Fully transitive p-groups with finite first Ulm subgroup

By

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Abstract

An abelian p-group G is called (fully) transitive if for all $x, y \in G$ with $U_G(x) = U_G(y)$ ($U_G(x) \leq U_G(y)$) there exists an automorphism (endomorphism) of G which maps x onto y. It is a long-standing problem of A.L.S. Corner whether there exist non-transitive but fully transitive p-groups with finite first Ulm subgroup. In this paper we restrict to p-groups of type A, this is to say p-groups satisfying $\mathrm{Aut}(G) \restriction_{p^\omega G} = U(\mathrm{End}(G) \restriction_{p^\omega G})$. We show that the answer to Corner's question is no if $p^\omega G$ is finite and G is of type A.

Introduction

The notions of transitivity and full transitivity originated in the book "Infinite Abelian Groups" by I. Kaplansky [11] and extensive classes of abelian p-groups which are both transitive and fully transitive were found in [8] and [10]. Moreover, p-groups with neither property were constructed in [13] and for larger cardinalities in [7]. In [5] Files and Goldsmith proved the surprising result that a p-group G is fully transitive if and only if its square $G \oplus G$ is transitive. Nevertheless, for $p = 2$ the independence of both concepts was shown by Corner in [4] and already Kaplansky had shown in [11] that for $p > 2$ transitivity always implies full transitivity. Therefore it is natural to ask which fully transitive non-transitive p-groups appear. By a fundamental observation of Corner [4] one can reduce the decision of whether or not a p-group G is (fully) transitive to its first Ulm subgroup $p^\omega G$. In [4] it was shown that G is (fully) transitive if and only if $\mathrm{End}(G) \restriction p^\omega G$ acts (fully) transitively on the first Ulm subgroup $p^\omega G$ of G, i.e., for any $x, y \in p^\omega G$ such that $U_{p^\omega G}(x) = U_{p^\omega G}(y)$ ($U_{p^\omega G}(x) \leq U_{p^\omega G}(y)$) there exists an automorphism (endomorphism) α of G such that $\alpha(x) = y$. Corner constructed a fully transitive non-transitive p-group with countable first Ulm subgroup in [4] and it is a long-standing problem whether there exists a fully transitive non-transitive p-group with finite first Ulm subgroup. Partial results were obtained by Carroll and Goldsmith in [2], [3] and by Hennecke in [9], but a general solution hasn't been found yet. It is the aim of this paper to show that the answer to Corner's question is no for a large class of p-groups, namely the class of p-groups of type A. Here a p-group G is

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of type A if $\mathrm{Aut}(G)\restriction_{p^\omega G}= U(\mathrm{End}(G)\restriction_{p^\omega G})$. Our approach is ring theoretic and uses the structure of the Jacobson radical of a ring which acts (fully) transitively on a finite p-group.

We follow standard notations found in Fuchs [6] and Kaplansky [11]. Multiplication by an integer n on some group H is denoted by $n * id\restriction_H$ while applying two endomorphisms φ and ψ successively is denoted by $\varphi\psi$.

1 A question by A.L.S. Corner

All groups considered will be abelian p-groups, where p is a prime. Let G be such a group. For each ordinal α we define $p^\alpha G$ to be $p^0 G = G$ and $p^\alpha G = \bigcap\limits_{\beta<\alpha} p(p^\beta G)$. Then the *height* $|x|_G$ in G of an element $x \in G$ is defined to be ∞ if $x \in p^\alpha G$ for all ordinals α and otherwise α if $x \in p^\alpha G \setminus p^{\alpha+1}G$, where infinity exceeds any ordinal. The *Ulm sequence* of an element $x \in G$ is given by $U_G(x) = (\alpha_0, \alpha_1, \alpha_2, \dots)$ where $\alpha_i = |p^i x|_G$. The Ulm sequences are partially ordered by agreeing that $U_G(x) \le U_G(y)$ if $|p^i x|_G \le |p^i y|_G$ for all $i \ge 0$. By $G(U_G(x))$ we denote the set of all elements $y \in G$ such that $U_G(y) \ge U_G(x)$.

Definition 1.1 Let G be a p-group and R a unital subring of $\mathrm{End}(G)$. We say that

(i) R acts *fully transitively* on G if for any $x, y \in G$ with $U_G(x) \le U_G(y)$ there exists an element of R which maps x to y;

(ii) R acts *transitively* on G if for any $x, y \in G$ with $U_G(x) = U_G(y)$ there exists a unit of R which maps x to y;

(iii) G is *(fully) transitive* if $\mathrm{End}(G)$ acts (fully) transitively on G.

We note that any direct summand of a fully transitive p-group G is again fully transitive. Moreover, the question of deciding when a group is (fully) transitive is often made easier by the following fundamental observation of Corner ([4]).

Lemma 1.2 *A p-group G is (fully) transitive if and only if* $\mathrm{End}(G)$ *acts (fully) transitively on the first Ulm subgroup $p^\omega G$ of G.*

In the same paper [4] Corner showed that transitivity implies full transitivity if the first Ulm subgroup is a direct sum of cyclic groups of one and the same order p^n. We call such groups homogeneous. Motivated by this result it was shown in [4] that the converse does not hold. In particular Corner constructed an example of a p-group which is fully transitive but not transitive and $p^\omega G$ is homogeneous of cardinality $\aleph_0$. Thus the following question is natural:

Question 1.3 *Is there a fully transitive non-transitive p-group with finite first Ulm subgroup?*

If G is a p-group then $\mathrm{End}(G)\restriction_{p^\omega G}$ is a subring R of $\mathrm{End}(p^\omega G)$ and similarly $S = \mathrm{Aut}(G)\restriction_{p^\omega G}$ is a subgroup of $\mathrm{Aut}(p^\omega G)$. Clearly $S \subseteq U(R)$, the units of R, but in general the inclusion may be strict. Hence Carroll and Goldsmith introduced the following notion in [3]

Definition 1.4 A p-group G is said to be of *type A* if $\mathrm{Aut}(G)\restriction_{p^\omega G}= U(\mathrm{End}(G)\restriction_{p^\omega G})$.

For instance any p-group with cyclic first Ulm subgroup is of type A and all examples constructed in [4] and [13] are of type A. Nevertheless, examples of p-groups which are not of type A may appear. The following result from [3] provides a helpful tool to handle p-groups of type A.

Lemma 1.5 *Let G be a p-group such that any unital subring of $\mathrm{End}(G)$ that acts fully transitively on G also acts transitively on G. Then every fully transitively p-group H of type A having G as its first Ulm subgroup is also transitive.*

The best results on Corner's Question 1.3 which are known up to now (to the authors knowledge) are the following which can be found in [2], [3] and [9].

Theorem 1.6 *Let G be a fully transitive p-group. Then G is transitive if one of the following conditions hold:*

(i) $p^\omega G = \mathbb{Z}(p^n) \oplus \mathbb{Z}(p^n)$ *for some* $n \in \mathbb{N}$*;*

(ii) $p^\omega G = \bigoplus_{i=1}^{n} \mathbb{Z}(p^{n_i})$ *for some* $n, n_i \in \mathbb{N}$ *such that* $n_i < n_j$ *if* $i < j$.

2 p-Groups with finite first Ulm subgroup

In this section we will mainly consider abelian p-groups G of type A with finite first Ulm subgroup, i.e., $p^\omega G = \bigoplus_{i=1}^{n} \mathbb{Z}(p^{m_i})$ for some $m_i, n \in \mathbb{N}$. We show that for this class of groups, full transitivity implies transitivity. If R is a ring then we denote by $J(R)$ the Jacobson radical of R and abbreviate $J(R)$ by J if there is no danger of confusion.

Lemma 2.1 *Let $n \in \mathbb{N}$ and $H = \bigoplus_{i=1}^{n} H_i$, where each H_i is homogeneous of the form $\bigoplus_{j=1}^{k_i} \mathbb{Z}(p^{m_i})$ for some fixed $m_i \in \mathbb{N}$ such that $m_i < m_j$ if $i < j$ and cardinals k_i. If $x, y \in H$ and R is a unital subring of $\mathrm{End}(H)$ which acts (fully) transitively on H, then the following hold:*

(i) *If* $n = 1$, *then:*

- *(a)* $\mathrm{ord}(x) = p^k$ *if and only if* $x \in p^{m_1-k}H \backslash p^{m_1-k+1}H$*;*
- *(b)* $\mathrm{ord}(x) \leq \mathrm{ord}(y)$ *if and only if* $U_H(y) \leq U_H(x)$*;*
- *(c)* *If* $\mathrm{ord}(x) = \mathrm{ord}(y)$, *then* $x \in J^kH$ *if and only if* $y \in J^kH$ *for* $k \in \mathbb{N}$*;*

(ii) *If* $U_H(x) = U_H(y)$ *then* $x \in J^kH$ *if and only if* $y \in J^kH$ *for* $k \in \mathbb{N}$*;*

(iii) $JH = H_1 \oplus \cdots \oplus H_{n-1} \oplus pH_n$ *if H is finite, i.e. all k_i are finite.*

Proof. If $n = 1$, then H is homogeneous, hence the proof of (a) is trivial. Moreover, if $x, y \in H$, then clearly $U_H(y) \leq U_H(x)$ implies $\operatorname{ord}(x) \leq \operatorname{ord}(y)$. Conversely, if $\operatorname{ord}(x) \leq \operatorname{ord}(y)$, then $|x| \leq |y|$ follows from (a). Thus $U_H(y) \leq U_H(x)$ using the fact that no Ulm sequence in H has a gap before the first infinite entry. Therefore (b) holds.
To show (c) let $\operatorname{ord}(x) = \operatorname{ord}(y)$. Then (b) implies that $U_H(x) = U_H(y)$, hence it suffices to prove (ii). But (ii) is trivial since each J^k is an ideal of R and R acts (fully) transitively on H. It remains to prove (iii). We have to show that $JH = H_1 \oplus \cdots \oplus H_{n-1} \oplus pH_n$. Since $p * id_H \in R$ is nilpotent we obtain $p * id_H \in J$ and hence $pH \subseteq JH$. Assume that there exists $x \in JH \backslash pH$ and write $x = (x_1, \cdots, x_n)$ with $x_i \in H_i$. Let k be maximal such that $x_k \neq 0$ and $x_k \notin pH_k$. Then any element in $H_1 \oplus \cdots \oplus H_k$ has Ulm sequence greater or equal to $U_H(x)$, hence $H_1 \oplus \cdots \oplus H_k \subseteq Rx \subseteq RJH \subseteq JH$ by full transitivity. Therefore $JH = H_1 \oplus \cdots H_{k-1} \oplus pH_k \oplus \cdots \oplus pH_n$ for some $1 \leq k \leq n$. Note that $H = JH$ would imply $H = 0$ by Nakayama's Lemma ([1], Corollary 15.13). If $k < n$, then $H/JH = (H_k \oplus \cdots \oplus H_n)/(pH_k \oplus \cdots \oplus pH_n)$ and we choose any element x in $H_k \backslash JH$. It follows that $U_H(x) = (0, 1, \cdots, m_k - 1, \infty, \cdots)$ and obviously any element $z \in H \backslash JH$ satisfies $U_H(z) \leq U_H(x)$. Hence there exists an endomorphism $r \in R$ such that $r(z) = x$ and thus $Rx \subseteq Rz$. This proves that any R/J submodule K/JH of H/JH contains the non-trivial module $(Rx + JH)/JH$. But by Theorem XII.5 from [12] H/JH decomposes as an R/J module into a direct sum of simple R/J modules, thus $H/JH = (Rx+JH)/JH$. By Nakayama's Lemma ([1], Corollary 15.13) it follows that $Rx = H$, a contradiction. Hence $k = n$ and $JH = H_1 \oplus \cdots \oplus H_{n-1} \oplus pH_n$. □

Next we will show that for the homogeneous case it is enough to consider groups of exponent p. Therefore we need the following Proposition which enables us to consider R/J acting on H/JH (Rx/Jx for some $x \in H$) instead of R acting on H.

Proposition 2.2 *Let $n \in \mathbb{N}$ and $H = \bigoplus_{i=1}^{n} H_i$, where each H_i is of the form $\bigoplus_{j=1}^{k_i} \mathbb{Z}(p^{m_i})$ for some fixed $m_i \in \mathbb{N}$ such that $m_i < m_j$ if $i < j$ and cardinals k_i. If R is a unital subring of* $\operatorname{End}(H)$*, then the following hold:*

(i) If H is finite and R acts (fully) transitively on H, then R/J acts (fully) transitively on H/JH;

(ii) If $n = 1$ and R acts fully transitively on H and R/J acts transitively on H/JH, then R acts transitively on H;

(iii) If $x, y \in H$ such that $U_H(x) = U_H(y)$, then $r(x) = y$ for some unit $r \in R$ if and only if $(s + J)(x + Jx) = y + Jx$ for some unit $(s + J) \in R/J$.

Here the action of R/J on H/JH (Rx/Jx) is given by $(r + J)(h + JH) = r(h) + JH$ ($(r + J)(h + Jx) = r(h) + Jx$).

Proof. First we prove (i) and assume that H is as stated, hence finite. Let R be a unital subring of $\operatorname{End}(H)$, then it is easy to check that the action $(r + J)(h + JH) = (r(h) + JH)$ of R/J on H/JH is well-defined. Now assume that R acts (fully) transitively on H and let $(h_1 + JH), (h_2 + JH) \in H/JH$ be two non-zero elements of H/JH. Then $U(h_1 + JH) = U(h_2 + JH)$, since $JH = H_1 \oplus \cdots \oplus H_{n-1} \oplus pH_n$ by Lemma 2.1 (iii) and hence H/JH is a

direct sum of copies of $\mathbb{Z}(p)$. Since $h_1, h_2 \notin JH = H_1 \oplus \cdots \oplus H_{n-1} \oplus pH_n$ we easily obtain $U(h_1) = U(h_2)$. Therefore there exists an (endomorphism) automorphism $r \in R$ such that $r(h_1) = h_2$. Thus $(r+J)(h_1 + JH) = r(h_1) + JH = h_2 + JH$ and $r + J$ is a non trivial (endomorphism) automorphism of R/J since otherwise $h_2 \in JH$, a contradiction. Therefore R/J acts (fully) transitively on H/JH.
To show (ii) let $n = 1$ and assume that R/J acts transitively on H/JH with the action given as above and R acts fully transitively on H. Let h_1, h_2 be two non-zero elements of H such that $U(h_1) = U(h_2)$. We have to distinguish between two cases. Note that $h_1 \in JH$ if and only if $h_2 \in JH$ by Lemma 2.1 (ii):
Case 1: If $h_1, h_2 \notin JH$, then $(h_i + JH) \neq 0$ for $i = 1, 2$, and hence $\mathrm{ord}(h_1) = \mathrm{ord}(h_2) = p^m$ by Lemma 2.1 (i) and $U(h_1 + JH) = U(h_2 + JH)$. Therefore there exists an automorphism $(r_0 + J) \in R/J$ such that $(r_0 + J)(h_1 + JH) = (h_2 + JH)$. Since automorphisms lift modulo the Jacobson radical, r_0 is an automorphism of H such that $r_0(h_1) - h_2 \in JH$. Note that $\mathrm{ord}(h_1) = p^m$ implies that $U(h_1) \leq U(x)$ for any $x \in H$ (Lemma 2.1 $(i)(b)$) and hence $Rh_1 = H$ since R acts fully transitively on H. Thus $JH = Jh_1$ and there exists $r_1 \in J$ such that $r_0(h_1) - h_2 = r_1(h_1)$. It follows that $(r_0 - r_1)(h_1) = h_2$ and $(r_0 - r_1) \in R^*$ since $(r_0 - r_1)r_0^{-1} = 1 - r_1 r_0^{-1} \in 1 - J \subseteq R^*$.
Case 2: If $h_1, h_2 \in JH$, then there exist $g_1, g_2 \notin pH = JH$ such that $h_i = p^{m-k} g_i$ for some $k \in \mathbb{N}$ and $U_H(g_1) = U_H(g_2)$ for $i = 1, 2$. By case 1 we obtain $r \in R^*$ such that $r(g_1) = g_2$, and hence $r(h_1) = h_2$.
Finally, (iii) is easy to check as above since units modulo the Jacobson radical lift and the action of R/J on Rx/Jx is well-defined. □

The following example shows that in Proposition 2.2 (ii) the assumption that R acts fully transitively on H can not be avoided.

Example 2.3 Let $p > 2$ be a prime, $H = \mathbb{Z}(p) \oplus \mathbb{Z}(p)$ and R be the unital subring of $\mathrm{End}(H)$ consisting of all lower triangular 2×2-matrices over $\mathbb{Z}(p)$. Then R acts neither fully transitively nor transitively on H but R/J acts (fully) transitively on H/JH.

Proof. Clearly no element of R can map the element $(0, 1)$ onto $(1, 0)$, hence R acts neither fully transitively nor transitively on H since $(1, 0)$ and $(0, 1)$ have the same Ulm sequence. Moreover, it is easy to see that J is the set of all 2×2 matrices over $\mathbb{Z}(p)$ which have a non-zero entry only in the left lower corner. Hence $JH = \{(0, x) : x \in \mathbb{Z}(p)\}$ and thus $H/JH \cong \mathbb{Z}(p)$. Now, if $(r, 0) + JH$ and $(s, 0) + JH$ are two non-zero elements in H/JH, then both elements have the same Ulm sequence and $\left(\begin{smallmatrix} sr^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) + J$ is an automorphism which maps $(r, 0) + JH$ onto $(s, 0) + JH$. Therefore R/J acts (fully) transitively on H/JH. □

By Proposition 2.2 it suffices to look at Rx/Jx for some $x \in H$. For this we prove a more general theorem on finite semisimple rings.

Theorem 2.4 *Let $1 \in R$ be a finite semisimple ring and M a finitely generated R-module. If $u, v \in M$ and $r, s \in R$ such that $ru = v$ and $sv = u$, then there exists a unit $t \in R$ such that $tu = v$.*

Proof. Since R is semisimple it follows by the Artin-Wedderburn Theorem ([1], Theorem 13.6) that R is the ring direct product $R \cong \prod_{i=1}^{k} \mathrm{Mat}_{n_i}(F_i)$ for some $k, n_i \in \mathbb{N}$ and some finite fields F_i $(1 \leq i \leq k)$. Since M is finitely generated over R it is projective ([12], Theorem XII.5). Hence [12][Theorem XII.4] implies that $M \cong \bigoplus_{i=1}^{l} Re_i$, where the $e_i \in R$ are minimal idempotents. Let $i \in \{1, \cdots, k\}$ and $E_j^i \in \mathrm{Mat}_{n_i}(F_i)$ be the matrix with 1 in the (j,j) entry and zeros elsewhere $(j = 1, \cdots, n_i)$. Then E_j^i is a minimal idempotent in $\mathrm{Mat}_{n_i}(F_i)$ and it is easy to see that $\{E_j^i : j = 1, \cdots, n_i\}$ is a complete set of minimal idempotents of $\mathrm{Mat}_{n_i}(F_i)$. Thus each Re_i in the decomposition of M is a copy of $F_j^{n_j}$ if $e_i \in \mathrm{Mat}_{n_j}(F_j)$. We let M_i be the collection of all summands Re_j belonging to $\mathrm{Mat}_{n_i}(F_i)$, i.e., $M_i = \bigoplus_{e_j \in \mathrm{Mat}_{n_i}(F_i)} Re_j$. Then any M_i is a direct sum of copies of $F_i^{n_i}$. Let $u, v \in M$ and $r, s \in R$ such that $ru = v$ and $sv = u$. We write $u = (u_1, \cdots, u_k)$ and $v = (v_1, \cdots, v_k)$ corresponding to the decomposition $M = \bigoplus_{i=1}^{k} M_i$. Moreover, we write $r = (r_1, \cdots, r_k)$ and $s = (s_1, \cdots, s_k)$ corresponding to the decomposition of $R = \prod_{i=1}^{k} \mathrm{Mat}_{n_i}(F_i)$. It follows that $ru = (r_1u_1, \cdots, r_ku_k) = v$ and $sv = (s_1v_1, \cdots, s_kv_k) = u$. Hence we obtain $r_iu_i = v_i$ and $s_iv_i = u_i$ for all $1 \leq i \leq k$. If we can find units $t_i \in \mathrm{Mat}_{n_i}(F_i)$ such that $t_iu_i = v_i$ for all i, then $t = (t_1, \cdots, t_k)$ would be a unit in R mapping u onto v and we are done. Thus we may assume w.l.o.g. that $r, s \in R = \mathrm{Mat}_n(F)$ for some finite field F and $u, v \in M \cong \bigoplus_{i=1}^{l} F^n$. Again we write $u = (u_1, \cdots, u_l)$ and $v = (v_1, \cdots, v_l)$ with $u_i, v_i \in F^n$ for all $1 \leq i \leq l$. Since r and s are matrices they preserve linear dependence of vectors. Let $\{u_i : i \in I\}$ be a maximal linearly independent set among the u_i's. Then $ru_i = v_i$ and $sv_i = u_i$ implies that $\{v_i : i \in I\}$ is also maximal linearly independent among the v_i's. Obviously there is a non-singular matrix $t \in \mathrm{Mat}_n(F)$ mapping u_i onto v_i for all $i \in I$ since both sets $\{u_i : i \in I\}$ and $\{v_i : i \in I\}$ can be extended to bases of F^n. It follows immediately that $tu_i = v_i$ for all $1 \leq i \leq l$ and this completes the proof. $\square$

The following corollary contrasts Corner's example of a fully transitive non-transitive p-group with countable first Ulm subgroup from [4].

Corollary 2.5 *Let $n \in \mathbb{N}$ and $H = \bigoplus_{i=1}^{n} \mathbb{Z}(p)$. If R is a unital subring of* $\mathrm{End}(H)$*, then R acts fully transitively on H if and only if R acts transitively on H.*

Proof. By Lemma 2.1 (*iii*) we get that $JH = pH = 0$ if R acts (fully) transitively on H. Hence $J = 0$ and R is semisimple. Thus the result follows by Theorem 2.4 since H is a finitely generated R-module. Note that transitivity implies full transitivity trivially in our situation since all non-zero elements have the same Ulm sequence. $\square$

Finally we obtain

Theorem 2.6 *Let H be a finite p-group. If R is a unital subring of* $\mathrm{End}(H)$ *which acts fully transitively on H, then R acts transitively on H. In particular every abelian p-group G of type A with finite first Ulm subgroup that is fully transitive must also be transitive.*

Proof. Let $x, y \in H$ such that $U_H(x) = U_H(y)$. Then there exist $r, s \in R$ such that $r(x) = y$ and $s(y) = x$. If $r \in J$, then $x = s(r(x)) \in Jx$, a contradiction, hence $r \notin J$ and similarly $s \notin J$. Thus $(r + J)(x + Jx) = (y + Jx)$ and $(s + J)(y + Jx) = (x + Jx)$ and by Theorem 2.4 it follows that $(t + J)(x + Jx) = (y + Jx)$ for some unit $(t + J) \in R/J$ since R/J is finite semi-simple. By Proposition 2.2 (*iii*) we get that $u(x) = y$ for some unit $u \in R$. The second claim follows by Lemma 1.5. □

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