Suggestions, extensions and corrections concerning our book "Approximations and Endomorphism Algebras of Modules"

Rüdiger Göbel and Jan Trlifaj
Email: ruediger.goebel@uni-due.de and trlifaj@karlin.mff.cuni.cz

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Open Problem Progress

- A positive solution to Open Problem 2 on p. 254 is given in a recent preprint by J. Trlifaj and D. Pospíšil called "Tilting and cotilting classes over Gorenstein rings." By Theorem 8.2.8 this also yields a positive solution to Open Problem 3 on p. 292.

In other words, tilting and cotilting modules over a commutative 1–Gorenstein ring R are characterized up to equivalence as the Bass tilting and cotilting modules defined in Examples 5.1.3 and 8.1.3, respectively. Moreover the Bass cotilting modules are classified up to equivalence as the modules of the form $Q \oplus \prod_{p \in P} J_p \oplus \bigoplus_{q \in P_1 \setminus P} E(R/q)$ for some $P \subseteq P_1$ where P_1 is the set of all prime ideals in R of height 1, and $J_p = \operatorname{End} E(R/p)$ denotes the p–adic module.

Corrections and Extensions[1]

- On page 27, line 13, replace "unions of chains" by "direct limits of chains". Similarly, in the statement of Lemma 1.2.10, replace "unions of well–ordered chains" by "direct limits of well–ordered chains". The point is that the canonical maps $g_{\alpha\beta}$ constructed on page 27, line -3, are not monic in general.

(It is not true that closure under unions of well–ordered chains implies closure under arbitrary direct limits: for example, consider the class $\mathcal{C}$ of all infinitely generated modules; then $\varinjlim \mathcal{C} = \text{Mod–}R$.)

- In Example 1.2.26 (part (a), item 1, page 41) replace "Simple modules $R_p = R/Rp$ ($p \in \operatorname{spec} R$)" by "Cyclic modules R/Rp^n ($p \in \operatorname{spec} R$, $1 \leq n < \omega$)". These modules are obviously endofinite.

- In Remark 2.2.7 (page 103) replace "R is a ring with a Morita duality" by "R is an artinian ring with a Morita self–duality".

- Page 103, line -5: Given a module M and $i < \omega$, the symbol $\Omega^i(M)$ ($\Omega^{-i}(M)$) is also used throughout the book in a different meaning, namely to denote the i–th syzygy (i–th cosyzygy) in a projective resolution (injective coresolution) of the module M.

- On page 120, line 6, replace 4.2.6 by 4.2.11.

- Lemma 3.3.3 on page 127 should be called 'Dual Eklof Lemma' or 'Lukas Lemma' since F. Lukas appears to have been the first to notice it in [314, §3].

- In the proof of Lemma 4.1.5 (page 136), replace "$\operatorname{Hom}_R(F, \nu)$ is surjective" by "$\operatorname{Hom}_R(F, \pi)$ is surjective" where $\pi : N \to N/\nu(M)$ is the projection.

[1] We are grateful to Lidia Angeleri Hügel, Silvana Bazzoni, Paul C. Eklof, Dolors Herbera, and Birge Huisgen–Zimmermann for some of the corrections listed below.

- In the proof of Theorem 4.1.7 on page 137, replace $\mathrm{Ext}^n_R(R/I, M)$ by $\mathrm{Ext}^{n+1}_R(R/I, M)$ (two instances).

- On page 158, line -7, replace M_λ by M_σ.

- On page 166, line 7, replace the second instance of $\mathrm{Hom}_R(Q, \widehat{M}/M)$ by $\mathrm{Hom}_R(R, \widehat{M}/M)$.

- The comment preceding Corollary 4.5.12 is misleading, since $\mathcal{P}_1 \subseteq \varinjlim \mathcal{P}_1^{<\omega}$ holds true for *any* ring R. This is just a particular case (when $\mathcal{A} = \mathcal{P}_1$) of Lemma 4.5.14.

- Comments on the last paragraph of Example 5.1.2 (page 191):

In S.B.Lee: "Semi–Baer modules over domains," Bull. Austral. Math. Soc. 64(2001), 21–26, the modules M satisfying $\mathrm{Ext}^1_R(M, D) = 0$ for each (torsion) divisible module are called *semi–Baer*. Bazzoni and Herbera have recently proved that semi–Baer modules coincide with the modules of projective dimension ≤ 1, so $\mathfrak{D} = (\mathcal{P}_1, \mathcal{DI})$ is the (1–tilting) cotorsion pair induced by δ for *any* domain R.

Moreover $\mathcal{F}_1 = \varinjlim \mathcal{P}_1 = \varinjlim \mathcal{P}_1^{<\omega}$ by Theorem 4.5.15, so the perfect cotorsion pair $(\mathcal{F}_1, \mathcal{F}_1^\perp)$ is the closure of $\mathfrak{D}$ in the sense of §4.5 for any domain R. Lee calls the modules in $\mathcal{F}_1^\perp$ *weak–injective*. They are studied in his papers in Comm. Algebra 34(2006), 361–370, and J. Algebra 299(2006), 854–862, and in a recent preprint by L.Fuchs and S.B.Lee called "Weak–injectivity and almost perfect domains."

- Comments on Example 5.1.3 (page 191):

— Here, Q denotes $\bigoplus_{q \in P_0} E(R/q)$ (= the middle term of the minimal injective coresolution of R).

— We also have to prove that $\mathrm{Ext}^1_R(R_P, R_P^{(\kappa)}) = 0$. This follows from the existence of the exact sequence $0 \to R \to R_P \to \bigoplus_{p \in P} E(R/p) \to 0$ and from $\mathrm{Ext}^1_R(E(R/p), R_P^{(\kappa)}) = 0$ for all $p \in P$.

- In Example 5.1.4 (page 192), Q should be G (two instances).

- In the statement of Proposition 5.2.4 (page 204), replace "closed under direct limits" by "closed under unions of well–ordered chains", and remove the first sentence of its proof.

- A correction to Corollary 5.2.5 (page 204):

The Corollary is restricted only to the filters $\mathcal{F}$ on an infinite set A of the form described on top of page 206, that is, such that there is a regular infinite cardinal κ and $\mathfrak{F}$ is the filter of all sets $I \subseteq A$ with $|A \setminus I| < \kappa$.

The proof of the restricted Corollary is by induction on $\lambda = |A|$. Since $\mathcal{B}$ is closed under arbitrary direct products, there is nothing to prove for $\lambda < \kappa$. Suppose that $\kappa \leq \mathrm{cf}(\lambda)$ and let $(M_\alpha \mid \alpha < \lambda)$ be a family of modules from $\mathcal{B}$. Then the $\mathfrak{F}$–product $\sum_{\mathfrak{F}} M = \bigcup_{\alpha<\lambda} N_\alpha$ where $N_\alpha = \sum_{\mathfrak{F}_\alpha} M$ and $\mathfrak{F}_\alpha$ is the filter on α consisting of all sets $J \subseteq \alpha$ with $|\alpha \setminus J| < \kappa$. Then $N_\alpha \in \mathcal{B}$ by the inductive hypothesis, so $\sum_{\mathfrak{F}} M \in \mathcal{B}$ by Proposition 5.2.4. Finally, assume $\mathrm{cf}(\lambda) < \kappa \leq \lambda$. Let $(\lambda_\beta \mid \beta < \mathrm{cf}(\lambda))$ be a strictly increasing continuous and cofinal sequence of ordinals in λ. Let A_β denote the ordinal interval $\langle \lambda_\beta, \lambda_{\beta+1})$ and $P_\beta = \sum_{\mathfrak{F}_\beta} M$ where $\mathfrak{F}_\beta$ is the filter on A_β consisting of all subsets J such that $|A_\beta \setminus J| < \kappa$. Then $P_\beta \in \mathcal{B}$ by the inductive hypothesis, and $\sum_{\mathfrak{F}} M \cong \prod_{\beta<\mathrm{cf}(\lambda)} P_\beta \in \mathcal{B}$.

- In the statement of Theorem 5.2.16 (page 212), replace "A a countably presented module and $\mathcal{B} = A^{\perp}$" by "$\mathcal{S}$ a class of countably presented modules and $\mathcal{B} = \mathcal{S}^{\perp}$". (This is the version used in the proof of Corollary 5.2.17.) Begin the proof of 5.2.16 by "Let A be an arbitrary element of $\mathcal{S}$."

- The following observation is missing from Lemma 5.2.22 (page 216):

For any ring R, $\mathcal{S} = \text{mod}–R$ is resolving. (This is obvious in the noetherian case; in general, it follows by induction from the classical Schanuel and Horseshoe Lemmas, and from Lemma 5.2.22.) This observation is used in the proof of Theorem 5.2.23.

- A remark on Lemma 6.2.1 (page 228):

Though the two infinitary versions of the Auslander–Reiten Formula are well–known, their proofs do not appear to be written down in any book or survey. A simple proof via Auslander's Defect Formula can be found in the paper by H. Krause: "A short proof for Auslander's defect formula," Linear Algebra and Its Appl. 365(2003), 267–270. The "Moreover ..." part in (a) and (b) follows from the general case by [29, IV.1.16] and its dual, respectively. (In contrast with the claim on page 228, both [108] and [379] only state these results without proof; in fact, [108] deals only with the hereditary case).

- Corrections to the proof of Lemma 6.3.15 (page 249):

— *Each* K_j is countably generated.
— Also the left hand column of the commutative diagram displayed is exact, and the middle column vertical map $F \to Q$ is just f.

- A comment on the proof of Theorem 6.3.16 (part (c) implies (a), in the middle of page 251):

To prove the first claim, it is enough to construct only the first countable multiplicative subset $S_0 \subseteq S$: since M_i is a direct summand in $S^{-1}R/R$ by assumption, M_i is obviously a direct summand in $S_0^{-1}R/R$.

- A remark on the proof of Theorem 7.1.12 (page 260):

By the Krull–Schmidt–Azumaya Theorem, the equality $\mathrm{Add}(T) = \mathcal{I}_0$ proved as a step here just says that each indecomposable injective module occurs as a direct summand in some term of the minimal injective coresolution of R.

- In the definition of Q_P in the middle of page 284, Q denotes the quotient field of R.

- On page 287, line 5, replace $\mathrm{Hom}_{R_{(p)}}(Q_{(p)}/R_{(p)}, Q_{(p)}/R_{(p)})$ by $\mathrm{Hom}_{R_{(p)}}(Q_{(p)}/R_{(p)}, (Q_{(p)}/R_{(p)})^{(\alpha_p)})$.

- On page 360, line -2, replace E_α by B_α.

- In the statement and proof of part (a) of Theorem 10.1.2 on page 361, the term κ–refinement refers to the notion defined in 3.2.6, but it should refer to the weaker notion *not* requiring the continuous chain to consist of pure submodules. This is needed for a direct application of Lemma 4.3.8 in the singular cardinal case of the proof on page 362, line 14. The variation of 4.3.8 for pure chains can thus be avoided.

- A correction to Corollary 14.5.7 (a) (page 579):

Corollary 14.5.7 (a) holds only under a weaker absolute notion under which rigid families of $\aleph_1$–free modules of size $\aleph_1$ remain rigid. Otherwise these modules can become countable in a different universe, hence free. The same holds for families of modules of larger size κ. Hence (a) is not true in general; see L. Fuchs, R. Göbel [176] for more details about this and the following correction and extension.

- A correction and extension to the definition of the module H on page 580, line 6:

6 can be replaced by 5 and H needs purity, i.e.

$$H = \langle M, \pi_j M^j \mid j < 5 \rangle_* \subseteq \widehat{M}.$$

This is needed to say "By continuity" in the proof of Lemma 14.5.8. Consequently, in Lemma 14.5.8, $\widehat{R}$ must be a domain.

In a forthcoming paper M. Droste, R. Göbel and S. Pokutta characterize graphs with prescribed absolute endomorphism monoid. It turns out that in this case $\kappa(\omega)$ is also the least upper bound. This is applied in [222] to characterize absolute endomorphism monoids of fields; compare page 566.

An Update of References

[4] has appeared in J. Algebra and Its Appl. 6(2006), 747-763.

[16] will appear in J. Pure Appl. Algebra.

The correct reference for [20] is L. Angeleri–Hügel, D. Herbera, J. Trlifaj: *Baer and Mittag–Leffler modules over tame hereditary algebras*, preprint (2007).

[43] will appear in Algebras and Repres. Theory.

[48] will appear in Proc. Amer. Math. Soc.

[152] has appeared in Algebras and Repres. Theory 9(2006), 423–430.

[296] will appear in Trans. Amer. Math. Soc.

[338] has appeared in Canad. Math. J. 58(2006), 180–224.

[379] has appeared in J. Algebra 311(2007), 299-318.

[380] has appeared in Bull. London Math. Soc. 39(2007), 121-132.

[389] has appeared in Vol. 332 of LMSLNS in Cambridge Univ. Press, Cambridge 2007, 279–321.