad \text{for all } t \in [0, T] \text{ and for all } s \in \mathbb{N}. \tag{24}$$

Consider $K := \{f_s; s \in \mathbb{N}\}$, $\widetilde{K} := \{g_s; s \in \mathbb{N}\}$, $M(K) := \{z_s; s \in \mathbb{N}\}$ and $M(\widetilde{K}) := \{w_s; s \in \mathbb{N}\}$. It follows by (24) that K and $\widetilde{K}$ are uniformly integrable subsets in $L^1(0, T; H)$.

Given $t \in (0, T]$ and $h > 0$ such that $t - h \in (0, T]$, we consider the operator $T_h : M(K)(t) \to H$ defined by $T_h z_s(t) = S^s(h) z_s(t - h)$. We can adapt the prove of the Statement 1, p.15 in [25] to the variable exponent case to prove that the operator $T_h : M(K)(t) \to H$ is compact.

Then, by Theorem 3.2 in [25], the set $M(K)$ is relatively compact in $C([0, T]; H)$ and so there are $z \in C([0, T]; H)$ and a subsequence $\{z_{s_j}\}$ of $\{z_s\}$ such that $z_{s_j} \to z$ in $C([0, T]; H)$.

As each z_{s_j} is a solution of (P_{f_s, u_0}) in $(0, T)$, then by Proposition 3.6 in [7], z_{s_j} verifies

$$\frac{1}{2} \| z_{s_j}(t) - \theta \|^2 \leq \frac{1}{2} \| z_{s_j}(\ell) - \theta \|^2 + \int_\ell^t \langle f_{s_j}(\tau) - y_j, z_{s_j}(\tau) - \theta \rangle d\tau \tag{25}$$

for all $\theta \in \mathcal{D}(A^{s_j}) \subset W^{1,p_{s_j}(\cdot)}(\Omega) \subset H$, $y_j = A^{s_j}(\theta)$ and for all $0 \le \ell \le t \le T$.

Analogously, we can show that there exists $w \in C([0,T];H)$ and there exists a subsequence $\{w_{s_j}\}$ of $\{w_s\}$ such that $w_{s_j} \to w$ in $C([0,T];H)$, verifying

$$\frac{1}{2} \parallel w_{s_j}(t) - \theta \parallel^2 \le \frac{1}{2} \parallel w_{s_j}(\ell) - \theta \parallel^2 + \int_\ell^t \langle g_{s_j}(\tau) - y_j, w_{s_j}(\tau) - \theta \rangle d\tau \tag{26}$$

for all $\theta \in \mathcal{D}(B^{s_j}) \subset W^{1,q_{s_j}(\cdot)}(\Omega) \subset H$, $y_j = B^{s_j}(\theta)$ and for all $0 \le \ell \le t \le T$.

As $\parallel f_{s_j}(\tau) \parallel_H \le D$ and $\parallel g_{s_j}(\tau) \parallel_H \le D$, for all $0 \le \tau \le T$ and for all $j \in \mathbb{N}$, we conclude that there exists a positive constant $\widetilde{D}$ such that

$$\parallel f_{s_j} \parallel_{L^2(0,T;H)} \le \widetilde{D} \qquad \text{and} \qquad \parallel g_{s_j} \parallel_{L^2(0,T;H)} \le \widetilde{D}, \quad \text{for all } j \in \mathbb{N}.$$

As $L^2(0,T;H)$ is a reflexive Banach space, there are $f, g \in L^2(0,T;H)$ and subsequences, which we do not relabel, $\{f_{s_j}\}$ and $\{g_{s_j}\}$ such that $f_{s_j} \rightharpoonup f$ and $g_{s_j} \rightharpoonup g$ in $L^2(0,T;H)$. Consequently $f_{s_j} \rightharpoonup f$ and $g_{s_j} \rightharpoonup g$ in $L^1(0,T;H)$.

Statement 1: $u_{s_j} \to z$ and $v_{s_j} \to w$ in $C([0,T];H)$ and moreover $f(t) \in F(z(t), w(t))$ and $g(t) \in G(z(t), w(t))$ a.e.in $[0,T]$.

In fact, let be $t \in [0,T]$. We have

$$\parallel u_{s_j}(t) - z(t) \parallel_H \le \parallel u_{s_j}(t) - z_{s_j}(t) \parallel_H + \parallel z_{s_j}(t) - z(t) \parallel_H .$$

Then,

$$\begin{aligned}\sup_{t\in[0,T]} \parallel u_{s_j}(t) - z(t) \parallel_H &\le \sup_{t\in[0,T]} \parallel I(u_{0s_j}) f_{s_j}(t) - I(u_0) f_{s_j}(t) \parallel_H \\ &+ \sup_{t\in[0,T]} \parallel z_{s_j}(t) - z(t) \parallel_H \\ &\le \parallel u_{0s_j} - u_0 \parallel_H + \sup_{t\in[0,T]} \parallel z_{s_j}(t) - z(t) \parallel_H \to 0\end{aligned}$$

as $j \to +\infty$. So $u_{s_j} \to z$ in $C([0,T];H)$ as $j \to +\infty$. Analogously we show that $v_{s_j} \to w$ in $C([0,T];H)$ as $j \to +\infty$. Then, by Theorem 3.3 in [14], $f(t) \in F(z(t), w(t))$ and $g(t) \in G(z(t), w(t))$ a.e. in $[0,T]$.

Observe that

$$f_{s_j} \rightharpoonup f \quad \text{in} \quad L^2(0,T;H) \Longrightarrow f_{s_j} \rightharpoonup f \quad \text{in} \quad L^2(\ell,t;H), \forall\, 0 \le \ell \le t \le T;$$

and

$$z_{s_j} \to z \text{ in } C([0,T];H) \Longrightarrow z_{s_j} \to z \text{ in } C([\ell,t];H)$$

and consequently

$$z_{s_j} \to z \quad \text{in} \quad L^2(\ell,t;H),\ \forall\, 0 \le \ell \le t \le T;$$

then

$$\langle f_{s_j} - \eta, z_{s_j} - \theta \rangle_{L^2(\ell,t;H)} \to \langle f - \eta, z - \theta \rangle_{L^2(\ell,t;H)}$$

for all $\theta, \eta \in H$.

Now consider $\overline{\theta} \in \mathbb{R} \subset H$ and let $\overline{h} := \phi_p(\overline{\theta}) \in \mathbb{R} \subset H$. We consider $y_j := A^{s_j}(\overline{\theta}) = -\mathrm{div}(D_{s_j}|\nabla\overline{\theta}|^{p_{s_j}(x)-2}\nabla\overline{\theta}) + |\overline{\theta}|^{p_{s_j}(x)-2}\overline{\theta}$. Note that $D(A^{s_j}) \supset \mathbb{R}$, $\forall\, j \in N$ and since $\overline{\theta}$ is a constant function $\nabla\overline{\theta} = 0$, so $y_j = |\overline{\theta}|^{p_{s_j}(x)-2}\overline{\theta}$. By (25) we know that

$$\begin{aligned}\frac{1}{2} \parallel z_{s_j}(t) - \overline{\theta} \parallel^2 \leq & \frac{1}{2} \parallel z_{s_j}(\ell) - \overline{\theta} \parallel^2 + \int_\ell^t \langle f_{s_j}(\tau) - y_j, z_{s_j}(\tau) - \overline{\theta}\rangle d\tau \\ = & \frac{1}{2} \parallel z_{s_j}(\ell) - \overline{\theta} \parallel^2 + \int_\ell^t \langle f_{s_j}(\tau) - \overline{h}, z_{s_j}(\tau) - \overline{\theta}\rangle d\tau \\ & + \int_\ell^t \langle \overline{h} - y_j, z_{s_j}(\tau) - \overline{\theta}\rangle d\tau.\end{aligned} \tag{27}$$

for all $0 \leq l \leq t \leq T$ and for all $j \in \mathbb{N}$. We claim that $\int_\ell^t \langle \overline{h} - y_j, z_{s_j}(\tau) - \overline{\theta}\rangle d\tau \to 0$ as $j \to \infty$. In fact, for $\overline{\theta} = 0$ this is immediate and if $\overline{\theta} \neq 0$ then for each $\tau > 0$

$$|\langle \overline{h} - y_j, z_{s_j}(\tau) - \overline{\theta}\rangle| \leq \int_\Omega \Big(\Big|\ |\overline{\theta}|^{p-1} - |\overline{\theta}|^{p_{s_j}(x)-1}\Big|\Big)|z_{s_j}(\tau)|dx + \int_\Omega \Big|\ |\overline{\theta}|^p - |\overline{\theta}|^{p_{s_j}(x)}\Big|dx.$$

Since $p_{s_j}(x) \to p$ in $L^\infty(\Omega)$ as $j \to \infty$ it follows by Dominated Convergence Theorem that

$$\int_\Omega \Big|\ |\overline{\theta}|^p - |\overline{\theta}|^{p_{s_j}(x)}\Big|dx \to 0 \ \text{ as } \ j \to \infty.$$

On the other hand, using the Mean Value Theorem we obtain

$$\int_\Omega \Big(\Big|\ |\overline{\theta}|^{p-1} - |\overline{\theta}|^{p_{s_j}(x)-1}\Big|\Big)|z_{s_j}(\tau)|dx \leq \int_\Omega |\overline{\theta}|^{\tau(s_j,x)} \ln(|\overline{\theta}|)(p_{s_j}(x) - p)|z_{s_j}(\tau)|dx$$

where $p < \tau(s_j, x) < p_{s_j}(x)$. Thus, considering $q_{s_j}(\cdot)$ such that $\frac{1}{p_{s_j}(x)} + \frac{1}{q_{s_j}(x)} = 1$, $\forall\, x \in \Omega$, we have

$$\begin{aligned}&\int_\Omega \Big(\Big|\ |\overline{\theta}|^{p-1} - |\overline{\theta}|^{p_{s_j}(x)-1}\Big|\Big)|z_{s_j}(\tau)|dx \leq \|p_{s_j} - p\|_\infty \int_\Omega |\overline{\theta}|^{\tau(s_j,x)+1}|z_{s_j}(\tau)|dx \\ &\leq \|p_{s_j} - p\|_\infty \left[\int_\Omega \frac{1}{q_{s_j}(x)}|\overline{\theta}|^{(\tau(s_j,x)+1)q_{s_j}(x)}dx + \int_\Omega \frac{1}{p_{s_j}(x)}|z_{s_j}(\tau)|^{p_{s_j}(x)}dx\right] \\ &\leq \|p_{s_j} - p\|_\infty \left[\int_\Omega |\overline{\theta}|^{(\tau(s_j,x)+1)q_{s_j}(x)}dx + \frac{1}{2}\int_\Omega |z_{s_j}(\tau)|^{p_{s_j}(x)}dx\right]\end{aligned}$$

By Lemma 13 there exists a constant $C > 0$ such that $\int_\Omega |z_{s_j}(\tau)|^{p_{s_j}(x)}dx \leq C$ for every $\tau \in (\ell, t)$ and $j \in \mathbb{N}$.

On the other hand, as $p+1 < \tau(s_j, x)+1 < p_{s_j}(x)+1 < L+1$ and $1 < q_{s_j}(x) < 2$ we obtain $\int_\Omega |\overline{\theta}|^{(\tau(s_j,x)+1)q_{s_j}(x)}dx \leq \tilde{C}$, $\forall\, j \in \mathbb{N}$. Thus, considering $\bar{C} := C + \tilde{C} > 0$, we have

$$\int_\Omega \Big(\Big|\ |\overline{\theta}|^{p-1} - |\overline{\theta}|^{p_{s_j}(x)-1}\Big|\Big)|z_{s_j}(\tau)|dx \leq \|p_{s_j} - p\|_\infty \bar{C} \to 0 \ \text{ as } \quad j \to \infty,$$

and we conclude that

$$\int_{\ell}^{t} \langle \overline{h} - y_j, z_{s_j}(\tau) - \overline{\theta} \rangle d\tau \to 0 \text{ as } j \to +\infty.$$

Thus, taking the limit as $j \to +\infty$, in (27) we obtain

$$\frac{1}{2} \parallel z(t) - \overline{\theta} \parallel^2 \leq \frac{1}{2} \parallel z(\ell) - \overline{\theta} \parallel^2 + \int_{\ell}^{t} \langle f(\tau) - \overline{h}, z(\tau) - \overline{\theta} \rangle d\tau \tag{28}$$

for all $\overline{\theta} \in \mathbb{R}$, $\overline{h} := \phi_p(\overline{\theta})$ and for all $0 \leq \ell \leq t \leq T$.

In the same way we can show that

$$\frac{1}{2} \parallel w(t) - \overline{\theta} \parallel^2 \leq \frac{1}{2} \parallel w(\ell) - \overline{\theta} \parallel^2 + \int_{\ell}^{t} \langle g(\tau) - \overline{h}, w(\tau) - \overline{\theta} \rangle d\tau$$

for all $\overline{\theta} \in \mathbb{R}$, $\overline{h} := \phi_q(\overline{\theta})$ and for all $0 \leq \ell \leq t \leq T$.

Statement 2: $z(t)$ and $w(t)$ are independents on x, for each $t > 0$.

In fact, let be $t > 0$. We already know that $z_{s_j}(t) \to z(t)$ in H. Since $z_{s_j}(0) = u_0$, $\forall\ j \in \mathbb{N}$, then by the Proposition 16 and Theorem 15 we have that $\|\nabla z_{s_j}(t)\|_H \to 0$ as $j \to +\infty$. We also have that $z_{s_j}(t) \in \mathcal{D}(A^{s_j}) \subset W^{1,p_{s_j}}(\Omega) \subset W^{1,2}(\Omega)$. Then, by the Poincaré-Wirtinger's inequality (see [6])

$$\|z_{s_j}(t) - \overline{z_{s_j}(t)}\|_H \leq C \|\nabla z_{s_j}(t)\|_H \to 0 \text{ as } j \to +\infty.$$

Then

$$\begin{aligned} \|z(t) - \overline{z(t)}\|_H &= \|z(t) - z_{s_j}(t) + z_{s_j}(t) - \overline{z_{s_j}(t)} + \overline{z_{s_j}(t)} - \overline{z(t)}\|_H \\ &\leq \|z(t) - z_{s_j}(t)\|_H + \|z_{s_j}(t) - \overline{z_{s_j}(t)}\|_H \\ &\quad + \|\overline{z_{s_j}(t)} - \overline{z(t)}\|_H \to 0 \text{ as } j \to +\infty. \end{aligned}$$

So $z(t) = \overline{z(t)} := \frac{1}{|\Omega|} \int_{\Omega} z(t)(x)\, dx$.

Analogously, we show that $w(t) = \overline{w(t)}$. As we would like to prove.

We already show in the Statement 1 that $f(t) \in F(z(t), w(t))$ and $g(t) \in G(z(t), w(t))$ a.e. in $(0, T)$. Therefore $f(t)$ and $g(t)$ are independents on x, t-a.e. in $(0, T)$.

Thus, from (28)

$$\frac{1}{2}|z(t) - \overline{\theta}|^2 |\Omega| \leq \frac{1}{2}|z(\ell) - \overline{\theta}|^2 |\Omega| + \int_{\ell}^{t} \int_{\Omega} (f(\tau) - \overline{h})(z(\tau) - \overline{\theta})\, dx\, d\tau.$$

So

$$\frac{1}{2}|z(t) - \overline{\theta}|^2 \leq \frac{1}{2}|z(\ell) - \overline{\theta}|^2 + \int_{\ell}^{t} (f(\tau) - \overline{h})(z(\tau) - \overline{\theta})\, d\tau$$

for all $\overline{\theta} \in \mathbb{R}$, $\overline{h} := \phi_p(\overline{\theta})$ and for all $0 \leq \ell < t \leq T$.
If $t = \ell = 0$, we have $z(0) = \lim_{j\to+\infty} z_{s_j}(0) = \lim_{j\to+\infty} u_0 = u_0$. Therefore $\frac{1}{2}|z(0) - \overline{\theta}|^2 = \frac{1}{2}|u_0 - \overline{\theta}|^2$.

So

$$\frac{1}{2}|z(t) - \overline{\theta}|^2 \leq \frac{1}{2}|z(\ell) - \overline{\theta}|^2 + \int_\ell^t (f(\tau) - \overline{h})(z(\tau) - \overline{\theta})\, d\tau$$

for all $\overline{\theta} \in \mathbb{R}$, $\overline{h} := \phi_p(\overline{\theta})$ and for all $0 \leq \ell \leq t \leq T$.

In the same way,

$$\frac{1}{2}|w(t) - \overline{\theta}|^2 \leq \frac{1}{2}|w(\ell) - \overline{\theta}|^2 + \int_\ell^t (g(\tau) - \overline{h})(w(\tau) - \overline{\theta})\, d\tau$$

for all $\overline{\theta} \in \mathbb{R}$, $\overline{h} := \phi_q(\overline{\theta})$ and for all $0 \leq \ell \leq t \leq T$.

So by the Proposition 3.6 in [7], we conclude that (z, w) is a weak solution of problem (18) with $(z(0), w(0)) = (u_0, v_0)$, but as $f, g \in L^2(0, T; H)$ we have in fact that (z, w) is a strong solution of problem (18). ■

Remark 20 *The Theorem 19 continues valid without the hypothesis* $(u_0, v_0) \in \mathbb{R} \times \mathbb{R}$, *whenever* $(u_{0s}, v_{0s}) \in \mathcal{A}_s$, $\forall\, s \in \mathbb{N}$, *because in this case we prove, analogously as was done in Lemma 4.1 in [27], that* u_0 *and* v_0 *are independents on* x.

The proof of the next result is completely analogous as in [26], but for convenience of the reader we put the proof.

Theorem 21 *The family of attractors* $\{\mathcal{A}_s\}_{s\in\mathbb{N}}$ *associated with the problem* (1) *is upper semicontinuous on infinity, on the topology of* $H \times H$.

Proof. Let $\{(u_{0s}, v_{0s})\}_{s\in\mathbb{N}}$ be an arbitrary sequence with

$$(u_{0s}, v_{0s}) \in \mathcal{A}_s,\ \forall\, s \in \mathbb{N} \text{ and } D_s \to +\infty \text{ as } s \to +\infty.$$

By Corollary 12c, there exists a subsequence, that we still denote the same, such that $(u_{0s}, v_{0s}) \to (u_0, v_0)$ in $H \times H$ as $s \to +\infty$. By [10], it is enough to prove that $(u_0, v_0) \in \mathcal{A}^\infty$.

Using the invariance of the attractors, Theorem 15 and Poincaré-Wirtinger's inequality, we can prove analogously to the Lemma 4.1 in [27], that $(u_0, v_0) \in \mathbb{R} \times \mathbb{R}$.

For each $s \in \mathbb{N}$, consider $t_s > s$, $t_1 < t_2 < \ldots < t_s < \ldots$. By invariance of the attractors, there are $(x_s, y_s) \in \mathcal{A}_s$ and solutions $\varphi^s = (\varphi_1^s, \varphi_2^s) \in \mathbb{G}_s$ with $\varphi^s(0) = (x_s, y_s)$ such that $\varphi^s(t_s) = (u_{0s}, v_{0s}) \to (u_0, v_0)$ in $H \times H$ as $s \to +\infty$. Note that

$$\varphi^s(t_s) \in T_s(t_s)(x_s, y_s) \in \mathcal{A}_s,\ \forall\, s \in \mathbb{N}.$$

By the definition of generalized semiflow, for each $s \in \mathbb{N}$, the translates $\left(\varphi^s\right)^{t_s}$ also are solutions, and we have $\left(\varphi^s\right)^{t_s}(0) \to (u_0, v_0)$ in $H \times H$ as $s \to +\infty$.

Using Theorem 19, we obtain that there exist a solution g_0 of the limit problem (18) with $g_0(0) = (u_0, v_0)$ and a subsequence of $\left\{ \left(\varphi^s\right)^{t_s} \right\}_s$, that we still denote the same, such that

$$\left(\varphi^s\right)^{t_s}(t) \to g_0(t) \text{ in } H \times H \text{ as } s \to +\infty, \ \forall\, t \geq 0.$$

Now we consider the sequence $\{\varphi^s(t_s - 1)\}$. Note that

$$\varphi^s(t_s - 1) \in T_s(t_s - 1)(x_s, y_s) \subset \bigcup_s \mathcal{A}_s$$

that is a precompact subset in $H \times H$, then, passing to a subsequence if necessary,

$$\left(\varphi^s\right)^{(t_s-1)}(0) = \varphi^s(t_s - 1) \to z_1 \text{ in } H \times H \text{ as } s \to +\infty.$$

As for each $s \in \mathbb{N}$, φ^s is a solution beginning on the attractor $\mathcal{A}_s$, we obtain by the invariance of the attractors that the sequence of initial values

$$\varphi^s(t_s - 1) \in \mathcal{A}_s, \ \forall\, s \in \mathbb{N}.$$

So using the Remark 20 and Theorem 19, we obtain that there exist a solution g_1 of the limit problem (18) with $g_1(0) = z_1$ and a subsequence of $\left\{ \left(\varphi^s\right)^{(t_s-1)} \right\}_s$, that we still denote in the same way, such that

$$\left(\varphi^s\right)^{(t_s-1)}(t) \to g_1(t) \text{ in } H \times H \text{ as } s \to +\infty, \ \forall\, t \geq 0.$$

Now note that $g_1^1 = g_0$, since for each $t \geq 0$, we have

$$g_1^1(t) = g_1(t+1) = \lim_{s\to+\infty} (\varphi^s)^{(t_s-1)}(t+1) = \lim_{s\to+\infty} (\varphi^s)^{t_s}(t) = g_0(t).$$

Proceeding so inductively , we find for each $r = 0, 1, 2, \cdots$, a solution $g_r \in \mathbb{G}^\infty$ with $g_r(0) = z_r$ such that $g_{r+1}^1 = g_r$. Given $t \in \mathbb{R}$, we define $g(t)$ as the common value of $g_r(t+r)$ for $r > -t$. Then we have that g is a complete orbit for $\mathbb{G}^\infty$ with $g(0) = g_0(0) = (u_0, v_0)$.

Note that for each $t \geq 0$, $r = 0, 1, 2, \cdots$, we have that each

$$g_r(t) = \lim_{s\to+\infty} \left(\varphi^s\right)^{(t_s-r)}(t) \text{ and } \left(\varphi^s\right)^{(t_s-r)}(t) \in \mathcal{A}_s, \ \forall\, s \in \mathbb{N}.$$

Working with the coordinated functions and using the invariance of the attractors, the Lemma 15 and the Poincaré-Wirtinger's inequality, we can prove, analogously to the Lemma 4.1 in [27], that each $g_r(t)$ independents on x. Consequently, we obtain that $g(t)$ is a constant function on x. As $\mathcal{A}_s \subset \overline{\bigcup_s \mathcal{A}_s}$, $\forall\, s \in \mathbb{N}$, we obtain that there exists a constant $C > 0$ such that $\|g_r(t)\|_{H\times H} \leq C$, $\forall\, t \geq 0$ and $r = 0, 1, 2, \cdots$. So, in particular, we have that $g(t)$ is bounded in $H \times H$. Then, there exists a constant $\tilde{C} > 0$ such that

$$|g(t)|_{\mathbb{R}\times\mathbb{R}} = \frac{1}{|\Omega|^{1/2}} \|g(t)\|_{H\times H} \leq \tilde{C}, \ \forall\, t \in \mathbb{R}.$$

So, we conclude that $g : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ is a complete bounded orbit for $\mathbb{G}^\infty$ through (u_0, v_0).

Using the Theorem 15 in [24], we obtain that $(u_0, v_0) \in \mathcal{A}^\infty$. ■

Remark 22 *Note that if $p_s(\cdot) \equiv p$ and $q_s(\cdot) \equiv q$ the family of attractors is also lower semicontinuous since each solution of (18) is also a solution of* (1). *For the general case of variable exponent, lower semicontinuity is an open problem.*

Acknoledgments.

This work has been partially supported by Science without Borders-CAPES-PVE-Process 88881.0303888/2013-01. The first author was supported with CNPq scholarship - process 202645/2014-2 (Brazil). The second author was supported with CAPES scholarship - process N. 99999.006893/2014-07 (Brazil).

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