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We study the extension of the Chambolle–Pock primal-dual algorithm to nonsmooth optimization problems involving nonlinear operators between function spaces. Local convergence is shown under technical conditions including metric regularity of the corresponding primal-dual optimality conditions. We also show convergence for a Nesterov-type accelerated variant provided one part of the functional is strongly convex. We show the applicability of the accelerated algorithm to examples of inverse problems with L^1- and L^∞-fitting terms as well as of state-constrained optimal control problems, where convergence can be guaranteed after introducing an (arbitrary small, still nonsmooth) Moreau–Yosida regularization. This is verified in numerical examples.

1 Introduction

This work is concerned with the numerical solution of optimization problems of the form

$$\min_{u\in X} F(K(u)) + G(u), \tag{1.1}$$

where $F : Y \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ and $G : X \to \overline{\mathbb{R}}$ are proper, convex and lower semicontinuous functionals, and $K : X \to Y$ is a bounded operator between two Hilbert spaces X and Y. Such problems arise for example in inverse problems with nonsmooth discrepancy or regularization terms or in optimal control problems subject to state or control constraints. We are particularly interested in the situation where K is a *nonlinear* operator involving the solution of a partial differential equation and F is a *nonsmooth* discrepancy or tracking term.

To fix ideas, a prototypical example is the L^1 fitting problem [7]

$$\min_{u\in L^2(\Omega)} \|S(u) - y^\delta\|_{L^1} + \frac{\alpha}{2}\|u\|^2_{L^2}, \tag{1.2}$$

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i.e., $G(u) = \frac{\alpha}{2}\|u\|_{L^2}^2$, $F(y) = \|y\|_{L^1}$, and $K(u) = S(u) - y^\delta$ where S maps u to the solution y of $-\Delta y + uy = f$ for given f and y^δ is a given noisy measurement. This problem occurs in parameter identification from data corrupted by impulsive noise instead of the usual Gaussian noise. For uniform noise, the constrained ("Morozov") formulation is more natural [6]:

$$\min_{u \in L^2(\Omega)} \frac{1}{2}\|u\|_{L^2} \quad \text{s.t.} \quad |S(u)(x) - y^\delta(x)| \leq \delta \quad \text{a.e. in } \Omega, \tag{1.3}$$

i.e., $F(y) = \iota_{\{|y(x)| \leq \delta\}}(y)$. (Note that since the noise level δ is used here explicitly, no regularization parameter needs to be chosen.) A related example is the state-constrained optimal control problem

$$\min_{u \in L^2(\Omega)} \frac{1}{2}\|S(u) - y^d\|_{L^2}^2 + \frac{\alpha}{2}\|u\|_{L^2}^2 \quad \text{s.t.} \quad S(u)(x) \leq c \quad \text{a.e. in } \Omega, \tag{1.4}$$

where $y^d \in L^2(\Omega)$ is a desired state and $c \in L^\infty(\Omega)$ is a given upper bound. (Lower bounds are also possible.)

One possible approach to solving (1.1) is to apply a Moreau–Yosida regularization to the nonsmooth functional F, which allows deriving classical first-order necessary optimality conditions that can be solved by a semismooth Newton method in function spaces; see, e.g. [12, 19] for semismooth Newton methods in general as well as [7, 6, 11] for their application to (1.2), (1.3) and (1.4), respectively. Such methods are very efficient due to their superlinear convergence and mesh independence; however, they suffer from local convergence, with the convergence region depending strongly on the choice of the Moreau–Yosida parameter. For this reason, usually continuation methods are employed, where a sequence of minimization problems with gradually diminishing parameter are solved.

An alternative approach which has become very popular in the context of imaging problems are primal-dual extragradient methods. One widely used example, introduced in [3] for linear operators and extended in [20] to nonlinear operators, applied to (1.1) reads as

$$\begin{cases} u^{k+1} = \operatorname{prox}_{\tau G}(u^k - \tau K'(u^k)^* y^k), \\ \bar{u}^{k+1} = 2u^{k+1} - u^k, \\ y^{k+1} = \operatorname{prox}_{\sigma F^*}(y^k + \sigma K(\bar{u}^{k+1})), \end{cases} \tag{1.5}$$

where $\sigma, \tau > 0$ are appropriately chosen step lengths, $K'(u)^*$ denotes the adjoint of the Fréchet derivative of K, and $\operatorname{prox}_{F^*}$ denotes the proximal mapping of the Fenchel conjugate of F; we postpone precise definitions to later and only remark that if F^* is the indicator function of a convex set C, the proximal mapping coincides with the metric projection onto C. Such methods do not suffer from local convergence (for a linear operator K) and, as first-order methods, do not require solving – possibly ill-conditioned – linear systems in each iteration. Consequently, they recently have received increasing interest also in the context of optimal control; see, e.g., [15, 13]. In addition, other proximal point methods for optimal control problems have been treated in [1] and [18]; in particular, the latter is concerned with classical forward-backward splitting for sparse control of linear elliptic PDEs. However, so far these methods have only been considered in the finite-dimensional setting, i.e., after discretizing (1.1), or for specific (linear) problems.

Our goal is therefore to show convergence of (1.5) in Hilbert spaces and to show that it can be applied to problems of the form (1.2), (1.3), and (1.4). To the best of our knowledge, this work is the first to consider (accelerated) primal-dual extragradient methods in function spaces as well as their application to PDE-constrained optimization problems.

While the general convergence theory is a straightforward extension of the analysis in [20] (in fact, the proof is virtually identical), it requires verifying a set-valued Lipschitz property – known as the *Aubin* or *pseudo-Lipschitz property* – on the inverse of a monotone operator $H_{\widehat{u}}$ encoding the optimality conditions. This is also called *metric regularity* of $H_{\widehat{u}}$. This verification is significantly more involved in infinite dimensions. For problems of the form (1.1) where F and G are given by integral functionals for regular integrands, we can apply the theory from [8] to obtain an explicit, verifiable, condition for the Aubin property to be satisfied. While our analysis will show that for (1.2) or (1.4), this condition in fact does not hold in general unless a Moreau–Yosida regularization is introduced, we do obtain convergence and mesh independence for arbitrarily small regularization parameter, and numerical examples show that this can be observed in practice. Similarly, although for nonlinear operators, the convergence is only local since smallness conditions on the distance to the solution enter via bounds on the nonlinearity of the operator, in contrast to semismooth Newton methods we actually observe convergence for any starting point and arbitrarily small regularization parameter without the need for continuation.

In addition, Moreau–Yosida regularization results in a strongly convex functional, which can be exploited for accelerating the iteration (1.5). The accelerated form differs from (1.5) in adaptive step length and extrapolation parameters. In particular, we consider for a given acceleration parameter $\bar{\gamma} \geq 0$ the iteration

$$\begin{cases} u^{k+1} = \operatorname{prox}_{\tau_k G}(u^k - \tau_k K'(u^k)^* v^k), \\ \quad \omega_k = 1/\sqrt{1+2\bar{\gamma}\tau_k}, \quad \tau_{k+1} = \tau_k/\omega_k, \quad \sigma_{k+1} = \sigma_k \omega_k, \\ \bar{u}^{k+1} = u^{k+1} + \omega_k(u^{k+1} - u^k), \\ v^{k+1} = \operatorname{prox}_{\sigma_{k+1}F^*}(v^k + \sigma_{k+1}K(\bar{u}^{k+1})). \end{cases} \tag{1.6}$$

Note that the choice $\bar{\gamma} = 0$ coincides with the unaccelerated version (1.5). The appropriate choice for $\bar{\gamma} > 0$ is related to the constant of strong convexity of F^*, and in the convex case yields a convergence rate of $\mathcal{O}(1/k^2)$ for the functional values rather than the rate $\mathcal{O}(1/k)$ for the original version, see [3, 4, 21]. A similar acceleration is possible if G is strongly convex. Such an acceleration was not considered in [20]; our main contribution is therefore to show convergence of a (finitely) accelerated primal-dual extragradient method in function spaces.

This work is organized as follows. In the remainder of this section, we summarize some notations and definitions necessary for what follows. Section 2 is concerned with the convergence analysis of the accelerated algorithm (1.6) in infinite-dimensional Hilbert spaces, where we discriminate the case of F^* (§ 2.1) or G (§ 2.2) being strongly convex. We also briefly address the verification of the Aubin property for functionals of the form (1.1) in § 2.3. A more detailed discussion for the specific case of the motivating problems (1.2), (1.3), and (1.4) is given in section 3, where we also derive the explicit form of the accelerated algorithm (1.6) in these cases. Section 4 concludes with numerical examples for the three model problems.

1.1 Notation and definitions

Convex analysis We assume $G : X \to \overline{\mathbb{R}}$ and $F : Y \to \overline{\mathbb{R}}$ to be convex, proper, lower semicontinuous functionals on Hilbert spaces X and Y, satisfying $\operatorname{int}\operatorname{dom} G \neq \emptyset$ and $\operatorname{int}\operatorname{dom} F \neq \emptyset$. We call, e.g., F strongly convex with constant $\gamma_F > 0$ if

$$F(v') - F(v) \geq \langle z, v' - v\rangle + \frac{\gamma_F}{2}\|v' - v\|^2 \quad (v, v' \in Y;\ z \in \partial F(v)), \tag{1.7}$$

where ∂F denotes the convex subdifferential of F. We denote by

$$F^* : Y^* \to \overline{\mathbb{R}}, \qquad F^*(p) = \sup_{y\in Y} \langle p, y\rangle_Y - F(y)$$

the *Fenchel conjugate* of F, which is convex, proper and lower semicontinous. As usual, we identify the topological dual Y^* of Y with itself. The *Moreau–Yosida* regularization of F for the parameter $\gamma > 0$ is defined as

$$F_\gamma(y) := \min_{y'\in Y} F(y') + \frac{1}{2\gamma}\|y' - y\|^2, \tag{1.8}$$

whose Fenchel conjugate is (cf., e.g., [2, Prop. 13.21 (i)])

$$F_\gamma^*(p) = F^*(p) + \frac{\gamma}{2}\|p\|^2. \tag{1.9}$$

Note that F_γ^* is strongly convex with constant at least γ.

Since G is convex and $K \in C^1(X; Y)$, we can apply the calculus of Clarke's generalized derivative (which reduces to the Fréchet derivative and convex subdifferential for differentiable and convex functions, respectively; see, e.g., [5, Chap. 2.3]) to deduce for (1.1) the overall system of critical point conditions

$$\begin{cases} K(\widehat{u}) \in \partial F^*(\widehat{v}), \\ -K'(\widehat{u})^*\widehat{v} \in \partial G(\widehat{u}). \end{cases} \tag{1.10}$$

The iterations (1.5) can be derived from these conditions with the help of the *proximal mapping* (or *resolvent*)

$$\operatorname{prox}_G(v) = \arg\min_{w\in X} \frac{1}{2}\|w - v\|_X^2 + G(w) = (\mathrm{Id} + \partial G)^{-1}(v)$$

and similarly for F^*. We recall the following useful calculus rules for proximal mappings, e.g., from [2, Prop. 23.29 (i), (viii)]:

(P1) For any $\sigma > 0$ it holds that

$$\operatorname{prox}_{\sigma F^*}(v) = v - \sigma \operatorname{prox}_{\sigma^{-1}F}(\sigma^{-1}v).$$

(P2) For any $\gamma > 0$ it holds that

$$\operatorname{prox}_{F_\gamma^*}(v) = \operatorname{prox}_{(1+\gamma)^{-1}F^*}((1+\gamma)^{-1}v).$$

Set-valued analysis We first define for $U \subset X$ the set of *Fréchet (or regular) normals* to U at $u \in U$ by

$$\widehat{N}(u; U) := \left\{ z \in X \,\middle|\, \limsup_{U \ni u' \to u} \frac{\langle z, u' - u \rangle}{\|u' - u\|} \leq 0 \right\}$$

and the set of *tangent vectors* by

$$T(u; U) := \left\{ z \in X \,\middle|\, \text{exist } \tau^i \searrow 0 \text{ and } u^i \in U \text{ such that } z = \lim_{i \to \infty} \frac{u^i - u}{\tau^i} \right\}.$$

For a convex set U, these coincide with the usual normal and tangent cones of convex analysis.

For any cone $V \subset X$, we also define the *polar cone*

$$V^\circ := \{ z \in X \mid \langle z, z' \rangle \leq 0 \text{ for all } z' \in V \}.$$

We use the notation $R : Q \rightrightarrows W$ to denote a set-valued mapping R from Q to W; i.e., for every $q \in Q$ holds $R(q) \subset W$. For $R : Q \rightrightarrows W$, we define the domain $\operatorname{dom} R := \{ q \in Q \mid R(q) \neq \emptyset \}$ and graph $\operatorname{Graph} R := \{ (q, w) \subset Q \times W \mid w \in R(q) \}$. The regular coderivatives of such maps are defined graphically with the help of the normal cones. Let Q and W be Hilbert spaces, and $R : Q \rightrightarrows W$ with $\operatorname{dom} R \neq \emptyset$. We then define the *regular coderivative* $\widehat{D}^* R(q|w) : W \rightrightarrows Q$ of R at $q \in Q$ for $w \in W$ as

$$\widehat{D}^* R(q|w)(\Delta w) := \left\{ \Delta q \in Q \mid (\Delta q, -\Delta w) \in \widehat{N}((q, w); \operatorname{Graph} R) \right\}.$$

We also define the *graphical derivative* $DR(q|w) : Q \rightrightarrows W$ by

$$DR(q|w)(\Delta q) := \{ \Delta w \in W \mid (\Delta q, \Delta w) \in T((q, w); \operatorname{Graph} R) \}$$

and its convexification $\widetilde{DR}(q|w)$ via

$$\operatorname{Graph} \widetilde{DR}(q|w) = \operatorname{conv} \operatorname{Graph}[DR(q|w)].$$

Finally, we say that the set-valued mapping $R : Q \rightrightarrows W$ is *metrically regular* at $\widehat{w}$ for $\widehat{q}$ if $\operatorname{Graph} R$ is locally closed and there exist $\rho, \delta, \ell > 0$ such that

$$\inf_{p \,:\, w \in R(p)} \|q - p\| \leq \ell \|w - R(q)\| \qquad \text{for any } q, w \text{ such that } \|q - \widehat{q}\| \leq \delta,\ \|w - \widehat{w}\| \leq \rho.$$

We denote the infimum over valid constants ℓ by $\ell_{R^{-1}}(\widehat{w}|\widehat{q})$, or $\ell_{R^{-1}}$ for short when there is no ambiguity about the point $(\widehat{w}, \widehat{q})$. If $\ell_{R^{-1}}(\widehat{w}|\widehat{q}) > 0$ holds, we say that R^{-1} has the *Aubin property*.

2 Convergence

We now demonstrate in infinite-dimensional Hilbert spaces the convergence of algorithm (1.6), where the acceleration is stopped at some iteration N. We do not attempt to prove any convergence rates, and acceleration is merely observed numerically.

The general outline of the proof of the convergence of (1.6) is nearly unchanged from the original proof in [20] for finite-dimensional spaces. For this reason, we merely give a sketch of the proof and only detail the modifications necessary to exploit the strong convexity. Here we distinguish between whether F^* or G is strongly convex. The former is always guaranteed by Moreau–Yosida regularization of F, while the latter – if it holds in addition, which is the case in the examples considered here – might allow stronger acceleration, independent of the (regularization or control cost) parameter α. The main difficulty in function spaces lies in the verification of the Aubin property required for the convergence result, which we will investigate based on the results of [8] at the end of the section.

We begin by observing from the definition of the proximal mapping that the algorithm (1.6) may be written in the form

$$0 \in H_{u^i}(q^{i+1}) + \nu^i + M_i(q^{i+1} - q^i)$$

for the monotone operator

$$H_{\bar u}(u, v) := \begin{pmatrix} \partial G(u) + K'(\bar u)^* v \\ \partial F^*(v) - K'(\bar u)u - c_{\bar u} \end{pmatrix}, \quad \text{where} \quad c_{\bar u} := K(\bar u) - K'(\bar u)\bar u,$$

the preconditioning operator

$$M_i := \begin{pmatrix} \tau_i^{-1}\,\mathrm{Id} & -K'(u^i)^* \\ -K'(u^i) & \sigma_i^{-1}\,\mathrm{Id} \end{pmatrix},$$

and the discrepancy term

$$\nu^i := \bar\nu^i + \nu^i_\omega := \begin{pmatrix} 0 \\ K'(u^i)\bar u^{i+1} + c_{u^i} - K(\bar u^{i+1}) \end{pmatrix} + \begin{pmatrix} 0 \\ (1-\omega_i)K'(u^i)(u^{i+1} - u^i) \end{pmatrix}.$$

Throughout, we set

$$q = (u, v) \in X \times Y, \quad \text{and} \quad w = (\xi, \eta) \in X \times Y,$$

extending this notation to $\widehat q$, etc., in the obvious way. Here we fix $R > 0$ such that there exists a solution $\widehat q$ to

$$0 \in H_{\widehat u}(\widehat q) \quad \text{with} \quad \|\widehat q\| \le R/2. \tag{2.1a}$$

The first part is the necessary optimality condition (1.10) for (1.1) for linear K, and can under suitable regularity assumptions be shown to be necessary for nonlinear K as well. Regarding the operator $K : X \to Y$ and the step length parameters $\sigma_i, \tau_i > 0$, we require that

$$K \in C^2(X;Y) \qquad \text{and} \qquad \sigma_i\tau_i \left(\sup_{\|u\| \le R} \|K'(u)\|^2 \right) < 1. \tag{2.1b}$$

Observe that $\sigma_i\tau_i = \sigma_0\tau_0$ is maintained under acceleration schemes such as the one in (1.6), so this condition only has to be satisfied for the initial choice. We then denote by L_2 the Lipschitz factor of $u \mapsto K'(u)$ on the closed ball $B(0,R) \subset X$, namely

$$L_2 := \sup_{\|u\| \le R} \|K''(u)\| \tag{2.1c}$$

in operator norm. By (2.1b), the supremum is bounded. (This can be replaced by $K \in C^1(X;Y)$ with K' locally Lipschitz.) We define the uniform condition number

$$\kappa := \Theta/\theta. \tag{2.1d}$$

based on Θ and θ from the condition

$$\theta^2 \operatorname{Id} \le M_i \le \Theta^2 \operatorname{Id}. \tag{2.2}$$

If $\tau_i, \sigma_i > 0$ are constant, $\|u^i\| \le R$, and (2.1b) holds, such θ and Θ exist [20]. Easily this extends to $0 < C_1 \le \tau_i, \sigma_i \le C_2 < \infty$.

Remark 2.1. *The bound (2.2), on which the analysis from [20] depends, is the reason we need to stop the acceleration; since $\tau_i \to 0$ and $\sigma_i \to \infty$, no uniform bound exists for M_i if the acceleration is not stopped. Possibly the convergence proofs from [20] can be extended to the fully accelerated case, but such an endeavour is outside the scope of the present work. In numerical practice, in any case, we stop the algorithm – and hence a fortiori the acceleration – at a suitable iteration N.*

2.1 Convergence for strongly convex F^*

We begin by considering the case of F^* being strongly convex, which is closest to the setting of [20]. In this case, we chose for $\bar\gamma \ge 0$ the acceleration sequence

$$\sigma_{i+1} := \omega_i \sigma_i \quad \text{and} \quad \tau_{i+1} := \tau_i/\omega_i \quad \text{for} \quad \omega_i := 1/\sqrt{1+2\bar\gamma\sigma_i}. \tag{2.3}$$

Under the above assumptions, and if the Aubin property holds for $H_{\widehat{u}}^{-1}$, algorithm (1.6) then converges to a solution of (2.1a).

Theorem 2.1. *Let (2.1) be satisfied with the corresponding constants R, Θ, κ and L_2, and suppose F^* is strongly convex with factor γ_{F^*}. Let $\widehat{q}$ solve $0 \in H_{\widehat{u}}(\widehat{q})$, and $H_{\widehat{u}}^{-1}$ have the Aubin property at 0 for $\widehat{q}$ with*

$$\ell_{H_{\widehat{u}}^{-1}} \kappa L_2 \|\widehat{v}\| < 1 - 1/\sqrt{1 + 1/(2\ell_{H_{\widehat{u}}^{-1}}^2 \Theta^4)}. \tag{2.4}$$

If $\bar\gamma \in [0, \gamma_{F^*})$ and we use the rule (2.3) for $i = 1, \ldots, N$ for some $N \ge 0$, after which $\tau_i = \tau_N$ and $\sigma_i = \sigma_N$ for $i > N$, there exists $\delta > 0$ such that for any $q^1 \in X \times Y$ with

$$\|q^1 - \widehat{q}\| \le \delta,$$

the iterates $q^{i+1} = (u^{i+1}, v^{i+1})$ generated by (1.6) converge to a solution $q^* = (u^*, v^*)$ of (2.1a).

Sketch of proof of Theorem 2.1. We follow [20, Theorem 3.2], which was only stated for finite-dimensional spaces X and Y. However, nothing in the proof depends on the finite-dimensionality, as all the arguments presented in [20] are entirely algebraic manipulations, estimating norms and inner products in terms of others through the axioms of Hilbert spaces. The inverse mapping theorem for set-valued functions [20, Lemma 3.8] is extracted from [9], but their results are also stated in general complete metric spaces. Therefore we conclude that the results of [20, Theorem 3.2] are valid in infinite-dimensional Hilbert spaces. We now outline the idea of the proof.

First, we need to show that for any fixed $i, C, \varepsilon > 0$, there exists $\rho > 0$ such that

$$\|v^i\| \le \varepsilon \|u^i - u^{i+1}\|, \quad (\|u^i - u^{i+1}\| \le \rho, \|u^i\| \le C). \tag{A-D^i}$$

But this has been shown for the component $\bar{v}^i$ in [20, Lemma 3.2], whereas for the acceleration component v^i_ω, the claim follows from (2.1b) and the fact that $\omega_i \to 1$. In fact we have more than that: $v^i_\omega = 0$ for $i \ge N$, as then $\omega_i = 1$.

We now use for a self-adjoint linear operator $M : X \to X$ the notation

$$\langle a, b \rangle_M := \langle Ma, b \rangle \qquad \text{and} \qquad \|a\|_M := \sqrt{\langle a, a \rangle_M}.$$

We define $\widetilde{q}^i \in X \times Y$ as the "local perturbed solution" at the current iteration u^i as satisfying at the base point u^i the condition

$$0 \in H_{u^i}(\widetilde{q}^i) + v^i.$$

To see why we call this the "perturbed" local solution, we observe that $\widehat{q}^i \in X \times Y$, defined as satisfying at the base point u^i the condition

$$0 \in H_{u^i}(\widehat{q}^i),$$

would be a solution to the linearized problem, where we replace in (1.1) the operator K by its linearization

$$u \mapsto K(u^i) + K'(u^i)(u - u^i) = K'(u^i)u + c_{u^i}.$$

With each step adding conditions on starting sufficiently close to a solution, the steps of the proof are now roughly the following, with details to be found in [20]:

1. [20, Lemma 2.1] for $V = Y$ shows the initial descent estimate

$$\|q^i - \widetilde{q}^i\|^2_{M_i} \ge \|q^{i+1} - q^i\|^2_{M_i} + \|q^{i+1} - \widetilde{q}^i\|^2_{M_i} + \gamma_{F^*}\|v^{i+1} - \widetilde{v}^i\|^2. \tag{$\widehat{\text{D}}^2$-loc-γ-F*}$$

2. In place of [20, Lemma 3.6], Lemma 2.3 below uses the strong convexity of F^* to update for the next iterate local norms from $\|\cdot\|_{M_i}$ to $\|\cdot\|_{M_{i+1}}$, namely to go from ($\widehat{\text{D}}^2$-loc-γ-F*) to

$$\|q^i - \widetilde{q}^i\|^2_{M_i} \ge \xi_1\|q^{i+1} - q^i\|^2_{M_i} + \|q^{i+1} - \widetilde{q}^i\|^2_{M_{i+1}}. \tag{$\widehat{\text{D}}^2$-M}$$

3. For technical reasons, we use the Aubin property [20, Lemma 3.11] to remove the squares,

$$\|\widetilde{q}^i - q^i\|_{M_i} \ge \xi_2\|q^{i+1} - q^i\|_{M_i} + \|\widetilde{q}^i - q^{i+1}\|_{M_{i+1}}. \tag{$\widehat{\text{D}}$-M}$$

4. Further, again using the Aubin property, [20, Lemma 3.12] bridges from the perturbed local solutions $\widetilde{q}^i$ to local solutions $\widehat{q}^i$,

$$\|q^i - \widehat{q}^i\|_{M_i} \ge \xi\|q^{i+1} - q^i\|_{M_i} + \|q^{i+1} - \widehat{q}^{i+1}\|_{M_{i+1}}. \tag{$\widehat{\text{D}}$}$$

5. Convergence follows from the general result [20, Theorem 2.1] on descent estimates of the type ($\widehat{\text{D}}$). ◻

Remark 2.2. *Strong convexity of F^* with factor γ is equivalent [10] to strong monotonicity of ∂F^* in the sense that*

$$\langle \partial F^*(v') - \partial F^*(v), v' - v\rangle \geq \gamma \|v' - v\|^2, \quad (v', v \in Y),$$

observing that there is no factor $1/2$ in the latter, unlike mistakenly written at [20, the end of page 7]. Hence the slight difference in the statement of ($\widehat{\text{D}^2}$-loc-γ-F) in comparison to the similarly-named equation in [20]. In the cited article, the exact factors make no difference however; in the present work they do for the acceleration.*

Remark 2.3. *Theorem 2.1 holds if F^* is merely strongly convex on the "nonlinear" subspace*

$$Y_{NL} := \{y \in Y : \langle z, K(\cdot)\rangle \in L(X, Y)\}^{\perp},$$

i.e., if (1.7) holds merely for all $v, v' \in Y_{NL}$. In this case, $\widehat{v}$ in (2.4) can be replaced by $P_{NL}\widehat{v}$, the orthogonal projection of $\widehat{v}$ on Y_{NL}. Indeed, Lemma 2.1 in [20] directly applies to $V = Y_{NL} \subsetneq Y$ to yield ($\widehat{\text{D}^2}$-loc-γ-F) for $P_{NL}(v^{i+1} - \widetilde{v}^i)$, and a straightforward modification of Lemma 2.3 below yields ($\widehat{\text{D}}^2$-M). Since the Moreau–Yosida regularization, required for the Aubin property in our examples, already implies strong convexity on the full space, we do not treat this more general case in detail.*

We still need to prove Lemma 2.3. This requires the following bound on the step lengths.

Lemma 2.2. *If $\{\sigma_i\}_{i\in\mathbb{N}}$ satisfies (2.3), then $\gamma + (\sigma_i^{-1} - \sigma_{i+1}^{-1}) \geq 0$.*

Proof. We first note that

$$2\gamma + (\sigma_i^{-1} - \sigma_{i+1}^{-1}) = \sigma_i^{-1}(2\gamma\sigma_i + 1 - \omega_i^{-1}).$$

Thus the claim holds if $c = \gamma$ satisfies

$$2\gamma\sigma_i + 1 - \omega_i^{-1} \geq c\sigma_i,$$

i.e., after multiplying both sides by ω_i^2 and using the definition of ω_i,

$$1 - \omega_i \geq c\omega_i^2\sigma_i.$$

In other words, we need to show

$$c \leq \frac{1-\omega_i}{\omega_i^2\sigma_i} = \frac{\omega_i^{-1} - 1}{\omega_i\sigma_i}. \tag{2.5}$$

But using the concavity of the square root, we can estimate

$$\omega_i^{-1} - 1 = (-\sqrt{1}) - (-\sqrt{1 + 2\gamma\sigma_i}) \geq -\frac{1}{2\sqrt{1+2\gamma\sigma_i}}(1 - (1 + 2\gamma\sigma_i)) = \gamma\sigma_i\omega_i.$$

This shows that (2.5) holds with $c = \gamma$. ◻

We finally provide the main lemma needed towards extending the results of [20] to the present setting.

Lemma 2.3. *Suppose (2.1) holds along with the assumption of Theorem 2.1 and assume that ($\widehat{\mathrm{D}^2}$-loc-γ-F*) holds. Let R, L_2, κ be as in (2.1), and choose $\xi_1 \in (0,1)$. If*

$$\|q^i - \widehat{q}\| \leq R/4 \quad \text{and} \quad \|q^i - \widetilde{q}^i\| \leq C \tag{2.6}$$

for a suitable constant $C = C(\gamma_{F^*}, \bar{\gamma}, \xi_1, \theta, L_2, \kappa, R)$, then ($\widehat{\mathrm{D}}^2$-M) holds.

Proof. Using (2.6) and the property $\|\widehat{q}\| \leq R/2$ from (2.1a), we have

$$\|q^i\| \leq \|q^i - \widehat{q}\| + \|\widehat{q}\| \leq 3R/4. \tag{2.7}$$

The estimate ($\widehat{\mathrm{D}^2}$-loc-γ-F*) implies

$$\|q^{i+1} - q^i\| \leq \kappa \|\widetilde{q}^i - q^i\|.$$

Choosing $C \leq R/(4\kappa)$ and using (2.6) and (2.7), we thus get

$$\|q^{i+1}\| \leq \|q^{i+1} - q^i\| + \|q^i\| \leq \kappa \|\widetilde{q}^i - q^i\| + \|q^i\| \leq R.$$

As both $\|q^i\| \leq R$ and $\|q^{i+1}\| \leq R$, by (2.1c) we have again

$$\|K'(u^{i+1}) - K'(u^i)\| \leq L_2 \|u^{i+1} - u^i\|. \tag{2.8}$$

We expand

$$\begin{aligned}\|q^{i+1} - \widetilde{q}^i\|^2_{M_i} - \|q^{i+1} - \widetilde{q}^i\|^2_{M_{i+1}} &= -2\langle v^{i+1} - \widetilde{v}^i, (K'(u^{i+1}) - K'(u^i))(u^{i+1} - \widetilde{u}^i)\rangle \\ &\quad + (\tau_i^{-1} - \tau_{i+1}^{-1})\|u^{i+1} - \widetilde{u}^i\|^2 + (\sigma_i^{-1} - \sigma_{i+1}^{-1})\|v^{i+1} - \widetilde{v}^i\|^2 \\ &\geq -2\langle v^{i+1} - \widetilde{v}^i, (K'(u^{i+1}) - K'(u^i))(u^{i+1} - \widetilde{u}^i)\rangle \\ &\quad + (\sigma_i^{-1} - \sigma_{i+1}^{-1})\|v^{i+1} - \widetilde{v}^i\|^2\end{aligned}$$

In the final step, we have used the fact that $\{\tau_i\}_{i\in\mathbb{N}}$ is non-decreasing. Using (2.8), we further derive by application of Young's inequality

$$\begin{aligned}\|q^{i+1} - \widetilde{q}^i\|^2_{M_i} - \|q^{i+1} - \widetilde{q}^i\|^2_{M_{i+1}} &\geq (\sigma_i^{-1} - \sigma_{i+1}^{-1})\|v^{i+1} - \widetilde{v}^i\|^2 \\ &\quad - 2L_2\|q^{i+1} - \widetilde{q}^i\|\|q^{i+1} - q^i\|\|u^{i+1} - \widetilde{u}^i\|.\end{aligned} \tag{2.9}$$

Using once more Young's inequality, (2.9), and Lemma 2.2, we now deduce

$$\begin{aligned}&\|q^{i+1} - \widetilde{q}^i\|^2_{M_i} - \|q^{i+1} - \widetilde{q}^i\|^2_{M_{i+1}} + \gamma_{F^*}\|v^{i+1} - \widetilde{v}^i\|^2 \\ &\qquad \geq (\bar{\gamma} + \sigma_i^{-1} - \sigma_{i+1}^{-1})\|v^{i+1} - \widetilde{v}^i\|^2 - \frac{L_2^2}{\gamma_{F^*} - \bar{\gamma}}\|q^{i+1} - q^i\|^2\|q^{i+1} - \widetilde{q}^i\|^2. \\ &\qquad \geq -\frac{L_2^2}{\gamma_{F^*} - \bar{\gamma}}\|q^{i+1} - q^i\|^2\|q^{i+1} - \widetilde{q}^i\|^2.\end{aligned} \tag{2.10}$$

By application of (2.2) and ($\widehat{\mathrm{D}^2}$-loc-γ-F*), we bound

$$\|q^{i+1} - \widetilde{q}^i\|^2 \le \theta^{-2}\|q^{i+1} - \widetilde{q}^i\|^2_{M_i} \le \kappa^2\|q^i - \widetilde{q}^i\|^2,$$

and

$$\|q^{i+1} - q^i\|^2 \le \theta^{-2}\|q^{i+1} - q^i\|^2_{M_i}.$$

Setting

$$c := \frac{L_2^2}{\gamma_{F^*} - \bar{\gamma}} \quad \text{and} \quad C := (1-\xi_1)\frac{\theta^2}{c\kappa^2},$$

and using (2.6) therefore yields

$$c\|q^{i+1} - q^i\|^2\|q^{i+1} - \widetilde{q}^i\|^2 \le \frac{c\kappa^2}{\theta^2}\|q^{i+1} - q^i\|^2_{M_i}\|q^i - \widetilde{q}^i\|^2 \le (1-\xi_1)\|q^{i+1} - q^i\|^2_{M_i}.$$

Using (2.10) and this estimate in ($\widehat{\mathrm{D}^2}$-loc-γ-F*), we obtain ($\widehat{\mathrm{D}}^2$-M). □

2.2 Convergence for strongly convex G

In this case, we chose for $\bar{\gamma} \ge 0$ the acceleration sequence

$$\sigma_{i+1} := \sigma_i/\omega_i \quad \text{and} \quad \tau_{i+1} := \tau_i\omega_i \quad \text{for} \quad \omega_i := 1/\sqrt{1 + 2\bar{\gamma}\tau_i}. \tag{2.11}$$

Under the above assumptions, and if the Aubin property holds for $H_{\widehat{u}}^{-1}$, algorithm (1.6) converges to a solution of (2.1a).

Theorem 2.4. *Let (2.1) be satisfied with the corresponding constants R, Θ, κ and L_2, and suppose G is strongly convex with factor γ_G. Let $\widehat{q}$ solve $0 \in H_{\widehat{u}}(\widehat{q})$, and $H_{\widehat{u}}^{-1}$ have the Aubin property at 0 for $\widehat{q}$ with*

$$\ell_{H_{\widehat{u}}^{-1}}\kappa L_2\|\widehat{v}\| < 1 - 1/\sqrt{1 + 1/(2\ell^2_{H_{\widehat{u}}^{-1}}\Theta^4)}.$$

If $\bar{\gamma} \in [0, \gamma_G)$ and we use the rule (2.11) for $i = 1, \ldots, N$ for some $N \ge 0$, after which $\tau_i = \tau_N$ and $\sigma_i = \sigma_N$ for $i > N$, there exists $\delta > 0$ such that for any $q^1 \in X \times Y$ with

$$\|q^1 - \widehat{q}\| \le \delta,$$

the iterates $q^{i+1} = (u^{i+1}, v^{i+1})$ generated by (1.6) converge to a solution $q^* = (u^*, v^*)$ of (2.1a).

Sketch of proof of Theorem 2.4. The proof follows that of Theorem 2.1, with the following modifications:

1. A trivial modification of [20, Lemma 2.1] employing the strong convexity assumptions shows

$$\|q^i - \widetilde{q}^i\|^2_{M_i} \ge \|q^{i+1} - q^i\|^2_{M_i} + \|q^{i+1} - \widetilde{q}^i\|^2_{M_i} + \gamma_G\|u^{i+1} - \widetilde{u}^i\|^2. \tag{$\widehat{\mathrm{D}^2}$-loc-γ-G}$$

2. In place of Lemma 2.3, we use Lemma 2.6 below to obtain ($\widehat{\mathrm{D}}^2$-M). □

It remains to prove Lemma 2.6. Analogously to Lemma 2.2, we first derive the following bounds.

Lemma 2.5. *Let $\{\tau_i\}_{i\in\mathbb{N}}$ satisfy (2.11). Then $\gamma + (\tau_i^{-1} - \tau_{i+1}^{-1}) \geq 0$.*

We can now show the main lemma in the case of strongly convex G.

Lemma 2.6. *Suppose (2.1) holds along with the assumption of Theorem 2.4, and assume that ($\widehat{\mathrm{D}^2}$-loc-γ-G) holds. Let R, L_2, κ be as in (2.1), and choose $\xi_1 \in (0,1)$. If*

$$\|q^i - \widehat{q}\| \leq R/4 \quad \text{and} \quad \|q^i - \widetilde{q}^i\| \leq C$$

for a suitable constant $C = C(\gamma_G, \bar{\gamma}, \xi_1, \theta, L_2, \kappa, R)$, then ($\widehat{\mathrm{D}}^2$-M) holds.

Proof. Proceeding as in the proof of Lemma 2.3, since now $\{\sigma_i\}_{i\in\mathbb{N}}$ is non-decreasing, we derive instead of (2.9) the estimate

$$\begin{aligned}\|q^{i+1} - \widetilde{q}^i\|_{M_i}^2 - \|q^{i+1} - \widetilde{q}^i\|_{M_{i+1}}^2 \geq{}& (\tau_i^{-1} - \tau_{i+1}^{-1})\|u^{i+1} - \widetilde{u}^i\|^2 \\ &- 2L_2\|q^{i+1} - \widetilde{q}^i\|\|q^{i+1} - q^i\|\|u^{i+1} - \widetilde{u}^i\|.\end{aligned}$$

Aplying Young's inequality, (2.9), and Lemma 2.2, we deduce

$$\begin{aligned}&\|q^{i+1} - \widetilde{q}^i\|_{M_i}^2 - \|q^{i+1} - \widetilde{q}^i\|_{M_{i+1}}^2 + \gamma_G\|u^{i+1} - \widetilde{u}^i\|^2 \\ &\qquad\geq (\bar{\gamma} + \tau_i^{-1} - \tau_{i+1}^{-1})\|u^{i+1} - \widetilde{u}^i\|^2 - \frac{L_2^2}{\gamma_G - \bar{\gamma}}\|q^{i+1} - q^i\|^2\|q^{i+1} - \widetilde{q}^i\|^2 \\ &\qquad\geq -\frac{L_2^2}{\gamma_G - \bar{\gamma}}\|q^{i+1} - q^i\|^2\|q^{i+1} - \widetilde{q}^i\|^2.\end{aligned}$$

We now conclude analogously to the proof of Lemma 2.3. □

2.3 Metric regularity

We finally address the verification of the Aubin property required for the convergence of the algorithm. Motivated by the problems considered in the next section, we assume that

$$F^*(v) = \int_\Omega f^*(v(x))\,dx$$

for a proper, convex, lower semicontinuous f^* and (after rescaling $F + G$, see below)

$$G(u) = \frac{1}{2}\|u\|_{L^2}^2.$$

We wish to apply the results from [8]. Towards this end, we consider the Moreau–Yosida regularization (1.8) of F for some parameter $\gamma > 0$, and assume (using (1.9)) that the regular coderivative of the regularized subdifferential satisfies at least at non-degenerate points for some cone $V_{\partial F^*}(v|\eta)$ the expression

$$\widetilde{D[\partial F_\gamma^*]}(v|\eta)(\Delta v) = \begin{cases} \gamma\Delta v + V_{\partial F^*}(v|\eta)^\circ, & \Delta v \in V_{\partial F^*}(v|\eta), \\ \emptyset, & \Delta v \notin V_{\partial F^*}(v|\eta). \end{cases} \tag{2.12}$$

We denote the corresponding operator $H_{\widehat{u}}$ by $H_{\gamma,\widehat{u}}$. We also assume $K \in C^1(X;Y)$. Then we have the following result.

Proposition 2.7 ([8, Prop. 4.6])**.** *Suppose $\widehat{q}$ solves $0 \in H_{\gamma,\widehat{u}}(\widehat{q})$ for some $\gamma \geq 0$. Then $w \mapsto H_{\gamma,\widehat{u}}^{-1}(w)$ has the Aubin property at $(0|\widehat{q})$ if and only if $\gamma > 0$ or*

$$\bar{b}(\widehat{q}|0;H_{\widehat{u}}) := \sup_{t>0} \inf \left\{ \frac{\|K'(\widehat{u})K'(\widehat{u})^*z - \nu\|}{\|z\|} \;\middle|\; \begin{array}{l} 0 \neq z \in V_{\partial F^*}(v'|\eta'),\ \nu \in V_{\partial F^*}(v'|\eta')^\circ, \\ \eta' \in \partial F^*(v'),\ \|v' - \widehat{v}\| < t, \\ \|\eta' - K(\widehat{u})\| < t \end{array} \right\} > 0.$$

This implies convergence for any choice of the Moreau–Yosida regularization parameter γ. On the other hand, if $\gamma = 0$, we have to prove existence of a lower bound for $\bar{b}$. This is significantly more difficult. We will address the issue of verifying – or disproving – the lower bound on $\bar{b}$ with specific examples in the next section.

3 Application to PDE-constrained optimization problems

We now discuss the application of the preceeding analysis in the context of the motivating problems (1.2), (1.3), and (1.4). Since this will depend on the specific structure of the mapping S, we consider as a concrete example the problem of recovering the potential term in an elliptic equation.

Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain with a Lipschitz boundary $\partial\Omega$. For a given coefficient $u \in \{v \in L^\infty(\Omega) : v \geq \varepsilon\} =: U \subset X := L^2(\Omega)$ and $f \in L^2(\Omega)$ fixed, denote by $S(u) := y \in H^1(\Omega) \subset L^2(\Omega) =: Y$ the weak solution of

$$\langle \nabla y, \nabla v\rangle + \langle uy, v\rangle = \langle f, v\rangle \qquad (v \in H^1(\Omega)). \tag{3.1}$$

This operator has the following useful properties [16]:

(A1) The operator S is uniformly bounded in $U \subset X$ and completely continuous: If for $u \in U$, the sequence $\{u_n\} \subset U$ satisfies $u_n \rightharpoonup u$ in X, then

$$S(u_n) \to S(u) \quad \text{in } Y.$$

(A2) S is twice Fréchet differentiable.

(A3) There exists a constant $C > 0$ such that

$$\|S'(u)h\|_{L^2} \leq C\|h\|_X \qquad (u \in U, h \in X).$$

(A4) There exists a constant $C > 0$ such that

$$\|S''(u)(h,h)\|_{L^2} \le C\|h\|_X^2 \qquad (u \in U, h \in X).$$

Furthermore, from the implicit function theorem, the directional Fréchet derivative $S'(u)h$ for given $h \in X$ can be computed as the solution $w \in H^1(\Omega)$ to

$$\langle \nabla w, \nabla v\rangle + \langle uw, v\rangle = \langle -yh, v\rangle \qquad (v \in H^1(\Omega)).$$

Similarly, the directional adjoint derivative $S'(u)^*h$ is given by yz, where $z \in H^1(\Omega)$ solves

$$\langle \nabla z, \nabla v\rangle + \langle uz, v\rangle = \langle -h, v\rangle \qquad (v \in H^1(\Omega)).$$

Similar expressions hold for $S''(u)(h_1, h_2)$ and $(S'(u)^*h_1)'h_2$. Hence, assumptions (A3–A4) hold for S'^* and $(S'(u)^*v)'$ for given v as well.

Other operators satisfying the above assumptions are mappings from a Robin or diffusion coefficient to the solution of the corresponding elliptic partial differential equation [7].

3.1 L^1 fitting

First, we consider the L^1 fitting problem (1.2). In order to make use of the strong convexity of the penalty term for the acceleration, we rewrite this equivalently as

$$\min_{u \in L^2} \frac{1}{\alpha}\|S(u) - y^\delta\|_{L^1} + \frac{1}{2}\|u\|_{L^2}^2,$$

i.e., we set $G(u) = \frac{1}{2}\|u\|_{L^2}^2$, $K(u) = S(u) - y^\delta$, and $F(y) = \frac{1}{\alpha}\|y\|_{L^1}$ in (1.1). Hence

$$[F^*(p)](x) = \iota_{[-\alpha^{-1}, \alpha^{-1}]}(p(x)) \qquad (\text{a.e. } x \in \Omega),$$

where ι_C denotes the indicator function of the convex set C in the sense of convex analysis [10].

To guarantee metric regularity, we replace F by its Moreau–Yosida regularization, which coincides with the well-known Huber norm, i.e.,

$$F_\gamma(y) = \int_\Omega |y(x)|_\gamma \, dx, \qquad |t|_\gamma = \begin{cases} \frac{1}{2\gamma}|t|^2 & \text{if } |t| \le \frac{\gamma}{\alpha}, \\ \frac{1}{\alpha}|t| - \frac{\gamma}{2\alpha} & \text{if } |t| > \frac{\gamma}{\alpha}. \end{cases}$$

Using the calculus of Clarke's generalized derivative and (1.9), i.e., $\partial F_\gamma^*(p) = \partial F^*(p) + \{\gamma p\}$, we obtain the corresponding regularized optimality conditions (cf. also [7, Theorem 2.7])

$$\begin{cases} S(u_\gamma) - y^\delta - \gamma p_\gamma \in \partial F^*(p_\gamma), \\ \qquad -S'(u_\gamma)^* p_\gamma = u_\gamma. \end{cases} \tag{3.2}$$

3.1.1 Algorithm

For G and F^* as above, the proximal mappings are given by

$$\begin{aligned}[\mathrm{prox}_{\tau G}(u)](x) &= \tfrac{1}{1+\tau}u(x),\\ [\mathrm{prox}_{\sigma F^*}(v)](x) &= \mathrm{proj}_{[-\alpha^{-1},\alpha^{-1}]}(v(x)).\end{aligned}$$

Using rule (P2) above, we thus obtain for the Moreau–Yosida regularization F^*_γ

$$[\mathrm{prox}_{\sigma F^*_\gamma}(v)](x) = \mathrm{proj}_{[-\alpha^{-1},\alpha^{-1}]}\left(\tfrac{1}{1+\sigma\gamma}v(x)\right).$$

Since G is strongly convex with constant $c_G = 1$, we can use the acceleration scheme (2.11) for any $\bar\gamma < 1$. The full algorithm thus consists in performing for $i = 1, \dots, N$ the steps

$$\text{(3.3)}\qquad \begin{cases} z^{i+1} = S'(u^i)^* v^i,\\ u^{i+1} = \frac{1}{1+\tau_i}(u^i - \tau_i z^{i+1}),\\ \omega_i = 1/\sqrt{1+2\bar\gamma\tau_i}, \quad \tau_{i+1} = \omega_i\tau_i, \quad \sigma_{i+1} = \sigma_i/\omega_i,\\ \bar u^{i+1} = u^{i+1} + \omega_i(u^{i+1} - u^i),\\ v^{i+1} = \mathrm{proj}_{[-\alpha^{-1},\alpha^{-1}]}\left(\frac{1}{1+\sigma_{i+1}\gamma}(v^i + \sigma_{i+1}(S(\bar u^{i+1}) - y^\delta))\right). \end{cases}$$

3.1.2 Metric regularity

To show convergence of algorithm (3.3) using Theorem 2.4 and Proposition 2.7, we have to verify the expression (2.12). This was shown in [8] using the pointwise expression of ∂F^*, which we summarize here for the sake of completeness.

Lemma 3.1 ([8, Lem. 2.10])**.** *Let $f^* : \mathbb{R} \to \overline{\mathbb{R}}$, $f^*(z) = \iota_{[-\alpha^{-1},\alpha^{-1}]}(z)$. Then*

$$\text{(3.4)}\qquad D(\partial f^*)(z|\zeta)(\Delta z) = \begin{cases} \mathbb{R}, & |z| = \alpha^{-1},\ \zeta \in (0,\infty)z,\ \Delta z = 0,\\ [0,\infty)z, & |z| = \alpha^{-1},\ \zeta = 0,\ \Delta z = 0,\\ \{0\}, & |z| = \alpha^{-1},\ \zeta = 0,\ z\Delta z < 0,\\ \{0\}, & |z| < \alpha^{-1},\ \zeta = 0,\\ \emptyset, & \textit{otherwise}, \end{cases}$$

as well as

$$\widetilde{D(\partial f^*)}(z|\zeta)(\Delta z) = \begin{cases} \mathbb{R}, & |z| = \alpha^{-1},\ \zeta \in (0,\infty)z,\ \Delta z = 0,\\ [0,\infty)z, & |z| = \alpha^{-1},\ \zeta = 0,\ z\Delta z \le 0,\\ \{0\}, & |z| < \alpha^{-1},\ \zeta = 0,\\ \emptyset, & \textit{otherwise}. \end{cases}$$

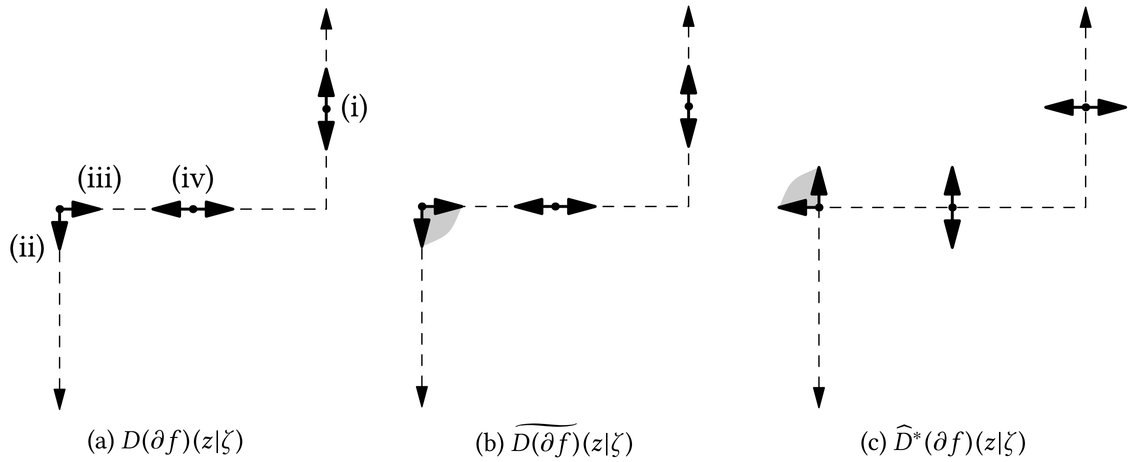


Figure 1: Illustration of the graphical derivative and regular coderivative for ∂f with $f = \iota_{[-1,1]}$. The dashed line is Graph ∂f. The dots indicate the base points (z, ζ) where the graphical derivative or coderivative is calculated, and the thick arrows and gray areas indicate the directions of $(\Delta z, \Delta\zeta)$ relative to the base point. The labels (i) etc. denote the corresponding case of (3.4). (Taken from [8].)

Corollary 3.2 ([8, Cor. 2.11]). *Let* $f^*(z) := \iota_{[-\alpha^{-1},\alpha^{-1}]}(z)$ *and*

$$F^*(v) := \int_\Omega f^*(v(x))\,dx \qquad (v \in L^2(\Omega)).$$

Then

$$D[\partial F^*](v|\eta)(\Delta v) = \begin{cases} V_{\partial F^*}(v|\eta)^\circ, & \Delta v \in V_{\partial F^*}(v|\eta) \text{ and } \eta \in \partial F^*(v), \\ \emptyset, & \text{otherwise}, \end{cases}$$

and

$$\widehat{D}^*[\partial F^*](v|\eta)(\Delta\eta) = \begin{cases} V_{\partial F^*}(v|\eta)^\circ, & -\Delta\eta \in V_{\partial F^*}(v|\eta) \text{ and } \eta \in \partial F^*(v), \\ \emptyset, & \text{otherwise}, \end{cases}$$

for the cone

$$V_{\partial F^*}(v|\eta) = \{z \in L^2(\Omega) \mid z(x)v(x) \le 0 \text{ if } |v(x)| = \alpha^{-1} \text{ and } z(x)\eta(x) \ge 0\}$$

and its polar

$$V_{\partial F^*}(v|\eta)^\circ = \{\nu \in L^2(\Omega) \mid \nu(x)v(x) \ge 0 \text{ if } \eta(x) = 0 \text{ and } \nu(x) = 0 \text{ if } |v(x)| < \alpha^{-1}\}.$$

Remark 3.1. *If* (v, η) *satisfy the strict complementarity condition* $|v(x)| < \alpha^{-1}$ *or* $|\eta(x)| > 0$ *for a. e.* $x \in \Omega$, *the degenerate second and third case in* (3.4) *(corresponding to the gray areas in Figure 1) do not occur, and the cone simplifies to*

$$V_{\partial F^*}(v|\eta) := \{z \in L^2(\Omega) \mid z(x) = 0 \text{ if } |v(x)| = \alpha^{-1},\ x \in \Omega\}.$$

Using a sum rule for regular coderivatives [8, Cor. 2.4], we deduce that (2.12) holds for F_γ^*. However, as discussed in [8, § 5.1], for $\gamma = 0$ (i.e., no regularization), we in general have $\bar{b}(\widehat{q}|0; H_{\widehat{u}}) = 0$. We remark that in the case of finite-dimensional data $y^\delta \in Y_h \subset Y$, replacing F by $F \circ P_h$, where P_h denotes the orthogonal projection onto Y_h, there exists a constant $c > 0$ such that $\bar{b}(\widehat{q}|0; H_{\widehat{u},h}) \geq c > 0$ holds; see [8, § 5.3].

The next corollary summarizes the convergence result for the present L^1 fitting problem.

Corollary 3.3. *Let $\gamma > 0$ and $\bar{\gamma} \in [0, 1)$ be arbitrary (setting $\bar{\gamma} = 0$ after a finite number of iterations). Let $(u_\gamma, p_\gamma) \in L^2(\Omega)^2$ be a solution to (3.2), and take $\tau_0, \sigma_0 > 0$ to satisfy (2.1b) for $K(u) = S(u) - y^\delta$. Then there exists $\delta > 0$ such that for any initial iterate $(u^1, p^1) \in L^2(\Omega)^2$ with $\|(u^1, p^1) - (u_\gamma, p_\gamma)\| \leq \delta$, the iterates (u^k, p^k) generated by algorithm (3.3) converge to a solution (u^*, p^*) to (3.2).*

Proof. Note that G is strongly convex with factor 1, while Moreau–Yosida regularization makes F_γ^* strongly convex with factor γ. By Proposition 2.7, $H_{\gamma,\widehat{u}}$ has the Aubin property at $(\widehat{q}, 0)$. The claim now follows from Theorem 2.4. □

3.2 L^∞ fitting

We next consider the L^∞ fitting ("Morozov") problem (1.3):

$$\min_u \frac{1}{2}\|u\|_{L^2} \quad \text{s. t.} \quad |S(u)(x) - y^\delta(x)| \leq \delta \quad \text{a. e. in } \Omega,$$

i.e., now $F(v) = \iota_{\{|v(x)| \leq \delta\}}(v)$ with G and K as before.

Again, it is well-known that the Moreau–Yosida regularization of pointwise constraints is given by its quadratic penalization, i.e.,

$$F_\gamma(y) = \frac{1}{2\gamma}\|\max\{0, |y| - \delta\}\|_{L^2}^2.$$

Hence,

$$\begin{cases} S(u_\gamma) - y^\delta - \gamma p_\gamma \in \partial F^*(p_\gamma), \\ \quad -S'(u_\gamma)^* p_\gamma = u_\gamma, \end{cases} \tag{3.5}$$

where now $F^*(v) = \delta\|v\|_{L^1}$.

3.2.1 Algorithm

In this case, the proximal mapping of F^* is given by

$$[\mathrm{prox}_{\sigma F^*}(v)](x) = (|v(x)| - \delta\sigma)^+ \operatorname{sign}(v(x)).$$

For the Moreau–Yosida regularization F_γ^*, we obtain after some simplification

$$[\mathrm{prox}_{\sigma F_\gamma^*}(v)](x) = \frac{1}{1 + \sigma\gamma}(|v(x)| - \delta\sigma)^+ \operatorname{sign}(v(x)).$$

Again, we use the acceleration scheme (2.11) for $\bar\gamma < c_G = 1$. The full algorithm thus consists in performing for $i = 1, \dots, N$ the steps

$$(3.6)\qquad \begin{cases} z^{i+1} = S'(u^i)^* v^i, \\ u^{i+1} = \frac{1}{1+\tau_i}(u^i - \tau_i z^{i+1}), \\ \omega_i = 1/\sqrt{1+2\bar\gamma\tau_i}, \quad \tau_{i+1} = \omega_i \tau_i, \quad \sigma_{i+1} = \sigma_i/\omega_i, \\ \bar u^{i+1} = u^{i+1} + \omega_i(u^{i+1} - u^i), \\ v^{i+1} = \frac{1}{1+\sigma_i\gamma}(|r^{i+1}| - \delta\sigma_i)^+ \operatorname{sign}(r^{i+1}). \end{cases}$$

3.2.2 Metric regularity

Convergence of algorithm (3.6) again rests on the pointwise analysis from [8] which we summarize below.

Lemma 3.4 ([8, Lem. 2.12]). *Let $f^* : \mathbb{R} \to \overline{\mathbb{R}}$, $f^*(z) = |z|$. Then*

$$(3.7)\qquad D(\partial f^*)(z|\zeta)(\Delta z) = \begin{cases} \{0\} & z \neq 0,\ \zeta = \operatorname{sign} z, \\ \{0\}, & z = 0,\ \Delta z \in (0,\infty)\zeta, \\ (-\infty, 0]\zeta, & z = 0,\ \Delta z = 0,\ |\zeta| = 1, \\ \mathbb{R}, & z = 0,\ \Delta z = 0,\ |\zeta| < 1, \\ \emptyset, & \text{otherwise,} \end{cases}$$

as well as

$$\widetilde{D(\partial f^*)}(z|\zeta)(\Delta z) = \begin{cases} \{0\} & z \neq 0,\ \zeta = \operatorname{sign} z, \\ (-\infty, 0]\zeta, & z = 0,\ \Delta z \in [0,\infty)\zeta,\ |\zeta| = 1, \\ \mathbb{R}, & z = 0,\ \Delta z = 0,\ |\zeta| < 1, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Corollary 3.5 ([8, Cor. 2.13]). *Let $f^*(z) := \delta|z|$ and*

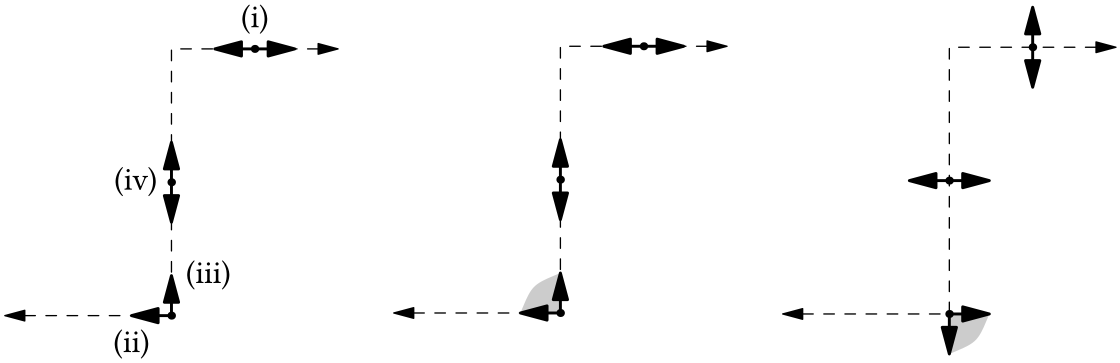
$$F^*(v) := \int_\Omega f^*(v(x))\,dx \qquad (v \in L^2(\Omega)).$$

Then

$$\widetilde{D[\partial F^*]}(v|\eta)(\Delta v) = \begin{cases} V_{\partial F^*}(v|\eta)^\circ, & \Delta v \in V_{\partial F^*}(v|\eta) \text{ and } \eta \in \partial F^*(v), \\ \emptyset, & \text{otherwise,} \end{cases}$$

and

$$\widehat{D}^*[\partial F^*](v|\eta)(\Delta\eta) = \begin{cases} V_{\partial F^*}(v|\eta)^\circ, & -\Delta\eta \in V_{\partial F^*}(v|\eta) \text{ and } \eta \in \partial F^*(v), \\ \emptyset, & \text{otherwise,} \end{cases}$$



(a) $D(\partial f^*)(z|\zeta)$ (b) $\widetilde{D(\partial f^*)}(z|\zeta)$ (c) $\widehat{D}^*(\partial f^*)(z|\zeta)$

Figure 2: Illustration of the graphical derivative and regular coderivative for ∂f with $f = |\cdot|$. The dashed line is Graph ∂f. The dots indicate the base points (z, ζ) where the graphical derivative or coderivative is calculated, and the thick arrows and gray areas indicate the directions of $(\Delta z, \Delta\zeta)$ relative to the base point. The labels (i) etc. denote the corresponding case of (3.7). (Taken from [8].)

for the cone

$$V_{\partial F^*}(v|\eta) = \{z \in L^2(\Omega) \mid z(x)\eta(x) \geq 0 \text{ if } v(x) = 0 \text{ and } (\delta - |\eta(x)|)z(x) = 0\},$$

and its polar

$$V_{\partial F^*}(v|\eta)^\circ = \{\nu \in L^2(\Omega) \mid \nu(x)\eta(x) \leq 0 \text{ if } |\eta(x)| = \delta \text{ and } v(x)\nu(x) = 0\}.$$

Remark 3.2. *If (v, η) satisfy the strict complementarity condition $v(x) \neq 0$ or $|\eta(x)| < \delta$ for a. e. $x \in \Omega$, the degenerate second and third case in (3.7) (corresponding to the gray areas in Figure 2) do not occur, and the cone simplifies to*

$$V_{\partial F^*}(v|\eta) := \{z \in L^2(\Omega) \mid z(x) = 0 \text{ if } v(x) = 0,\ x \in \Omega\}.$$

As before, we deduce that (2.12) holds for F_γ^*, while the discussion in [8, § 5.2] shows that metric regularity only holds for $\gamma > 0$ (or finite-dimensional data). Summarizing, we similarly have the following convergence result.

Corollary 3.6. *Let $\gamma > 0$ and $\bar\gamma \in [0, 1)$ be arbitrary (setting $\bar\gamma = 0$ after a finite number of iterations). Furthermore, let $(u_\gamma, p_\gamma) \in L^2(\Omega)^2$ be a solution to (3.5), and take $\tau_0, \sigma_0 > 0$ to satisfy (2.1b) for $K(u) = S(u) - y^\delta$. Then there exists $\delta > 0$ such that for any initial iterate $(u^1, p^1) \in L^2(\Omega)^2$ with $\|(u^1, p^1) - (u_\gamma, p_\gamma)\| \leq \delta$, the iterates (u^k, p^k) generated by algorithm (3.6) converge to a solution (u^*, p^*) to (3.5).*

3.3 State constraints

Finally, we address the state-constrained optimal control problem (1.4), which we again rewrite as

$$\min_{u\in L^2} \frac{1}{2\alpha}\|S(u)-y^d\|_{L^2}^2 + \frac{1}{2}\|u\|_{L^2}^2 \quad \text{s.t.} \quad S(u)(x)\le c \quad \text{a.e. in } \Omega.$$

In this case, G is as before and $F(y)=\frac{1}{2\alpha}\|v-y^d\|_{L^2}^2+\iota_{v\le c}(y)$ with $K(u)=S(u)$. For simplicity, we assume here that the upper bound c is constant; the extension to variable $c\in L^\infty(\Omega)$ (as well as lower bounds) is straightforward.

For F_γ, we directly use the definition (1.8) to compute pointwise

$$f_\gamma(x,v)=\begin{cases}\frac{1}{2\alpha}|c-y^d(x)|^2+\frac{1}{2\gamma}|v-c|^2 & \text{if } v>(1+\frac{\alpha}{\gamma})c-\frac{\alpha}{\gamma}y^d(x),\\ \frac{1}{2(\alpha+\gamma)}|v-y^d(x)|^2 & \text{if } v\le(1+\frac{\alpha}{\gamma})c-\frac{\alpha}{\gamma}y^d(x),\end{cases}$$

and obtain

$$F_\gamma(y)=\int_\Omega f_\gamma(x,y(x))\,dx.$$

The corresponding regularized optimality conditions are again given by

$$\begin{cases} S(u_\gamma)-y^\delta-\gamma p_\gamma\in\partial F^*(p_\gamma),\\ -S'(u_\gamma)^*p_\gamma=u_\gamma.\end{cases} \tag{3.8}$$

It remains to compute F^*. Since $y^d\in L^2(\Omega)$ is measurable,

$$f(x,v)=\frac{1}{2\alpha}|v-y^d(x)|^2+\iota_{(-\infty,c]}(v)$$

is a proper, convex, and normal integrand, and hence we can proceed by pointwise computation.

Let $x\in\Omega$ be arbitrary. For the Fenchel conjugate with respect to y,

$$f^*(x,z)=\sup_{v\le c} vz-\frac{1}{2\alpha}|v-y^d(x)|^2,$$

we consider the first-order necessary conditions for the maximizer

$$\bar v=\operatorname{proj}_{(-\infty,c]}\left(\alpha z+y^d(x)\right).$$

Inserting this into the definition and making the case distinction $\alpha v+y^d(x)\le c$ yields

$$f^*(x,z)=\begin{cases} cz-\frac{1}{2\alpha}|c-y^d(x)|^2 & z>\alpha^{-1}(c-y^d(x)),\\ \frac{\alpha}{2}|z|^2+zy^d(x) & z\le\alpha^{-1}(c-y^d(x)).\end{cases}$$

The subdifferential (with respect to z) is given by

$$\partial f^*(x,z)=\begin{cases}\{c\} & z>\alpha^{-1}(c-y^d(x)),\\ \{\alpha z+y^d(x)\} & z\le\alpha^{-1}(c-y^d(x)).\end{cases} \tag{3.9}$$

Note that the cases agree for $z=\alpha c-y^d(x)$, i.e., $z\mapsto\partial f^*(x,z)$ is single-valued and hence $z\mapsto f^*(x,z)$ is continuously differentiable for almost every $x\in\Omega$.

3.3.1 Algorithm

To derive an explicit algorithm, we still need to compute the pointwise proximal mapping $\operatorname{prox}_{\sigma f^*(x,\cdot)}(v)$ for given $x \in \Omega$. Here we use the resolvent formula

$$\operatorname{prox}_{\sigma f^*(x,\cdot)}(v) = (\operatorname{Id} + \sigma\partial f^*(x,\cdot))^{-1}(v) =: w,$$

i.e., $v \in \{w\} + \sigma\partial f^*(x,w)$, together with the case distinction to obtain

(i) $v = w + \sigma c$, i.e., $w = v - \sigma c$, if $w > \alpha^{-1}(c - y^d(x))$, i.e., if $v > \alpha^{-1}(c - y^d(x)) + \sigma c$.

(ii) $v = w + \sigma(\alpha w + y^d(x))$, i.e., $w = (1+\sigma\alpha)^{-1}(v - \sigma y^d(x))$, if $w \le \alpha^{-1}(c - y^d(x))$, i.e., if

$$v \le \frac{1+\sigma\alpha}{\alpha}(c - y^d(x)) + \sigma z = \alpha^{-1}(c - y^d(x)) + \sigma c.$$

Together we obtain

$$[\operatorname{prox}_{\sigma F^*}(v)](x) = \begin{cases} v(x) - \sigma c & v(x) > \frac{1}{\alpha}(c - y^d(x)) + \sigma c, \\ (1+\sigma\alpha)^{-1}(v(x) - \sigma y^d(x)) & v(x) \le \frac{1}{\alpha}(c - y^d(x)) + \sigma c. \end{cases}$$

For the Moreau–Yosida regularization $f^*_\gamma(x,v) = f^*(x,v) + \frac{\gamma}{2}|v|^2$, we similarly obtain

$$[\operatorname{prox}_{\sigma F^*_\gamma}(v)](x) = \begin{cases} (1+\sigma\gamma)^{-1}(v(x) - \sigma c) & v(x) > \frac{1+\sigma\gamma}{\alpha}(c - y^d(x)) + \sigma c, \\ (1+\sigma(\alpha+\gamma))^{-1}(v(x) - \sigma y^d(x)) & v(x) \le \frac{1+\sigma\gamma}{\alpha}(c - y^d(x)) + \sigma c. \end{cases}$$

Again, we use the acceleration scheme (2.11) for $\bar\gamma < c_G = 1$. The full algorithm thus consists in performing for $i = 1, \dots, N$ the steps

$$\tag{3.10} \begin{cases} u^{i+1} = \frac{1}{1+\tau_i}(u^i - \tau_i S'(u^i)^* v^i), \\ \omega_i = 1/\sqrt{1+2\bar\gamma\tau^i}, \quad \tau^{i+1} = \omega_i\tau^i, \quad \sigma_{i+1} = \sigma_i/\omega_i, \\ \bar u^{i+1} = u^{i+1} + \omega_i(u^{i+1} - u^i), \\ r^{i+1} = v^i + \sigma_{i+1}(S(\bar u^{i+1}) - y^\delta), \\ \chi^{i+1} = \left[\!\left[r^{i+1} > \frac{1+\sigma_{i+1}\gamma}{\alpha}(c - y^d) + \sigma_{i+1}c \right]\!\right], \\ v^{i+1} = \frac{1}{1+\sigma_{i+1}\gamma}\chi^{i+1}\left(r^{i+1} - \sigma_{i+1}c\right) + \frac{1}{1+\sigma_{i+1}(\alpha+\gamma)}(1-\chi^{i+1})\left(r^{i+1} - \sigma_{i+1}y^d\right), \end{cases}$$

where $[\![P]\!]$ for a logical proposition depending on x denotes the pointwise *Iverson bracket*, i.e., $[\![P]\!](x) = 1$ if $P(x)$ is true and 0 else.

3.3.2 Metric regularity

The verification of the Aubin property rests on the following explicit characterization of the regular coderivative of $\partial f^*(x,\cdot)$, where we suppress the dependence on x for the sake of presentation.

Lemma 3.7. *For f^* as in (3.9), we have*

$$D(\partial f^*)(v|\zeta)(\Delta v) = \begin{cases} 0, & \alpha v > c - y^d,\ \zeta = c, \\ \alpha\Delta v, & \alpha v < c - y^d,\ \zeta = \alpha v + y^d, \\ 0, & \alpha v = c - y^d,\ \zeta = c,\ \Delta v \geq 0, \\ \alpha\Delta v, & \alpha v = c - y^d,\ \zeta = c,\ \Delta v < 0, \end{cases} \tag{3.11}$$

and

$$\widetilde{D(\partial f^*)}(v|\zeta)(\Delta v) = \begin{cases} 0, & \alpha v > c - y^d,\ \zeta = c, \\ \alpha\Delta v, & \alpha v < c - y^d,\ \zeta = \alpha v + y^d, \\ (-\infty, 0], & \alpha v = c - y^d,\ \zeta = c,\ \Delta v \geq 0, \\ \alpha\Delta v + (-\infty, 0], & \alpha v = c - y^d,\ \zeta = c,\ \Delta v < 0. \end{cases} \tag{3.12}$$

Proof. The claim is best seen by inspecting Figure 3. For completeness we however sketch the (somewhat tedious) proof based on casewise inspection of (3.9).

(i) If $\alpha v \neq c - y^d$, we have $\partial f^*(v) = \{(f^*)'(v)\}$ with $(f^*)'(v)$ differentiable. Computing these differentials yields the first two cases of (3.11), where the constraints on ζ come from $\zeta = (f^*)'(v)$.

(ii) If $\alpha v = c - y^d$, we have $\partial f^*(v) = \{c\}$, so we need $\zeta = c$. Approaching v with $v^i = v + t^i\Delta v$ with $\Delta v \geq 0$ and $t^i \searrow 0$, we have

$$\limsup_{i\to\infty} \frac{\partial f^*(v^i) - \zeta}{t^i} = \limsup_{i\to\infty} \frac{c - c}{t^i} = \{0\}.$$

This gives the third case of (3.11).

(iii) If $\Delta v < 0$, we obtain

$$\limsup_{i\to\infty} \frac{\partial f^*(v^i) - \zeta}{t^i} = \limsup_{i\to\infty} \frac{\alpha(v + t^i\Delta v) + y^d - c}{t^i} \limsup_{i\to\infty} \frac{\alpha t^i \Delta v}{t^i} = \{\alpha\Delta v\}.$$

This gives the fourth case of (3.11).

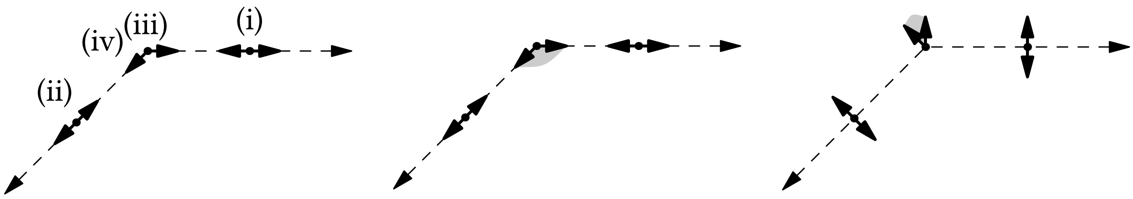
Finally, the first two cases of the convexification (3.12) correspond directly to those of (3.11), while the last two cases come from taking the convex hull of the set

$$A := ([0,\infty) \times \{0\}) \cup \{(\Delta v, \alpha\Delta v) \mid \Delta v < 0\},$$

corresponding to the last two cases of (3.11), which is given by

$$\operatorname{conv} A = ([0,\infty) \times (-\infty, 0]) \cup \{\{\Delta v\} \times (-\infty, \alpha\Delta v] \mid \Delta v < 0\}. \qquad \square$$

Since f is proper, convex, and normal, so is f^*; see, e.g., [17, Thm. 14.50] for the former. Furthermore, for almost every $x \in \Omega$, the functional $f^*(x,\cdot)$ is piecewise affine, and hence $\partial f^*(x,\cdot)$ is proto-differentiable; see [17, Prop. 13.9, Thm. 13.40]. We can thus apply [8, Cor. 2.7] to obtain the following pointwise characterization of the second-order generalized derivatives of F^*.



(a) $D(\partial f^*)(z|\zeta)$ (b) $\widetilde{D(\partial f^*)}(z|\zeta)$ (c) $\widehat{D}^*(\partial f^*)(z|\zeta)$

Figure 3: Illustration of the graphical derivative and Fréchet coderivative for ∂f^* with f^* as in (3.9). The dashed line is Graph ∂f. The dots indicate the base points (z, ζ) where the graphical derivative or coderivative is calculated, and the thick arrows and gray areas indicate the directions of $(\Delta z, \Delta\zeta)$ relative to the base point. The labels (i) etc. denote the corresponding case of (3.11).

Corollary 3.8. *Let f^* be as in (3.9), and*

$$F^*(v) := \int_\Omega f^*(v(x))\,dx \qquad (v \in L^2(\Omega)).$$

Suppose $\alpha v(x) \neq c - y^d(x)$ for a.e. $x \in \Omega$. Then

$$\widetilde{D[\partial F^*]}(v|\eta)(\Delta v) = \begin{cases} T_{F^*,v}\Delta v + V_{\partial F^*}(v|\eta)^\circ, & \Delta v \in V_{\partial F^*}(v|\eta) \text{ and } \eta \in \partial F^*(v), \\ \emptyset, & \text{otherwise}, \end{cases} \tag{3.13}$$

and

$$\widehat{D}^*[\partial F^*](v|\eta)(\Delta\eta) = \begin{cases} T^*_{F^*,v}\Delta\eta + V_{\partial F^*}(v|\eta)^\circ, & -\Delta\eta \in V_{\partial F^*}(v|\eta) \text{ and } \eta \in \partial F^*(v), \\ \emptyset, & \text{otherwise}, \end{cases} \tag{3.14}$$

for the cone

$$V_{\partial F^*}(v|\eta) = L^2(\Omega),$$

its polar

$$V_{\partial F^*}(v|\eta)^\circ = \{0\} \subset L^2(\Omega),$$

and the linear operator $T_{F^*,v}$ defined by

$$[T_{F^*,v}\Delta v](x) := t_v(x)\Delta v(x), \quad t_v(x) := \begin{cases} 0, & \alpha v(x) > c - y^d(x), \\ \alpha, & \alpha v(x) < c - y^d(x). \end{cases}$$

Remark 3.3. *We have excluded $\alpha v(x) = c - y^d(x)$ – which amounts to a strict complementarity assumption for v – because the calculations of [8] only apply when the polarity relationships in (3.13) and (3.14) regarding V hold. We have verified that the calculations could be improved to handle this non-strictly complementary case. However, since non-strictly complementary solutions can be replaced by strictly complementary solutions by infinitesimal modifications of v, we have decided for conciness to simply exclude the case.*

Let us assume that strict complementarity holds, i.e., $\alpha v(x) \neq c - y^d(x)$ for a.e. $x \in \Omega$. Then $t_v(x) \in \{0, \alpha\}$ for a.e. $x \in \Omega$. Since $V_{\partial F^*}(v|\eta) = L^2(\Omega)$ and $V_{\partial F^*}(v|\eta)^\circ = \{0\}$, we deduce

$$\bar{b}(\widehat{q}|0; H_{\widehat{u}}) = \sup_{t>0} \inf \left\{ \frac{\|S'(\widehat{u})S'(\widehat{u})^* z\|}{\|z\|} \,\middle|\, 0 \neq z \in L^2(\Omega) \right\}.$$

However, the lower bound

$$\|S'(\widehat{u})^* z\| \geq c\|z\| \qquad (z \in L^2(E))$$

does not hold in general. This can be seen by taking any orthonormal basis of $L^2(E)$, which converges weakly but not strongly to zero, and use the fact that $S'(u)$ is a compact operator from $L^2(\Omega)$ to $L^2(\Omega)$ due to the Rellich–Kondrachev embedding theorem. Therefore, also $\bar{b}(\widehat{q}|0; H_{\widehat{u}}) = 0$. By Proposition 2.7, there is thus no metric regularity without regularization ($\gamma > 0$). Similarly to L^1 fitting, if the state constraints are only prescribed at a finite number of points, it is possible to show metric regularity for $\gamma = 0$ as well.

The next corollary, which follows similarly to Corollary 3.3, summarizes the convergence results Theorem 2.1 and Theorem 2.4 for the present state-constrained problem.

Corollary 3.9. *Let $\gamma > 0$ and $\bar{\gamma} \in [0, 1)$ be arbitrary (setting $\bar{\gamma} = 0$ after a finite number of iterations). Furthermore, let $(u_\gamma, p_\gamma) \in L^2(\Omega)^2$ be a solution to (3.8), and take $\tau_0, \sigma_0 > 0$ to satisfy (2.1b) for $K(u) = S(u) - y^\delta$. Then there exists $\delta > 0$ such that for any initial iterate $(u^1, p^1) \in L^2(\Omega)^2$ with $\|(u^1, p^1) - (u_\gamma, p_\gamma)\| \leq \delta$, the iterates (u^k, p^k) generated by algorithm (3.10) converge to a solution (u^*, p^*) to (3.8).*

4 Numerical results

We now illustrate the convergence behavior of the primal-dual extragradient method for the three model problems in section 3. In each case, the operator S corresponds to the solution of (3.1) for $\Omega = [-1, 1]$ and constant right-hand side $f \equiv 1$. For the implementation, we use a finite element approximation of (3.1) on a uniform grid with (unless stated otherwise) $n = 1000$ elements with a piecewise constant discretization of u and a piecewise linear discretization of y as in [7]. The functional values

$$J_\gamma(u^i) = F_\gamma(K(u^i)) + G(u^i)$$

are are computed using an approximation of the integrals by mass lumping, which amounts to a proper scaling of the corresponding discrete sums. In this way, the functional values are independent of the mesh size.

The parameters in the primal-dual extragradient method are chosen as follows: The Moreau–Yosida parameter is fixed at $\gamma = 10^{-12}$, and we compare the two cases of $\bar{\gamma} = 0$ (no acceleration) and $\bar{\gamma} = 1 - 10^{-16}$ (full acceleration). As a starting value, we take in each case $u^1 \equiv 1$ and $p^1 \equiv 0$. The (initial) step sizes are set to $\sigma_1 = \tilde{L}^{-1}$ and $\tau_1 = 0.99\tilde{L}^{-1}$, where $\tilde{L} = \max\{1, \|S''(u^1)u^1\|/\|u^1\|\}$ is a very simple estimate of the Lipschitz constant of $K' = S'$. The algorithm (and the acceleration) is terminated after a prescribed number N of iterations. The MATLAB implementation used to generate these results can be downloaded from https://github.com/clason/nlpdegm.

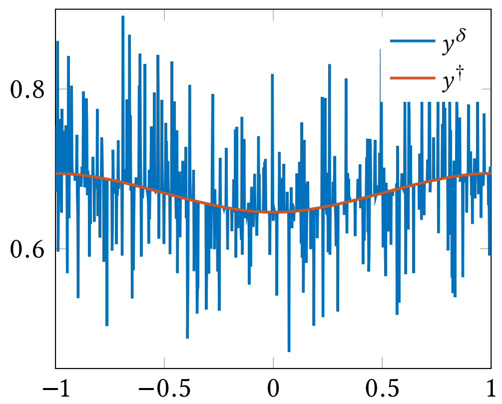


Figure 4: L^1 fitting: noisy and exact data

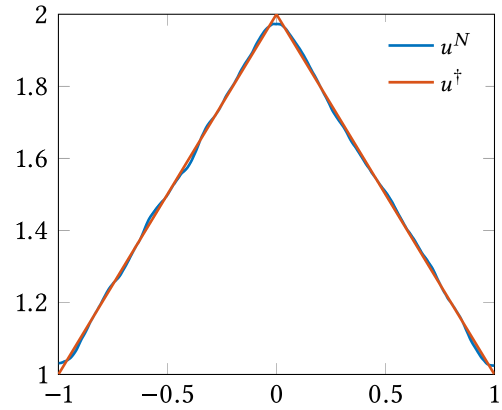


Figure 5: L^1 fitting: reconstruction and true solution

4.1 L^1 fitting

We first consider the L^1 fitting problem (1.2) using the example from [7]: We choose the exact parameter $u^\dagger(x) = 2 - |x|$ and corresponding exact data $y^\dagger = S(u^\dagger)$ and add random-valued impulsive noise by setting

$$y^\delta(x) = \begin{cases} y^\dagger(x) + \|y^\dagger\|\xi(x) & \text{with probability } r, \\ y^\dagger(x) & \text{with probability } 1 - r, \end{cases}$$

where for each $x \in \Omega$, $\xi(x)$ is an independent normally distributed random value with mean 0 and variance d^2. For the results shown, we take $r = 0.3$ and $d = 0.1$, i.e., 30% of data points are corrupted by at least 10% noise. Figure 4 shows a typical realization. We then apply algorithm 3.3 with $N = 1000$ iterations and $\alpha = 10^{-2}$ fixed; the final iterate u^N (with $\bar\gamma \approx 1$) is shown in Figure 5 together with $u^\dagger$.

Figure 6 compares the convergence behavior of the functional values with $\bar\gamma = 0$ and $\bar\gamma \approx 1$ (for the same data y^δ). The effect of acceleration can be seen clearly. Note that the convergence is nonmonotone due to the acceleration (and the rather aggressive choice of step lengths). Note also that due to the compactness of the forward operator S, the functional value changes very little over most of the iteration even though there are still significant changes in the iterates u^i.

The convergence behavior for different mesh sizes is illustrated in Figure 7, which shows the functional values for $n \in \{100, 1000, 1000\}$ (as averages over 10 different realizations of y^δ in order to mitigate the influence of the random data). As can be observed, the number of iterations to reach a given functional value is virtually independent of the mesh size. This property – shared by many function-space algorithms – is often referred to as *mesh independence*.

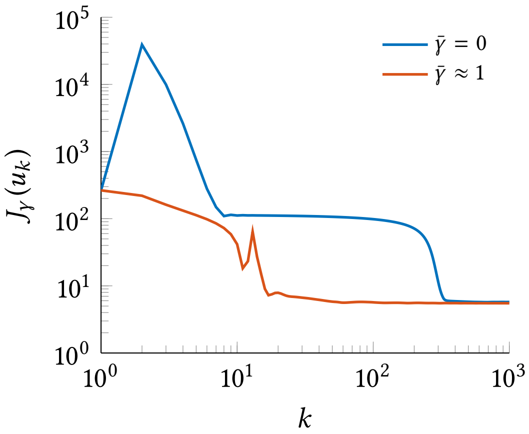


Figure 6: L^1 fitting: convergence without ($\bar{\gamma} = 0$) and with ($\bar{\gamma} \approx 1$) acceleration

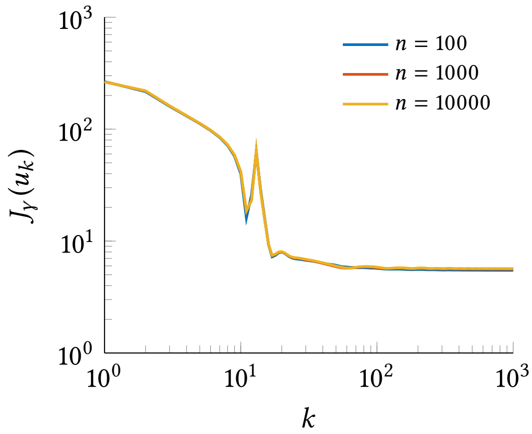


Figure 7: L^1 fitting: convergence for different mesh sizes n (average of 10 realizations)

4.2 L^∞ fitting

For the L^∞-fitting problem (1.3), we choose a test problem from [6], where y^δ is obtained from $y^\dagger = S(u^\dagger)$ (with $u^\dagger$ as above) by quantization. Specifically, we set

$$y^\delta(x) = y_s \left[\frac{y^\dagger(x)}{y_s}\right], \qquad y_s = n_b^{-1}\left(\sup_{x\in\overline{\Omega}}\left(y^\dagger(x)\right) - \inf_{x\in\overline{\Omega}}\left(y^\dagger(x)\right)\right),$$

where n_b denotes the number of bins and $[s]$ denoting the nearest integer to $s \in \mathbb{R}$ (i.e., the data are rounded to n_b discrete equidistant values). Here we take $n_b = 11$; see Figure 8. Applying algorithm 3.6 for $N = 10000$ iterations (with full acceleration) yields the reconstruction u^N shown in Figure 9.

Again, Figure 10 compares the functional values over the iteration without and with acceleration and demonstrates the significantly better performance of the latter. Similarly, the comparison of different mesh sizes in Figure 11 illustrates the mesh independence of the algorithm (with slightly faster convergence for $n = 100$, which can be explained by the effect of coarse discretization on the rounding procedure).

4.3 State constraints

Finally, we consider the state-constrained optimal control problem (1.4). Here, we choose the desired state $y^d = S(u^\dagger)$ (with $u^\dagger$ again as before) and the constraint $c = 0.68$. The control costs are set to $\alpha = 10^{-12}$. Figure 12 shows target and constraint together with the state $y^N = S(u^N)$ reached after $N = 10000$ (accelerated) iterations; the corresponding control u^N is shown in Figure 13.

As before, Figure 14 and Figure 15 illustrate the benefit of acceleration and the mesh independence of the algorithm, respectively.

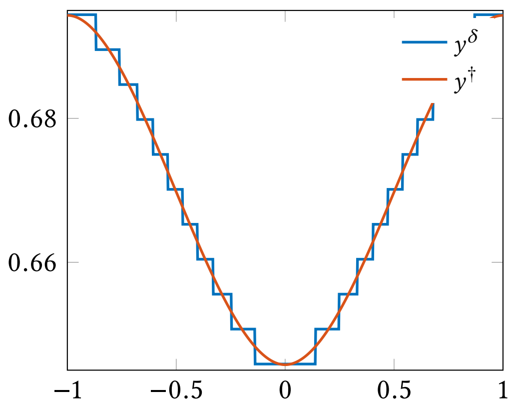


Figure 8: L^∞ fitting: noisy and exact data

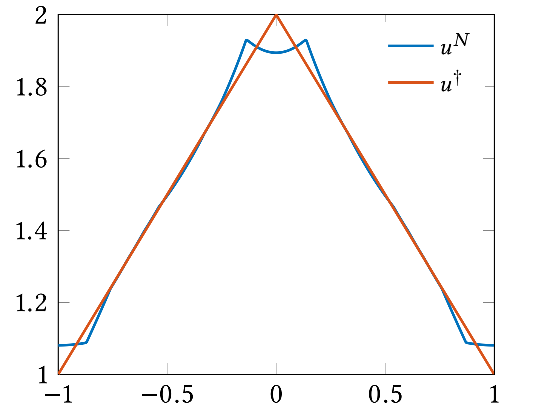


Figure 9: L^∞ fitting: reconstruction and true solution

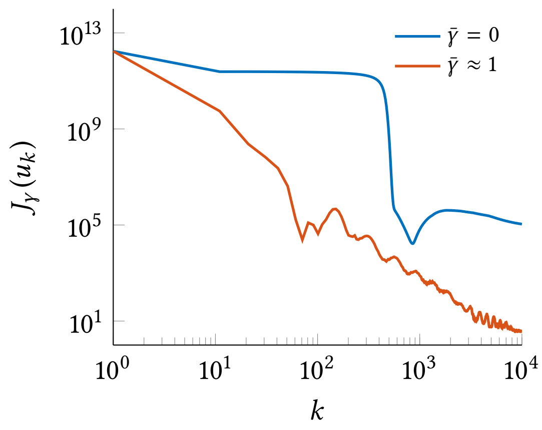


Figure 10: L^∞ fitting: convergence without ($\bar\gamma = 0$) and with ($\bar\gamma \approx 1$) acceleration

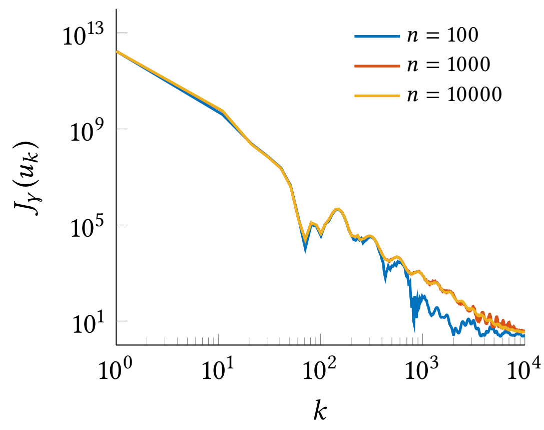


Figure 11: L^∞ fitting: convergence for different mesh sizes n

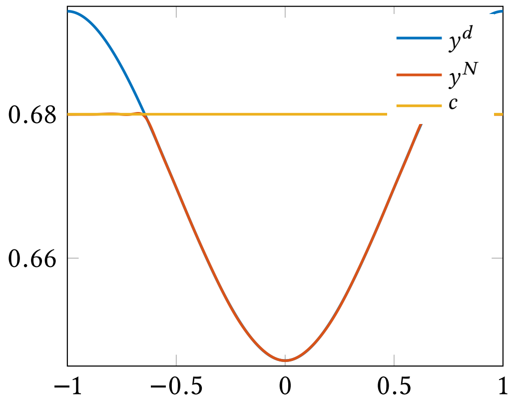


Figure 12: State constraints: target, final state, constraint

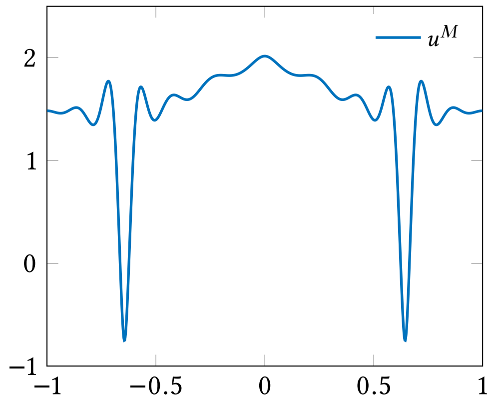


Figure 13: State constraints: final control

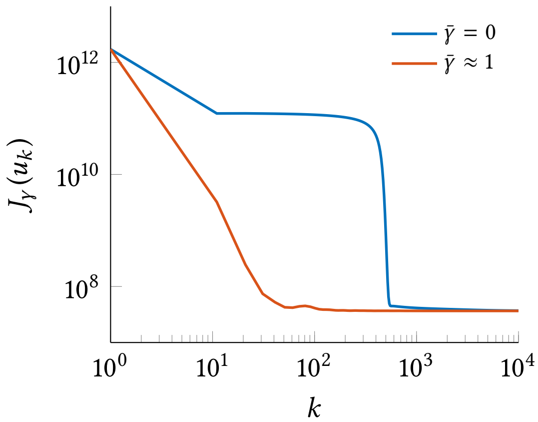


Figure 14: State constraints: convergence without ($\bar{\gamma} = 0$) and with ($\bar{\gamma} \approx 1$) acceleration

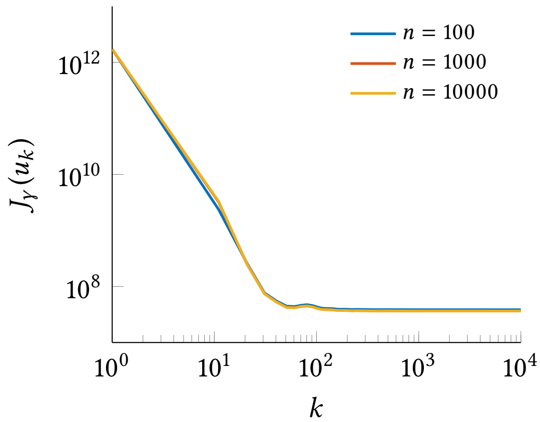


Figure 15: State constraints: convergence for different mesh sizes n

5 Conclusion

Accelerated primal-dual extragradient methods with nonlinear operators can be formulated and analyzed in function space. Their convergence rests on metric regularity of the corresponding saddle-point inclusion, which can be verified for the class of PDE-constrained optimization problems considered here after introducing a Moreau–Yosida regularization. Unlike semismooth Newton methods (which also require Moreau–Yosida regularization in function space, cf., e.g., [7, 6]), however, in practice it is not necessary for convergence to choose γ sufficiently large. Hence, no continuation or warm starts are required. In addition, formulating and analyzing the algorithm in function space leads to mesh independence. These properties are observed in our numerical examples.

This work can be extended in a number of directions. We plan to investigate the possibility of obtaining convergence estimates on the primal variable alone under lesser assumptions. An alternative would be to exploit the uniform stability with respect to regularization for fixed discretization, and with respect to discretization for fixed regularization, to obtain a combined convergence for a suitably chosen net $(\gamma, h) \to (0, 0)$. This is related to the adaptive regularization and discretization of inverse problems [14]. Furthermore, it would be of interest to extend our analysis to include nonsmooth regularizers G, which were excluded in the current work for the sake of the presentation.

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A data statement for the EPSRC

To be filled when the code is archived.

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