Classes of partially isomorphic modules

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'Ancient' History - the classical theorem

Theorem (Ulm)

For G and H countable, reduced p-groups the following are equivalent:

1. $G \cong H$;
2. $u_p(\alpha, G) = u_p(\alpha, H)$ *for all ordinals* α.

Definition

For a group (or module) G, $\alpha \in ON$ and p a prime define

$$u_p(\alpha, G) := dim(p^\alpha G[p]/p^{\alpha+1}G[p]) \text{ (as a } \mathbb{Z}_p\text{-vector space)},$$

the (α^{th}) **Ulm-Kaplansky invariant** of G and

$$u_p(\infty, G) := dim(p^\infty G[p]).$$

History - the generalization

Theorem (Barwise-Eklof)

For G and H p-groups and $\delta \in ON$: If

(a) $\hat{u}_p(\alpha, G) = \hat{u}_p(\alpha, H)$ *for all* $\alpha < \omega\delta$,

(b) $l(G) < \omega\delta \Leftrightarrow \hat{r}(G_d) = \hat{r}(H_d)$,

then $G \equiv_\delta H$.
If $\delta \in LORD$ *the converse also holds.*

$$\hat{g}(x) := \begin{cases} g(x) & \text{if } g(x) < \aleph_0, \\ \infty & \text{if } g(x) \geq \aleph_0. \end{cases}$$

Proof ideas

The proofs of both Theorems consider sets I of **height-preserving** ($h_G(x) = h_H(f(x))$) partial isomorphisms (for the generalization: height-preserving up to a certain ordinal) between G and H and show that these possess the

Definition

Back-and-forth property: For any $f \in I$ and $a \in G$ (resp. $b \in H$), there is $g \in I$ such that $f \subseteq g$ and $a \in dom(g)$ (resp. $b \in rng(g)$).
If such a set I exists, then G and H are called **partially isomorphic**, $G \cong_p H$.

Linking Algebra and Logic

J. Barwise, *Back and Forth through Infinitary Logic*, 1973.

Theorem (Karp)

$$G \equiv_\infty H \Leftrightarrow G \cong_p H.$$

Theorem (Barwise)

If G and H are countable or countably generated,

$$G \cong_p H \Leftrightarrow G \cong H.$$

Question: How closely related are uncountable structures if they are partially isomorphic?

Constructing modules in V=L: The Step-Lemma

Setting:
Let A be a ring satisfying the Pierce-condition:

$\Leftrightarrow$ A^+ is the p-adic completion of a free J_p-module;

$\Leftrightarrow$ there exists a p-group G with $\mathrm{End}(G) = A \oplus \mathrm{Small}(G)$.

Call G an unbounded A-module, if $G = \bigoplus_{k<\omega} G_k$ with $G_k = (A/p^{i_k}A)^{(\kappa_k)} \cong \bigoplus \mathbb{Z}_{p^{i_k}}$. Assume $\kappa_k > |A|$ for all k.

Call $i_0 < i_1 < i_2 < \dots$ the **chain** of G.

$d_k := i_{k+1} - i_k$. Then $p^{i_k} p^{d_k} = p^{i_{k+1}}$.

Constructing modules in V=L: The Step-Lemma

Definition

Let G be an unbounded A-module as above.

- Call $(g_k)_{k<\omega} \subseteq \overline{G}$ a d_k-**chain** iff $g_k - p^{d_k} g_{k+1} \in G$ and $p^{i_k} g_k = 0$ for all k.
- Call $(g_k)_{k<\omega} \subseteq \overline{G}$ a **basic** d_k-**chain** iff $g_k - p^{d_k} g_{k+1} \in B$ and $\operatorname{Ann} g_k = p^{i_k} A$ for all k where B is some (fixed) basis of $G = \bigoplus_{k<\omega} G_k$.

Lemma (Göbel-Herden-L)

Let G be an unbounded A-module as above.

a) *There exists a basic d_k-chain $(g_k)_{k<\omega}$ of elements $g_k \in \overline{G} \setminus G$.*

b) *For every element $g \in \overline{G}$ and every $n < \omega$ with $p^{i_n} g = 0$ there exists a d_k-chain $(g_k)_{k<\omega}$ with $g_n = g$.*

Constructing modules in V=L: The Step-Lemma

Proposition (Göbel-Herden-L)

Let G be an unbounded A-module as above.
If $\varphi \in \mathit{End}(G) \setminus A \oplus \mathit{Ines}(G)$, then there exists a decomposition $G = P \oplus H$ with P, H both unbounded A-modules, $|P| \leq |A|, P\varphi \subseteq P$ and such that

$$\textit{there is } n < \omega \textit{ such that } \overline{P}[p^n](\varphi - a) \nsubseteq G \textit{ for all } a \in A.$$

One is able to secure that G, P and H all have the same chain.

Extension will be realised by $G' := \langle G, h_k A : k < \omega \rangle$ or $G' := \langle G, (p_k + h_k)A : k < \omega \rangle$ where $(p_k)_{k<\omega}$ a d_k-chain in $\overline{P}$, $(h_k)_{k<\omega}$ a basic d_k-chain in $\overline{H}$.

G' unbounded with same chain as G, p-pure in $\overline{G}$, $G'/G \cong A(p^\infty)$.

Constructing modules in V=L: The Step-Lemma

Step-Lemma (Göbel-Herden-L)

Let A be a ring which satisfies the Pierce-condition. Moreover, let G be an unbounded A-module and let $G = \bigcup_{l<\omega} G_l$ be a chain of unbounded A-modules such that $G_{l+1} = G_l \oplus H_l$ for suitable unbounded A-modules H_l with chain $i_0 < i_1 < \ldots$ and $H_l \cong \bigoplus_{n<\omega} (A/p^{i_n}A)^{(\kappa_{ln})}$ with $\kappa_{ln} > |A|$ for all $l, n < \omega$. If $\varphi \in$ End$(G) \setminus A \oplus$Ines(G), then there exists an extension G' of G such that the following hold.

1. *G' is a p-pure A-submodule of $\overline{G}$.*
2. *G' is an unbounded A-module with chain $i_0 < i_1 < \ldots$.*
3. *$G_l \sqsubset G'$ for all $l < \omega$.*
4. *$G'/G \cong A(p^\infty)$.*
5. *φ does not extend to an endomorphism of G'. More precisely there exists some d_k-chain $(g_k)_{k<\omega} \subseteq \overline{G}$ inside G' with $g_k\varphi \notin G'$ for some $k < \omega$.*

Constructing modules in V=L: The Diamond Principle

Let E be a stationary subset of $\lambda \in CARD$ ($C \subseteq \lambda$ cub: $E \cap C \neq \emptyset$), where λ is regular and uncountable.

The Diamond Principle

Suppose $\diamondsuit_\lambda E$. For any two λ-filtrations $A = \bigcup_{\alpha<\lambda} A_\alpha$ and $B = \bigcup_{\alpha<\lambda} B_\alpha$ there are **Jensen-functions**

$$g_\alpha : A_\alpha \to B_\alpha (\alpha \in E)$$

such that, for any function $g : A \to B$,

$$\{\alpha \in E : g_\alpha = g \restriction A_\alpha\} \text{ is stationary in } \lambda.$$

In the following E will be non-reflecting ($\alpha \in \lambda, cf(\alpha) > \omega : \exists$ cub $C \subseteq \alpha : C \cap E = \emptyset$), too.

Constructing modules in V=L: The Realization Theorem

Theorem (Göbel-Herden-L)

Let A be a ring with Pierce-condition and $i_0 < i_1 < ...$ a sequence of positive integers. If $\lambda > |A|^+$ and we assume $\diamondsuit_\lambda E$ for a non-reflecting, stationary subset E of λ consisting of limit ordinals of cofinality ω, then there exists a strongly-λ-direct A-module M of cardinality λ with End$(M) = A \oplus$Ines(M).

Construction inductively. Fix λ-filtration $M = \bigcup_{\nu<\lambda} M_\nu$.

Constructing modules in V=L: The Realization Theorem

For all $\nu < \lambda$:

- $|M_\nu| = |\nu| + |A|^+ = |M_{\nu+1}/M_\nu|$;
- M_ν is an unbounded A-module with chain $i_0 < i_1 < \ldots$;
- if $\rho \in \nu \setminus E$, then $M_\rho \sqsubset M_\nu$ with $M_\nu = M_\rho \oplus H_{\nu\rho}$, where $H_{\nu\rho}$ is also unbounded with chain $i_0 < i_1 < \ldots$;
- if $\nu = \rho + 1$ with $\rho \in E$ and $g_\rho \in \mathrm{End}(M_\rho) \setminus A \oplus \mathrm{Ines}(M_\rho)$, then g_ρ does not extend to an endomorphism of M_ν.

Set $M_0 := \bigoplus_{n<\omega} (A/p^{i_n}A)^{(|A|^+)}, |M_0| = |A|^+$.

Calculating Ulm-invariants

Let $G = \bigoplus_{n<\omega} \mathbb{Z}_{p^{i_n}}^{(\kappa_n)}$, an unbounded A-module.

Then $p^k G = \bigoplus_{n<\omega} p^k \mathbb{Z}_{p^{i_n}}^{(\kappa_n)} = \bigoplus_{i_n>k} p^k \mathbb{Z}_{p^{i_n}}^{(\kappa_n)}$ and therefore
$p^k G[p] = \bigoplus_{i_n>k} p^{i_n-1} \mathbb{Z}_{p^{i_n}}^{(\kappa_n)}$ and
$p^{k+1} G[p] = \bigoplus_{i_n>k+1} p^{i_n-1} \mathbb{Z}_{p^{i_n}}^{(\kappa_n)}$,thus

$$p^k G[p]/p^{k+1}G[p] \cong p^k \mathbb{Z}_{p^{k+1}}^{(\kappa_l)} \cong \mathbb{Z}_p^{(\kappa_l)} \text{ (if } k+1 = i_l) \ .$$

$$u_p(\beta, G) := \begin{cases} \kappa_l, & \text{if } \beta < \omega \text{ and } \beta + 1 = i_l, \\ 0, & \text{else.} \end{cases}$$

$$\hat{u}_p(\beta, G) := \begin{cases} \infty, & \text{if } \beta < \omega \text{ and } \beta + 1 = i_l, \\ 0, & \text{else.} \end{cases}$$

Calculating Ulm-invariants

Now let $G = \bigcup_{\alpha<\lambda} G_\alpha$, as realized with Step-Lemma and Diamond.

All modules have the same chain!

$$p^k G[p]/p^{k+1}G[p] \cong (\bigcup_{\alpha<\lambda} p^k G_\alpha[p])/p^{k+1}G[p]$$

$$\cong \bigcup_{\alpha<\lambda}(p^k G_\alpha[p]+p^{k+1}G[p]/p^{k+1}G[p]) \cong \bigcup_{\alpha<\lambda}(p^k G_\alpha[p]/p^{k+1}G_\alpha[p])$$

Thus $u_p(k,G) = \sup_{\alpha<\lambda} \dim\ (p^k G_\alpha[p]/p^{k+1}G_\alpha[p])$.

We then have $u_p(k,G) = \sup_{\alpha<\lambda} u_p(k,G_\alpha)$ and thus

$$\hat{u}_p(\beta,G) = \hat{u}_p(\beta,G_\alpha) = \begin{cases} \infty, & \text{if } \beta<\omega \text{ and } \beta+1 = i_l, \\ 0, & \text{else.} \end{cases}$$

Conclusion

We are able to construct classes of L_∞-equivalent modules which have the following generalized Ulm-Kaplansky invariants:

$$\hat{u}_p(\beta, G) = \begin{cases} \infty, & \text{for infinitely many } \beta < \omega \text{ (chain)}, \\ 0, & \text{for } \beta \geq \omega, \\ \text{arbitrary}, & \text{else.} \end{cases}$$

The equivalence classes thus contain partially isomorphic modules, which might either be unbounded (G_α) or not (G), they might have a prescribed endomorphism ring (G) or not (G_α).