Forum Math. **19** (2007), 749–760
DOI 10.1515/FORUM.2007.029

Forum
Mathematicum

Completeness of cotorsion pairs

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(Communicated by Rüdiger Göbel)

Abstract. Complete cotorsion pairs are among the main sources of module approximations. Given a ring R and a cotorsion pair $\mathfrak{C} = (\mathscr{A}, \mathscr{B})$, we consider closure properties of the classes $\mathscr{A}$ and $\mathscr{B}$ that imply completeness of $\mathfrak{C}$.

Assuming Gödel's Axiom of Constructibility (V = L) we prove that $\mathfrak{C}$ is complete provided $\mathfrak{C}$ is generated by a set, and either (i) $\mathscr{A}$ is closed under pure submodules, or (ii) $\mathfrak{C}$ is hereditary and $\mathscr{B}$ consists of modules of finite injective dimension. These two results are independent of ZFC + GCH. However, (i) or (ii) implies completeness of $\mathfrak{C}$ in ZFC provided $\mathscr{B}$ is closed under arbitrary direct sums.

In ZFC, we also show that $\mathfrak{C}$ is complete whenever $\mathfrak{C}$ is hereditary, $\mathscr{A}$ closed under arbitrary direct products, and $\mathscr{B}$ consists of modules of finite injective dimension. This yields a characterization of n-cotilting cotorsion pairs as the hereditary cotorsion pairs $(\mathscr{C}, \mathscr{D})$ such that $\mathscr{C}$ is closed under arbitrary direct products and $\mathscr{D}$ consists of modules of injective dimension $\leq n$.

1991 Mathematics Subject Classification: 16D90, 16E30, 18E40, 03E35, 03E45.

Introduction

Given an arbitrary ring R, it is generally not possible to classify all R-modules. A way to overcome this obstacle consists in selecting appropriate classes of R-modules, $\mathscr{C}$, and studying $\mathscr{C}$-approximations (envelopes and covers) of R-modules. This approach has successfully been used in module theory starting from classical works on injective envelopes and projective covers by Matlis, Bass et al. in the 1960's, over applications in commutative algebra and representation theory of artin algebras by Auslander's school, to constructions of flat covers by Enochs, Xu et al., and recent applications to tilting theory and finitistic dimension conjectures.

Many approximation classes of modules come from complete cotorsion pairs. However, not all cotorsion pairs are complete, so it is essential to have criteria of completeness available. In the present paper, we investigate closure properties of the classes $\mathscr{A}$ and $\mathscr{B}$ that imply completeness of the cotorsion pair $\mathfrak{C} = (\mathscr{A}, \mathscr{B})$ both in ZFC, and in the extension of ZFC with the Axiom of Constructibility (V = L).

Supported by GAČR 201/03/0937, and by the research project MSM 0021620839.

Given an arbitrary ring R, a pair of classes of right R-modules, $\mathfrak{C} = (\mathscr{A}, \mathscr{B})$, is a *cotorsion pair* provided $\mathscr{A} = {}^{\perp_1}\mathscr{B}$ $(= \{M \in \text{Mod-}R \mid \text{Ext}^1_R(M, B) = 0 \ \forall B \in \mathscr{B}\})$ and $\mathscr{B} = \mathscr{A}^{\perp_1}$ $(= \{N \in \text{Mod-}R \mid \text{Ext}^1_R(A, N) = 0 \ \forall A \in \mathscr{A}\})$.

A cotorsion pair $\mathfrak{C}$ is *complete* provided that the class $\mathscr{A}$ is special precovering (see below for unexplained terminology on module approximations). Salce [Sa] observed that this is equivalent to $\mathscr{B}$ being a special preenveloping class. So special $\mathscr{A}$-precovers of all modules exist iff special $\mathscr{B}$-preenvelopes do. In this way, complete cotorsion pairs are also helpful to proving dual results in the category Mod-R where no categorial duality is available.

In Section 1, we first work under the assumption of V = L. We prove that $\mathfrak{C}$ is complete whenever $\mathfrak{C}$ is generated by a set and $\mathscr{A}$ is closed under pure submodules (Theorem 1.3). We also show that $\mathfrak{C}$ is complete whenever $\mathfrak{C}$ is hereditary, generated by a set, and $\mathscr{B}$ consists of modules of finite injective dimension (Theorem 1.7). These results generalize [ET, Theorem 14] which says that (under V = L) any cotorsion pair generated by a set is complete in the particular setting of right hereditary rings. However, by [ES], these results are independent of ZFC + GCH.

In the rest of the paper, we work in ZFC and prove analogous results, but replacing V = L by further closure properties of the classes $\mathscr{A}$ and $\mathscr{B}$. First, we show that $\mathfrak{C}$ is complete whenever $\mathscr{B}$ is closed under arbitrary direct sums and either (i) $\mathscr{A}$ is closed under pure submodules, or (ii) $\mathfrak{C}$ is hereditary and $\mathscr{B}$ consists of modules of finite injective dimension (Theorem 1.9). Moreover, if R is right $\aleph_0$-noetherian and $\mathfrak{C}$ is a hereditary cotorsion pair such that $\mathscr{B}$ is closed under arbitrary direct sums and $\mathscr{B}$ consists of modules of finite injective dimension, then $\mathfrak{C}$ is of countable type (Corollary 1.10).

In Section 2, we employ cotorsion pairs for dualizing to cotilting modules some of the recent results on (infinitely generated) tilting modules. Recall that a cotorsion pair $\mathfrak{C} = (\mathscr{A}, \mathscr{B})$ is *n-tilting* provided there is a tilting module T of projective dimension $\leq n$ such that $\mathscr{B} = \{T\}^{\perp} (= \bigcap_{i \geq 1} \text{KerExt}^i_R(T, -))$. By [ST, Theorem 2], a cotorsion pair $\mathfrak{C}$ is n-tilting iff $\mathfrak{C}$ is hereditary, $\mathscr{A}$ consists of modules of projective dimension $\leq n$, and $\mathscr{B}$ is closed under arbitrary direct sums. Dually, a cotorsion pair $\mathfrak{C} = (\mathscr{A}, \mathscr{B})$ is called *n-cotilting* provided there is a cotilting module C of injective dimension $\leq n$ such that $\mathscr{A} = {}^{\perp}\{C\} (= \bigcap_{i \geq 1} \text{KerExt}^i_R(-, C))$.

The main result of Section 2 says that a cotorsion pair $\mathfrak{C}$ is n-cotilting iff $\mathfrak{C}$ is hereditary, $\mathscr{A}$ is closed under arbitrary direct products, and $\mathscr{B}$ consists of modules of injective dimension $\leq n$ (Theorem 2.4). In particular, 1-cotilting cotorsion pairs are characterized as the cotorsion pairs $(\mathscr{A}, \mathscr{B})$ such that $\mathscr{A}$ is closed under arbitrary direct products and $\mathscr{B}$ consists of modules of injective dimension ≤ 1.

For a ring R, denote by Mod-R the category of all (unitary right R-) modules. For a module M, gen(M) denotes the minimal cardinality of an R-generating subset in M, and $E(M)$ the injective envelope of M. For $n \geq 0$, the class of all modules of injective dimension $\leq n$ is denoted by $\mathscr{I}_n$.

A module T is a *tilting module* provided T has finite projective dimension, $\text{Ext}^i_R(T, T^{(I)}) = 0$ for any set I and any $i > 0$, and there is a short exact sequence

$0 \to R \to T_0 \to \cdots \to T_r \to 0$ for some $r \geq 0$ such that T_i is a direct summand in a (possibly infinite) direct sum of copies of T for each $i = 0, \ldots, r$. A tilting module of projective dimension $\leq n$ is called *n-tilting*.

Dually, a module C is a *cotilting module* provided C has finite injective dimension, $\mathrm{Ext}_R^i(C^I, C) = 0$ for any set I and any $i > 0$, and there is a short exact sequence $0 \to C_r \to \cdots \to C_0 \to W \to 0$ for some $r \geq 0$ and some injective cogenerator W of Mod-R such that $C_i \in \mathrm{Prod}(C)$ for each $i = 0, \ldots, r$ where $\mathrm{Prod}(C)$ denotes the class of all direct summands in a (possibly infinite) direct product of copies of C. Cotilting modules of injective dimension $\leq n$ are called *n-cotilting*.

For a class of modules $\mathscr{C}$, let $\mathscr{C}^\perp = \bigcap_{i>0} \mathscr{C}^{\perp_i}$ where $\mathscr{C}^{\perp_i} = \{M \in \text{Mod-}R \mid \mathrm{Ext}_R^i(C, M) = 0 \text{ for all } C \in \mathscr{C}\}$. Similarly, ${}^\perp\mathscr{C} = \bigcap_{i>0} {}^{\perp_i}\mathscr{C}$ where ${}^{\perp_i}\mathscr{C} = \{M \in \text{Mod-}R \mid \mathrm{Ext}_R^i(M, C) = 0 \text{ for all } C \in \mathscr{C}\}$.

Note that $\mathfrak{D} = ({}^{\perp_1}\mathscr{C}, ({}^{\perp_1}\mathscr{C})^{\perp_1})$ and $\mathfrak{C} = ({}^{\perp_1}(\mathscr{C}^{\perp_1}), \mathscr{C}^{\perp_1})$ are cotorsion pairs called the cotorsion pairs *generated* and *cogenerated* by $\mathscr{C}$ respectively. If $\mathscr{C}$ has a representative set of elements, we say that $\mathfrak{D}$ is *generated by a set*, and $\mathfrak{C}$ is *cogenerated by a set*; in this case, $\mathfrak{C}$ is a complete cotorsion pair by [ET1].

A cotorsion pair $(\mathscr{A}, \mathscr{B})$ is *hereditary* provided $\mathscr{A} = {}^\perp\mathscr{B}$ and $\mathscr{B} = \mathscr{A}^\perp$ (that is, $\mathrm{Ext}_R^i(A, B) = 0$ for all $A \in \mathscr{A}$, $B \in \mathscr{B}$ and $i \geq 1$).

A class $\mathscr{C}$ is *n-cotilting* provided there is an n-cotilting module C such that $\mathscr{C} = {}^\perp\{C\}$.

A class $\mathscr{C}$ of modules is *special preenveloping* provided for each module M there are $C \in \mathscr{C}$, $D \in {}^{\perp_1}\mathscr{C}$, and an exact sequence $0 \to M \to C \to D \to 0$ (the monomorphism $M \to C$ is called a *special $\mathscr{C}$-preenvelope* of the module M). Dually, $\mathscr{C}$ is *special precovering* if for each module M there are $C \in \mathscr{C}$, $D \in \mathscr{C}^{\perp_1}$, and an exact sequence $0 \to D \to C \to M \to 0$ (the epimorphism $C \to M$ is a *special $\mathscr{C}$-precover* of M).

For unexplained terminology, we refer to [EJ] and [EM].

1 Completeness under (V = L)

Let R be a ring, M a module, and $\sigma > 0$ an ordinal. A sequence of submodules of M, $(M_\alpha \mid \alpha < \sigma)$ is a *σ-filtration* of M provided that $M_0 = 0$, $M_\alpha \subseteq M_{\alpha+1}$ for all $\alpha + 1 < \sigma$, $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for all limit ordinals $\alpha < \sigma$, and $M = \bigcup_{\alpha<\sigma} M_\alpha$.

Following [ET2], given a cardinal κ, we define a *κ-refinement* of M (of length σ) as a (σ-)filtration of M such that M_α is a pure submodule of M for all $\alpha < \sigma$, and $\mathrm{card}\, M_{\alpha+1}/M_\alpha \leq \kappa$ for all $\alpha + 1 < \sigma$.

We start with a generalization of [T1, Lemma 3.7]:

Lemma 1.1 (V = L). *Let κ be a regular uncountable cardinal, and E a stationary subset of κ. Let R be a ring such that* $\mathrm{card}\, R \leq \kappa$, *and N a module with* $\mathrm{card}\, E(N) \leq \kappa$. *Let M be a module of cardinality κ with a κ-filtration $(C_\alpha \mid \alpha < \kappa)$ such that* $\mathrm{gen}(C_\alpha) < \kappa$ *and* $\mathrm{Ext}_R^1(C_\alpha, N) = 0$ *for all $\alpha < \kappa$, and $E = \{\alpha < \kappa \mid \mathrm{Ext}_R^1(C_{\alpha+1}/C_\alpha, N) \neq 0\}$. Then* $\mathrm{Ext}_R^1(M, N) \neq 0$.

Proof. The proof is as for [T1, Lemma 3.7], except that [T1, Lemma 3.7] has the additional assumption of M being κ-projective (that is, all $<\kappa$-generated submodules of M being projective) rather than $\mathrm{Ext}_R^1(C_\alpha, N) = 0$ for all $\alpha < \kappa$. We explain how to overcome the two points where κ-projectivity was originally used:

(i) For existence of the maps $f_\alpha \in \mathrm{Hom}_R(C_\alpha, N) \setminus \mathrm{Im}\,\mathrm{Hom}_R(\nu_\alpha, N)$ $(\alpha \in E)$ in the second paragraph on p. 1529. Since $\mathrm{Ext}_R^1(C_{\alpha+1}/C_\alpha, N) \neq 0$, the existence of f_α follows already from our assumption of $\mathrm{Ext}_R^1(C_{\alpha+1}, N) = 0$.

(ii) For the definition of the map $g_{\alpha+1}$ extending g_α in case (II) on p. 1530. The projectivity of C_α was used to factorize g_α through the projection $\pi : I \to I/N$. However, the existence of a factorization follows already from $\mathrm{Ext}_R^1(C_\alpha, N) = 0$. □

Lemma 1.2 (V = L). *Let N be a module such that ${}^{\perp_1}N$ is closed under pure submodules, and κ be a cardinal with $\kappa \geq \mathrm{card}\, R + \mathrm{card}\, E(N) + \aleph_0$.*

Then for each module $M \in {}^{\perp_1}N$ there are an ordinal σ and a κ-refinement of M of length σ, $(M_\alpha \mid \alpha < \sigma)$, such that $M_{\alpha+1}/M_\alpha \in {}^{\perp_1}N$ for all $\alpha + 1 < \sigma$.

Proof. The existence of the κ-refinement of M is proved by induction on the cardinality λ of M. It is clear for $\lambda \leq \kappa$.

Let λ be a regular cardinal $> \kappa$. By induction, we construct a κ-refinement $\mathscr{R}$ of M. First, we enumerate the elements of M, $M = \{m_\alpha \mid \alpha < \lambda\}$, and let $M_0 = 0$.

Let $\alpha < \lambda$. Since $\kappa \geq \mathrm{card}\, R + \aleph_0$, there is a pure submodule P/M_α of M/M_α containing $m_\alpha + M_\alpha$ such that $\mathrm{card}\, P/M_\alpha \leq \kappa$ (P/M_α is the purification of $\{m_\alpha + M_\alpha\}$ in M/M_α). Since M_α is pure in M by inductive assumption, also P is pure in M, and we let $M_{\alpha+1} = P$. If α is a limit ordinal, we let $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ which is again a pure submodule in M. Since λ is a regular cardinal, $\mathscr{R}$ has length λ, and $\mathrm{card}\, M_\alpha < \lambda$ for all $\alpha < \lambda$.

Possibly taking a λ-subfiltration, we can w.l.o.g. assume that $\mathscr{R}$ is a λ-filtration with the following property: if $\alpha < \beta < \lambda$ are such that $\mathrm{Ext}_R^1(M_\beta/M_\alpha, N) \neq 0$, then also $\mathrm{Ext}_R^1(M_{\alpha+1}/M_\alpha, N) \neq 0$.

Since $M \in {}^{\perp_1}N$ and ${}^{\perp_1}N$ is closed under pure submodules, $\mathrm{Ext}_R^1(M_\alpha, N) = 0$ for every $\alpha < \lambda$, and Lemma 1.1 yields that the set

$$E = \{\alpha < \lambda \mid \mathrm{Ext}_R^1(M_{\alpha+1}/M_\alpha, N) \neq 0\}$$

is not stationary in λ. So there is a closed and unbounded subset U of λ such that $U \cap E = \emptyset$. Taking the λ-subfiltration of $\mathscr{R}$ indexed by the elements of U, we obtain a λ-filtration, $(F_\alpha \mid \alpha < \lambda)$, of M such that $F_{\alpha+1}/F_\alpha \in {}^{\perp_1}N$ for all $\alpha < \lambda$. By inductive assumption, we can refine this λ-filtration into a σ-filtration which is a κ-refinement of M.

If λ is singular $> \kappa$, we use the version of Shelah's singular compactness theorem from [EM, Theorem IV.3.7]. We call a module M "free" if M has a κ-refinement as in the claim of the Theorem. In order to prove that M is "free", it suffices to show that M is ρ-"free" for any regular cardinal $\kappa < \rho < \lambda$, and apply [EM, Lemma XII.1.14] (with $\mu = \kappa$). For the system witnessing the ρ-"freeness" of M, we take the set, $\mathscr{W}$, of all pure submodules of M of cardinality $< \rho$. Since ${}^{\perp_1}N$ is closed under pure submodules, each element of $\mathscr{W}$ is "free" by inductive assumption. Moreover, any subset

X of M of cardinality $< \rho$ is contained in an element of $\mathscr{W}$ (e.g., in the purification of X in M). Finally, pure submodules of M are closed under unions of arbitrary well-ordered chains. So $\mathscr{W}$ witnesses ρ-"freeness" of M in the sense of [EM, Definition IV.1.1]. □

Now, we are in a position to prove our first main result, generalizing [ET2, Theorem 14] to arbitrary rings:

Theorem 1.3 (V = L). *Let R be a ring and $\mathfrak{C} = (\mathscr{A}, \mathscr{B})$ be a cotorsion pair such that $\mathfrak{C}$ is generated by a set, and $\mathscr{A}$ is closed under pure submodules. Then $\mathfrak{C}$ is complete.*

Proof. By assumption, there is a module N such that $\mathscr{A} = {}^{\perp_1}N$. By Lemma 1.2 and [EM, Proposition XII.1.5], $\mathscr{B} = \mathscr{S}^{\perp_1}$ where $\mathscr{S}$ is a representative set of those modules from $\mathscr{A}$ that have cardinality $\leq \operatorname{card} R + \operatorname{card} E(N) + \aleph_0$. So $\mathfrak{C}$ is cogenerated by a set, and hence $\mathfrak{C}$ is complete by [ET1, Theorem 10]. □

For the other generalization of [ET2, Theorem 14], we will need the following refinement of a result of Fuchs and Lee [FL, Theorem 2.1] (which in turn is based on a construction of Hill [H]) and another two lemmas. These results will be proved in ZFC.

Recall that given an infinite cardinal κ, a ring R is right κ-*noetherian* provided that each right ideal of R is $\leq \kappa$-generated.

Lemma 1.4. *Let R be a ring and M be a module possessing a σ-filtration $\mathscr{M} = (M_\alpha \mid \alpha < \sigma)$ such that for each $\alpha + 1 < \sigma$, $M_{\alpha+1} = M_\alpha + A_\alpha$ for a module A_α.*

Then there is a family, $\mathscr{F}$, consisting of submodules of M such that

(i) *$0, M \in \mathscr{F}$;*

(ii) *$\mathscr{F}$ is closed under arbitrary sums;*

(iii) *Let κ be an infinite cardinal such that R is right κ-noetherian and $\operatorname{gen}(A_\alpha) \leq \kappa$ for all $\alpha + 1 < \sigma$. Assume $B \in \mathscr{F}$ and X is a subset of M of cardinality $\leq \kappa$. Then there exists $C \in \mathscr{F}$ such that $B \cup X \subseteq C$ and $\operatorname{gen}(C/B) \leq \kappa$;*

(iv) *Let $N, P \in \mathscr{F}$ be such that $N \subseteq P$. Then there exists a τ-filtration $\bar{\mathscr{P}} = (\bar{P}_\gamma \mid \gamma < \tau)$ of the module $\bar{P} = P/N$ such that $\tau \leq \sigma$, for each $\gamma + 1 < \tau$ there is $\alpha + 1 < \sigma$ with $\bar{P}_{\gamma+1} = \bar{P}_\gamma + (A_\alpha + N)/N$, and each consecutive factor in $\bar{\mathscr{P}}$ is isomorphic to some consecutive factor in $\mathscr{M}$.*

Proof. We call a subset $S \subseteq \sigma$ 'closed' provided that each $\beta \in S$ satisfies $M_\beta \cap A_\beta \subseteq \sum_{\alpha \in S, \alpha < \beta} A_\alpha$. Let $\mathscr{F}$ be the family of all modules of the form $M(S) = \sum_{\alpha \in S} A_\alpha$ where $S \subseteq \sigma$ is 'closed'. By [FL, Theorem 2.1], $\mathscr{F}$ satisfies conditions (i)–(iii), and also condition (iv) in the particular case of $N = 0$.

In order to prove (iv) in full generality, let $N = M(S)$ and $P = M(T)$ for some 'closed' subsets $S, T \subseteq \sigma$. Since $S \cup T$ is also 'closed', we can w.l.o.g. assume that $S \subseteq T$. For each $\beta < \sigma$, let $F_\beta = (N + \sum_{\alpha \in T \setminus S, \alpha < \beta} A_\alpha)/N$. Clearly, $(\bar{\mathscr{F}} = (F_\beta \mid \beta < \sigma)$ is a σ-filtration of $\bar{P}$ such that $F_{\beta+1} = F_\beta$ for $\beta \notin T \setminus S$, and $F_{\beta+1} = F_\beta + (A_\beta + N)/N$ otherwise.

Let $\beta \in T \setminus S$. Then $F_{\beta+1}/F_\beta \cong A_\beta/(A_\beta \cap B_\beta)$ where $B_\beta = N + \sum_{\alpha \in T \setminus S, \alpha < \beta} A_\alpha$. However, $B_\beta = C_\beta + \sum_{\alpha \in S, \beta < \alpha} A_\alpha$, where $C_\beta = \sum_{\alpha \in T, \alpha < \beta} A_\alpha$, so $A_\beta \cap B_\beta \supseteq A_\beta \cap C_\beta = A_\beta \cap M_\beta$ (because $\beta \in T$, so $A_\beta \cap M_\beta \subseteq C_\beta$).

Conversely, if $a \in A_\beta \cap B_\beta$, then $a = c + a_{\alpha_0} + \cdots + a_{\alpha_k}$ where $c \in C_\beta$ $(\subseteq M_\beta)$, $\alpha_i \in S$ and $a_{\alpha_i} \in A_{\alpha_i}$ for all $i \leq k$, and $\alpha_i > \alpha_{i+1}$ for all $i < k$. W.l.o.g., we can assume that α_0 is minimal possible. If $\alpha_0 > \beta$, then $a_{\alpha_0} = a - c - a_{\alpha_1} \cdots - a_{\alpha_k} \in M_{\alpha_0} \cap A_{\alpha_0} \subseteq \sum_{\alpha \in S, \alpha < \alpha_0} A_\alpha$ (since $\alpha_0 \in S$) in contradiction with the minimality of α_0. So $\alpha_0 < \beta$, and hence $a \in M_\beta$. This proves that $A_\beta \cap B_\beta = A_\beta \cap M_\beta$.

So $\beta \in T \setminus S$ implies $F_{\beta+1}/F_\beta \cong A_\beta/(M_\beta \cap A_\beta) \cong M_{\beta+1}/M_\beta$ which is a consecutive factor in $\mathscr{M}$. Finally, $\overline{\mathscr{P}}$ is obtained from $\overline{\overline{\mathscr{F}}}$ by removing repetitions. □

Lemma 1.5 [ST, Theorem 8]. *Let R be a ring and $\mathscr{A} = {}^{\perp_1}\mathscr{C}$ with $\mathscr{C}$ closed under arbitrary direct sums. Let κ be a regular uncountable cardinal, and assume that M is a module in $\mathscr{A}$ possessing a κ-filtration $\mathscr{M} = (M_\alpha \mid \alpha < \kappa)$ such that $M_\alpha \in \mathscr{A}$ and M_α is $<\kappa$-generated for all $\alpha < \kappa$. Then there is a subfiltration of $\mathscr{M}$ with all consecutive factors in $\mathscr{A}$.*

Going through the proof of [ST, Theorem 8], it is easy to see that the hypothesis "$\mathscr{C}$ is closed under direct sums" can be replaced by "$\mathscr{C}$ is closed under κ-bounded products". By a *κ-bounded product*, we mean $\prod_{\iota \in I}^{<\kappa} C_\iota$, i.e. the submodule of $\prod_{\iota \in I} C_\iota$ consisting of elements with support of cardinality less than κ. The only change to be made in the original proof of [ST, Theorem 8] is to define the module D as $\prod_{\alpha<\kappa}^{<\kappa} B_\alpha$ instead of $\bigoplus_{\alpha<\kappa} B_\alpha$. It yields $\mathrm{Ker}(\pi_\alpha) \cong \prod_{\beta<\alpha} B_\beta \in \mathscr{B}$ for $\alpha < \kappa$. Everything else remains unchanged. We will refer to this alternative version of Lemma 1.5 as to "modified Lemma 1.5".

Lemma 1.6. *Let μ be an infinite cardinal, R a right μ-noetherian ring, and n be a positive integer. Then every module $M \in {}^{\perp_1}\mathscr{I}_n$ has a σ-filtration whose consecutive factors are $\leq \mu$-generated elements of ${}^{\perp_1}\mathscr{I}_n$.*

Proof. For $1 \leq i \leq n$, consider the cotorsion pairs $\mathscr{C}_i = (\mathscr{A}_i, \mathscr{B}_i)$ such that $\mathscr{A}_i = {}^{\perp_i}\mathscr{I}_n$; otherwise said, $\mathscr{C}_i$ is generated by the class of $(i-1)$-th cosyzygies of modules in $\mathscr{I}_n$. Let $\mathscr{Q}_i$ be a representative set of all $\leq \mu$-generated modules in $\mathscr{A}_i$. By downward induction on i, we will prove the following claim for each $M \in \mathscr{A}_i$: "M has a σ-filtration $(M_\alpha \mid \alpha < \sigma)$ such that for each $\alpha + 1 < \sigma$, $M_{\alpha+1}/M_\alpha$ is isomorphic to an element of $\mathscr{Q}_i$".

Let $i = n$. We prove this step by induction on $\lambda = \mathrm{gen}(M)$. The claim is clear for $\lambda \leq \mu$. Let $\lambda > \mu$ be regular. Since $\mathscr{A}_n$ is closed under arbitrary submodules, M has a λ-filtration $\mathscr{M} = (M_\alpha \mid \alpha < \lambda)$ such that for each $\alpha < \lambda$, $M_\alpha \in \mathscr{A}_n$ and M_α is $<\lambda$-generated. R right μ-noetherian implies $\prod_{\iota \in I}^{<\nu} E_\iota$ is injective provided $\nu > \mu$ and E_ι is injective for every $\iota \in I$. Thus a ν-bounded product of $(i-1)$-th cosyzygies of modules in $\mathscr{I}_n$ is again an $(i-1)$-th cosyzygy of a module in $\mathscr{I}_n$, for all $i = 1, 2, \ldots, n$ and $\nu > \mu$.

By modified Lemma 1.5, possibly passing to a subfiltration of $\mathscr{M}$, we can w.l.o.g. assume that for each $\alpha < \lambda$, $M_{\alpha+1}/M_\alpha \in \mathscr{A}_n$, so by inductive premise, $M_{\alpha+1}/M_\alpha$ satisfies the claim, and so does M. If $\lambda > \mu$ is singular, the singular compactness theorem applies, cf. [EM, Theorem IV.3.7 and Lemma XII.1.14], since $\mathscr{A}_n$ is closed under arbitrary submodules, and the claim follows.

Let $0 < i < n$. Again, the claim is clear when $\lambda = \mathrm{gen}(M) \leq \mu$. If $\lambda > \mu$, by inductive premise, M has a λ-filtration $\mathscr{M} = (M_\alpha \mid \alpha < \lambda)$ such that for each $\alpha < \lambda$, $M_{\alpha+1}/M_\alpha$ is isomorphic to an element of $\mathscr{Q}_{i+1}$, and hence $\mathrm{gen}(M_\alpha) < \lambda$. By [EM, Proposition XII.1.5], also $M_\alpha \in \mathscr{A}_{i+1}$, $M/M_\alpha \in \mathscr{A}_{i+1}$, and the exact sequence for $N \in \mathscr{I}_n$

$$0 = \mathrm{Ext}_R^i(M,N) \to \mathrm{Ext}_R^i(M_\alpha,N) \to \mathrm{Ext}_R^{i+1}(M/M_\alpha,N) = 0$$

gives $M_\alpha \in \mathscr{A}_i$ for each $\alpha < \lambda$. If λ is regular, we use again modified Lemma 1.5 to obtain a subfiltration of $\mathscr{M}$ whose consecutive factors are $<\lambda$-generated elements of $\mathscr{A}_i$. By inductive premise, the consecutive factors satisfy the claim, hence so does M. For λ singular, Lemma 1.4 applies to the filtration $\mathscr{M}$ and yields the family $\mathscr{F}$. If $K \in \mathscr{F}$ then K and M/K have filtrations whose consecutive factors are $\leq \mu$-generated elements of $\mathscr{A}_{i+1}$ (by 1.4(i) and (iv)). In particular, $K, M/K \in \mathscr{A}_{i+1}$, and the argument above yields $K \in \mathscr{A}_i$. Now, for each regular cardinal κ such that $\mu < \kappa < \lambda$, each $<\kappa$-generated module $K \in \mathscr{F}$ satisfies the claim by inductive premise. By 1.4(i)–(iii), and by the singular compactness theorem [EM, Theorem IV.3.7 and Lemma XII.1.14], we conclude that also M satisfies the claim.

Finally, for $i = 1$, we get that each module $M \in \mathscr{A}$ has a σ-filtration with consecutive factors isomorphic to elements from $\mathscr{Q}_1$. □

Theorem 1.7 (V = L). *Let R be a ring, $\mathfrak{C} = (\mathscr{A}, \mathscr{B})$ be a cotorsion pair such that $\mathfrak{C}$ is hereditary and generated by a set, and $\mathscr{B}$ consists of modules of finite injective dimension. Then $\mathfrak{C}$ is complete.*

Proof. By the assumptions, there are an $n < \omega$ and a module $N \in \mathscr{I}_n$ such that $\mathscr{A} = {}^{\perp_1}N$. Let $\mu = \mathrm{card}\, R + \mathrm{card}\, E(N) + \aleph_0$ (for the use of Lemma 1.1 later; see also Theorem 1.9(ii)). For $0 \leq i \leq n$, let $\mathfrak{C}_i = (\mathscr{A}_i, \mathscr{B}_i)$ be a (hereditary) cotorsion pairs with $\mathscr{A}_i = {}^{\perp_1}(\mathscr{B} \cup \mathscr{I}_i) = {}^{\perp_1}\mathscr{B} \cap {}^{\perp_1}\mathscr{I}_i$. As in the proof of the preceding lemma, let $\mathscr{Q}_i$ be a representative set of all $\leq \mu$-generated modules in $\mathscr{A}_i$. And again, by downward induction on i, we will prove that every module $M \in \mathscr{A}_i$ has a σ-filtration $(M_\alpha \mid \alpha < \sigma)$ such that for each $\alpha + 1 < \sigma$, $M_{\alpha+1}/M_\alpha$ is isomorphic to an element of $\mathscr{Q}_i$. However, note the difference between the situation in Lemma 1.6 and here: while in the previous proof one had $\mathscr{A}_i \subseteq \mathscr{A}_{i+1}$, it is vice versa in this case.

Let $i = n$. Since $\mathscr{B} \subseteq \mathscr{I}_n$, we have $\mathscr{A}_n = {}^{\perp_1}\mathscr{I}_n$, and the claim follows directly from Lemma 1.6 because R is clearly μ-noetherian. Let $0 \leq i < n$. We will proceed by induction on $\lambda = \mathrm{gen}(M)$, similarly as in [ST, Lemma 14]. There is nothing to prove for $\lambda \leq \mu$. Let $\lambda > \mu$ and $M \in \mathscr{A}_i$. Consider the exact sequence

$$0 \to K \overset{\subseteq}{\to} R^{(\lambda)} \overset{\pi}{\to} M \to 0.$$

We may suppose that $\mathrm{gen}(K) = \lambda$. Obviously $K \in \mathscr{A}_i$ since $\mathfrak{C}_i$ is hereditary. Moreover, K is a syzygy of a module from ${}^{\perp_1}\mathscr{I}_i$, hence $K \in {}^{\perp_1}\mathscr{I}_{i+1}$. So we actually have $K \in \mathscr{A}_{i+1}$. By inductive premise, there is a σ-filtration $\mathscr{K}$ of K whose consecutive factors are isomorphic to elements of $\mathcal{Q}_{i+1}$. Using Lemma 1.4 we obtain the family $\mathcal{F}$ for $\mathcal{K}$. Let us define $\mathcal{G} = \{L \subseteq M \mid (\exists A_L \subseteq \lambda)(\pi(R^{(A_L)}) = L \ \& \ K \cap R^{(A_L)} \in \mathcal{F})\}$.

We claim that $\mathcal{G} \subseteq \mathcal{A}_i$. Indeed, let L be a module from $\mathcal{G}$. Then for $B \in \mathcal{B}_i$, we have $0 = \operatorname{Ext}^1_R(M,B) \to \operatorname{Ext}^1_R(L,B) \to \operatorname{Ext}^2_R(M/L,B) \cong \operatorname{Ext}^1_R(K/K \cap R^{(A_L)}, B) = 0$. The last equality follows from $\mathcal{B}_i \subseteq \mathcal{B}_{i+1}$ and the fact that $K/K \cap R^{(A_L)} \in \mathcal{A}_{i+1}$ (use (iv) from Lemma 1.4 and $K \cap R^{(A_L)} \in \mathcal{F}$).

It is obvious that $0 \in \mathcal{G}$, and that $\mathcal{G}$ is closed under well-ordered unions of chains. As the next step, we show that for every regular $\kappa \leq \lambda$ with $\kappa > \mu$, and a subset $X \subseteq M$ of cardinality $< \kappa$, there is a $<\kappa$-generated module $L \in \mathcal{G}$ containing X.

Choose a subset $A_0 \subseteq \lambda$ of cardinality $< \kappa$ such that $X \subseteq \pi(R^{(A_0)})$. By (iii) from Lemma 1.4, there is a $<\kappa$-generated module $K_0 \in \mathcal{F}$ such that $K \cap R^{(A_0)} \subseteq K_0$. Take $A_1 \supseteq A_0$ with $K_0 \subseteq R^{(A_1)}$ and $\operatorname{card}(A_1) < \kappa$. Iterating the process, we obtain a chain $K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots$ of $<\kappa$-generated modules from $\mathcal{F}$ and a chain $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$ of subsets of λ of cardinality $< \kappa$. Let us define $L = \pi(R^{(\bigcup_{k<\omega} A_k)})$. Then L is a module from $\mathcal{G}$ we have been looking for.

Let λ be regular. By the previous step, we can select from $\mathcal{G}$ a λ-filtration $\mathcal{M}$ of M consisting of $<\lambda$-generated modules. Applying the modified Lemma 1.5 to $\mathcal{C} = \mathcal{I}_i$ and then Lemma 1.1 for $M \in {}^{\perp_1}N$, we obtain a subfiltration $\mathcal{M}'$ of $\mathcal{M}$ with consecutive factors from ${}^{\perp_1}\mathcal{I}_i$, and then a subfiltration $\hat{\mathcal{M}}$ of $\mathcal{M}'$ whose consecutive factors are in $\mathcal{A}$. However, by [EM, Proposition XII.1.5], these factors even belong to $\mathcal{A}_i$, and they are clearly $<\lambda$-generated. Hence, by inductive premise, they possess σ-filtrations whose consecutive factors are isomorphic to elements from $\mathcal{Q}_i$. It follows that M has the same property.

If λ is singular, the properties of $\mathcal{G}$ proved above make it possible to apply the singular compactness theorem and conclude that M has a σ-filtration whose consecutive factors are isomorphic to elements from $\mathcal{Q}_i$.

Finally, using [EM, Proposition XII.1.5], we obtain that $\mathcal{B} = \mathcal{Q}_0^{\perp_1}$, and [ET1, Theorem 10] finishes our proof of completeness of $\mathfrak{C}$. □

Remarks 1.8. (i) Neither Theorem 1.3 nor Theorem 1.7 can be proved in ZFC or ZFC + GCH. Eklof and Shelah constructed in [ES] a model of ZFC + GCH such that the class of all Whitehead groups (that is, ${}^{\perp_1}\mathbb{Z}$) is not a (special) precovering class of abelian groups. In particular, Lemma 1.2, and Theorems 1.3 and 1.7 fail in that model, so they are independent of ZFC + GCH.

(ii) The proof of Lemma 1.2 relies on three basic properties of pure extensions: the existence of purifications, the fact that if C is pure in A and B/C is pure in A/C then B is pure in A, and on the union of a chain of pure submodules being a pure submodule, cf. [ET2, Lemma 6]. We could alternatively assume that ${}^{\perp_1}N$ is closed under elementary substructures and use the corresponding three basic properties of elementary embeddings: the downward Löwenheim-Skolem Theorem, [P, Proposition 2.25], and the fact that a union of a chain of elementary substructures is again an elementary substructure, respectively, to conclude that ${}^{\perp_1}N$ is a special precovering class.

Assuming $\mathcal{B}$ closed under arbitrary direct sums, we can prove analogs of Theorems 1.3 and 1.7 in ZFC:

Theorem 1.9. *Let R be a ring, μ an infinite cardinal, and $\mathfrak{C} = (\mathscr{A}, \mathscr{B})$ a cotorsion pair such that $\mathscr{B}$ is closed under arbitrary direct sums. Let $\mathscr{Q}$ be a representative set of all $\leq \mu$-generated modules in $\mathscr{A}$. Assume that either*

(i) *$\mathscr{A}$ is closed under pure submodules and $\mu \geq \operatorname{card} R$; or*

(ii) *R is right μ-noetherian, $\mathfrak{C}$ is hereditary, and $\mathscr{B}$ consists of modules of finite injective dimension.*

Then $\mathscr{B} = \mathscr{Q}^{\perp_1}$. In particular, $\mathfrak{C}$ is complete.

Proof. This is proved as in Theorems 1.3 and 1.7, with Lemma 1.5 replacing Lemma 1.1 in the regular case. (For (ii), we do not define μ in the beginning of the proof of Theorem 1.7—we just use that R is right μ-noetherian. Also, the definition of N in the proof of Theorem 1.7 is omitted, and ${}^{\perp_1}N$ is replaced by ${}^{\perp_1}\mathscr{B}$.) $\square$

Recall that a cotorsion pair $\mathfrak{C} = (\mathscr{A}, \mathscr{B})$ is of *countable type* provided there is a set of modules, $\mathscr{C}$, possessing a projective resolution consisting of countably generated projective modules such that $\mathscr{B} = \mathscr{C}^{\perp}$.

Corollary 1.10. *Let R be a right $\aleph_0$-noetherian ring and $\mathfrak{C} = (\mathscr{A}, \mathscr{B})$ be a hereditary cotorsion pair such that $\mathscr{B}$ is closed under arbitrary direct sums and consists of modules of finite injective dimension. Then $\mathfrak{C}$ is of countable type and $\mathscr{B}$ is definable (i.e. closed under direct products, direct limits and pure submodules).*

Proof. $\mathfrak{C}$ is of countable type by Theorem 1.9(ii) and $\mathscr{B}$ is always closed under direct products. The closure under pure submodules follows from [BH, Theorem 2.5], and $\mathfrak{C}$ hereditary implies $\mathscr{B}$ closed under pure epimorphic images. In particular, $\mathscr{B}$ is closed under direct limits. $\square$

Remark 1.11. There are many cotorsion pairs $(\mathscr{A}, \mathscr{B})$ without $\mathscr{B}$ closed under direct sums. In general, it is difficult to verify this closure property, and the property may depend on the set theory we are working in. For example, assuming V = L, every Whitehead group is free, hence $({}^{\perp_1}\mathbb{Z})^{\perp_1} = \text{Mod-}\mathbb{Z}$ is closed under direct sums. By Theorem 1.9, this is not the case in the model constructed in [ES], see Remark 1.8(i).

2 Cotilting cotorsion pairs

Let R be a ring, $n \geq 0$, and $\mathfrak{C} = (\mathscr{A}, \mathscr{B})$ be a cotorsion pair. In [AC], Angeleri and Coelho proved that $\mathfrak{C}$ is n-tilting iff $\mathfrak{C}$ is complete and hereditary, $\mathscr{B}$ is closed under arbitrary direct sums, and $\mathscr{A}$ consists of modules of projective dimension $\leq n$. Recently, Šťovíček and the second author have shown that completeness is redundant in this characterization of tilting cotorsion pairs, cf. [ST, Theorem 2].

Making use of [AB], Angeleri and Coelho also proved the following dual result:

Lemma 2.1 [AC, Theorem 4.2] and [B2, Lemma 3.2]. *Let R be a ring, $n \geq 0$, and $\mathfrak{C} = (\mathscr{A}, \mathscr{B})$ be a cotorsion pair. Then $\mathfrak{C}$ is n-cotilting iff $\mathfrak{C}$ is complete and hereditary, $\mathscr{A}$ is closed under arbitrary direct products, and $\mathscr{B}$ consists of modules of injective dimension $\leq n$.*

In order to prove that completeness is redundant also in the latter characterization, we cannot proceed dually to [ST] for several reasons: while filtrations (directed unions) naturally arise from generating sets, co-filtrations (= well-ordered inverse systems of epimorphisms) do not, the inverse limit functor is not exact (while $\varinjlim$ is), and any map from a "small" module into a "big" directed union factorizes through a term of the union, while the dual result fails in general, cf. [T2]. We rather make use of the recent proof of pure injectivity of cotilting modules by Šťovíček [S] (which in turn relies on [B1–3] and [BGS]). The fact proved in [S] needed here is the following stronger version of Bazzoni's "Conjectures A and B" from [B3, §5]:

Lemma 2.2 [S, Theorem 11] and [B3, Proposition 5.4]. *Let R be a ring and $\mathfrak{C} = (\mathscr{A}, \mathscr{B})$ be a cotorsion pair such that $\mathscr{A}$ is closed under pure submodules and arbitrary direct products. Then $\mathscr{A}$ is closed under pure epimorphic images, and $\mathfrak{C}$ is complete.*

Another ingredient needed is the following lemma:

Lemma 2.3. *Let R be a ring, and $(\mathscr{A}, \mathscr{B})$ be a cotorsion pair such that $\mathscr{A}$ is closed under arbitrary direct products and ${}^{\perp}\mathscr{B}$ contains all direct products of projective modules. Then ${}^{\perp_n}\mathscr{B}$ is closed under arbitrary direct products for each $n \geq 1$.*

If moreover $\mathscr{B} \subseteq \mathscr{I}_n$ for some $n \geq 0$, then $\mathscr{A}_k = \bigcap_{i \geq k} {}^{\perp_i}\mathscr{B}$ is an $(n-k+1)$-cotilting class for each $1 \leq k \leq n+1$.

Proof. The first claim is proved by induction on n. The case of $n = 1$ is our assumption on $\mathscr{A}$. Let $(M_\alpha \mid \alpha < \kappa)$ be a family of modules in ${}^{\perp_{n+1}}\mathscr{B}$. Consider the short exact sequences $0 \to K_\alpha \to P_\alpha \to M_\alpha \to 0$ with P_α projective for each $\alpha < \kappa$. Since $\mathrm{Ext}_R^{n+1}(M_\alpha, B) \cong \mathrm{Ext}_R^n(K_\alpha, B) = 0$ for all $B \in \mathscr{B}$, the inductive premise gives $\prod_{\alpha<\kappa} K_\alpha \in {}^{\perp_n}\mathscr{B}$, so our assumption on ${}^{\perp}\mathscr{B}$ yields $\prod_{\alpha<\kappa} M_\alpha \in {}^{\perp_{n+1}}\mathscr{B}$.

For the second claim, we first prove by reverse induction on $1 \leq k \leq n+1$ that $\mathscr{A}_k$ is closed under pure submodules. The case of $k = n+1$ is clear since $\mathscr{A}_{n+1} = \text{Mod-}R$ by the assumption on $\mathscr{B}$. Let $M \in \mathscr{A}_{k-1}$, P be a pure submodule in M, and $B \in \mathscr{B}$. Since $\mathscr{A}_{k-1} \subseteq \mathscr{A}_k$, we have $\mathrm{Ext}_R^{k-1}(P, B) \cong \mathrm{Ext}_R^k(M/P, B)$, so it suffices to prove that $\mathscr{A}_k$ is closed under pure epimorphic images. By Lemma 2.2, it is enough to show that $\mathscr{A}_k$ is closed under pure submodules and arbitrary direct products. However, this is the case by the inductive premise and by the first claim.

So Lemma 2.2 applies, and for each $1 \leq k \leq n+1$, $\mathfrak{C}_k = (\mathscr{A}_k, \mathscr{A}_k^{\perp})$ is a complete hereditary cotorsion pair such that $\mathscr{A}_k$ is closed under arbitrary direct products. Moreover, $\mathscr{A}_k^{\perp} \subseteq \mathscr{I}_{n-k+1}$ since $\mathscr{B} \subseteq \mathscr{I}_n$. So $\mathfrak{C}_k$ is $(n-k+1)$-cotilting by Lemma 2.1. □

Now, we can drop the completeness assumption in the characterization of n-cotilting cotorsion pairs in Lemma 2.1:

Theorem 2.4. *Let R be a ring, $n \geq 0$, and $\mathfrak{C} = (\mathscr{A}, \mathscr{B})$ be a cotorsion pair. Then $\mathfrak{C}$ is n-cotilting iff $\mathfrak{C}$ is hereditary, $\mathscr{A}$ is closed under arbitrary direct products, and $\mathscr{B}$ consists of modules of injective dimension $\leq n$.*

Proof. The direct implication is clear by Lemma 2.1. For the reverse one, just note that $\mathscr{A} = \mathscr{A}_1$ in the second claim of Lemma 2.3. □

Remark 2.5. By Lemma 2.3, the conjunction of the following three conditions: (1) $\mathscr{A}$ closed under arbitrary products, (2) $\mathscr{B} \subseteq \mathscr{I}_n$, and (3) $\mathfrak{C}$ hereditary, implies that $\mathscr{A}$ is closed under pure submodules and $\mathfrak{C}$ is generated by a set (namely, by the set of all cosyzygies in a fixed injective coresolution of the cotilting module C satisfying $\mathscr{A} = {}^{\perp}C$). That is, the conjuction is stronger than the assumptions of Theorems 1.3 and 1.7; in fact, it implies completeness of $\mathfrak{C}$ in ZFC without any extra set-theoretic assumptions.

The characterization of cotilting cotorsion pairs in Theorem 2.4 cannot be simplified further (of course, except for the trivial case of $n = 0$ where condition (2) implies (1) and (3), and the case of $n = 1$ where condition (2) implies (3), so 1-cotilting cotorsion pairs are exactly the cotorsion pairs satifying (1) and (2)).

For $n > 1$, (3) is independent from (1) and (2): consider a ring R and a tilting left R-module T of projective dimension n possessing a projective resolution consisting of finitely generated modules (if R is left coherent, this just says that T is a finitely presented tilting left R-module of projective dimension n). Let $C = \mathrm{Hom}_{\mathbb{Z}}(T, \mathbb{Q}/\mathbb{Z})$ be the character module of T. By [AHT, Proposition 2.3], C is a cotilting module of injective dimension n. Consider the cotorsion pair $\mathfrak{C} = (\mathscr{A}, \mathscr{B})$ where $\mathscr{A} = {}^{\perp_1}C$. Then (1) holds by [EJ, Theorem 3.2.26] since $\mathscr{A} = \mathrm{Ker}\,\mathrm{Tor}_1^R(-, T)$ and T is finitely presented, and (2) is clear. However, (3) fails: otherwise $\mathscr{A} = {}^{\perp}C$, and by [T1, Lemma 6.9], for any injective cogenerator $W \in \text{Mod-}R$, there is a short exact sequence $0 \to C^{\lambda} \to A \to W \to 0$ where λ is a cardinal, and $A \in \mathscr{A} \cap \mathscr{B} = {}^{\perp}C \cap ({}^{\perp}C)^{\perp} = \mathrm{Prod}(C)$ by [AC]. By [B2, Proposition 3.5], this means that C has injective dimension ≤ 1, a contradiction.

To see that (1) and (3) do not imply (2) for $n \geq 0$, consider a right perfect and left coherent ring R of global dimenson $> n$, and the trivial cotorsion pair $(\mathscr{P}_0, \text{Mod-}R)$: then (1) and (3) hold, but (2) does not. Similarly, for any $n > 0$, if R is non-left coherent or non-right perfect and R has right global dimension n, then (2) and (3) hold for the trivial cotorsion pair, but (1) does not.

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Received 12 September 2005; in final form 11 December 2005

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