Infinitary equivalence of abelian groups and modules

Katrin Leistner

University of Duisburg-Essen

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History - Ulm's Theorem

Definition

For a group (or module) G, $\alpha \in ON$ and p a prime define

$$u_p(\alpha, G) := dim(p^\alpha G[p]/p^{\alpha+1}G[p]) \text{ (as a } \mathbb{Z}_p\text{-vector space)},$$

the (α^{th}) **Ulm-Kaplansky invariant** of G and

$$u_p(\infty, G) := dim(p^\infty G[p]).$$

Theorem (Ulm)

For G and H countable, reduced p-groups the following are equivalent:

1. $G \cong H$;
2. $u_p(\alpha, G) = u_p(\alpha, H)$ *for all ordinals* α.

Model-theory

We consider the language $L_{\infty\omega}$.

Definition

$L_\delta := \{\varphi \in L_{\infty\omega} \mid qr(\varphi) \leq \delta\}$
Contains atomic formulas; closed under (L1)-(L3):
(L1) $\varphi \in L_\delta \Rightarrow \neg\varphi \in L_\delta$;
(L2) $\Phi \subseteq L_\delta \Rightarrow \bigwedge \Phi, \bigvee \Phi \in L_\delta$;
(L3) $\varphi \in L_\beta (\beta < \delta), \nu$ a variable $\Rightarrow \exists\nu\ \varphi, \forall\nu\ \varphi \in L_\delta$.

$\mathfrak{A} \equiv_\delta \mathfrak{B}$ iff $\mathfrak{A} \models \varphi \Rightarrow \mathfrak{B} \models \varphi$ for all sentences $\varphi \in L_\delta$ and vice versa.

Lemma (Barwise-Eklof)

$\mathfrak{A} \equiv_{\omega_1} \mathfrak{B} \Rightarrow \mathfrak{A} \cong \mathfrak{B}$.

more Model-theory

Theorem (Karp)

For models $\mathfrak{A}$ and $\mathfrak{B}$ of the same structure, the following are equivalent:

1. $\mathfrak{A} \equiv_\delta \mathfrak{B}$;
2. *For each ordinal $\alpha \leq \delta$ exists a non-empty set I_α of isomorphisms on finitely generated substructures of $\mathcal{A}$ into finitely generated substructures of $\mathcal{B}$ such that:*
   (a) $I_\alpha \subseteq I_\beta$ *for* $\beta \leq \alpha$;
   (b) *If $\alpha + 1 \leq \delta$ and given $a \in A$ and $f \in I_{\alpha+1}$, there exists $g \in I_\alpha$ such that $f \subseteq g$ and $a \in dom(g)$;*
   (c) *If $\alpha + 1 \leq \delta$ and given $b \in B$ and $f \in I_{\alpha+1}$, there exists $g \in I_\alpha$ such that $f \subseteq g$ and $b \in rng(g)$.*

((b) & (c): "back-and-forth property")

The generalization by Barwise-Eklof

$g : X \to$ cardinals; $\hat{g}(x) := g(x)$, if $g(x) < \aleph_0$, $\hat{g}(x) = \infty$ otherwise.

Theorem (Barwise-Eklof)

For G and H p-groups and $\delta \in ON$: If

(a) $\hat{u}_p(\alpha, G) = \hat{u}_p(\alpha, H)$ *for all* $\alpha < \omega\delta$,

(b) $l(G) < \omega\delta \Leftrightarrow \hat{r}(G_d) = \hat{r}(H_d)$,

then $G \equiv_\delta H$.
If $\delta \in LORD$ *the converse also holds.*

Preserving heights "up to α":
$h_G(x) < \alpha \Rightarrow h_H(f(x)) = h_G(x)$
$h_G(x) \geq \alpha \Rightarrow h_H(f(x)) \geq \alpha$.

History - Warfield's Theorem

We consider $\mathbb{Z}_p$-modules.

Theorem (Warfield)

For two countable Warfield modules M and N the following are equivalent:

1. $M \cong N$;
2. $u_p(\alpha, M) = u_p(\alpha, N)$ *for all* $\alpha \in ON$, $u_p(\infty, M) = u_p(\infty, N)$, *and* $w_M(\bar{\beta}) = w_N(\bar{\beta})$ *for all Ulm-sequences* $\bar{\beta}$.

History - Warfield's Theorem

Definition

- For a module M, $X = \{x_i\}_{i \in I} \subseteq M$ is called a **decomposition basis for** M, if
  1. (i) X is independent;
  2. (ii) $o(x_i) = \infty$ for all i;
  3. (iii) $M/\langle X \rangle$ is torsion;
  4. (iv) $\langle X \rangle = \bigoplus_{i \in I} \langle x_i \rangle$ is a valuated coproduct in M.
- $\langle X \rangle$ is a **nice** submodule of M (and thus X a nice decomposition basis), if $p^\alpha(M/\langle X \rangle) = (p^\alpha M + \langle X \rangle)/\langle X \rangle$.
- For two decomposition bases X, X' of M, call X' a **subordinate** of X, if every element of X' is a nonzero multiple of an element of X.
- A module is called **simply presented** if it can be defined in terms of generators and relations so that the only relations are of the forms $px = 0$ and $px = y$.

History - Warfield's Theorem

A **Warfield module**, M, is a direct summand of a simply presented module
$\Leftrightarrow$ possesses a nice decomposition basis X with $M/\langle X\rangle$ simply presented.

- **Ulm-sequence** $\bar{\beta} = (\beta_i : i < \omega), \beta_i \in ON \cup \infty, \beta_i \leq \beta_{i+1}$
- $p^k\bar{\beta} = (\beta_{i+k} : i < \omega)$
- $\bar{\beta} \sim \bar{\gamma}$, if $\exists k, l < \omega : \; p^k\bar{\beta} = p^l\bar{\gamma}$
- $u_{(p)}(x) = \{|p^i x| : i < \omega\}$

- $M(\bar{\beta}) = \{x \in M : |p^i x| \geq \beta_i$ for all $i < \omega\}$
- $M(\bar{\beta}^*) = \langle x \in M(\bar{\beta}) : |p^i x| > \beta_i$ for infinitely many values of $i\rangle$, if $\beta_i \neq \infty$ for all i, otherwise $M(\bar{\beta}^*) = tM(\bar{\beta})$.

History - Warfield's Theorem

Definition

$w_M(\bar{\beta}) = dim(M(\bar{\beta})/M(\bar{\beta}^*))$, the $\bar{\beta}$-**Warfield invariant of** M $(= |\{x \in X : u_{(p)}(x) \sim \bar{\beta}\}|$, X a decomposition basis)

Theorem (Warfield)

For two countable Warfield modules M and N the following are equivalent:

1. $M \cong N$ *;*
2. $u_p(\alpha, M) = u_p(\alpha, N)$ *for all* $\alpha \in ON$, $u_p(\infty, M) = u_p(\infty, N)$, *and* $w_M(\bar{\beta}) = w_N(\bar{\beta})$ *for all Ulm-sequences* $\bar{\beta}$*.*

An extension result by Warfield

Extension Theorem (Warfield)

Let X and Y be any decompositon bases of modules with identical Warfield invariants.
There exist subordinates X' of X and Y' of Y such that there exists a bijection $X' \to Y'$, which induces a height-preserving isomorphism $\langle X' \rangle \to \langle Y' \rangle$. It follows:
If $A \subseteq X', B \subseteq Y'$ finite and $f : \langle A \rangle \to \langle B \rangle$ is a height-preserving isomorphism with $f(A) = B$, then:
for all $x \in X'$ ($y \in Y'$) there exists $y \in Y'$ ($x \in X'$), such that f extends to a height-preserving isomorphism $\langle A, x \rangle \to \langle B, y \rangle$.

A generalized extension result

- $\bar{\beta} =_\alpha \bar{\gamma}$ if $min\{\beta_i, \alpha\} = min\{\gamma_i, \alpha\}$
- $w^\alpha_M(\bar{\beta}) = \sum_{\bar{\eta} =_\alpha \bar{\beta}, \bar{\eta} \in U} w_M(\bar{\eta}) = |\{x \in X : u_{(p)}(x) =_\alpha \bar{\beta}\}|$

Extension Theorem (Göbel-L-Loth-Strüngmann)

Let M, N be modules with decomposition bases X and Y, such that $\tilde{w}^\alpha_M(\bar{\beta}) = \tilde{w}^\alpha_N(\bar{\beta})$ for all Ulm-sequences $\bar{\beta}$ where α is a fixed ordinal or the symbol ∞. Then there exist subordinates X' of X and Y' of Y such that:
If A and B are countable subsets of X' and Y' and $f : \langle A \rangle \longrightarrow \langle B \rangle$ is an α-height-preserving isomorphism with $f(A) = B$, then for every $x \in X'$ ($y \in Y'$) there is $y \in Y'$ ($x \in X'$) such that f extends to an α-height-preserving isomorphism $f' : \langle A, x \rangle \rightarrow \langle B, y \rangle$ by sending x onto y.

A model-theoretic Lemma

Lemma (Göbel-L-Loth-Strüngmann)

Let M and N be modules and $M \equiv_\lambda N$ where $\lambda \in LORD$. Then:

- (a) $\hat{u}_p(\alpha, M) = \hat{u}_p(\alpha, N)$, *if* $\alpha < \omega\lambda$.
  (b) *If* $l(M) < \omega\lambda$ *and* $l(N) < \omega\lambda$, *then* $\hat{u}_p(\infty, M) = \hat{u}_p(\infty, N)$.
- *If M and N have decomposition bases and if $\alpha < \omega\lambda$ where $\lambda = \omega\gamma$ and $\gamma \in LORD$, then $\hat{w}^\alpha_M(\bar{\beta}) = \hat{w}^\alpha_N(\bar{\beta})$ for all Ulm-sequences $\bar{\beta}$.*

A first generalized classfication result

Theorem (Göbel-L-Loth-Strüngmann)

Let M *and* N *be modules with nice decomposition bases and let* $\delta \in ON$. *If*

1. $\hat{u}_p(\alpha, M) = \hat{u}_p(\alpha, N)$ *for all ordinals* $\alpha < \omega\delta$;
2. $l(tM) < \omega\delta \Rightarrow \hat{u}_p(\infty, M) = \hat{u}_p(\infty, N)$;
3. $\tilde{w}_M^{\omega\nu}(\bar{\beta}) = \tilde{w}_N^{\omega\nu}(\bar{\beta})$ *for all ordinals* $\nu \leq \delta$ *and all Ulm-sequences* $\bar{\beta}$,

then $M \equiv_\delta N$.
The converse holds if $\delta = \omega\gamma$ *where* $\gamma \in LORD$ *and if* $\tilde{w}_M^{\omega\delta}(\bar{\beta}) = \tilde{w}_N^{\omega\delta}(\bar{\beta})$ *and for all* $\nu < \delta$ *and all Ulm-sequences* $\bar{\beta}$ *holds* $w_M^{\omega\nu}(\bar{\beta}) \leq \aleph_0 \Leftrightarrow w_N^{\omega\nu}(\bar{\beta}) \leq \aleph_0$.

Further invariants and another classification

Let G be a module over $\mathbb{Z}$, a group.

Definition

A system $\mathcal{C}$ is called **partial decomposition basis for** G if

1. $\mathcal{C} \neq \emptyset$, $X \in \mathcal{C} \Rightarrow |X| < \aleph_0$;
2. $X \in \mathcal{C} \Rightarrow X$ is a decomposition set;
3. $X \in \mathcal{C}$, $x \in G \Rightarrow \ \exists Y \in \mathcal{C} : \ X \subseteq Y, x \in \langle Y \rangle^0$.

- **Ulm-matrix of** x: $U(x) = (u_{(p)}(x))_{p \in P}$
- A and B are **compatible** if $\exists m, n \in \mathbb{N} : \ mB \geq A$ and $nA \geq B$

Further invariants and another classification

Definition

Warfield invariant:
$w(c, p, e, G) := |\{x \in X : U(x) \in c, u_{(p)}(x) \in e\}|$
(c a compatibility class of Ulm-matrices, p a prime, e an equivalence class of Ulm-sequences, X a decomposition basis of G)

- $A =_\alpha B$ if $min\{a_{p,i}, \alpha\} = min\{b_{p,i}, \alpha\}$ for all primes p and $i < \omega$
- $c \sim_\alpha c'$ if $\exists A \in c, B \in c' : A =_\alpha B$
- $e \sim_\alpha e'$ if $\exists \bar{\beta} \in e, \bar{\gamma} \in e' : \bar{\beta} =_\alpha \bar{\gamma}$
- $\hat{w}_\alpha(c, p, e, G) = min\{\sum_{e' \sim_\alpha e, c' \sim_\alpha c} \hat{w}(c', p, e', G), \omega\}$

A second generalized classification theorem

Theorem (Jacoby-L-Loth-Strüngmann)

Let G and H be groups with partial decomposition bases $\mathcal{C}$ and $\mathcal{D}$ and let $\delta \in ON$. If

1. $\hat{u}_p(\alpha, G) = \hat{u}_p(\alpha, H)$ *for all primes* p *and* $\alpha < \omega\delta$,
2. $\hat{w}_{\omega(\nu+1)}(c, p, e, G) = \hat{w}_{\omega(\nu+1)}(c, p, e, H)$ *for every compatibility class* c *of Ulm-matrices, prime* p, *equivalence class* e *of Ulm-sequences and* $\nu < \delta$,
3. $l(t(G_p)) < \omega\delta \Rightarrow \hat{u}_p(\infty, G) = \hat{u}_p(\infty, H)$,

then $G \equiv_\delta H$.