Kulikov's problem on universal torsion-free abelian groups

Saharon Shelah and Lutz Strüngmann

ABSTRACT

Let T be an abelian group and λ an uncountable regular cardinal. We consider the question of whether there is a λ-universal group U among all torsion-free abelian groups G of cardinality less than or equal to λ satisfying $\mathrm{Ext}(G, T) = 0$. Here U is said to be λ-universal for T if, whenever a torsion-free abelian group G of cardinality at most λ satisfies $\mathrm{Ext}(G, T) = 0$, then there is an embedding of G into U. For large classes of abelian groups T and cardinals λ it is shown that the answer is consistently no, that is to say there is a model of ZFC in which for pairs T and λ there is no universal group. In particular, for T torsion, this solves a problem by Kulikov.

1. *Introduction*

Given a class $\mathcal{C}$ of objects it is natural to ask for universal objects in $\mathcal{C}$. A universal object is an element $C \in \mathcal{C}$ such that every other object of the class $\mathcal{C}$ can be embedded into C. The existence of universal objects clearly simplifies the structure theory for $\mathcal{C}$.

On the other hand, if there are no universal objects in $\mathcal{C}$, this indicates that the class $\mathcal{C}$ has a complicated structure. Since the definition of universal objects is formulated categorically the search for universal objects appears - as well known - in any field of mathematics.

In the present paper we focus on abelian groups and begin with the class $\mathcal{TF}_\lambda$ of all torsion-free abelian groups G of rank less than or equal to λ, where λ is a fixed cardinal. We consider the subclass $\mathcal{C} = \mathcal{TF}_\lambda(T)$ of all $G \in \mathcal{TF}_\lambda$ with $Ext(G, T) = 0$ for some fixed abelian group T. Here $\mathrm{Ext}(\_, T)$ denotes the first derived functor of the functor $\mathrm{Hom}(\_, T)$. In 1969 Kulikov raised the problem whether or not there are universal groups in $\mathcal{TF}_\lambda(T)$ for all (uncountable) cardinals λ and torsion abelian groups T. If the group T is cotorsion, that is $\mathrm{Ext}(\mathbb{Q}, T) = 0$, then for any λ there is a universal group in $\mathcal{TF}_\lambda(T)$, namely the torsion-free divisible group of rank λ. Surely, the restriction to classes of groups bounded by some fixed cardinal λ is necessary to find universal objects. We want to consider Kulikov's problem and its solution in context of recently investigated cotorsion theories.

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Cotorsion theories for abelian groups have been introduced by Salce in 1979 [**10**]. Following his notation we call a pair $(\mathcal{F}, \mathcal{C})$ a cotorsion theory if $\mathcal{F}$ and $\mathcal{C}$ are classes of abelian groups which are maximal with respect to the property that $\mathrm{Ext}(F, C) = 0$ for all $F \in \mathcal{F}$, $C \in \mathcal{C}$.
Salce [**10**] has shown that every cotorsion theory is cogenerated by a class of torsion and torsion–free groups where $(\mathcal{F}, \mathcal{C})$ is said to be cogenerated by the class $\mathcal{A}$ if $\mathcal{C} = \mathcal{A}^{\perp} = \{X \in \text{Mod–}\mathbb{Z} \mid \mathrm{Ext}(A, X) = 0 \quad \text{for all } A \in \mathcal{A}\}$ and $\mathcal{F} = {}^{\perp}(\mathcal{A}^{\perp}) = \{Y \in \text{Mod–}\mathbb{Z} \mid \mathrm{Ext}(Y, X) = 0 \text{ for all } X \in \mathcal{A}^{\perp}\}$. Examples for cotorsion theories are $(\mathcal{L}, \text{Mod–}\mathbb{Z}) = ({}^{\perp}(\mathbb{Z}^{\perp}), \mathbb{Z}^{\perp})$ where $\mathcal{L}$ is the class of all free groups, and the classical cotorsion theory $(\mathcal{TF}, \mathcal{CO}) = ({}^{\perp}(\mathbb{Q}^{\perp}), \mathbb{Q}^{\perp})$ where $\mathcal{TF}$ is the class of all torsion–free groups and $\mathcal{CO}$ is the class of all cotorsion groups. If $(\mathcal{F}, \mathcal{C})$ is a cotorsion theory, then $\mathcal{C}$ is called cotorsion class and $\mathcal{F}$ is called torsion-free class. We put $(\mathcal{F}, \mathcal{C}) \leq (\mathcal{F}', \mathcal{C}')$ for cotorsion theories $(\mathcal{F}, \mathcal{C})$ and $(\mathcal{F}', \mathcal{C}')$ if $\mathcal{C} \subseteq \mathcal{C}'$. We say that $(\mathcal{F}, \mathcal{C})$ is singly cogenerated if $\mathcal{C} = G^{\perp}$ for some group G. Clearly $\mathbb{Z}^{\perp}$ and $\mathbb{Q}^{\perp}$ give rise to the maximal and minimal cotorsion theories. Moreover, Göbel, Wallutis and Shelah have shown in [**4**] that any partially ordered set can be embedded into the lattice of all cotorsion classes. Hence there is no hope to characterize cotorsion theories. But if we restrict ourselves to torsion-free groups in $\mathcal{TF}_{\lambda}(T)$, then the existence of a universal group provides a step towards classification. The existence of universal groups also contributes information about the size of singly cogenerated cotorsion classes $G^{\perp}$. If G is torsion-free of rank at most λ and $T \in G^{\perp}$ provides a universal group C_T for $\mathcal{TF}_{\lambda}(T)$ then $C_T^{\perp} \subseteq G^{\perp}$.
In a first contribution to Kulikov's problem Strüngmann [**16**] showed that in Gödel's universe $(V = L)$ for every cardinal λ and torsion abelian group T there exists a λ-universal group $G \in \mathcal{TF}_{\lambda}(T)$ if T has only finitely many non-trivial bounded primary components. Moreover, if λ is finite, then this characterizes those T's which give rise to universal groups even in in ZFC.
In this paper we shall prove that there is a model of ZFC in which the generalized continuum hypothesis GCH holds with the property that for every abelian group T which is not cotorsion and every uncountable regular cardinal κ there is a cardinal $\lambda \geq \kappa$ such that the class $\mathcal{TF}_{\lambda}(T)$ has no universal object. Moreover, for torsion abelian groups T (not cotorsion) of cardinality less than or equal to $\aleph_1$ there is no uncountable regular cardinal λ such that $\mathcal{TF}_{\lambda}(T)$ has universal groups. This shows that the result in [**16**] is not provable under ZFC and it answers Kulikov's problem consistently in the negative.

The notations are standard and for unexplained notions in abelian group theory and set theory we refer to [**3**] and [**1**], [**7**]. For uniformization see [**9**] or [**12**].

2. *λ-universal groups*

In this section we introduce the notions of λ-universal groups for a given group T and obtain some basic properties. We are mainly interested in the case when our group T is torsion but leave T arbitrary whenever this is possible. Let $\mathcal{TF}$ be the class of all torsion-free groups. For a cardinal λ we denote by $\mathcal{TF}_{\lambda}(T)$ the class of all torsion-free groups G of rank at most λ such that $\mathrm{Ext}(G, T) = 0$. Moreover, we let $\mathcal{TF}(T) = \bigcup\limits_{\lambda} \mathcal{TF}_{\lambda}(T)$ be the class of all torsion-free groups G satisfying $\mathrm{Ext}(G, T) = 0$.

We recall from the introduction that $G \in \mathcal{C}$ is universal in the class $\mathcal{C}$ if any group in $\mathcal{C}$ embeds into G. In particular we shall use the following

DEFINITION 2.1. If T is a group and λ a cardinal, then a group G is called *λ-universal for T* if G is universal in the class $\mathcal{TF}_\lambda(T)$.

Kulikov [**8**, Question 1.66] asked the following problem.

PROBLEM 2.2. *Do λ-universal groups exist for all uncountable cardinals and for all torsion groups ?*

The starting point of this paper are the following results obtained recently by Strüngmann [**16**]. For this we recall that a completely decomposable group is just a direct sum of subgroups of the rational numbers $\mathbb{Q}$.

PROPOSITION 2.3 ([**16**]). *Let T be a torsion group and λ a natural number. Then there exists a λ-universal group G for T if and only if T has only finitely many non-trivial bounded primary components. In this case, G is completely decomposable.*

THEOREM 2.4 ([**16**], $V = L$). *If T is a torsion group with only finitely many non-trivial bounded primary components and λ is a cardinal, then there is a λ-universal group for T which is completely decomposable.*

We want to omit the case $\lambda = \aleph_0$ and therefore restrict ourselves to uncountable (regular) cardinals λ.
It is well-known that any torsion group T has a basic subgroup $B \subseteq T$ which is pure, a direct sum of cyclic groups and has divisible quotient T/B; We have the following immediate

LEMMA 2.5. *Let T be a torsion group, B a basic subgroup of T and λ a cardinal. A torsion-free group G is λ-universal for T if and only if G is λ-universal for B.*

PROOF. Since $\mathrm{Ext}(H, T) = 0$ if and only if $\mathrm{Ext}(H, B) = 0$ for any torsion-free group H the lemma follows straight (see for example [**16**, Lemma 1.2]). $\square$

PROPOSITION 2.6. *If λ is a cardinal, then a torsion-free group G is λ-universal for T if and only if G is λ-universal for the reduced part of T.*

PROOF. Decompose $T = D \oplus R$ into the maximal divisible subgroup D and the reduced part R. Then the lemma follows from $\mathrm{Ext}(H, D) = 0$ and $\mathrm{Ext}(H, T) = \mathrm{Ext}(H, R)$ for any group H. $\square$

Thus it is enough to consider reduced groups. We may reduce the question about the existence of λ-universal groups for T further and by Lemma 2.7 we also assume that T is not cotorsion. Recall that a group T is called *cotorsion* if $\mathrm{Ext}(\mathbb{Q}, T) = 0$.

LEMMA 2.7. *There is a λ-universal group for every cardinal λ and any cotorsion group T.*

PROOF. We have $D = \bigoplus_\lambda \mathbb{Q} \in \mathcal{TF}_\lambda(T)$ for any cotorsion group T and cardinal λ, and D is λ-universal because any torsion-free group of rank at most λ embeds into D. $\square$

From [**16**] we also note.

LEMMA 2.8. *Let G be any group and T a torsion group. Then* $\mathrm{Ext}(G, T) = 0$ *if and only if* $\mathrm{Ext}(G, T') = 0$ *for all pure subgroups T' of T such that $|T'| \leq |G|$.*

Thus it is no restriction to assume $|T| \leq \lambda$ and we will even assume $|T| < \lambda$ in the sequel of this paper.

3. (T, λ, γ)-*suitable groups*

Let $\gamma < \lambda$ be fixed regular infinite cardinals.

Definition 3.1. Let T be a group with $|T| < \lambda$. Then a group G is called (T, λ, γ)-*suitable* if the following conditions are satisfied

(i) $\mathrm{Ext}(G, T) \neq 0$;

(ii) There is an increasing chain $(F_i : i \leq \gamma)$ of free groups such that

  (a) $|F_i| \leq |i| + \aleph_0$ for all $i < \gamma$;

  (b) F_γ / F_i is free for all $i < \gamma$;

  (c) $F_\gamma / \bigcup\limits_{i<\gamma} F_i \cong G$.

Our next lemma shows that for any group T not cotorsion there is a (T, λ, ω)-suitable group.

Lemma 3.2. *Let T be a group with $|T| < \lambda$ and G a countable group such that $\mathrm{Ext}(G, T) \neq 0$. Then G is (T, λ, ω)-suitable. If T is not cotorsion then $\mathbb{Q}$ is (T, λ, ω)-suitable.*

Proof. Choose a free resolution

$$0 \to K \overset{id}{\to} F \to G \to 0$$

of G. We may assume that K and F have countable rank and write $K = \bigoplus\limits_{i<\omega} \mathbb{Z} e_i$. Then each $F_n = \bigoplus\limits_{i \leq n} e_i \mathbb{Z}$ is a direct summand of F and hence G is (T, λ, ω)-suitable. If T is not cotorsion, then $\mathrm{Ext}(\mathbb{Q}, T) \neq 0$ and hence the same arguments show that $\mathbb{Q}$ is (T, λ, ω)-suitable. □

Next we show the existence of (T, λ, γ)-suitable groups for uncountable cardinals γ. Recall that a group G is *almost-free* if all subgroups of smaller cardinality are free.

Lemma 3.3. *If T is a group with $|T| < \lambda$ and G is almost-free with $|G| = \gamma$ such that $\mathrm{Ext}(G, T) \neq 0$, then G is (T, λ, γ)-suitable.*

Proof. Choose a γ-filtration $G = \bigcup\limits_{\alpha<\gamma} G_\alpha$ of G such that each G_α is free. By [**1**, Lemma 1.4, page 346] there is a free resolution associated with this filtration. This is to say there are free groups $F = \bigoplus\limits_{\alpha<\gamma} F_\alpha$ and $K = \bigoplus\limits_{\alpha<\gamma} K_\alpha$ such that the short sequences

$$0 \longrightarrow K \longrightarrow F \longrightarrow G \longrightarrow 0$$

and

$$0 \longrightarrow \bigoplus_{\alpha<\beta} K_\alpha \longrightarrow \bigoplus_{\alpha<\beta} F_\alpha \longrightarrow G_\beta \longrightarrow 0$$

are exact for all $\beta < \gamma$. Since each G_β is free it follows that G is (T, λ, γ)-suitable. □

Lemma 3.4. *Let H be an epimorphic image of the group T. If G is (H, λ, γ)-suitable then G is (T, λ, γ)-suitable.*

PROOF. The claim follows immediately noting that $\mathrm{Ext}(G,T) = 0$ implies $\mathrm{Ext}(G,H) = 0$. □

PROPOSITION 3.5. *Let $S \subseteq \gamma$ be stationary non-reflecting such that $\mathrm{cf}(\alpha) = \omega$ for all $\alpha \in S$ and assume that $\diamondsuit_S$ holds. Let T be a group which has an epimorphic image of size at most γ that is not cotorsion. Then there is a strongly γ-free group of size γ which is (T, λ, γ)-suitable.*

The proposition has an immediate corollary.

COROLLARY 3.6. *Suppose γ and S satisfy the conditions in Proposition 3.5. If T is not cotorsion and torsion or $|T| \leq \gamma$ then there is a (T, λ, γ)-suitable torsion-free group.*

PROOF. If $|T| \leq \gamma$ then the corollary follows from Proposition 3.5. If T is torsion then we choose a basic subgroup B' of T and a countable unbounded direct summand B of B' which is therefore not cotorsion. Note that B exists since T is not cotorsion. It is well-known that B is an epimorphic image of T (see [**3**, Theorem 36.1]), hence Lemma 3.4 applies and it is enough to construct a (B, λ, γ)-suitable group which follows from Proposition 3.5. □

PROOF. (of Proposition 3.5) Let B be the epimorphic image of T as in the Proposition. Then by Lemma 3.4 it is enough to construct a (B, λ, γ)-suitable group. Therefore, we may assume that $|T| \leq \gamma$. By Lemma 3.2 there is a countable torsion-free group R which is (T, λ, ω)-suitable. If $\lambda_n \geq \aleph_0$ $(n \in \omega)$ are cardinals then as in [**1**, Corollary 1.2, page 182] there exist free abelian groups $K \subseteq F$ and an ascending chain $K_n (n \in \omega)$ of subgroups with $K = \bigcup\limits_{n \in \omega} K_n$ and F/K_n is free for all $n \in \omega$, K_0 is free of rank λ_0 and K_{n+1}/K_n is free of rank λ_{n+1}. From $F/K \cong R$ follows $\mathrm{Ext}(F/K, T) \neq 0$. As in the proof of [**1**, Theorem 1.4, page 185] we can construct a torsion-free group G of cardinality γ which has a γ-filtration $G = \bigcup\limits_{\alpha < \gamma} G_\alpha$ satisfying the following conditions for all $\alpha < \beta < \gamma$.

(i) G_α is free of rank $|\alpha| + \aleph_0$;
(ii) if α is a limit ordinal, then $G_\alpha = \bigcup\limits_{\delta < \alpha} G_\delta$;
(iii) if $\alpha \notin S$, then G_β / G_α is free of rank $|\beta| + \aleph_0$;
(iv) if $\alpha \in S$, then $\mathrm{Ext}(G_\beta / G_\alpha, T) \neq 0$.

From $\diamondsuit_S$ follows that $\mathrm{Ext}(G, T) \neq 0$ (see [**1**, Theorem 1.15, page 353]), hence G is (T, λ, γ)-suitable by Lemma 3.3. □

4. *The uniformization*

Besides regular cardinals $\aleph_0 \leq \gamma < \lambda$ we also fix a stationary subset S of λ consisting of limit ordinals of cofinality γ. To prove Theorem 4.6 and Theorem 4.9 we shall use a construction developed in [**2**]. Thus we shall concentrate on the basic steps.

DEFINITION 4.1. A *ladder system $\bar{\eta}$ on S* is a family of functions $\bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$ such that $\eta_\delta : \gamma \to \delta$ is strictly increasing with $\sup(\mathrm{rg}(\eta_\delta)) = \delta$, where $\mathrm{rg}(\eta_\delta)$ denotes the range of η_δ. We call the ladder system *tree-like* if for all $\delta, \nu \in S$ and every $\alpha, \beta \in \gamma$, $\eta_\delta(\alpha) = \eta_\nu(\beta)$ implies $\alpha = \beta$ and $\eta_\delta(\rho) = \eta_\nu(\rho)$ for all $\rho \leq \alpha$.

For a ladder system $\bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$ on S we can form a *tree* $B_{\bar{\eta}} \subseteq {}^{\leq \gamma}\lambda$ of height γ: Let $B_{\bar{\eta}} = \{\eta_\delta \restriction_\alpha : \delta \in S, \alpha \leq \mathrm{l}(\eta_\delta)\}$, where $\mathrm{l}(\eta_\delta)$ denotes the length of η_δ. Note that

$B_{\bar{\eta}}$ is partially ordered by defining $\eta \leq \nu$ if and only if $\eta = \nu \restriction_{1(\eta)}$.
From the ladder system $\bar{\eta}$ and a group G which is (T, λ, γ)-suitable for some group T we want to find a new group $H_{\bar{\eta}}$. Fix a chain $\langle F_\alpha : \alpha \leq \gamma \rangle$ for G as in Definition 3.1. Hence $F_\gamma / \bigcup\limits_{\alpha<\gamma} F_\alpha \cong G$. For each $\eta \in B_{\bar{\eta}}$ we let $H_\eta = F_{1(\eta)}$ and if $\eta \leq \nu \in B_{\bar{\eta}}$ then let $i_{\eta,\nu}$ be the inclusion map of H_η into H_ν. Finally, let $H_{\bar{\eta}}$ be the direct limit of $(H_\eta, i_{\eta,\nu} : \eta \leq \nu \in B_{\bar{\eta}})$. More precisely, $H_{\bar{\eta}}$ equals $\bigoplus\{H_\eta : \eta \in B_{\bar{\eta}}\}/K$ where K is the subgroup generated by all elements of the form $x_\eta - y_\nu$ where $y_\nu \in H_\nu, x_\eta \in H_\eta$, $\eta \leq \nu$ and $i_{\eta,\nu}(x_\eta) = y_\nu$. Canonically we can embed H_η into $H_{\bar{\eta}}$ and we shall therefore regard H_η as a subgroup of $H_{\bar{\eta}}$ in the sequel.

Definition 4.2. Let κ be an uncountable regular cardinal. The tree $B_{\bar{\eta}}$ is called *κ-free* if for every $X \subseteq S$ with $|X| < \kappa$ there is a function $\Psi : X \to \gamma$ such that

$$\{\{\eta_\delta \restriction_\alpha : \Psi(\delta) < \alpha \leq \gamma\} : \delta \in X\}$$

is a family of pairwise disjoint sets. The ladder system $\bar{\eta}$ is called *κ-free* if $B_{\bar{\eta}}$ is κ-free.

We now state some properties of the constructed group $H_{\bar{\eta}}$.

Lemma 4.3. *If κ is an uncountable regular cardinal and $B_{\bar{\eta}}$ is a κ-free tree, then $H_{\bar{\eta}}$ is a κ-free group.*

Proof. See [**2**, Lemma 1.4]. □

Lemma 4.4. *If S is non-reflecting, then $H_{\bar{\eta}}$ is λ-free but not free.*

Proof. Since S is non-reflecting $\bar{\eta}$ is λ-free and by Lemma 4.3 the group $H_{\bar{\eta}}$ is λ-free. However, if $\delta \in S$ there exists $\nu \geq \delta$ such that for all $\mu < \gamma$, $\eta_\nu \restriction_\mu \in \{\eta_\alpha \restriction_\mu : \alpha < \delta\}$. Thus [2, Lemma 1.5] applies and $H_{\bar{\eta}}$ is not free. □

We recall μ-uniformization for a ladder system $\bar{\eta}$ and a cardinal μ.

Definition 4.5. If μ is a cardinal and $\bar{\eta}$ is a ladder system on S we say that $\bar{\eta}$ has *μ-uniformization* if for every family $\{c_\delta : \delta \in S\}$, where $c_\delta : \mathrm{rg}(\eta_\delta) \to \mu$, there exist $\Psi : \lambda \to \mu$ and $\Psi^* : S \to \mu$ such that $\Psi(\eta_\delta(\alpha)) = c_\delta(\eta_\delta(\alpha))$ with $\Psi^*(\delta) \leq \alpha < \gamma$ for all $\delta \in S$.

Theorem 4.6. *Let T be a group with $|T| < \lambda$ and G be (T, λ, γ)-suitable. If S is non-reflecting and $\bar{\eta}$ is a tree-like ladder system on S with $2^{(|T|^\gamma)}$-uniformization, then there exists a torsion-free group H of size λ such that*

(i) *H has a λ-filtration $\langle \bar{H}_\alpha : \alpha < \lambda \rangle$;*
(ii) *if $\alpha \in S$, then $\bar{H}_{\alpha+1}/\bar{H}_\alpha \cong G$;*
(iii) *if $\alpha \notin S$, then $\bar{H}_\beta/\bar{H}_\alpha$ is free for all $\alpha \leq \beta$;*
(iv) $\mathrm{Ext}(H, W) = 0$ *for all groups W with $|W| \leq |T|$.*

Proof. Let T, S, $\bar{\eta}$ and G be as stated and choose the group $H_{\bar{\eta}} = \bigoplus\{H_\eta : \eta \in B_{\bar{\eta}}\}/K$ as constructed above. Then $H_{\bar{\eta}}$ is almost-free but not free by Lemma 4.4. From [**2**, Theorem 1.7] follows that $H_{\bar{\eta}}$ satisfies $\mathrm{Ext}(H_{\bar{\eta}}, W) = 0$ for every group W of size at most $|T|$ and it is easy to see that $H_{\bar{\eta}}$ has a λ-filtration as stated letting $\bar{H}_\alpha = \bigcup\{(H_\eta + K)/K : \eta \in B_{\bar{\eta}}, \sup(\mathrm{rg}(\eta)) < \alpha\}$. □

We want to apply Theorem 4.6 in models of ZFC to ladder systems which have μ-uniformization for all $\mu < \lambda$. This is possible for many regular cardinals as long as $|T|$ is small enough to obtain $2^{(|T|^\gamma)} < \lambda$, which is the case when λ is strongly inaccessible or the successor of a singular cardinal. The case $\lambda = \kappa^+$ with $\gamma = \kappa$ regular is not covered by Theorem 4.6, which explains our intention to show Theorem 4.9 next.

DEFINITION 4.7. Let κ be regular and $\lambda = \kappa^+$. A ladder system $\bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$ has *strong κ-uniformization* if for every system $\bar{P} = \langle P_\alpha : \alpha < \lambda \rangle$ such that

(i) $\emptyset \neq P_\delta \subseteq \{f | f : \mathrm{rg}(\eta_\delta) \to \delta \cap \kappa\}$ if $\delta \in S$;

(ii) if $\delta \in S$ and $i < \gamma$, then $P_{\eta_\delta(i)} = \{f \restriction_{\mathrm{rg}(\eta_\delta \restriction_{(i+1)})} | f \in P_\delta\}$;

(iii) if $\delta \in S$ and $i < \gamma$ is a limit ordinal, then for every increasing sequence $\langle f_j : j < i \rangle$, $f_j \in P_{\eta_\delta(j)}$ there exists $f_i \in P_{\eta_\delta(i)}$ which extends the union $\bigcup_{j<i} f_j$

there exists a function $f : \lambda \to \kappa$ such that $f \restriction_{\mathrm{rg}(\eta_\delta)} \in P_\delta$ for all $\delta \in S$.

PROPOSITION 4.8. *Let $\lambda = \gamma^+$ for some regular cardinal γ and let $\bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$ be a tree-like ladder system on S such that $\bar{\eta}$ has γ-uniformization and $\diamondsuit_\gamma$ holds. Then $\bar{\eta}$ has strong γ-uniformization.*

PROOF. Let $\bar{P}$ be given as in Definition 4.7 stated and let J be a stationary subset of γ such that $\diamondsuit_\gamma(J)$ holds. We may assume that $J = \gamma$. Thus there exists a system of diamond functions $\bar{h} = \langle h_\delta : \delta \to \gamma | \delta < \gamma \rangle$ such that for every function $h : \gamma \to \gamma$ the set $\{\delta < \gamma : h \restriction_\delta = h_\delta\}$ is stationary in γ. For each $\delta \in S$ and $i < \gamma$ we define

$$h_i^\delta : \mathrm{rg}(\eta_\delta \restriction_i) \to \gamma, (\eta_\delta(j) \mapsto h_i(j)) .$$

If $\delta \in S$ then let $E_\delta = \{i < \gamma : h_i^\delta \subseteq f \text{ for some } f \in P_\delta\}$ and choose $g_i^\delta \in P_\delta$ such that $g_i^\delta \restriction_{\mathrm{rg}(\eta_\delta \restriction_i)} = h_i^\delta$ for $i \in E_\delta$. Define $f_\delta : \mathrm{rg}(\eta_\delta) \to H(\gamma)$ for $\delta \in S$ as follows

$$f_\delta(\eta_\delta(i)) = \left\langle g_j^\delta \restriction_{\mathrm{rg}(\eta_\delta \restriction_{(i+1)})} : j \leq i, j \in E_\delta \right\rangle .$$

Here $H(\gamma)$ denotes the class of sets hereditarily of cardinality $< \gamma$. Note that $H(\gamma)$ has size $\leq \gamma$. By the γ-uniformization of $\bar{\eta}$ we can find $F : \lambda \to H(\gamma)$ such that for all $\delta \in S$ there exists $\alpha_\delta < \gamma$ with $f_\delta(\eta_\delta(i)) = F(\eta_\delta(i))$ for all $\alpha_\delta \leq i < \gamma$. For $i < \gamma$ let

$$F(\eta_\delta(i)) = \left\langle G_j^{\eta_\delta(i)} : j \leq i, j \in E_{\eta_\delta(i)} \right\rangle$$

for some $E_{\eta_\delta(i)} \subseteq \gamma$. Note, that $F(\eta_\delta(i))$ depends only on the value $\eta_\delta(i)$ and not on δ. Moreover, F is well-defined since $\bar{\eta}$ is tree-like.

We now define $f : \lambda \to \gamma$ on $\bigcup_{\delta \in S} \mathrm{rg}(\eta_\delta)$ and arbitrarily on the complement. We use induction on $i < \gamma$. For $i = 0$ choose any member $u \in P_{\eta_\delta(0)}$ and put $f(\eta_\delta(0)) = u(\eta_\delta(0))$. Now assume that $f(\eta_\delta(j))$ has been defined for $j < i$ and $\delta \in S$ such that for $j < i$, $f \restriction_{\mathrm{rg}(\eta_\delta \restriction_{(j+1)})} \in P_{\eta_\delta(j)}$. Put $\bar{f}_\delta = \{f(\eta_\delta(j)) : j < i\}$ and let

$$J_i^\delta = \{j \in E_{\eta_\delta(i)} : G_j^{\eta_\delta(i)} \restriction_{(\mathrm{rg}(\eta_\delta \restriction_i))} \subseteq \bar{f}_\delta\}.$$

If $j_i^\delta = \min(J_i^\delta)$ exists then let $f(\eta_\delta(i)) = G_{j_i^\delta}^{\eta_\delta(i)}(\eta_\delta(i))$. If $\min(J_i^\delta)$ does not exist, then we distinguish between two cases: if i is a limit ordinal, then Definition 4.7 (iii) implies that there is $f_i \in P_{\eta_\delta(i)}$ which extends $\bigcup_{j<i} f \restriction_{\mathrm{rg}(\eta_\delta \restriction_{(j+1)})}$. If i is a successor ordinal, then Definition 4.7 (ii) ensures that there is $f_i \in P_{\eta_\delta(i)}$ extending $f \restriction_{\mathrm{rg}(\eta_\delta \restriction_i)}$. In both cases put $f(\eta_\delta(i)) = f_i(\eta_\delta(i))$. Note that f is well-defined since $\bar{\eta}$ is tree-like, hence $\min(J_i^\delta)$ exists if and only if $\min(J_j^\nu)$ exists for $\eta_\delta(i) = \eta_\nu(j)$

$(\delta, \nu \in S,\ i, j < \gamma)$. It remains to check that $f \restriction_{\mathrm{rg}(\eta_\delta)} \in P_\delta$. By the uniformization we have $G_j^{\eta_\delta(i)} = g_j^\delta \restriction_{\mathrm{rg}(\eta_\delta \restriction_{(i+1)})}$ for all $j \leq i$, $j \in E_{\eta_\delta(i)}$, $i \geq \alpha_\delta$ and $\delta \in S$. We define

$$h : \gamma \to \gamma, (j \mapsto f(\eta_\delta(j)))\,.$$

By $\diamondsuit_\gamma$ there exists $\beta_\delta \geq \alpha_\delta$ such that $h \restriction_{\beta_\delta} = h_{\beta_\delta}$ and hence

$$f \restriction_{\mathrm{rg}(\eta_\delta \restriction_{\beta_\delta})} = h^\delta_{\beta_\delta} \restriction_{\mathrm{rg}(\eta_\delta \restriction_{\beta_\delta})} = h^\delta_{\beta_\delta}.$$

Thus $\beta_\delta \in J^\delta_{\beta_\delta}$ and $f(\eta_\delta(i)) = G_j^{\eta_\delta(i)}$ as above. Therefore

$$f \restriction_{\mathrm{rg}(\eta_\delta \restriction_{(\beta_\delta+1)})} = g^\delta_{\beta_\delta} \restriction_{\mathrm{rg}(\eta_\delta \restriction_{(\beta_\delta+1)})}$$

by definition of f. By induction on $i \geq \beta_\delta$ it follows that $f \restriction_{\mathrm{rg}(\eta_\delta)} = g^\delta_{\beta_\delta} \in P_\delta$ and this finishes the proof. $\square$

The extension of Theorem 4.6 is now immediate.

THEOREM 4.9. *Let $\lambda = \gamma^+$ and assume $\diamondsuit_\gamma$ holds. Moreover let $\bar{\eta}$ be a tree-like ladder system on a non-reflecting stationary subset S of λ whose elements have cofinality γ. If T is a group with $|T| < \lambda$ and G is (T, λ, γ)-suitable, then there is a torsion-free group H of size λ such that*

(i) *H has a λ-filtration $\langle \bar{H}_\alpha : \alpha < \lambda \rangle$;*
(ii) *if $\alpha \in S$, then $\bar{H}_{\alpha+1}/\bar{H}_\alpha \cong G$;*
(iii) *if $\alpha \notin S$, then $\bar{H}_\beta/\bar{H}_\alpha$ is free for all $\alpha \leq \beta$;*
(iv) $\mathrm{Ext}(W, T) = 0$ *for all groups W with $|W| \leq |T|$.*

PROOF. (which is similar to [2, Proposition 1.8]) Let S, $\bar{\eta}$ and G be as in Theorem 4.9 and choose the direct limit $H_{\bar{\eta}} = \bigoplus\{H_\eta : \eta \in B_{\bar{\eta}}\}/K$ as constructed above. Inspecting the proof of Theorem 4.6 we see that $H_{\bar{\eta}}$ is an almost-free non-free group of size λ which has the desired λ-filtration. It remains to show that $\mathrm{Ext}(H_{\bar{\eta}}, W) = 0$ for all groups W with $|W| \leq |T|$. We must show that any short exact sequence

$$(*) \qquad 0 \to W \stackrel{id}{\to} N \stackrel{\pi}{\to} H_{\bar{\eta}} \to 0$$

has a splitting map $\varphi : H_{\bar{\eta}} \to N$ such that $\varphi\pi = id \restriction_{H_{\bar{\eta}}}$. Choose any set function $u : H_{\bar{\eta}} \to N$ such that $u\pi = id \restriction_{H_{\bar{\eta}}}$. As in [**2**, Proposition 1.8] the splitting maps φ of π are in one-one correspondence with set mappings $h : H_{\bar{\eta}} \to W$ with $h(0) = 0$ such that for all $x, y \in H_{\bar{\eta}}$ and $z \in \mathbb{Z}$ the following hold.

(i) $zh(x) - h(zx) = zu(x) - u(zx)$ and
(ii) $h(x) + h(y) - h(x + y) = u(x) + u(y) - u(x + y)$

It is customary to denote $\mathrm{Trans}(H, W)$ to be the set of all these maps h for a fixed subgroup $H \subseteq H_{\bar{\eta}}$. Thus $(*)$ splits if and only if $\mathrm{Trans}(H_{\bar{\eta}}, W)$ is non-empty.
Fix a chain $\langle F_\alpha : \alpha \leq \gamma \rangle$ for G as in Definition 3.1. For $\delta \in S$, $i < \gamma$ and $h \in \mathrm{Trans}(H_{\eta_\delta \restriction_i}, W)$ let

$$\mathrm{seq}(h) : \mathrm{rg}(\eta_\delta \restriction_i) \to W \left(\eta_\delta(j) \mapsto h \restriction_{F_{1(\eta_\delta \restriction_j)}} \ (j < i) \right).$$

For $\delta \in S$ let $P_\delta = \{\mathrm{seq}(h) : h \in \mathrm{Trans}(H_{\eta_\delta}, W)\}$ and for $i < \gamma$ put $P_{\eta_\delta(i)} = \{\mathrm{seq}(h) : h \in \mathrm{Trans}(H_{\eta_\delta \restriction_{(i+1)}}, W)\}$. Let $P_\alpha = \emptyset$ if it has not been defined yet $(\alpha < \lambda)$. By Proposition 4.8 the ladder system $\bar{\eta}$ has strong γ-uniformization and it is easy to check that the system $\bar{P} = \langle P_\alpha : \alpha \in \lambda \rangle$ satisfies the conditions of Definition 4.7 since F_n and F_n/F_m are free for $m < n \leq \gamma$. Since $|W| \leq \gamma$ there exists a function $f : \lambda \to W$ such that $f \restriction_{\mathrm{rg}(\eta_\delta)} \in P_\delta$ for all $\delta \in S$. We now define $h : H_{\bar{\eta}} \to W$ by putting $h \restriction_{H_{\eta_\delta}} = f \restriction_{\mathrm{rg}(\eta_\delta)}$ and clearly h is well-defined and belongs to $\mathrm{Trans}(H_{\bar{\eta}}, W)$ and therefore $(*)$ splits. $\square$

5. *The Forcing Theorem*

Before we state the main theorem of this section we describe our strategy. Using class forcing we shall construct a model of ZFC satisfying GCH in which for every regular cardinal λ there exists a sequence of stationary non-reflecting subsets S_α of λ of length λ^+ on which we have "enough" uniformization for some ladder system. Using this and the existence of (T, λ, γ)-suitable groups (for some particular γ) we can then construct, for a given group T, a sequence of torsion-free groups G_α ($\alpha < \lambda^+$) of cardinality λ satisfying $\mathrm{Ext}(G_\alpha, T) = 0$. These G_α will have λ-filtrations $\langle G_{\alpha,\delta} : \delta < \lambda \rangle$ whose quotients satisfy $\mathrm{Ext}(G_{\alpha,\delta+1}/G_{\alpha,\delta}, T) \neq 0$ for $\delta \in S_\alpha$. We shall show that all these groups are not subgroups of a single group $G \in \mathcal{TF}_\lambda(T)$ since this would force $\mathrm{Ext}(G, T) \neq 0$. Thus there can not be any λ-universal group for the group T.

THEOREM 5.1. *Let V be a model of ZFC in which GCH holds. Then for some class forcing $\mathcal{P}$ not collapsing cardinals and preserving GCH the following is true in $V^{\mathcal{P}}$:*
Case A: If $\lambda > \gamma$ are infinite regular cardinals such that $\gamma = \mathrm{cf}(\mu)$ for $\lambda = \mu^+$ then

(i) *there is a normal ideal $J = J^\lambda_\gamma$ on λ;*
(ii) *there is a stationary subset $S = S^\lambda_\gamma$ of λ such that $S \notin J$;*
(iii) *if $\delta \in S$, then $\mathrm{cf}(\delta) = \gamma$;*
(iv) *S is non-reflecting, i.e. $S \cap \alpha$ is not stationary in α for every $\alpha < \lambda$;*
(v) *if $S' \subseteq S$ is stationary in λ, then there is a stationary $S^* \in J$ such that $S^* \subseteq S'$;*
(vi) *if $S' \subseteq S$ and $S' \notin J$, then $\diamondsuit_{S'}$ holds;*
(vii) *if $S' \subseteq S$ is stationary and $S' \in J$, then there exists a tree-like ladder system on S' which has μ-uniformization for all $\mu < \kappa$ if $\lambda = \kappa^+$ and κ is singular, and for all $\mu < \lambda$ otherwise;*
(viii) *there are $S_\epsilon = S^\lambda_{\gamma,\epsilon} \in J$ for $\epsilon < \lambda^+$ such that*
  (a) *if $\eta < \epsilon < \lambda^+$, then $S_\eta \setminus S_\epsilon$ is bounded;*
  (b) *if $\epsilon < \lambda^+$, then $S_{\epsilon+1} \setminus S_\epsilon$ is stationary;*
  (c) *$J = \{S' \subseteq S : \exists \epsilon < \lambda^+ \forall \epsilon < \nu < \lambda^+ \ S' \setminus S_\nu$ is not stationary $\}$.*

Case B: If $\lambda = \mathrm{cf}(\lambda) > \gamma$, then there is a stationary $S^* = S^{\lambda,*}_\gamma$ such that

(1) *if $\alpha \in S^*$, then $\mathrm{cf}(\alpha) = \gamma$;*
(2) *S^* is non-reflecting;*
(3) *$\diamondsuit_{S^*}$ holds.*

The proof of Theorem 5.1 will be divided into several steps. First we deal with a fixed regular cardinal λ and then use Easton-support iteration to put the forcings together. We assume knowledge on forcing and our notation will follow [**9**] with the exception that $p \leq q$ means that the condition q is stronger than the condition p. Recall that a poset P is called λ-*complete* if for every cardinal $\kappa < \lambda$, every ascending chain

$$p_0 \leq p_1 \leq \cdots \leq p_\alpha \quad (\alpha < \kappa)$$

has an upper bound. Moreover, P is said to be λ-*strategically complete* if Player I has a winning strategy in the following game of length κ for every $\kappa < \lambda$. Players I and II alternately choose an ascending sequence

$$p_0 \leq p_1 \leq \cdots \leq p_\alpha \quad (\alpha < \kappa)$$

of elements of P, where Player I chooses at the even ordinals; Player I wins if and only if at each stage there is a legal move and the whole sequence, $\langle p_\alpha : \alpha < \kappa \rangle$ has an upper bound (see also [**13**, Definition A1.1]). Note that, if P is λ-strategically complete and G is generic over P, then $V[G]$ has no new functions from κ into V for all $\kappa < \lambda$, hence cardinals $\leq \lambda$ and their cofinalities are preserved.

PROPOSITION 5.2. *Let λ be a regular cardinal with $\lambda^{<\lambda} = \lambda$. For any regular $\kappa < \lambda$, there exists a poset Q of cardinality $\leq \lambda$ which is λ-strategically complete (and hence preserves all cardinals and preserves cofinalities $\leq \lambda$) with the following property: For G generic over P, in $V[G]$ there exists a non-reflecting stationary and co-stationary subset S of λ such that every member of S has cofinality κ.*

PROOF. The proof is similar to the proof of [**2**, Lemma 2.3]. We let Q be the set of all functions $q : \alpha \to 2 = \{0, 1\}$ $(\alpha < \lambda)$ such that $q(\mu) = 1$ implies that $\mathrm{cf}(\mu) = \kappa$ and such that for all limits $\delta \leq \alpha$, the intersection of $q^{-1}[1]$ with δ is not stationary in δ. Then, for G generic over P,

$$S = \bigcup \{q^{-1}[1] : q \in G\}$$

will be the desired set. We have to prove that S is stationary and co-stationary in λ. Hence, assume that q forces f is the name of a continuous increasing function $\bar{f} : \lambda \to \lambda$; choose an ascending chain

$$q_o \leq q_1 \leq \cdots \leq q_\alpha \quad (\alpha < \kappa)$$

such that for each α there exist $\beta_\alpha, \gamma_\alpha$ such that $q_\alpha \Vdash \bar{f}(\beta_\alpha) = \gamma_\alpha$ and

$$\mathrm{dom}(q_\alpha) \geq \gamma_\alpha > \mathrm{dom}(q_\mu)$$

for all $\mu < \alpha$. Let $\delta = \sup\{\gamma_\alpha : \alpha < \kappa\} = \sup\{\mathrm{dom}(q_\alpha) : \alpha < \kappa\}$ and let

$$q_i = \bigcup \{q_\alpha : \alpha < \kappa\} \cup \{(\delta, i)\},$$

for $i = 0, 1$. Then $q_i \in Q$ $(i = 0, 1)$ since $q_i^{-1}[1]$ is not stationary in δ, because δ has cofinality κ. Moreover, $q_1 \Vdash \delta \in \mathrm{rg}(f) \cap S$ and $q_0 \Vdash \delta \in \mathrm{rg}(f) \cap (\lambda \backslash S)$.
Since Q has cardinality $\leq \lambda$, it preserves cardinals $> \lambda$. To show that all cardinals $\leq \lambda$ are preserved (and their cofinalities), it suffices to prove that Q is λ-strategically complete. Let $\tau < \lambda$ be a limit ordinal. Let Player I choose q_α for even α such that $\mathrm{dom}(q_\alpha)$ is a successor ordinal, say $\delta_\alpha + 1$, and $q_\alpha(\delta_\alpha) = 0$. Moreover, at limit ordinals α he chooses q_α to have domain $= \sup\{\delta_\beta : \beta < \alpha\} + 1$. Then $q = \bigcup\{q_\alpha : \alpha < \mu\}$ is a member of Q because $\{\delta_\alpha : \alpha < \mu, \alpha \text{ even }\}$ is a cub in $\mathrm{dom}(q)$ which misses $q^{-1}[1]$. This is a winning strategy for Player I and thus Q is λ-strategically complete. □

The next proposition is a collection of results from [**12**], [**13**] and [**14**] (see also [**15**]).

PROPOSITION 5.3. *Let $\lambda > \gamma$ be regular infinite cardinals. Moreover, assume $\lambda^{<\lambda} = \lambda$, $2^\lambda = \lambda^+$ and let S be a non-reflecting, stationary and co-stationary subset of λ such that each member of S has cofinality γ. Furthermore, let $\gamma = \mathrm{cf}(\kappa)$ if $\lambda = \kappa^+$.*

Then there exists a poset P of cardinality $\leq \lambda^+$ which is λ-strategically complete, satisfies the λ^+-chain condition, adds no new sequences of length $< \lambda$ and has the following properties:

(i) *S is non-reflecting, stationary and co-stationary in λ as element in V^P;*
(ii) *if λ is inaccessible, then every ladder system on S has μ-uniformization for all $\mu < \lambda$; in particular, there exists a tree-like ladder system on S;*
(iii) *if $\aleph_2 \leq \lambda = \kappa^+$ and κ is regular, then every ladder system on S has μ-uniformization for all $\mu < \lambda$; in particular, there exists a tree-like ladder system on S;*
(iv) *if $\lambda = \aleph_1$, then there is a tree-like ladder system on S which has μ-uniformization for all $\mu < \lambda$;*
(v) *if $\lambda = \kappa^+$ and κ is singular, then there is a tree-like ladder system on S which has μ-uniformization for all $\mu < \kappa$.*

Proof. For λ inaccessible the proof is contained in [**13**, Case A] and also for the case of $\lambda = \kappa^+$, κ regular (see [**13**, Case B]). For $\lambda = \aleph_1$ see [**12**, Theorem 1.7] and for $\lambda = \kappa^+$, κ singular see [**14**, Theorem 2.10, Theorem 2.12]. Moreover, simpler versions with less complicated and comprehensive proofs can be found in [**11**] for all cases if we drop the requirements "for every ladder system..." which is in fact not needed for our purposes. Finally, let us remark that co-stationarity is only needed when λ is inaccessible or a successor of a regular cardinal. $\square$

Theorem 5.4. *Let λ be a regular cardinal such that $\lambda^{<\lambda} = \lambda$ and $2^\lambda = \lambda^+$. Then there is a poset P of cardinality $\leq \lambda^+$ satisfying the λ^+-chain condition which is λ-strategically complete and adds no new sequences of length $< \lambda$ such that in V^P for every regular $\gamma < \lambda$ with $\gamma = \mathrm{cf}(\kappa)$ if $\lambda = \kappa^+$ the statements of Case A and Case B of Theorem 5.1 hold.*

First we deduce Theorem 5.1 from Theorem 5.4.

Proof. Let V be a model of ZFC with GCH. For any ordinal α let $P_\alpha = \left\langle P_j, \dot{Q}_i : j \leq \alpha, i < \alpha \right\rangle$ be an iteration with Easton support; this is to say we take direct limits when $\aleph_\alpha$ is regular and inverse limits elsewhere or equivalently we have bounded support below inaccessibles and full support below non-inaccessibles. For any ordinal i, let $\dot{Q}_i$ be the forcing notion in V^{P_i} described in Theorem 5.4 for $\lambda = \aleph_i$ if $\aleph_i$ is regular and let Q be 0 elsewhere. Let P be the direct limit of the P_α (α an ordinal). We claim that P has the desired properties. The proof is very similar to [**2**, Theorem 2.1] and hence we shall only state the main ingredients which are needed.

(i) For every κ and Easton support iteration $\left\langle P_j, \dot{Q}_i : \kappa \leq j \leq \alpha, \kappa \leq i < \alpha \right\rangle$, if each $\dot{Q}_i$ is κ-strategically complete, then so is P_α.
(ii) $P = P_\alpha * P_{\geq \alpha}$, where, in V^{P_α}, $P_{\geq\alpha}$ is the direct limit of P^α_β (β an ordinal), with P^α_β the Easton support iteration $\left\langle P^\alpha_j, \dot{Q}^\alpha_i : j \leq \beta, i < \beta \right\rangle$ where $\dot{Q}^\alpha_i = \dot{Q}_{\alpha+i}$.
(iii) $|P_n| = 1$ (for $n \in \omega$); if $\aleph_\delta$ is singular, $|P_\delta| \leq \aleph_\delta$ and $|P_\delta| \leq \aleph_{\delta+1}$ if $\aleph_\delta$ is regular, hence inaccessible.
(iv) $P_{\geq\alpha}$ is $\aleph_\alpha$-strategically complete, and $P_{\geq\alpha+1}$ is even $\aleph_{\alpha+n}$- strategically complete for all $n \in \omega$.

By construction (i) to (viii) and (1) to (3) of Theorem 5.1 are now satisfied in V^P. Note, that stationarity is preserved in the iteration because $P_{\geq\alpha}$ is $\aleph_\alpha$-strategically complete. It remains to prove that V^P is a model of ZFC satisfying GCH and preserving cofinalities (and hence cardinals). This follows as in [**2**, Theorem 2.1] and hence we will omit the proof. $\square$

It remains to prove Theorem 5.4.

Proof. The proof follows from results in [**13**] and [**14**] but for the convenience of the reader we shall give some details. If $\lambda = \kappa^+$ is a successor cardinal, then the proof is immediate. There is only one γ with $\gamma = \mathrm{cf}(\kappa)$. We choose P to be the two step iterated forcing of the two forcings from Proposition 5.2 and from Proposition 5.3 with $\gamma = \mathrm{cf}(\kappa)$. Moreover, we may assume that P also forces the sets $S^* = S^{\lambda,*}_\gamma$ satisfying Theorem 5.1 (1) to (3) by an initial forcing. Note that the assumptions on λ in Theorem 5.4 are satisfied by [**6**, Exercise 12, page 70]. If λ is inaccessible, then the argument is more complicated since we have to deal with all regular cardinals $\gamma < \lambda$. But this was established in [**13**, Case B] where a stronger version of Proposition 5.3 was shown. It was proved that there is a forcing notion P such that for all regular $\gamma < \lambda$ and given non-reflecting stationary, co-stationary subsets S_γ of λ consisting of ordinals of cofinality γ, every ladder system on S_γ has μ-uniformization for all $\mu < \lambda$. Using this stronger result and again forcing the sets $S^{\lambda,*}_\gamma$ satisfying Theorem 5.1 (1) to (3) it remains to show that we can define the ideal $J = J^\lambda_\gamma$ satisfying Theorem 5.1 (vi) and (viii) (Theorem 5.1 (vii) is clear).
Our forcing P (from [**13**] and [**14**]) is the result of a ($< \lambda$)-support iteration of length λ^+, say $\left\langle P_i, \dot{Q}_j : i \leq \lambda^+, j < \lambda^+ \right\rangle$. Let us assume that P_γ forces the set $S = S^\lambda_\gamma$ and the tree-like ladder system $\bar{\eta}$ on S. In V^{P_γ} there exists a sequence $\left\langle S_\epsilon = S^\lambda_{\gamma,\epsilon} : \epsilon < \lambda^+ \right\rangle$ such that

(i) $S_\epsilon \subseteq S$;
(ii) $\eta < \epsilon < \lambda^+$ implies $S_\eta \backslash S_\epsilon$ is bounded;
(iii) $\epsilon < \lambda^+$ implies $S_{\epsilon+1} \backslash S_\epsilon$ is stationary.

Now we define $J = J^\lambda_\gamma$ as $J = \{S' \subseteq S : \exists \epsilon < \lambda^+ \forall \epsilon < \nu < \lambda^+ \ S' \backslash S_\nu$ is not stationary $\}$. For each $i < \lambda^+$, $\dot{Q}_i$ forces μ_i-uniformization for the ladder system $(\dot{A}_i, \dot{f}_i)$ where $\dot{f}_i = \left\langle \dot{f}^i_\delta : \delta \in \dot{S}_i \right\rangle$. Here $\dot{S}_i$ and $\dot{f}^i_\delta$ are P_i-names for a member of ${}^{A_\delta}(\mu_i)$. Thus we obtain $\Vdash$ " $\dot{S}_i \subseteq \dot{S}$ is stationary and there is $\epsilon < \lambda^+$ such that $(\forall \epsilon < \eta < \lambda^+)(\dot{S}_i \cap \dot{S}_\eta \backslash \dot{S}_\epsilon)$ is not stationary ". A condition in $\dot{Q}_i$ is for instance given by $g : \alpha \to \mu_i$ such that $\delta \in \dot{S}_i$, $\delta \leq \alpha$ implies $f^i_\delta \subseteq^* g$.
It remains to show that $\diamondsuit_{S'}$ holds for $S' \notin J$. Choose $i < \lambda^+$ such that S' comes from V^{P_i}. For some $j \in (i, \lambda^+)$, Q_j is adding λ Cohen reals and we can interprete it as adding a diamond sequence $\langle \rho_\epsilon : \epsilon \in S' \rangle$ by initial segments. Trivially, in $V^{P_{j+1}}$, $\diamondsuit_{S'}$ holds and we may work in $V^{P_{j+1}}$ now. For χ large enough we can find for every $x \in H(\chi(\lambda))$ an increasing continuous sequence $\bar{N} = \langle N_i : i < \lambda \rangle$ of elementary submodels of $H(\chi(\lambda), \epsilon, <^*)$ of cardinality less than λ such that $x \in N_0$, $S' \in N_0$ and $\bar{N} \restriction_{(i+1)} \in N_{i+1}$ for all $i < \lambda$. Let $E = \{\delta < \lambda : N_\delta \cap \lambda = \delta\}$ which is a cub in λ. Thus, for $\delta \in E$, for every $p \in (P/P_{j+1}) \cap N_\delta$ there is a condition $p \leq q \in P/P_{j+1}$ which is $(N_\delta, P/P_{j+1})$ generic and forces a value to $G \cap N_\delta$. It is known that we can now replace the diamond sequence on S' which we have in $V^{P/P_{j+1}}$ by one that is preserved by forcing with P/P_{j+1} since P/P_{j+1} adds no

new subsets of λ of length less than λ and by the strategically completeness. This finishes the proof. $\square$

6. *Application to Kulikov's question*

Finally we show that there is a model of ZFC and GCH in which Kulikov's question has a negative answer, this is to say that for a given group T and any cardinal λ large enough there is no λ-universal group for T. This is in contrast to the results in Gödel's universe $V = L$, see [**16**].

Definition 6.1. Let T, H be groups, H torsion-free with $|T| < \lambda = |H|$ where H has a λ-filtration $\{H_\alpha : \alpha < \lambda\}$. Moreover, let S be as in Theorem 5.1 (ii). Then

$$S^\lambda_\gamma[H, T] = \{\delta \in S : \mathrm{Ext}(H_{\delta+1}/H_\delta, T) \neq 0\}.$$

Note that $S^\lambda_\gamma[H, T]$ depends on the filtration of H, which could be traded into an invariant of H, namely $\Gamma^S_H(T) = \{E \subseteq S :$ there exists a cub $C \subseteq \lambda$ such that $E \cap C = S^\lambda_\gamma[H, T] \cap C\}$.

Theorem 6.2. *Let T, H be groups, H torsion-free with $|T| < \lambda = |H|$. If $S = S^\lambda_\gamma[H, T]$ and $\diamondsuit_S$ holds, then* $\mathrm{Ext}(H, T) \neq 0$.

Proof. The proof is standard and can be found for instance in [**1**, Theorem 1.15, page 353]. $\square$

We shall now work in the model $V^{\mathcal{P}}$ obtained in Theorem 5.1. Thus all results shall be consistent with ZFC and GCH. The symbol $(V^{\mathcal{P}})$ indicates that the statement holds in our model $V^{\mathcal{P}}$.

Theorem 6.3 $(V^{\mathcal{P}})$. *Let T be an abelian group with $|T| < \lambda$ and G be (T, λ, γ)-suitable for $\gamma < \lambda$ regular, $\gamma = \mathrm{cf}(\mu)$ for $\lambda = \mu^+$. If $\lambda > 2^{(|T|^\gamma)}$ or μ is regular, then there is no λ-universal group for T.*

Proof. Assume that there is a λ-universal group U for T. If $\lambda > 2^{(|T|^\gamma)}$, then for $\epsilon < \lambda^+$ we apply Theorem 4.6 to $S^\lambda_{\gamma,\epsilon}$, T, G and the tree-like ladder system $\bar\eta_\epsilon$ on $S^\lambda_{\gamma,\epsilon}$ which comes from Theorem 5.1 (vii). Note that no $S^\lambda_{\gamma,\epsilon}$ reflects in any $\alpha < \lambda$ and that $\bar\eta_\epsilon$ has $2^{(|T|^\gamma)}$-uniformization since $\lambda > 2^{(|T|^\gamma)}$ (and $\mu > 2^{(|T|^\gamma)}$ if $\lambda = \mu^+$, μ singular). If μ is regular, then we apply Theorem 4.9 instead of Theorem 4.6. Note that $\diamondsuit_\mu$ holds in $V^{\mathcal{P}}$ by Theorem 5.1 (1) to (3). Hence for each $\epsilon < \lambda^+$ we obtain a torsion-free group $H_\epsilon = \bigcup_{\alpha\in\lambda} H_{\epsilon,\alpha}$ satisfying $\mathrm{Ext}(H_\epsilon, T) = 0$ and $\mathrm{Ext}(H_{\epsilon,\alpha+1}/H_{\epsilon,\alpha}, T) \neq 0$ for all $\alpha \in S^\lambda_\gamma[H_\epsilon, T] = S^\lambda_{\gamma,\epsilon}$. The universal group U allows an embedding $i_\epsilon : H_\epsilon \longrightarrow U$ for all $\epsilon \leq \lambda^+$. We claim that $S^\lambda_{\gamma,\epsilon}[H_\epsilon, T] \subseteq S^\lambda_\gamma[U, T]$ modulo a non-stationary set for each $\epsilon < \lambda^+$. To see this choose a λ-filtration $U = \bigcup_{\alpha\in\lambda} U_\alpha$ of U such that $\mathrm{Ext}(U_{\alpha+1}/U_\alpha, T) = 0$ if and only if for some $\beta > \alpha$ we have $\mathrm{Ext}(U_\beta/U_\alpha, T) = 0$. Fix $\epsilon < \lambda^+$, then there is a cub $C_\epsilon \subseteq \lambda$ such that for all $\alpha \in C_\epsilon$ we have $H_{\epsilon,\alpha} = U_\alpha \cap H_\epsilon$. Thus for $\alpha < \beta \in C_\epsilon$ it follows that $H_{\epsilon,\beta}/H_{\epsilon,\alpha} = (U_\beta \cap H_\epsilon)/(U_\alpha \cap H_\epsilon) \subseteq U_\beta/U_\alpha$ and hence $\mathrm{Ext}(H_{\epsilon,\beta}/H_{\epsilon,\alpha}, T) \neq 0$ implies $\mathrm{Ext}(U_\beta/U_\alpha, T) \neq 0$. Therefore also $\mathrm{Ext}(U_{\alpha+1}/U_\alpha, T) \neq 0$ and $C_\epsilon \subseteq S^\lambda_\gamma[U, T]$. Thus by the definition of the normal ideal J (see Theorem 5.1 (viii)) we have $\bar S = S^\lambda_\gamma[U, T] \notin J$ and therefore $\diamondsuit_{\bar S}$ holds by Theorem 5.1 (vi). Hence $\mathrm{Ext}(U, T) \neq 0$ by Theorem 6.2 - a contradiction. $\square$

From $(V^P \Vdash GCH)$ follows $2^{(|T|^\gamma)} \leq max\{\gamma^{++}, |T|^{++}\}$ for any group T.

Corollary 6.4 ($V^{\mathcal{P}}$). *Let T be a group not cotorsion with $|T| < \lambda$. If λ is strongly inaccessible, then there is no λ-universal group for T.*

Proof. Since λ is strongly inaccessible it is a limit ordinal and we may choose $\gamma = \omega$. Moreover, for every $\alpha < \lambda$ we have $2^\alpha < \lambda$, hence $\lambda > 2^{(|T|^\omega)}$. Lemma 3.2 implies that there is a (T, λ, ω)-suitable group for T and hence Theorem 6.3 shows that there is no λ-universal group for T. □

Corollary 6.5 ($V^{\mathcal{P}}$). *Let T be a group, not cotorsion such that $|T|^+ < \lambda$. If $\lambda = \mu^+$ and $\mathrm{cf}(\mu) = \omega$, then there is no λ-universal group for T.*

Proof. There is a (T, λ, ω)-suitable group by Lemma 3.2. Since $\mathrm{cf}(\mu) = \omega$, we have $\lambda > 2^{(|T|^\omega)}$ and hence we may choose $\gamma = \omega$ and apply Theorem 6.3 to see that there is no λ-universal group for T. □

Corollary 6.6 ($V^{\mathcal{P}}$). *Let T be a group, not cotorsion and H an epimorphic image of T such that $|T| < \lambda$ and $|H| \leq \mathrm{cf}(\mu)$ if $\lambda = \mu^+$. If μ exists and is singular let $2^{(|T|^{\mathrm{cf}(\mu)})} < \lambda$ and $2^{(|T|^\omega)} < \lambda$ otherwise. Then T has no λ-universal group.*

Proof. We choose $\gamma = \omega$ if λ is a limit ordinal and $\gamma = \mathrm{cf}(\mu)$ if $\lambda = \mu^+$. By Theorem 5.1 (1) to (3) there exists a stationary non-reflecting set $S \subseteq \gamma$ consisting of limit ordinals of cofinality ω such that $\diamondsuit_S$ holds. By assumption we may apply Proposition 3.5 to S, H and λ, γ to obtain a (T, λ, γ)-suitable group G. Since $\lambda > 2^{(|T|^\gamma)}$ or μ is regular we apply Theorem 6.3 to see that there is no λ-universal group for T. □

Corollary 6.7 ($V^{\mathcal{P}}$). *For any group T, not cotorsion and any cardinal λ there exists a regular uncountable cardinal $\delta \geq \lambda$ such that there is no δ-universal group for T.*

Corollary 6.8 ($V^{\mathcal{P}}$). *If T is a torsion group, not cotorsion with $|T|$ regular, then there is no λ-universal group for T for any regular cardinal $\lambda > |T|$.*

Proof. Let T and λ be as stated. If λ is strongly inaccessible , then Corollary 6.4 applies. Hence assume that $\lambda = \kappa^+$. Thus $|T| \leq \kappa$ and by Proposition 3.5 there exists a $(T, \lambda, \mathrm{cf}(\kappa))$-suitable group G for T. If κ is regular then Theorem 6.3 shows that there is no λ-universal group for T. If κ is singular, then $\kappa > |T|^{++}$ since $|T|$ is regular. Hence $\lambda > 2^{(|T|^{\mathrm{cf}(\kappa)})}$ and again Corollary 6.8 follows from Theorem 6.3. □

Corollary 6.9 ($V^{\mathcal{P}}$). *There is no λ-universal Whitehead group for all $\lambda > \aleph_2$.*

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Institute of Mathematics, The Hebrew University, Givat Ram, Jerusalem 91904, Israel and, Rutgers University, Newbrunswick, NJ U.S.A.
E-mail address: `Shelah@math.huji.ac.il`

Institute of Mathematics, The Hebrew University, Givat Ram, Jerusalem 91904, Israel
E-mail address: `lutz@math.huji.ac.il`