SURPRISINGLY SHORT

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ABSTRACT. This note is a compilation of several results in set theory which has surprisingly short proofs. From time to time (e.g., whenever I write down notes for my tutor lectures), more results will be added to this compilation.

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1. Jensen's diamond principle is (nearly) a cardinal arithmetic statement

Recall that for a cardinal θ (of uncountable cofinality) and a stationary set $S \subseteq \theta$, $\diamondsuit_S$ is said to hold iff there exists a collection $\{S_\delta \mid \delta \in S\}$ such that for any $Z \subseteq \theta$, the set $\{\delta \in S \mid S_\delta = Z \cap \delta\}$ is stationary.

Evidently, if λ is a cardinal and $\{S_\delta \mid \delta \in \lambda^+\}$ is a collection witnessing $\diamondsuit_{\lambda^+}$, then $[\lambda^+]^{\leq\lambda} = \{X \subseteq \lambda^+ \mid |Z| \leq \lambda\} \subseteq \{S_\delta \mid \delta \in \lambda^+\}$. In particular, $\diamondsuit_{\lambda^+}$ implies $2^\lambda = \lambda^+$.

The result of this section was announced in [3] and deals with the inverse implication. The simplified presentation is due to Péter Komjáth.

Theorem (Shelah)**.** *Suppose λ is a cardinal satisfying $2^\lambda = \lambda^+$.*
Then $\diamondsuit_S$ holds for any stationary $S \subseteq \{\delta < \lambda^+ \mid \mathrm{cf}(\delta) \neq \mathrm{cf}(\lambda)\}$.

Proof. Fix a stationary set S as above. In particular, λ is uncountable. To avoid trivialities, we may also assume that $S \cap \lambda = \emptyset$ and that S contains no successor ordinals. Set $\kappa := \mathrm{cf}(\lambda)$. For each $\delta \in S$, let $\{A_i^\delta \mid i < \kappa\}$ be an increasing chain of elements of $[\delta]^{<\lambda}$ satisfying $\delta = \bigcup_{i<\kappa} A_i^\delta$.

For all $\delta \in S$, since $\mathrm{cf}(\delta) < \lambda$, we may also assume that $\sup(A_0^\delta) = \delta$.

Notation. For $X \subseteq I \times Y$ and $i \in I$, write $(X)_i = \{y \mid (i, y) \in X\}$.

Lemma 1.1. *Suppose $\{X_\beta \mid \beta < \lambda^+\}$ is an enumeration of $[\kappa \times (\lambda \times \lambda^+)]^{\leq\lambda}$.*
Then there exists some $i < \kappa$ such that for all $Z \subseteq \lambda \times \lambda^+$, the following is stationary:

$$S_{i,Z} := \left\{\delta \in S \mid \sup\{\alpha \in A_i^\delta \mid \exists\beta \in A_i^\delta (Z \cap (\lambda \times \alpha) = (X_\beta)_i)\} = \delta\right\}.$$

Proof. Suppose not. Then for all $i < \kappa$, we may find some $Z_i \subseteq \lambda \times \lambda^+$ and a club $D_i \subseteq \lambda^+$ that avoids S_{i,Z_i}. Define $f : \lambda^+ \to \lambda^+$ by:

$$f(\alpha) := \min\{\beta < \lambda^+ \mid X_\beta = \bigcup_{j<\kappa} \{j\} \times (Z_j \cap (\lambda \times \alpha))\}.$$

Let $D \subseteq \bigcap_{i<\kappa} D_i$ be a club such that $f(\alpha) < \delta$ for all $\alpha < \delta \in D$.

Clearly, for $\delta \in D$:

$$A_0^\delta = \{\alpha \in A_0^\delta \mid \exists\beta < \delta \forall j < \kappa (Z_j \cap (\lambda \times \alpha) = (X_\beta)_j)\}.$$

Fix $\delta \in D \cap S$. For $i < \kappa$, write:

$$B_i^\delta := \{\alpha \in A_0^\delta \mid \exists\beta \in A_i^\delta \forall j < \kappa (Z_j \cap (\lambda \times \alpha) = (X_\beta)_j)\}.$$

By $A_0^\delta = \bigcup_{i<\kappa} B_i^\delta$, $\sup A_0^\delta = \delta$ and $\mathrm{cf}(\delta) \neq \kappa$, there must exist some $i < \kappa$ with $\sup(B_i^\delta) = \delta$. In particular:

$$\sup\{\alpha \in A_i^\delta \mid \exists\beta \in A_i^\delta (Z_i \cap (\lambda \times \alpha) = (X_\beta)_i)\} = \delta,$$

i.e., $\delta \in S_{i,Z_i}$. A contradiction to $\delta \in D_i$. $\square$

Corollary 1.2. *There exists a sequence* $\langle A^\delta \in [\delta]^{<\lambda} \mid \delta \in S\rangle$*, and an enumeration* $\{X_\beta \mid \beta < \lambda^+\} = [\lambda \times \lambda^+]^{\leq\lambda}$ *such that for all* $Z \subseteq \lambda \times \lambda^+$*, the following set is stationary:*

$$S_Z := \left\{\delta \in S \mid \sup\{\alpha \in A^\delta \mid \exists\beta \in A^\delta(Z \cap (\lambda \times \alpha) = X_\beta)\} = \delta\right\}.$$

Proof. Take i as above, and consider $\langle A^\delta_i \mid \delta \in S\rangle$ and $\{(X_\beta)_i \mid \beta < \lambda^+\}$. □

Let us fix such sequence $\langle A^\delta \mid \delta \in S\rangle$ and enumeration $\{X_\beta \mid \beta < \lambda^+\}$.

We shall now recursively define a sequence of subsets of λ^+, $\langle Y_\tau \mid \tau < \lambda\rangle$, and a $\subseteq$-decreasing sequence of clubs of λ^+, $\langle E_\tau \mid \tau < \lambda\rangle$.

Notation. Whenever $\langle Y_\tau \mid \tau < \gamma\rangle$ is defined, we shall denote for $\delta \in S$:

$$V^\delta_\gamma = \left\{(\alpha, \beta) \in A^\delta \times A^\delta \mid \forall\tau < \gamma\,(Y_\tau \cap \alpha = (X_\beta)_\tau)\right\}.$$

We start the recursion by letting $E_0 = Y_0 = \lambda^+$. Suppose now $\langle (Y_\tau, E_\tau) \mid \tau < \gamma\rangle$ has been defined for some $\gamma < \lambda$. Clearly, for any set Y_γ, and any $\delta \in S$, we would have $V^\delta_\gamma \supseteq V^\delta_{\gamma+1}$. If there exists a set $Y_\gamma \subseteq \lambda^+$ and a club $E_\gamma \subseteq \bigcap_{\tau<\gamma} E_\tau$ such that for all $\delta \in E_\gamma \cap S$:

$$\sup\{\alpha < \delta \mid \exists\beta < \delta\left((\alpha, \beta) \in V^\delta_\gamma\right)\} = \delta \text{ implies } V^\delta_\gamma \neq V^\delta_{\gamma+1},$$

then continue the recursion with such Y_γ and E_γ. Otherwise, terminate the recursion.

Claim 1.3. *The recursion must terminate at some* $\gamma^* < \lambda$.

Proof. Suppose not, and let $\langle Y_\tau \mid \tau < \lambda\rangle$, $\langle E_\tau \mid \tau < \lambda\rangle$ be the output sequences. Put $E = \bigcap_{\tau<\lambda} E_\tau$ and $Z = \bigcup_{\tau<\lambda}\{\tau\} \times Y_\tau$.

Fix $\delta \in E \cap S_Z$. Then by definition of S_Z:

$$\sup\{\alpha \in A^\delta \mid \exists\beta \in A^\delta(Z \cap (\lambda \times \alpha) = X_\beta)\} = \delta.$$

In other words:

$$\sup\{\alpha \in A^\delta \mid \exists\beta \in A^\delta\forall\tau < \lambda(Y_\tau \cap \alpha = (X_\beta)_\tau)\} = \delta.$$

It follows that $\sup\{\alpha < \delta \mid \exists\beta < \delta\left((\alpha, \beta) \in V^\delta_\gamma\right)\} = \delta$ for all $\gamma < \lambda$. Since $S_Z \subseteq S$, the recursive construction gives that $\langle V^\delta_\gamma \mid \gamma < \lambda\rangle$ is a strictly $\subseteq$-decreasing sequence of subsets $A^\delta \times A^\delta$, contradicting the fact that $|A^\delta| < \lambda$. □

Thus, let γ^* be the point at which the recursion terminates, and let $\langle Y_\tau \mid \tau < \gamma^*\rangle$,$\langle E_\tau \mid \tau < \gamma^*\rangle$ be the resulted sequences. Set $E = \bigcap_{\tau<\gamma^*} E_\tau$.

For every $\delta \in S \cap E$, put:

$$S_\delta := \bigcup\{(X_\beta)_{\gamma^*} \mid (\alpha, \beta) \in V^\delta_{\gamma^*}\}.$$

Claim 1.4. $\{S_\delta \mid \delta \in E \cap S\}$ *exemplify* $\diamondsuit_S$.

Proof. Assume towards a contradiction that there exists a set $Y \subseteq \lambda^+$ and a club $C \subseteq E$ such that $S_\delta \neq Y \cap \delta$ for all $\delta \in C \cap S$.

Following the notation of the recursion, write $Y_{\gamma^*} := Y$.

Let $Z = \bigcup_{\tau \leq \gamma^*} \{\tau\} \times Y_\tau$. Then, for $\delta \in C \cap S_Z$, we have:

$$\sup\{\alpha \in A^\delta \mid \exists \beta \in A^\delta \forall \tau \leq \gamma^* (Y_\tau \cap \alpha = (X_\beta)_\tau)\} = \delta.$$

So, $\sup\{\alpha < \delta \mid \exists \beta < \delta \left((\alpha, \beta) \in V^\delta_{\gamma^*}\right)\} = \delta$, and also:

$$Y \cap \delta = \bigcup \{(X_\beta)_{\gamma^*} \mid (\alpha, \beta) \in V^\delta_{\gamma^*+1}\}.$$

It follows that if $V^\delta_{\gamma^*+1} = V^\delta_{\gamma^*}$, then $Y \cap \delta = S_\delta$. However, by the choice of Y and $\delta \in C$, this is not the case, i.e., $V^\delta_{\gamma^*+1} \neq V^\delta_{\gamma^*}$.

But if $\sup\{\alpha < \delta \mid \exists \beta < \delta \left((\alpha, \beta) \in V^{\delta^*}_\gamma\right)\} = \delta$ and $V^\delta_{\gamma^*+1} \neq V^\delta_{\gamma^*}$ for all $\delta \in S \cap C$, this means that the recursion could have been continued using Y and C, while it was terminated at γ^*. A contradiction. $\square$

$\square$

2. SCH holds above a strongly-compact cardinal

Fix a regular cardinal λ. Recall that for a cardinal κ, $\mathcal{P}_\kappa(\lambda)$ denotes the family of subsets of λ of cardinality less than κ. An ultrafilter U over $\mathcal{P}_\kappa(\lambda)$ is *fine* iff for every $\alpha < \lambda$, the 'final segment' $\hat{\alpha} := \{X \in \mathcal{P}_\kappa(\lambda) \mid \alpha \in X\}$ is a member of U.

Definition 2.1. *A cardinal κ is* λ-strongly compact *iff there exists a κ-complete fine ultrafilter over $\mathcal{P}_\kappa(\lambda)$.*

A cardinal κ is strongly compact *iff it is λ-compact for all cardinals $\lambda \geq \kappa$.*

This section will be dedicated to proving the following theorem from [4]:

Theorem (Solovay)**.** *The Singular Cardinal Hypothesis (SCH) holds above the first strongly compact cardinal (if exists).*

By a celebrated result of Silver from [2], to show that SCH holds above a cardinal κ, it suffices to prove that $\lambda^{\aleph_0} = \lambda$ for all regular $\lambda \geq \kappa$. The latter will be established in this section. The simplified proof is due to ???

Lemma 2.2. *Suppose κ is a λ-strongly compact cardinal, then every collection of less than κ many stationary subsets of λ mutually reflects at some $\delta < \lambda$ of cofinality $< \kappa$.*

Proof. Let U is an ultrafilter witnessing that κ is λ-strongly compact. Consider ${}^{\mathcal{P}_\kappa(\lambda)}V/U$, an ultrapower of the universe. Since U is κ-complete, the ultrapower is well-founded, and we may identify it with its transitive collapse, M. For each x in V, let C_x denote the constant function $C_x : \mathcal{P}_\kappa(\lambda) \to \{x\}$. Now, defining $j : V \to M$ by letting $j(x) := [C_x]_U$ for all $x \in V$ yields an elementary embedding from the universe to a transitive proper class.

Consider the identity function, $\mathrm{id} : \mathcal{P}_\kappa(\lambda) \to \mathcal{P}_\kappa(\lambda)$. Since $\{X \in \mathcal{P}_\kappa(\lambda) \mid |\,\mathrm{id}(X)| < C_\kappa(X)\} = \mathcal{P}_\kappa(\lambda) \in U$, we get that $|[\mathrm{id}]_U| < j(\kappa)$. Since U is fine, for all $\alpha < \lambda$, we have $\{X \in \mathcal{P}_\kappa(\lambda) \mid C_\alpha(X) \in \mathrm{id}(X)\} \in U$, and hence $j(\alpha) \in [\mathrm{id}]_U$. Thus $j``\lambda \subseteq [\mathrm{id}]_U$ and $|j``\lambda| < j(\kappa)$. Put $\delta := \sup(j``\lambda)$. So $\mathrm{cf}(\delta) < j(\kappa)$. Also, since $\sup(X) < \lambda$ for all $X \in \mathcal{P}_\kappa(\lambda)$, we get from $j``\lambda \subseteq [\mathrm{id}]_U$ that $\delta < j(\lambda)$.

Suppose $\mathcal{S} = \{S_\alpha \mid \alpha < \mu\}$ is a collection of stationary subsets of λ, for some $\mu < \kappa$. Since U is κ-complete, we get that $j(\mu) = \mu$ and so $j(\mathcal{S}) = \{j(S_\alpha) \mid \alpha < \mu\}$.

Let $\alpha < \mu$ be arbitrary. We have $j``S_\alpha \subseteq j(S_\alpha)$ and actually $j``S_\alpha \subseteq j(S_\alpha) \cap \delta$. Suppose now C is a club in δ. Since j is elementary, it is continuous, so $j``\lambda$ is a club in $\delta$, and hence $C \cap j``\lambda$ is a club. Let $D \subseteq \lambda$ in V be such that $j``D = C \cap j``\lambda$. Again, since j is elementary, D is a club, so there exists some $\gamma \in S_\alpha \cap D$, and then, $j(S_\alpha) \cap C \neq \emptyset$.

Thus, we have shown that:

$$M \models \exists \delta < j(\lambda)\,(\mathrm{cf}(\delta) < j(\kappa), \forall S \in j(\mathcal{S})(S \cap \delta \text{ is stationary}))\,.$$

Consequently:

$$V \models \exists\delta < \lambda \left(\mathrm{cf}(\delta) < \kappa, \forall S \in \mathcal{S}(S \cap \delta \text{ is stationary})\right). \quad \square$$

Corollary 2.3. *Suppose $\lambda \geq \kappa$ are regular cardinals.*
If κ is strongly compact, then $\lambda^{\aleph_0} = \lambda$.

Proof. Clearly, it suffices to prove that $\lambda^{<\kappa} = \lambda$. Let $\langle S_\alpha \mid \alpha < \lambda\rangle$ be a partition of λ into λ many mutually-disjoint stationary sets. For each $\delta < \lambda$, consider the set $A_\delta = \{\alpha < \lambda \mid S_\alpha \cap \delta \text{ is stationary }\}$. Notice that if $\delta < \lambda$ and $\mathrm{cf}(\delta) < \kappa$, then there exists a club subset of δ of cardinality $< \kappa$. Consequently, $|A_\delta| < \kappa$ for such δ (because the stationary subsets are mutually-disjoint), and also $|\mathcal{P}(A_\delta)| < \kappa$, because κ is strongly inaccessible.

Finally, by the previous lemma, for all $a \in [\lambda]^{<\kappa}$, there exists some $\delta < \lambda$ such that $a \subseteq A_\delta$. Thus, $[\lambda]^{<\kappa} \subseteq \bigcup\{\mathcal{P}(A_\delta) \mid \delta < \lambda, \mathrm{cf}(\delta) < \kappa\}$, and so $\lambda^{<\kappa} = \lambda$. $\square$

3. Partition relations

Recall the arrow notation: for cardinals $\kappa \leq \lambda$, $(\lambda) \to (\kappa)^2_2$ asserts that for any function $f : [\lambda]^2 \to 2$, there exists some $H \subseteq \lambda$ of cardinality κ such that $|f``[H]^2| = 1$.[1]

A well-known theorem of Frank Ramsey states that $(\omega) \to (\omega)^2_2$, and its proof can be found in any relevant textbook. Consequently, $(\omega_1) \to (\omega)^2_2$.

In the next subsection it is shown that $(\omega_1) \to (\omega_1)^2_2$ does not hold, and actually a strong negation of it holds.

So, how about $(\omega_2) \to (\omega_1)^2_2$? In section 3.2. it is proved that the latter happens to be equivalent to the Continuum Hypothesis.

3.1. **Todorcevic's anti-Ramsey theorem.** This subsection will be dedicated to proving the following surprising theorem.

Theorem (Todorčević, [5])**.** *There exists a coloring $h : [\omega_1]^2 \to \omega_1$ such that every uncountable $Z \subseteq \omega_1$ satisfies $h``[Z]^2 = \omega_1$.*

The simplified presentation given here is due to Dan Velleman [6].

Definition 3.1. *For every $B \subseteq \omega_1$, denote:*

$$C^B := \{\delta < \omega_1 \mid B \subseteq \delta\} \cup \{\delta < \omega_1 \mid |B| = \aleph_1, \delta = \sup(B \cap \delta)\}.$$

Lemma 3.2. *For all $B \subseteq \omega_1$, C^B is a club.*

Proof. Fix $B \subseteq \omega_1$. To see that C^B is closed, suppose $\langle \alpha_n \mid n < \omega \rangle$ is an increasing sequence of elements of C^B, and let $\alpha := \sup_{n<\omega} \alpha_n$. If $B \subseteq \alpha$, then we are done. Otherwise, $B \nsubseteq \alpha_n$ for all $n < \omega$, then $|B| = \aleph_1$, and $\alpha_n = \sup(B \cap \alpha_n)$ for all $n < \omega$. To see that $\alpha = \sup(B \cap \alpha)$, we need to show that for all $\beta < \alpha$ there exists some $\gamma \in B \cap \alpha$ above β. Now, simply notice that if $\beta < \alpha$, then there exists some $n < \omega$ with $\beta < \alpha_n < \alpha$ and by $\alpha_n = \sup(B \cap \alpha_n)$, we may find some $\gamma \in B \cap \alpha_n$ above β.

To see that C^B is unbounded, pick $\beta < \omega_1$. If B is countable, then we may find some $\delta < \omega_1$ such that $B \subseteq \delta$, and then $\beta + \delta \in C^B$ as well. If B is uncountable, then just pick an increasing sequence of elements $\langle \alpha_n \mid n < \omega \rangle$ from B with $\alpha_0 > \beta$. Thus $\alpha := \sup_{n<\omega} \alpha_n$ satisfies $\alpha \in C^B$. $\square$

For the sake of this proof, fix ω_1 many distinct functions $\{r_\alpha \mid \alpha < \omega_1\} \subseteq {}^{\omega}2$. For every $Z \subseteq \omega_1$, $n < \omega$ and $g : n \to 2$, denote $B^Z_g := \{\alpha \in Z \mid g \subseteq r_\alpha\}$.

Corollary 3.3. *For any $Z \subseteq \omega_1$, the following set is a club:*

$$C_Z := \left\{\delta < \omega_1 \mid \forall g \in {}^{<\omega}2 \left((B^Z_g \subseteq \delta) \text{ or } (|B^Z_g| = \aleph_1 \text{ and } \delta = \sup(B^Z_g \cap \delta))\right)\right\}.$$

Proof. Since $C_Z = \bigcap_{g \in {}^{<\omega}2} C^{B^Z_g}$ and the countable intersection of clubs is a club. $\square$

[1] For a function g and a set $A \subseteq \mathrm{dom}(g)$, $g``A$ denotes the set $\{g(a) \mid a \in A\}$.

Theorem 3.4. *There exists a function* $f : [\omega_1]^2 \to \omega_1$ *such that* $f``[Z]^2$ *contains a club for any unbounded* $Z \subseteq \omega_1$.

Proof. For each $\alpha < \omega_1$, by $|\alpha| \leq \aleph_0$, let us fix some injection $e_\alpha : \alpha \to \omega$.

For any $\alpha < \beta < \omega_1$, let:

$$\Delta(\alpha, \beta) := \min\{n < \omega \mid r_\alpha(n) \neq r_\beta(n)\},$$

$$\Gamma(\alpha, \beta) := \{\gamma < \beta \mid e_\beta(\gamma) \leq \Delta(\alpha, \beta)\} \setminus \alpha.$$

Thus $\Gamma(\alpha, \beta) \subseteq [\alpha, \beta)$. If $\Gamma(\alpha, \beta)$ is empty, let $f(\alpha, \beta) := 0$, otherwise, let $f(\alpha, \beta) := \min \Gamma(\alpha, \beta)$. We claim that $C_Z \subseteq f[Z]$ for any unbounded $Z \subseteq \omega_1$.

Indeed, fix an unbounded $Z \subseteq \omega_1$ and some $\delta \in C_Z$. Since Z is unbounded, let us pick an arbitrary $\beta \in Z$ satisfying $\beta > \delta$. We now aim at finding some $\alpha < \beta$ with $\alpha \in Z$ such that $f(\alpha, \beta) = \delta$. Put $n := e_\beta(\delta)$, $g := r_\beta \restriction n$. Then $g \subseteq r_\beta$ and $\beta \in B^Z_g$. In particular, $B^Z_g \not\subseteq \delta < \beta$, so by $\delta \in C_Z$, we have $|B^Z_g| = \aleph_1$.

For each $\gamma \in B^Z_g$, put $m_\gamma := \Delta(\gamma, \beta)$. Clearly, $m_\gamma \geq n$. Evidently, there exists some uncountable subset $B \subseteq B^Z_g$ such that m_γ equals to some fixed $m < \omega$ for all $\gamma \in B$.

By shrinking further, we may also assume that $r_\gamma \restriction m + 1 = h$ for some fixed $h : m + 1 \to \omega$ for all $\gamma \in B$.

Thus, for B^Z_h, we have $B \subseteq B^Z_h$, i.e., $|B^Z_h| = \aleph_1$, and by $\delta \in C_Z$, we get that $\sup(B^Z_h \cap \delta) = \delta$. Let $F = \{\gamma < \delta \mid e_\beta(\gamma) \leq m\}$. Since e_β is an injection, F is finite and $\sup(F) < \delta$, and so there exists some $\alpha \in B^Z_h \cap \delta$ with $F \subseteq \alpha$. We claim that α works.

Indeed, by $\alpha \in B^Z_h$, we have $\Delta(\alpha, \beta) \geq m \geq n = e_\beta(\delta)$, i.e., $\delta \in \Gamma(\alpha, \beta)$. We need to show that δ is minimal in that sense. Pick $\gamma \in \Gamma(\alpha, \beta)$. Then $e_\beta(\gamma) \leq m$. If, in addition, $\gamma < \delta$, then $\gamma \in F$, but then $\gamma < \alpha$, contradicting the fact that $\Gamma(\alpha, \beta) \cap \alpha = \emptyset$. $\square$

Lemma 3.5. *There exists a function* $\psi : \omega_1 \to \omega_1$ *such that* $\psi[C] = \omega_1$ *for any club* $C \subseteq \omega_1$.

Proof. Let $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ be a partition of ω_1 into mutually disjoint stationary sets. Now, for $\alpha < \omega_1$, let $\psi(\alpha) = \beta$ where β is the unique ordinal such that $\alpha \in S_\beta$.

Finally, notice that if C is a club, then $C \cap S_\beta \neq \emptyset$ for all $\beta < \omega_1$. $\square$

Corollary 3.6. *There exists a function* $h : [\omega_1]^2 \to \omega_1$ *such that* $h``[Z]^2 = \omega_1$ *for any unbounded* $Z \subseteq \omega_1$.

Proof. Put $h = \psi \circ f$, for some f as in Theorem 3.4 and ψ as in Lemma 3.5. $\square$

3.2. **The Continuum Hypothesis is a Ramsey statement.**

Theorem 3.7 (Erdös-Rado, Sierpinski)**.** *The following are equivalent:*

(1) *CH;*
(2) $(\aleph_2) \to (\aleph_1)^2_2$.

Proof. (1)⇒(2) The simplified proof given here is due to Simpson. Suppose $f : [\omega_2]^2 \to 2$ is given. For every countable $X \subseteq \omega_2$ and $y \in \omega_2 \setminus X$, define $f^X_y : X \to 2$ by letting $f^X_y(x) := f(x, y)$ for all $x \in X$. Notice that by CH, the following set is of cardinality $\leq \aleph_1$:

$$\phi^X := \{f^X_y \mid y \in \omega_2 \setminus \sup(X)\}.$$

We now define a continuous and increasing chain $\{M_\alpha \mid \alpha < \omega_1\} \subseteq [\omega_2]^{\aleph_1}$ by induction on $\alpha < \omega_1$.

Let M_0 be an arbitrary subset of ω_2 of cardinality $\aleph_1$. Given M_α, by $|[M_\alpha]^{\leq\aleph_0}| \leq \aleph_1$ and the above remark, pick some $M_{\alpha+1} \in [\omega_2]^{\aleph_1}$ extending M_α such that $\phi^X = \{f^X_y \mid y \in M_{\alpha+1} \setminus \sup(X)\}$ for every countable $X \subseteq M_\alpha$.

Finally, let $M = \bigcup_{\alpha<\omega_1} M_\alpha$. Since $|M| = \aleph_1$, let us pick some $y^* \in \omega_2 \setminus \sup(M)$. We shall now define an increasing sequence of ordinals $\langle y_\alpha \mid \alpha < \omega_1\rangle \in \prod_{\alpha<\omega_1} M_{\alpha+1}$.

Let y_0 be an arbitrary element of M_1. Suppose $X_\beta := \{y_\alpha \mid \alpha < \beta\}$ is defined for some $\beta < \omega_1$. By the construction of $M_{\beta+1}$, there exists some $y_\beta \in M_{\beta+1} \setminus \sup(M_\beta)$ such that $f^{X_\beta}_{y_\beta} = f^{X_\beta}_{y^*}$, so let y_β be such element. This completes the description of the construction.

Let $i < 2$, put $H_i := \{y_\alpha \mid \alpha < \omega_1, f(y_\alpha, y^*) = i\}$. Notice that if $\alpha < \beta$ and $y_\alpha, y_\beta \in H_i$, then $f(y_\alpha, y_\beta) = f(y_\alpha, y^*) = i$. Consequently, $f[[H_i]^2] = \{i\}$.

The proof is complete noticing that $\aleph_1 \in \{|H_0|, |H_1|\}$.

$(\neg 1) \Rightarrow (\neg 2)$ Pick $X \subseteq \mathbb{R}$ of cardinality $\aleph_2$. Let $\leq$ denote the natural ordering of the real line and let $\trianglelefteq$ be some well-ordering of X.

Define $f : [X]^2 \to 2$ by letting $f(x, y) = 1$ iff $\{(x, y), (y, x)\} \cap \leq \cap \trianglelefteq \neq \emptyset$, that is, iff $\leq$ and $\trianglelefteq$ agrees on the relation of x and y. Now assume towards a contradiction that $Y \subseteq X$ is a set of cardinality $\aleph_1$, such that $f``[Y]^2 = \{1\}$.[2]

By thinning out, we may assume that Y has no maximal element, so for all $y \in Y$, define the *successor of* y to be the minimal element above it:

$$\hat{y} := \min_{\trianglelefteq}\{z \in Y \mid z \neq y, y \trianglelefteq z\}.$$

But $y \trianglelefteq \hat{y}$ implies that $y \leq \hat{y}$, and so $\{(y, \hat{y}) \mid y \in Y\}$ is an uncountable family of mutually-disjoint open intervals of the real line, but this is impossible, because each such interval contains a rational point, and there are only countably many rational numbers. □

[2]The other case is handled in the same fashion.

4. The consistency of the negation of Borel's conjecture

Recall that a subset $X \subseteq \mathbb{R}$ is said to be a *null set* iff for every positive $\varepsilon \in \mathbb{R}$ there exists a sequence of open intervals $\langle I_n \mid n < \omega\rangle$ such that $X \subseteq \bigcup_{n<\omega} I_n$ and also $\sum_{n<\omega} |I_n| < \varepsilon$.[3]

A subset $X \subseteq \mathbb{R}$ is said to be a *strongly null set* iff for every sequence of positive real numbers, $\langle \varepsilon_n \mid n < \omega\rangle$, there exists a sequence of open intervals $\langle I_n \mid n < \omega\rangle$ such that $X \subseteq \bigcup_{n<\omega} I_n$ and also $|I_n| < \varepsilon_n$ for all $n < \omega$.

Clearly, any strongly null set is a null set, and any countable subset of $\mathbb{R}$ is strongly null. Also, there are well-known examples of uncountable null sets, but what about uncountable strongly null sets?

Émile Borel conjectured that a set is strongly null iff it is countable. In [1], Richard Laver proved the consistency of this conjecture. In this section, we shall include a proof for the consistency of the negation of Borel's conjecture.

This result is considered as folklore, but its ingredients are due to Luzin.

Theorem. *$\mathsf{CH} \Rightarrow$ There exists an uncountable strongly null set.*

Proof. By $2^{\aleph_0} = \aleph_1$, let $\mathcal{F} = \{F_\alpha \mid \alpha < \omega_1\}$ be an enumeration of all closed nowhere dense subsets of $\mathbb{R}$.

Fix $\alpha < \omega_1$. Since α is countable, by Baire category theorem, $\bigcup_{\beta<\alpha} F_\beta \neq \mathbb{R}$, so let us pick some $x_\alpha \in \mathbb{R} \setminus \bigcup_{\beta<\alpha} F_\beta$.

Let $X := \{x_\alpha \mid \alpha < \omega_1\}$. Clearly, for all $\alpha < \omega_1$, $X \cap F_\alpha \subseteq \{x_\beta \mid \beta < \alpha\}$. In particular, since all singletons are in $\mathcal{F}$, the set X is uncountable.

Finally, suppose $\langle \varepsilon_n \mid n < \omega\rangle$ is a sequence of positive real numbers. Let $\{q_{2n} \mid n < \omega\}$ be some enumeration of $\mathbb{Q}$. For all $n < \omega$, pick an interval I_{2n} such that $q_{2n} \in I_{2n}$ and $|I_{2n}| < \varepsilon_{2n}$.

Consider the set $G = \bigcup_{n<\omega} I_{2n}$; this is an open and dense set, and hence there exists some $\alpha < \omega_1$ such that $\mathbb{R} \setminus G = F_\alpha$. By $|X \cap F_\alpha| \leq \aleph_0$, let $\{y_{2n+1} \mid n < \omega\}$ be some enumeration of $X \setminus G$. For all $n < \omega$, pick an interval I_{2n+1} such that $y_{2n+1} \in I_{2n+1}$ and $|I_{2n+1}| < \varepsilon_{2n+1}$.

Clearly, $X \subseteq \bigcup_{n<\omega} I_n$. □

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[3] Here, $|I|$ denotes the diameter of the open interval I.

References

[1] R. Laver, *On the consistency of Borel's conjecture*, Acta Math. 137(3-4): 151–169, 1977.

[2] J. Silver, *On the singular cardinals problem* , Proc. 1974 Intl Congress of Mathematicians.

[3] S. Shelah, *Diamonds*, preprint of paper #922.

[4] R. Solovay, *Strongly Compact Cardinals and the GCH*, In **Proceedings of the Tarski Symposium**, Proceedings of Symposia in Pure Mathematics 25, American Mathematical Society, Providence (1974), 365-372.

[5] S. Todorčević, *Partitioning pairs of countable ordinals*, Acta Math. 159(3-4): 261–294, 1987.

[6] D. Velleman, *Partitioning pairs of countable sets of ordinals*, J. Symbolic Logic 55(3): 1019–1021, 1990.

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