On problems by Baer and Kulikov using $V = L$

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Abstract. Let T be a torsion abelian group and λ a cardinal. Among all torsion-free abelian groups H of rank less than or equal to λ satisfying $\text{Ext}(H,T) = 0$ a group G is called λ-universal for T if it is universal with respect to group-embedding. We show that in Gödel's constructible universe ($V = L$) there always exists a λ-universal group for T if T has only finitely many non-trivial bounded p-components. This answers a question by Kulikov in the affirmative. Moreover, we prove that in $V = L$ for a large class of torsion-free abelian groups G there exists a completely decomposable group C such that $\text{Ext}(G,T') = 0$ if and only if $\text{Ext}(C,T') = 0$ for any torsion abelian group T'. This is related to a question by Baer.

Introduction

In 1936 R. Baer [**B**] asked to characterize all pairs of torsion-free abelian groups G and torsion abelian groups T satisfying $\text{Ext}(G,T) = 0$. This is just a simpler way to ask for pairs G and T such that any mixed abelian group M with torsion subgroup T and torsion-free quotient $M/T \cong G$ has to split, i.e. $M \cong T \oplus G$ in the natural way. Baer [**B**] himself gave a characterization for countable G and the first time this question was considered again was by Wallutis and the author in [**SW**]. In the framework of cotorsion theories singly cogenerated by a torsion-free abelian group G (as they were introduced by Salce in [**S**]) Wallutis and the author introduced the class $\mathcal{TC}(G)$ as the class of all torsion abelian groups T satisfying $\text{Ext}(G,T) = 0$. The characterization of the class $\mathcal{TC}(G)$ for torsion-free abelian G is obviously closely related to Griffith's solution of the Baer problem from [**G**]. In this terminology the result from [**G**] reads as follows: A torsion-free abelian group G is free if and only if $\mathcal{TC}(G)$ is the class of all torsion groups, i.e. G is free if and only if every mixed abelian group M with $M/t(M) \cong G$ splits. For rational groups $R \subseteq \mathbb{Q}$ and, hence for completely decomposable groups C, a complete description was found for $\mathcal{TC}(R)$ and $\mathcal{TC}(C)$, respectively (see [**SW**]). Moreover, a necessary and sufficient criterion was given when a class of torsion abelian groups $\mathfrak{C}$ can appear as $\mathcal{TC}(C)$ for some completely decomposable group C. A similar result was shown by the author [**St**] replacing completely decomposable by rational group and it was proved that any finite rank torsion-free abelian group G satisfies

The author was supported by a MINERVA fellowship.
2000 Mathematics Subject Classification. Primary 20K15, 20K20, 20K35, 20K40; Secondary 18E99, 20J05.

$\mathcal{TC}(G) = \mathcal{TC}(R)$ for some rational group R. We shall show in section 2 of this paper that the criterion from [**SW**] is satisfied for a large class of torsion-free abelian groups assuming Goedel's universe of constructibility. Hence, for a group G in this class, $\mathcal{TC}(G) = \mathcal{TC}(C)$ for some completely decomposable group C. Since $\mathcal{TC}(C)$ is well-understood this gives a characterization of $\mathcal{TC}(G)$ for torsion-free abelian G in this class. However, we can not show that in $V = L$ every torsion-free abelian group is of this kind and we leave it as the $\mathcal{TC}$-Conjecture that it is in fact the case. In section 3 we consider Kulikov's problem on the existence of λ-universal groups. As defined in the abstract a torsion-free abelian group G is λ-universal for a torsion abelian group T and a cardinal λ if G is of rank less than or equal to λ, $\mathrm{Ext}(G, T) = 0$ and every torsion-free abelian group H of rank less than or equal to λ satisfying $\mathrm{Ext}(H, T) = 0$ embeds into G. Clearly, the existence of a λ-universal group G for T answers the question of which torsion-free H (of cardinality at most λ) satisfy $\mathrm{Ext}(H, T) = 0$: namely, precisely the subgroups of G. Kulikov [**KN**] asked if for uncountable λ and arbitrary T there is always a λ-universal group for T. We first deal with the case λ being a positive integer or ω and obtain satisfying classification results. Moreover, we can show that in $V = L$ for every λ and every torsion abelian group T with only finitely many non-trivial bounded p-components there is always a λ-universal group for T.

Our notations are standard and for unexplained notions we refer to [**F**] or [**EM**]. All groups under consideration are abelian, the set of primes is denoted by Π and all rational groups $R \subseteq \mathbb{Q}$ are assumed to contain the 1. Moreover, we identify rational groups with their types since it does not cause any confusion. However, if S and R are rational groups we write $S \subseteq R$ if we mean set inclusion and $S \leq R$ if we mean inequality as types.

1. Preliminaries

Let us first recall a definition from [**SW**].

Definition 1.1. *Let G be a (torsion-free) group. By $\mathcal{TC}(G)$ we denote the class of all torsion groups T such that* $\mathrm{Ext}(G, T) = 0$.

It is easy to see that $\mathcal{TC}(G)$ is closed under taking epimorphic images and contains all finite groups. Moreover, a torsion group T is in $\mathcal{TC}(G)$ if and only if its reduced part is in $\mathcal{TC}(G)$.
For our investigations we shall need the following lemma which is well known. However, we include the proof for the convenience of the reader.
First recall that a basic subgroup B of a torsion group T is the direct sum $B = \bigoplus_{p\in\Pi} B_p$ of basic subgroups B_p of the p–components T_p; for each prime p, B_p is a direct sum of cyclic p–groups, B_p is a pure subgroup of T_p and the quotient T_p/B_p is divisible (see [**F**]).

Lemma 1.2. *Let T be a torsion group and $B \subseteq T$ a basic subgroup of T. Then, for any group G, T is an element of $\mathcal{TC}(G)$ if and only if B is.*

Proof. The short exact sequence $0 \to B \to T \to T/B \to 0$ induces the exact sequence $\mathrm{Ext}(G, B) \to \mathrm{Ext}(G, T) \to \mathrm{Ext}(G, T/B) = 0$ where the last term is zero since T/B is divisible. Thus $\mathrm{Ext}(G, B) = 0$ implies $\mathrm{Ext}(G, T) = 0$, i.e. if $B \in \mathcal{TC}(G)$ then $T \in \mathcal{TC}(G)$.

Conversely, assume $T \in \mathcal{TC}(G)$. By [**F**, Theorem 36.1] B is an epimorphic image of T and thus B also belongs to $\mathcal{TC}(G)$. $\square$

It is well-known that the functor $\mathrm{Ext}(G, -)$ is closed under taking epimorphic images but usually not under taking (pure) subgroups. However, if we restrict to $\mathcal{TC}(G)$ then this holds.

Lemma 1.3. *Let G be any group and T a torsion group. Then $T \in \mathcal{TC}(G)$ if and only if $T' \in \mathcal{TC}(G)$ for all pure subgroups T' of T such that $|T'| \leq |G|$. Moreover, $T' \in \mathcal{TC}(G)$ for all pure subgroups T' of T.*

Proof. Assume that $T \in \mathcal{TC}(G)$ and let T' be a pure subgroup of T. Choose a basic subgroup B' of T', then B' is pure in T and by [**F**, Corollary 36.2] B' is an epimorphic image of T. Thus $B' \in \mathcal{TC}(G)$ and hence $T' \in \mathcal{TC}(G)$ by Lemma 1.2. Conversely, assume that $T \notin \mathcal{TC}(G)$ but $T' \in \mathcal{TC}(G)$ for all pure subgroups T' of T of cardinality less than or equal to $|G|$. Since $T \notin \mathcal{TC}(G)$, T is infinite. Let

$$0 \longrightarrow T \xrightarrow{id_T} M \xrightarrow{\varphi} G \longrightarrow 0 \tag{1.1}$$

be a non-splitting short exact sequence. For $g \in G$ choose $m_g \in \varphi^{-1}(\{g\})$ and put $M' = \langle m_g : g \in G \rangle \subseteq M$. Note that $|M'| = |G|$. By [**F**, Proposition 26.2] there exists a pure subgroup $M^* \subseteq M$ such that $M' \subseteq M^*$ and $|M^*| = |M'| = |G|$. We obtain the short exact sequence

$$0 \longrightarrow T' \xrightarrow{id_{T'}} M^* \xrightarrow{\varphi \restriction_{M^*}} G \longrightarrow 0, \tag{1.2}$$

where $T' = M^* \cap T$. Since T' is pure in T the sequence splits and we obtain $\psi \in \mathrm{Hom}(G, M^*)$ such that $\varphi \restriction_{M^*} \circ \psi = id_G$. Thus also $\varphi \circ \psi = id_G$ and therefore (1.1) splits - a contradiction. $\square$

To conclude this section let us recall some of the basic results which were obtained in [**St**, Theorem 2.5] and [**SW**, Proposition 2.2, Corollary 3.7] and which we shall need in the next sections. For a torsion-free group G we denote by $\mathrm{OT}(G)$ its *outer type* (see [**A**, page 84]).

Proposition 1.4 ([**St**], [**SW**]). *Let G be a torsion-free group and T a reduced torsion group with T_p its p-component (p a prime). Then the following are true:*

- (i) *If $G \subseteq \mathbb{Q}$, then $T \in \mathcal{TC}(G)$ if and only if the following conditions are satisfied, where $r_p = \chi_p^G(1)$:*
  - (a) *T_p is bounded for all p such that $r_p = \infty$;*
  - (b) *$T_p = 0$ for almost all p such that $r_p \neq 0$.*
- (ii) *If G is countable, then there exists a completely decomposable group C such that $\mathcal{TC}(G) = \mathcal{TC}(C)$;*
- (iii) *If G is of finite rank, then $\mathcal{TC}(G) = \mathcal{TC}(\mathrm{OT}(G))$.*

2. Characterization of $\mathcal{TC}(G)$ in $V = L$

In [**B**] R. Baer asked to characterize $\mathcal{TC}(G)$ for all torsion-free groups G. For countable groups Proposition 1.4 gives a satisfying description of $\mathcal{TC}(G)$ since the structure of $\mathcal{TC}(C)$ is fully described for completely decomposable groups C (see [**SW**]). In this section we shall show that assuming $V = L$ a large class of torsion-free groups G satisfy $\mathcal{TC}(G) = \mathcal{TC}(C)$ for some completely decomposable group C. Let us first recall Theorem 3.6 from [**SW**] which characterizes the classes of torsion groups that can appear as $\mathcal{TC}(C)$ for some completely decomposable group C.

Theorem 2.1 ([**SW**]). *Let $\mathfrak{C}$ be a class of torsion groups. Then $\mathfrak{C} = \mathcal{TC}(C)$ for some completely decomposable group C if and only if the following conditions are satisfied:*

(i) *$\mathfrak{C}$ contains all torsion cotorsion groups;*
(ii) *$\mathfrak{C}$ is closed under epimorphic images;*
(iii) *For all primes p, $\bigoplus\limits_{n\in\omega} \mathbb{Z}(p^n) \in \mathfrak{C}$ if and only if $\mathfrak{C}$ contains all p–groups;*
(iv) *If P is an infinite set of primes then, $\bigoplus\limits_{p\in P} \mathbb{Z}(p) \in \mathfrak{C}$ if and only if $\bigoplus\limits_{p\in P} T_p \in \mathfrak{C}$ for all p–groups $T_p \in \mathfrak{C}$;*
(v) *If P is an infinite set of primes such that $\bigoplus\limits_{p\in P} \mathbb{Z}(p) \notin \mathfrak{C}$ then there exists an infinite subset P' of P such that $\bigoplus\limits_{p\in X} \mathbb{Z}(p) \notin \mathfrak{C}$ for all infinite $X \subseteq P'$.*

Note that for countable torsion-free groups G, $\mathcal{TC}(G)$ satisfies all the conditions in Theorem 2.1 by Proposition 1.4. Moreover, it was shown in [**SW**, Corollary 3.9] that under the assumption of $V = L$, Theorem 2.1 (iii) is always satisfied for $\mathcal{TC}(G)$, where G is a torsion-free group.

Lemma 2.2 ($V = L$, [**SW**]). *Let G be a torsion–free group and let p be any prime number. Then the following are equivalent:*

(i) *$\mathcal{TC}(G)$ contains all p–groups;*
(ii) *$\mathcal{TC}(G)$ contains $\bigoplus\limits_{n\in\omega} \mathbb{Z}(p^n)$;*
(iii) *$\mathbb{Z}_{(p)} \otimes G$ is a free $\mathbb{Z}_{(p)}$–module.*

Hence, Theorem 2.1 (iii) *holds for $\mathcal{TC}(G)$.*

The following example of a Shelah group shows that Lemma 2.2 does not hold in ZFC (see [**SW**, Lemma 3.10]).

Example 2.3 (MA + ¬ CH, [**SW**]). *For any prime number p there exists a non–free $\mathbb{Z}_{(p)}$–module G of cardinality $\aleph_1$ such that* $\mathrm{Ext}(G, \bigoplus\limits_{n\in\omega} \mathbb{Z}(p^n)) = 0$.

Next we prove condition Theorem 2.1 (iv) under the assumption of $V = L$. We need a basic lemma from [**ET**, Lemma 1] on the vanishing of Ext.

Lemma 2.4 ([**ET**]). *Let T be a torsion group. Suppose that the torsion-free group G is the union of a continuous ascending chain of subgroups G_α $(\alpha < \lambda)$ such that $T \in \mathcal{TC}(G_0)$ and $T \in \mathcal{TC}(G_{\alpha+1}/G_\alpha)$ for all $\alpha < \lambda$, then $T \in \mathcal{TC}(G)$.*

Using Lemma 1.3 and Theorem XII.1.15 from [**EM**] it is now easy to prove that a torsion-free group G satisfies $\mathrm{Ext}(G, T) = 0$ (T a torsion group) if and only if G is the union of a continuous well-ordered ascending chain $\{G_\alpha : \alpha < \lambda\}$ of subgroups ($G_0 = 0$) such that $|G_\alpha| < |G|$ and $\mathrm{Ext}(G_{\alpha+1}/G_\alpha, T) = 0$ for all $\alpha < \lambda$. But we can do even better using results from [**BFS**, Theorem 3.1].

Proposition 2.5 ($V = L$). *Let G be a torsion-free group of infinite rank and T a torsion group. Then $\mathrm{Ext}(G, T) = 0$ if and only if G is the union of a continuous well-ordered ascending chain $\{G_\alpha : \alpha < \lambda\}$ of subgroups ($G_0 = 0$) such that $G_{\alpha+1}/G_\alpha$ is countable, $|G_\alpha| < |G|$ and $\mathrm{Ext}(G_{\alpha+1}/G_\alpha, T) = 0$ for all $\alpha < \lambda$.*

Proof. The proof of [**BFS**, Theorem 3.1] carries over word by word to our situation. All one has to do is to replace the property of being Whitehead by satisfying $\mathrm{Ext}(G, T) = 0$. □

PROPOSITION 2.6 ($V = L$). *Let G be a torsion-free group and T be a torsion group. If G is of singular cardinality λ, then* $\operatorname{Ext}(G, T) = 0$ *if and only if* $\operatorname{Ext}(H, T) = 0$ *for all $H \subset G$ of smaller cardinality than λ.*

PROOF. See [**E**, Theorem 5.5] or the proof of [**BFS**, Theorem 3.1] using Shelah's Singular Compactness Theorem from [**Sh**]. The notion of freeness is defined in the following way: G is "free" if and only if there exists a chain as in Proposition 2.5. Then it is readily checked that this definition satisfies the assumptions of Shelah's Singular Compactness Theorem. □

Let us remark that Proposition 2.6 does not hold in ZFC as shall be shown in [**SS1**].

THEOREM 2.7 ($V = L$). *Let G be a torsion-free abelian group. If P is an infinite set of primes then, $\bigoplus_{p \in P} \mathbb{Z}(p) \in \mathcal{TC}(G)$ if and only if $\bigoplus_{p \in P} T_p \in \mathcal{TC}(G)$ for all p–groups $T_p \in \mathcal{TC}(G)$. In particular, Theorem 2.1* (iv) *holds for $\mathcal{TC}(G)$.*

PROOF. First note, that one implication is trivial and we only have to prove that $\bigoplus_{p \in P} \mathbb{Z}(p) \in \mathcal{TC}(G)$ implies $\bigoplus_{p \in P} T_p \in \mathcal{TC}(G)$ for all p–groups $T_p \in \mathcal{TC}(G)$. We induct on the cardinality of G. If G is countable, then the claim is true by Proposition 1.4. Hence assume that $\lambda = |G|$ is greater than or equal to $\aleph_1$. If λ is singular, then $\bigoplus_{p \in P} \mathbb{Z}(p) \in \mathcal{TC}(G)$ implies that $\bigoplus_{p \in P} \mathbb{Z}(p) \in \mathcal{TC}(H)$ for all subgroups H of G of smaller cardinality. Moreover, $\mathcal{TC}(G) \subseteq \mathcal{TC}(H)$ and hence the induction hypothesis implies that $\bigoplus_{p \in P} T_p \in \mathcal{TC}(H)$ for all p–groups $T_p \in \mathcal{TC}(G) \subseteq \mathcal{TC}(H)$. Thus Proposition 2.6 shows that $\bigoplus_{p \in P} T_p \in \mathcal{TC}(G)$ for all p–groups $T_p \in \mathcal{TC}(G)$. Finally, assume that λ is regular. Fix $T_p \in \mathcal{TC}(G)$ for $p \in P$ and without loss of generality we may assume that T_p is of cardinality less than or equal to λ by Lemma 1.3. Let $G = \bigcup_{\beta < \lambda} G_\beta$ be an appropriate λ-filtration of G as in Proposition 2.5, i.e. $\bigoplus_{p \in P} \mathbb{Z}(p) \in \mathcal{TC}(G_{\alpha+1}/G_\alpha)$ for all $\alpha < \lambda$. Similarly, choose λ-filtrations $G = \bigcup_{\beta < \lambda} G_{\beta,p}$ of G for each $p \in P$ such that $T_p \in \mathcal{TC}(G_{\alpha+1,p}/G_{\alpha,p})$ for all $\alpha < \lambda$. It is well-known that for each $p \in P$ there is a cub D_p of λ such that $G_\beta = G_{\beta,p}$ for all $\beta \in D_p$. Since $\lambda = cf(\lambda) > \aleph_0$ the intersection $D = \bigcap_{p \in P} D_p$ is still a cub in λ. Thus for $\alpha < \gamma \in D$ we have $\bigoplus_{p \in P} \mathbb{Z}(p) \in \mathcal{TC}(G_\gamma/G_\alpha)$ and $T_p \in \mathcal{TC}(G_\gamma/G_\alpha)$ for all $p \in P$. Therefore, by induction hypothesis, for $\alpha < \gamma \in D$ we obtain $\bigoplus_{p \in P} T_p \in \mathcal{TC}(G_\gamma/G_\alpha)$. Now $\bigoplus_{p \in P} T_p \in \mathcal{TC}(G)$ by [**EM**, Proposition XII.1.14] or Lemma 2.4. □

Note, that Theorem 2.7 already implies that for a large class of torsion-free groups G there exists a completely decomposable group C such that $\mathcal{TC}(G) = \mathcal{TC}(C)$ if we assume $V = L$. In fact, this is true for all torsion-free groups G satisfying Theorem 2.1 (v).

However, we shall consider condition (v) in $V = L$ and we shall prove that the smallest torsion-free group G violating Theorem 2.1 (v) (if it exists) must be of size $\aleph_1$ in $V = L$.

THEOREM 2.8 ($V = L$). *Let G be a torsion-free group and assume that for all groups H of size less than or equal to $\aleph_1$ condition Theorem 2.1* (v) *is satisfied.*

If P is an infinite set of primes such that $\bigoplus_{p\in P}\mathbb{Z}(p)\notin\mathcal{TC}(G)$ then there exists an infinite subset P' of P such that $\bigoplus_{p\in X}\mathbb{Z}(p)\notin\mathcal{TC}(G)$ for all infinite $X\subseteq P'$. Hence, condition Theorem 2.1 (v) *is satisfied for G.*

Proof. Assume that the claim is not true and let G be a counterexample of smallest cardinality. By assumption $|G|=\lambda\geq\aleph_2$ and for all groups H of smaller cardinality than λ Theorem 2.1 (v) holds. If λ is singular, then $\bigoplus_{p\in P}\mathbb{Z}(p)\notin\mathcal{TC}(G)$ implies that $\bigoplus_{p\in P}\mathbb{Z}(p)\notin\mathcal{TC}(H)$ for some subgroup H of G of smaller cardinality by Proposition 2.6. Thus, by assumption, our claim holds for H and therefore also holds for G. Finally, assume that λ is regular. For any infinite subset X of P put $T_X=\bigoplus_{p\in X}\mathbb{Z}(p)$, hence $T_P\notin\mathcal{TC}(G)$. Choose a λ-filtration $G=\bigcup_{\alpha\in\lambda}G_\alpha$ such that $T_P\notin\mathcal{TC}(G_{\alpha+1}/G_\alpha)$ if for some $\beta>\alpha$ $T_P\notin\mathcal{TC}(G_\beta/G_\alpha)$. Let $W=\{P_\alpha:\alpha\in 2^{\aleph_0}\}$ be an enumeration of all infinite subsets of P. Note that $|W|=2^{\aleph_0}=\aleph_1$ since we work in $V=L$. Since G is a counterexample to Theorem 2.1 (v) for every $P_\alpha\in W$ there exists an $X_\alpha\in W$ such that $X_\alpha\subseteq P_\alpha$ and $T_{X_\alpha}\in\mathcal{TC}(G)$. Choose λ-filtrations $G=\bigcup_{\beta<\lambda}G_{\beta,\rho}$ of G for each $P_\rho\in W$ ($\rho<\aleph_1$) as in Proposition 2.5 such that $T_{X_\rho}\in\mathcal{TC}(G_{\alpha+1,\rho}/G_{\alpha,\rho})$ for all $\alpha<\lambda$. Note that for all $\mu<\nu<\lambda$ we therefore have $T_{X_\rho}\in\mathcal{TC}(G_{\nu,\rho}/G_{\mu,\rho})$. It is well-known that for each $\rho<\aleph_1$ there is a cub D_ρ of λ such that $G_{\beta,\rho}=G_\beta$ for all $\beta\in D_\rho$. Since $\lambda=cf(\lambda)>\aleph_1$ the intersection $C=\bigcap_{\rho<\aleph_1}D_\rho$ is still a cub in λ. Assume that there exists $\beta\in C$ such that $T_P\notin\mathcal{TC}(G_{\beta+1}/G_\beta)$. Since $G_{\beta+1}/G_\beta$ is of smaller cardinality than G there exists an infinite subset $X\subseteq P$ such that $T_Y\notin\mathcal{TC}(G_{\beta+1}/G_\beta)$ for all infinite subsets $Y\subseteq X$. Choose $\beta+1\leq\gamma\in C$. Then also $T_Y\notin\mathcal{TC}(G_\gamma/G_\beta)$ for all infinite subsets $Y\subseteq X$. But then $X=P_\rho$ for some $P_\rho\in W$, hence $T_{X_\rho}\in\mathcal{TC}(G_\gamma/G_\beta)=\mathcal{TC}(G_{\gamma,\rho}/G_{\beta,\rho})$ - a contradiction. Thus for all $\beta\in C$ we have $T_P\in\mathcal{TC}(G_{\beta+1}/G_\beta)$ and thus the relative Γ-invariant $\Gamma_{T_P}(G)=0$ and therefore $T_P\in\mathcal{TC}(G)$ by [**EM**, Proposition XII.1.14] - a contradiction. □

Corollary 2.9 ($V=L$). *Assume that for all torsion-free groups H of size less than or equal to $\aleph_1$ there exists a completely decomposable group C_H such that $\mathcal{TC}(H)=\mathcal{TC}(C_H)$. Then all torsion-free groups G satisfy $\mathcal{TC}(G)=\mathcal{TC}(C)$ for some completely decomposable group C.*

Proof. Let G be any torsion-free group. By Lemma 2.2 and Theorem 2.7 conditions (i) to (iv) of Theorem 2.1 are satisfied for $\mathcal{TC}(G)$. Moreover, Theorem 2.8 shows that also condition Theorem 2.1 (v) is satisfied and hence $\mathcal{TC}(G)=\mathcal{TC}(C)$ for some completely decomposable group C by Theorem 2.1. □

We obtain a special case of [**EFS**, Theorem C].

Corollary 2.10 ($V=L$). *Let G be a torsion-free abelian group. Then G is free if and only if $\bigoplus_{p\in\Pi}\bigoplus_{n\in\omega}\mathbb{Z}(p^n)\in\mathcal{TC}(G)$. Hence there exists a countable test-group for freeness.*

Proof. If G is free, then trivially $\bigoplus_{p\in\Pi}\bigoplus_{n\in\omega}\mathbb{Z}(p^n)\in\mathcal{TC}(G)$. Hence assume that $\bigoplus_{p\in\Pi}\bigoplus_{n\in\omega}\mathbb{Z}(p^n)\in\mathcal{TC}(G)$. Since we assume $V=L$ we know that conditions

(i) to (iv) of Theorem 2.1 are satisfied for $\mathcal{TC}(G)$. Therefore, $\bigoplus_{n\in\omega} \mathbb{Z}(p^n)$ implies that $\mathcal{TC}(G)$ contains all p-groups for all primes $p \in \Pi$ by Theorem 2.1 (iii). Thus $\bigoplus_{p\in\Pi} \mathbb{Z}(p) \in \mathcal{TC}(G)$ and Theorem 2.1 (iv) imply that $\mathcal{TC}(G)$ contains all direct sums of arbitrary p-groups ($p \in \Pi$) and hence contains all torsion groups. Thus $\mathcal{TC}(G)$ is the class of all torsion groups and by Griffith's solution of the Baer problem (see [**G**]) it follows that G is free. □

Remark 2.11. *The Example 2.3 of a Shelah group under Martin's axiom shows that the above corollary does not hold in ZFC.*

We could not show that in $V = L$ every torsion-free group G satisfies $\mathcal{TC}(G) = \mathcal{TC}(C)$ for some completely decomposable group C. Hence we conclude this section with a conjecture.

$\mathcal{TC}$-Conjecture 2.12. *In Goedel's constructible universe $V = L$, for every torsion-free group G there exists a completely decomposable group C such that $\mathcal{TC}(G) = \mathcal{TC}(C)$.*

3. Kulikov's problem

Let T be any torsion group and λ any cardinal. A torsion-free group G of rank λ is called *λ-universal for T* if it satisfies $\mathrm{Ext}(G, T) = 0$ and every torsion-free group H of rank less than or equal to λ satisfying $\mathrm{Ext}(H, T) = 0$ can be embedded into G. Kulikov [**KN**, Question 1.66] had asked if there is always a λ-universal group for arbitrary T and uncountable λ. For a large class of torsion groups we shall give a positive answer in $V = L$. Since to the authors knowledge there are no published results on Kulikov's question for λ countable or finite we begin with the case λ being an integer. Let us first note that it is easy to see that G is λ-universal for T if and only if G is λ-universal for the reduced part of T, hence we may always assume that T is reduced. Recall that a group T is called *cotorsion* if $\mathrm{Ext}(\mathbb{Q}, T) = 0$.

Lemma 3.1. *If T is a torsion cotorsion group, then there exists for any cardinal λ a λ-universal group G for T.*

Proof. Clearly the direct sum of λ copies of the rationals $\mathbb{Q}$ forms a λ-universal group for T since every torsion-free group can be embedded into its divisible hull. □

Lemma 3.2. *Let T be a torsion group and G be n-universal for T for some $n > 0$. Then G is homogeneous completely decomposable.*

Proof. Since G is of finite rank n it follows from Proposition 1.4 (iii) that the outer type $R = OT(G)$ satisfies $\mathcal{TC}(R) = \mathcal{TC}(G)$, hence $\mathrm{Ext}(R, T) = 0$. Thus $\bigoplus_{i\leq n} R$ can be embedded into G by universality. Therefore, there exists a maximal linearly independent set $\{x_1, \cdots, x_n\}$ of elements of G having type greater than or equal to R. Thus, the inner type $IT(G)$ is greater than or equal to R (for inner type see [**A**, page 84]). Hence $R = OT(G) = IT(G)$ and it follows from [**A**, Proposition 3.1.13] that G must be homogeneous completely decomposable of type R. □

Theorem 3.3. *Let T be a torsion group and $n \in \mathbb{N}$. Then there exists an n-universal group G for T if and only if T has only finitely many non-trivial bounded p-components. In this case, G is completely decomposable.*

Proof. Without loss of generality we may assume that T is reduced. Assume that G is n-universal for T for some positive integer n. Then G must be homogeneous completely decomposable by Lemma 3.2, hence $\mathcal{TC}(G) = \mathcal{TC}(S)$ for some rational group $S \subseteq \mathbb{Q}$. Assume that T has infinitely many bounded p-components, say T_p is bounded and non-trivial for $p \in P$, P an infinite set of primes. Then $\mathrm{Ext}(\mathbb{Q}^{(p)}, T) = 0$ for all $p \in P$, hence $\mathbb{Q}^{(p)}$ embeds into G and thus $\chi_p^S(1) = \infty$ for all $p \in P$ by Proposition 1.4 (i). But then $\mathrm{Ext}(S, T) \neq 0$ again by Proposition 1.4 (i)- a contradiction. Thus T can only have finitely many bounded non-trivial p-components.
Conversely, assume that T has only finitely many non-trivial bounded p-components. Let $P = \{p \in \Pi : T_p \textit{ is unbounded }\}$ and put $R = \langle 1/p^\infty : p \notin P\rangle \subseteq \mathbb{Q}$. Then clearly $C = \bigoplus\limits_{i \leq n} R$ satisfies $\mathrm{Ext}(C, T) = 0$ by Proposition 1.4 (i). Let G be any torsion-free group of rank less than or equal to n satisfying $\mathrm{Ext}(G, T) = 0$. We shall show that G can be embedded into C. Since $\mathrm{Ext}(G, T) = 0$ it follows that for any type S in the typeset of G we have $\mathrm{Ext}(S, T) = 0$. Hence $S \leq R$ since R is idempotent. Thus the tensor product $G \otimes R$ is homogeneous of type R. Moreover, the short exact sequence

$$0 \to \mathbb{Z} \to R \to D \to 0 \tag{3.1}$$

with D torsion divisible induces the short exact sequence

$$0 \to G \to G \otimes R \to D \otimes R \to 0 \tag{3.2}$$

and thus G is embedable into $G \otimes R$. Note that D has non-trivial p-components only for $p \notin P$. Let $T = T_1 \oplus T_2$ with T_1 finite and $T_2 = \bigoplus\limits_{p \in P} T_p$. Then $\mathrm{Ext}(G \otimes R, T) = 0$ if and only if $\mathrm{Ext}(G \otimes R, T_2) = 0$. Applying the Hom-functor to the sequence (3.2) we obtain

$$\mathrm{Ext}(D \otimes R, T_2) \to \mathrm{Ext}(G \otimes R, T_2) \to \mathrm{Ext}(G, T_2) \to 0 \tag{3.3}$$

and since D is q-divisible for all $q \in P$ it follows that $\mathrm{Ext}(D \otimes R, T_2) = 0$, hence $\mathrm{Ext}(G \otimes R, T) = 0$. By Proposition 1.4 (iii) there exists a rational group $S \subseteq \mathbb{Q}$ such that $\mathcal{TC}(G \otimes R) = \mathcal{TC}(S)$. It follows that $S \geq R$ and since R is idempotent and $\mathrm{Ext}(G \otimes R, T) = 0$ we obtain $S = R$. Thus [**St**, Corollary 2.11] implies that $G \otimes R$ is completely decomposable and hence embedable into C. $\square$

In the case of ω-universal groups the situation is more delicate.

Lemma 3.4. *If T is a torsion group with only finitely many non-trivial bounded p-components, then there is a ω-universal group C for T which is completely decomposable.*

Proof. Without loss of generality assume that T is reduced torsion and has only finitely many non-trivial bounded components. As in in the proof of Theorem 3.3 we define $P = \{p \in \Pi : T_p \textit{ is unbounded }\}$ and put $R = \langle 1/p^\infty : p \notin P\rangle \subseteq \mathbb{Q}$. We shall show that $C = \bigoplus\limits_{n \in \omega} R$ is ω-universal for T. Let G be countable torsion-free such that $\mathrm{Ext}(G, T) = 0$. The same arguments as in the proof of Theorem 3.3 show that $G \otimes R$ is homogeneous of type R and that $\mathrm{Ext}(G \otimes R, T) = 0$. By Proposition 1.4 (ii) there exists a completely decomposable group H such that $\mathcal{TC}(G \otimes R) = \mathcal{TC}(H)$. It follows that $S \leq R$ for all types S in the typeset of H since R is idempotent and $\mathrm{Ext}(G \otimes R, T) = 0$. Therefore $\mathcal{TC}(R) \subseteq \mathcal{TC}(H) = \mathcal{TC}(G \otimes R)$. Moreover, by homogenity we have $\mathcal{TC}(G \otimes R) \subseteq \mathcal{TC}(R)$ and hence $\mathcal{TC}(G \otimes R) = \mathcal{TC}(R)$. Now Griffith's solution of the Baer problem (see [**G**]) implies that $G \otimes R$ is completely decomposable of type R and therefore embeds into C. $\square$

If T has infinitely many non-trivial bounded p-components then we can find at least a completely decomposable $(< \omega)$-universal group for T, i.e. a completely decomposable countable torsion-free group G satisfying $\mathrm{Ext}(G, T) = 0$ and every finite rank torsion-free group H such that $\mathrm{Ext}(H, T) = 0$ is embedable into G.

Lemma 3.5. *Let T be a torsion group. Then there exists a $(< \omega)$-universal group for T which is completely decomposable.*

Proof. Let T be torsion and define P_1 to be the set of all primes p such that the p-component T_p of T is bounded but non-trivial. Moreover, let P_2 contain all primes such that $T_p = 0$ and $P_3 = \Pi \backslash (P_1 \cup P_2)$. For a finite subset $Q \subseteq P_1$ we put

$$R_Q = \langle 1/p^\infty : p \in (Q \cup P_3) \rangle$$

and put $C_Q = \bigoplus_{n \in \omega} R_Q$. Finally, let $C = \bigoplus_{Q \text{ (finite) } \subseteq P_1} C_Q$. Then $\mathrm{Ext}(C, T) = 0$ by Proposition 1.4 (i) and we shall show that every finite rank torsion-free group H such that $\mathrm{Ext}(H, T) = 0$ is embedable into C. Let H be such a group. Then $\mathcal{TC}(H) = \mathcal{TC}(K)$ for some rational group $K \subseteq \mathbb{Q}$ by Proposition 1.4 (iii). Note that K is the outer type of G. Thus, if S is in the typeset of H then clearly $S \le R_{Q_S}$ for some finite subset Q_S of P_1 by Proposition 1.4 (i). Let $Q = \bigcup_{S \in Tst(H)} Q_S \subseteq P_1$. Then Q must be finite since Q infinite would force $\chi_p^K(1) = \infty$ for infinitely many primes $p \in P_1$ - a contradiction since $\mathrm{Ext}(K, T) = 0$. We let $R = \sup\{R_Q, K\}$ and conclude that $H \otimes R$ must be homogeneous of type R. Notice that R is idempotent and that $\mathrm{Ext}(R, T) = 0$. We consider the following short exact sequence

$$0 \to \mathbb{Z} \to R \to D \to 0 \tag{3.4}$$

where D is torsion divisible with non-trivial p-components for $\chi_p^R(1) = \infty$, say $U = \{p \in \Pi : D_p \neq 0\}$. Note that $U \cap P_1$ is finite by the choice of R. By applying first the $\otimes$-functor and then the Hom-functor to (3.4) we obtain the short exact sequence

$$\mathrm{Ext}(D \otimes H, T) \to \mathrm{Ext}(H \otimes R, T) \to \mathrm{Ext}(H, T) = 0. \tag{3.5}$$

By elementary properties of Ext it follows that $\mathrm{Ext}(D \otimes H, T) \cong \mathrm{Ext}(\bigoplus_{p \in U \cap P_1} D_p \otimes H, \bigoplus_{p \in U \cap P_1} T_p)$ but by the choice of P_1 and the finiteness of $U \cap P_1$ we obtain that $\bigoplus_{p \in U \cap P_1} T_p$ is bounded, hence also $\mathrm{Ext}(D \otimes H, T)$ is bounded. But $\mathrm{Ext}(H \otimes R, T)$ is divisible and an epimorphic image of $\mathrm{Ext}(D \otimes R, T)$, hence trivial. Thus $\mathrm{Ext}(H \otimes R, T) = 0$. By Proposition 1.4 (iii) there exists a rational group $S \subseteq \mathbb{Q}$ such that $\mathcal{TC}(H \otimes R) = \mathcal{TC}(S)$, hence $R \le S$. Note that S must be idempotent. Assume that $S > R$, then there exists $p \in P_1$ such that $\chi_p^S(1) = \infty$ and $\chi_p^R(1) = 0$. Since $K \le R$ it follows that there exists an unbounded p-group $T_1 \in \mathcal{TC}(H)$. We now prove by induction on the rank of H that $T_1 \in \mathcal{TC}(H \otimes R) = \mathcal{TC}(S)$ - a contradiction. If H is of rank one then clearly $\chi_p^{H \otimes R}(1) < \infty$ and hence $\mathrm{Ext}(H \otimes R, T_1) = 0$. Assume that H is of rank n and choose a pure subgroup M of H of rank $n-1$. Let

$N = H/M \subseteq \mathbb{Q}$. From the exact sequence

$$0 \to M \to H \to N \to 0 \tag{3.6}$$

we obtain the short exact sequence

$$\mathrm{Ext}(N \otimes R, T_1) \to \mathrm{Ext}(H \otimes R, T_1) \to \mathrm{Ext}(M \otimes R, T_1) \to 0. \tag{3.7}$$

By induction hypothesis $\mathrm{Ext}(M \otimes R, T_1) = 0$. Moreover, $N \leq K$ since $K = OT(G)$ and therefore $\mathrm{Ext}(G, T_1) = 0$ implies $\mathrm{Ext}(K, T_1) = 0 = \mathrm{Ext}(N, T_1)$. Thus, $\chi_p^N(1) < \infty$ and hence also $\chi_p^{N \otimes R}(1) < \infty$. We conclude $\mathrm{Ext}(N \otimes R, T_1) = 0$ and hence $\mathrm{Ext}(H \otimes R, T_1) = 0$ - a contradiction. Thus $S = R$ and $\mathcal{TC}(H \otimes R) = \mathcal{TC}(R)$. By Griffith's solution of the Baer problem or [**St**, Corollary 2.11] it follows that $H \otimes R$ is completely decomposable of type R and hence $H \subseteq H \otimes R$ embeds into C. $\square$

Note that Lemma 3.2 can not hold anymore for ω-universal groups, for if G is ω-universal for T and H is countable torsion-free satisfying $\mathrm{Ext}(H, T) = 0$, then also $G \oplus H$ is ω-universal for T. We can even show that no ω-universal group for T (if it exists) can be completely decomposable if T has infinitely many non-trivial bounded p-components. Recall that a torsion-free group B is called a *B_1-group* if $\mathrm{Bext}(G, T) = 0$ for all torsion groups T.

Lemma 3.6 ([**SW**]). *Let B be a B_1-group. Then $\mathcal{TC}(B) = \mathcal{TC}(\bigoplus_{R \in \mathrm{Tst}(B)} R)$, where $\mathrm{Tst}(B)$ denotes the type-set of B.*

Proposition 3.7. *Let T be a torsion group with infinitely many non-trivial bounded p-components. Then there exists a countable torsion-free group G with $\mathrm{Ext}(G, T) = 0$ and G is not embedable into any completely decomposable group C satisfying $\mathrm{Ext}(C, T) = 0$.*

Proof. The proof is very similar to the one in [**A**, Example 3.4.2] and hence we will just outline it briefly. Let P be the set of primes such that T_p is non-trivial and bounded. Let S be the set of words on the alphabet $\{0, 1\}$ and denote by $\emptyset$ the trivial word. We divide P into two disjoint infinite subsets P_0 and P_1 and enumerate them by ω, e.g. $P_i = \{p_{i,j} : j < \omega\}$ $(i = 0, 1)$. For $s \in S$ let $X_s = \langle 1/p_{i,j}^\infty : s(j) = i \rangle \subseteq \mathbb{Q}$. Then the X_s $(s \in S)$ form a tree, i.e. $X_s = X_{s0} \cap X_{s1}$ for all $s \in S$ and clearly $\mathrm{Ext}(X_s, T) = 0$. Moreover, if R is a type and for each $n \in \mathbb{N}$ there is $s \in S$, $l(s) = n$ such that $R \geq X_s$, then $\chi_p^R(1) = \infty$ for infinitely many primes $p \in P$ and hence $\mathrm{Ext}(R, T) \neq 0$. For $n \in \mathbb{N}$ now define $G_n = \bigoplus\{X_s : s \in S, l(s) = n\}$, hence $\mathrm{Ext}(G_n, T) = 0$, and let G be the direct limit of $\{G_n, f_n : n \geq 1\}$, where $f_n : G_n \to G_{n+1}$, $x \mapsto (x, x)$. Then G is a B_1-group and the type-set $\mathrm{Tst}(G)$ of the group G is contained in the (even finite) meet closure of the X_s's. Since $\mathrm{Ext}(X_s, T) = 0$ for all $s \in S$ it follows that $T \in \mathcal{TC}(\bigoplus_{s \in S} X_s) = \mathcal{TC}(G)$ and thus $\mathrm{Ext}(G, T) = 0$ by Lemma 3.6. Finally, assume that G is a subgroup of a completely decomposable group $C = \bigoplus_{i \in \omega} C_i$ satisfying $\mathrm{Ext}(C, T) = 0$. Since G is reduced we may assume that C is reduced. There exists $m \geq 1$ such that $X_\emptyset = \mathbb{Z} \subseteq C_1 \oplus \cdots \oplus C_m$. Let $0 \neq x \in X_\emptyset$ and $n \geq 1$. Then $x = \bigoplus\{x_s : l(s) = n\} \in G_n$ with each $x_s \neq 0$. Moreover, $x_s = c(1, s) \oplus \cdots \oplus c(k, s)$ for some $k \geq m$ and $c(i, s) \in C_i$. Hence,

$$X_s = \mathrm{type}(x_s) \leq \cap\{\mathrm{type}(c(i, s)) : 1 \leq i \leq k\} \leq \cap\{\mathrm{type}(c(i, s)) : 1 \leq i \leq m\}$$

and since $x \neq 0$ there is some $1 \leq j \leq m$ with $c(j,s) \neq 0$. Thus, for every $n \geq 1$ there exists $s \in S$ with $l(s) = n$ and some $1 \leq j \leq m$ with $X_s \leq C_j$. By the choice of the X_s's it follows that $\mathrm{Ext}(C_j, T) \neq 0$ - a contradiction. $\square$

Corollary 3.8. *If T has infinitely many non-trivial bounded p-components and λ is an infinite cardinal, then no λ-universal group G for T can be completely decomposable.*

To conclude this section we shall show that in $V = L$ every torsion group with only finitely many non-trivial p-components has λ-universal groups for every λ. This contrasts the consistency result from [**SS2**] showing that there is a model of ZFC in which for every torsion, not cotorsion group T there is a class of cardinals λ such that there exist no λ-universal groups for T.

Theorem 3.9 ($V = L$). *If T is a torsion group with only finitely many non-trivial bounded p-components and λ is a cardinal, then there is a λ-universal group C for T which is completely decomposable.*

Proof. Assume that T is reduced torsion and has only finitely many non-trivial p-bounded components. We define $P = \{p \in \Pi : T_p \textit{ is unbounded } \}$ and put $R = \langle 1/p^{\infty} : p \notin P \rangle \subseteq \mathbb{Q}$. Then clearly $C = \bigoplus\limits_{\lambda} R$ satisfies $\mathrm{Ext}(C, T) = 0$ by Proposition 1.4 (i). We will show that C is λ-universal for T. Let G be torsion-free of cardinality less than or equal to λ satisfying $\mathrm{Ext}(G, T) = 0$. Since $\mathrm{Ext}(G, T) = 0$ it follows that for any type S in the typeset of G we have $\mathrm{Ext}(S, T) = 0$. Hence $S \leq R$ since R is idempotent. Thus the tensor product $G \otimes R$ is homogeneous of type R. As in the proof of Theorem 3.3 it follows that $\mathrm{Ext}(G \otimes R, T) = 0$. Since we don't know whether or not the $\mathcal{TC}$-Conjecture holds we show directly that Theorem 2.1 (v) holds for $\mathcal{TC}(G \otimes R)$. Assume that Q is an infinite set of primes such that $\bigoplus\limits_{p \in Q} \mathbb{Z}(p) \notin \mathcal{TC}(G \otimes R)$. Then $Q \backslash P$ must be infinite since $T' = \bigoplus\limits_{p \in Q \cap P} \mathbb{Z}(p)$ is an epimorphic image of T and hence $T' \in \mathcal{TC}(G \otimes R)$. It now follows that Theorem 2.1 (v) holds for $\mathcal{TC}(G \otimes R)$ since it holds for $\mathcal{TC}(R)$. Thus $V = L$ and the results from section 2 imply that there exists a completely decomposable group H such that $\mathcal{TC}(G) = \mathcal{TC}(H)$. As in the proof of Theorem 3.4 we conclude that $\mathcal{TC}(G \otimes R) = \mathcal{TC}(R)$ and thus Griffith's solution of the Baer problem (see [**G**]) implies that $G \otimes R$ is completely decomposable of type R and therefore embeds into C. $\square$

The author would like to remark that to his knowledge it is known whether or not under $V = L$ for every torsion group T and every infinite cardinal λ there exists a λ-universal group for T. However, the author strongly conjectures that the answer is yes but the techniques developed in this article are not sufficient to prove a complete solution. Therefore we pose the following open question.

Question 3.10 (V=L). *Let T be a torsion group (with infinitely many non-zero p-components) and λ an infinite cardinal. Does there exist a λ-universal group for T ?*

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