On Ext-universal modules in Gödel's universe

Lutz Strüngmann

Abstract. Let R be an associative unital ring, T a left R-module and λ an infinite cardinal. We consider the class ${}^{\perp}T$ of all left R-modules M satisfying $\mathrm{Ext}^1_R(M,T) = 0$ and search for λ-universal objects in suitable subclasses $\mathfrak{C}$ of ${}^{\perp}T$. Here, a R-module $U \in \mathfrak{C}$ is λ-universal for T if $|U| \leq \lambda$ and every R-module $M \in \mathfrak{C}$ of cardinality less than or equal to λ embeds into U. We show that the existence of $|T|$-universal objects which are strong splitters implies the existence of λ-universal objects for sufficiently large λ if we assume $(V = L)$. We then apply our results to module classes over small Dedekind domains to partially solve a generalized problem by Kulikov.

Introduction

In 1936 R. Baer [**Ba**] asked to characterize all pairs of torsion-free abelian groups G and torsion abelian groups T satisfying $\mathrm{Ext}(G, T) = 0$. This is just a simpler way to ask for pairs G and T such that any mixed abelian group M with torsion subgroup T and torsion-free quotient $M/T \cong G$ has to split, i.e. $M \cong T \oplus G$ in the natural way. Baer [**Ba**] himself gave a characterization for countable G and this question was considered again by Wallutis and the author in [**StWa**]. In the framework of cotorsion theories (as they were introduced by Salce in [**Sa**]) either singly cogenerated by a torsion-free abelian group G or singly generated by a torsion group T Baer's question has been solved partially using various additional set-theoretic assumptions, e.g. working in Gödel's constructible universe $(V = L)$ (see [**StWa**], [**St1**],

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[**St2**], [**St2**], [**St3**], [**St4**] and the references given there). In particular, solving Baer's problem goes hand in hand with another problem due to Kulikov [**KN**]. As defined in the abstract a torsion-free abelian group G is λ-universal for a torsion abelian group T and a cardinal λ if G is of cardinality less than or equal to λ, $\mathrm{Ext}(G,T) = 0$ and every torsion-free abelian group H of cardinality less than or equal to λ satisfying $\mathrm{Ext}(H,T) = 0$ embeds into G. Clearly, the existence of a λ-universal group G for T answers the question of which torsion-free H (of cardinality at most λ) satisfy $\mathrm{Ext}(H,T) = 0$: namely, precisely the subgroups of G. Kulikov [**KN**] asked if for uncountable λ and arbitrary T there is always a λ-universal group for T. By [**St1**] the answer is yes for large classes of torsion abelian groups in $(V = L)$.
In this paper we generalize the notion of universality to R-modules T over arbitrary associative unital rings R. Let λ and κ be infinite cardinals and $\mathfrak{C}(T)$ be a subclass of ${}^{\perp}T$ closed under arbitrary direct sums. A R-module $U \in \mathfrak{C}(T)$ is called $(\lambda, \kappa, \mathfrak{C}(T))$-*universal for* T if $|U| \leq \kappa$ and every R-module $H \in \mathfrak{C}(T)$ of size less than or equal to λ can be embedded into U. In the first section we prove a *lifting theorem* which shows that under the assumption of Gödel's constructible universe, the existence of small $(\lambda, \lambda, \mathfrak{C}(T))$-universal R-modules for T implies the existence of arbitrarily large $(\lambda', \lambda', \mathfrak{C}(T))$-universal R-modules for T. This is then applied in Section 2 to R-modules over Dedekind domains. In particular, we prove that the answer to Kulikov's question (formulated over Dedekind domains) is yes whenever the torsion R-module has only finitely many non-trivial bounded primary components. Finally, in section 3 we apply our lifting theorem to torsion-free abelian groups of finite rank.

Our notations are standard and for unexplained notions we refer to [**Fu**] for abelian groups, [**EkMe**] for set-theoretic methods, and [**Ma**] and [**FuSa**] for module theory.

1. Universal objects in $(V = L)$

Let R be any associative unital ring and T a left R-module. Moreover, let λ and κ be infinite cardinals and $\mathfrak{C}(T)$ be a subclass of ${}^{\perp}T$ closed under arbitrary direct sums. As usual ${}^{\perp}T$ denotes the class of all left R-modules M satisfying $\mathrm{Ext}^1_R(M,T) = 0$. Note that ${}^{\perp}T$ itself is closed under direct sums. Dually, $T^{\perp}$ consists of all left R-modules M satisfying $\mathrm{Ext}^1_R(T,M) = 0$. In the sequel all R-modules will be left R-modules. A R-module $U \in \mathfrak{C}(T)$ is called $(\lambda, \kappa, \mathfrak{C}(T))$-*universal for* T if $|U| \leq \kappa$ and every R-module $H \in \mathfrak{C}(T)$ of size less than or equal to λ can be embedded into U. Obviously, $(\lambda, \kappa, \mathfrak{C}(T))$-universal R-modules always exist for sufficiently large κ since all R-modules of cardinality $\leq \lambda$ form a set. For instance, $\bigoplus_{M \in \mathfrak{C}(T), |M| \leq \lambda} M$ is $(\lambda, \kappa, \mathfrak{C}(T))$-universal for T with $\kappa = |\{M \in \mathfrak{C}(T) : |M| \leq \lambda\}|$. On the other hand, $(\lambda, \kappa, \mathfrak{C}(T))$-universal R-modules do not exist if $\kappa < \lambda$. Thus we say that U is $(\lambda, \mathfrak{C}(T))$-*universal for* T if it is $(\lambda, \kappa, \mathfrak{C}(T))$-universal for T for some $\lambda \leq \kappa$. Our main interest is the search for $(\lambda, \lambda, \mathfrak{C}(T))$-universal R-modules for T. At some places it is therefore natural to restrict ourselves to infinite cardinals $\lambda \geq |R|$. The aim is to show that the existence of certain $(|T|, \mathfrak{C}(T))$-universal R-modules for T implies the existence of $(\lambda, \lambda, \mathfrak{C}(T))$-universal R-modules for T for all sufficiently large λ if we assume Gödel's constructible universe.

We shall use the following results on the vanishing of Ext^1_R which are by now standard. For an R-module T we denote by $E(T)$ the injective envelope of T.

Lemma 1.1. *Suppose that* $M = \bigcup_{\alpha<\lambda} M_\alpha$ *is the union of a continuous chain of submodules such that* $\mathrm{Ext}^1_R(M_0, T) = 0$ *and for all* $\alpha + 1 < \lambda$, $\mathrm{Ext}^1_R(M_{\alpha+1}/M_\alpha, T) = 0$. *Then* $\mathrm{Ext}^1_R(M, T) = 0$.

Proof. See [**EkMe**, Proposition XII 1.5]. $\square$

It is well-known that the converse of the above Lemma 1.1 does not hold in Zermelo-Fraenkel set theory. However, assuming Gödel's constructible universe we obtain

Lemma 1.2 ($V = L$). *Let* T *be any* R*-module such that* ${}^\perp T$ *is closed under pure submodules, and* κ *be a cardinal with* $\kappa \geq |R| + |E(T)| + \aleph_0$. *Assume that* M *is an* R*-module satisfying* $\mathrm{Ext}^1_R(M, T) = 0$. *Then* M *can be written as the union of a continuous well-ordered ascending chain* $\{M_\alpha : \alpha < \lambda\}$ *of pure submodules (*$M_0 = 0$*) such that* $M_{\alpha+1}/M_\alpha$ *is of size at most* κ *and* $\mathrm{Ext}^1_R(M_{\alpha+1}/M_\alpha, T) = 0$ *for all* $\alpha + 1 < \lambda$.

Proof. See [**EkMe**, Corollary XII 1.13] for left hereditary rings and [**SaTr**, Lemma 1.2] for the general case. $\square$

For our purposes we shall define suitable subclasses of ${}^\perp T$ which are adjusted to Lemma 1.2 and also to the subsequent Corollary 1.4 and Theorem 1.8.

Definition 1.3. *Let T be an R-module. A subclass $\mathfrak{C}(T) \subseteq {}^{\perp}T$ is called* suitable *if $\mathfrak{C}(T)$ is, as a subclass of ${}^{\perp}T$, closed under the following operations*

(i) *direct sums;*
(ii) *pure submodules;*
(iii) *pure epimorphic images;*
(iv) *extensions;*
(v) *unions of continuous ascending chains with successive factors in $\mathfrak{C}(T)$.*

Note that suitable does not mean that all extensions (pure submodules, unions of chains etc.) of elements of $\mathfrak{C}(T)$ are in $\mathfrak{C}(T)$ again but only those which are already in ${}^{\perp}T$. Since ${}^{\perp}T$ is always closed under direct sums, extensions and chains as in Definition 1.3 (v), it is natural to demand this also on $\mathfrak{C}(T)$. Moreover, Lemma 1.2 forces us to require also the closure under pure submodules and pure epimorphic images.

Corollary 1.4 (V=L). *Let T be any R-module and $\mathfrak{C}(T) \subseteq {}^{\perp}T$ a suitable subclass. Moreover, let κ be a cardinal with $\kappa \geq |R| + |E(T)| + \aleph_0$ and assume that ${}^{\perp}T$ is closed under pure submodules. Then any R-module $M \in \mathfrak{C}(T)$ can be written as the union of a continuous well-ordered ascending chain $\{M_\alpha : \alpha < \lambda\}$ of pure submodules ($M_0 = 0$) such that $M_{\alpha+1}/M_\alpha$ is of size at most κ and $M_{\alpha+1}/M_\alpha \in \mathfrak{C}(T)$ for all $\alpha + 1 < \lambda$.*

Proof. Follows immediately from Lemma 1.2 and the closure properties of the suitable class $\mathfrak{C}(T)$. □

In order to find universal R-modules we shall restrict ourselves to suitable classes $\mathfrak{C}(T)$ in the sequel. Hence we give first examples of suitable classes which will serve as our main application. Recall that a R-module M is called *flat* if short exact sequences remain short exact when tensored by M. The R-module M is called *torsion-free* if $Tor^1_R(M, R/rR) = 0$ for all $r \in R$.

Example 1.5. *Assume that R is any left hereditary ring. Then*

(i) *The class of all torsion R-modules is suitable;*
(ii) *The class of all flat R-modules is suitable;*
(iii) *The class of all torsion-free R-modules is suitable.*

Proof. Clearly, the class of all torsion R-modules is suitable. Thus we concentrate on (ii) and (iii). By [**EkMe**, Example XVI 2.11] both classes are left perpendicular classes of a class of pure injective R-modules. Hence both classes are closed under direct sums, extensions, unions of chains and pure epimorphic images. Finally, the closure under pure submodules follows since R is left hereditary. $\square$

REMARK 1.6. *Being suitable is a very strong assumption on the class $\mathfrak{C}(T)$. However, in order to obtain our main theorems of this section we only need that every R-module in the class $\mathfrak{C}(T)$ satisfies Corollary 1.4.*

We can now easily prove the following result which needs the definition of a *good* R-module. Recall that an R-module M is *U-filtered* for some R-module U if M posesses a filtration with successive factors isomorphic to U.

DEFINITION 1.7. *Let R be any ring and U a left R-module. Then U is called* good *if one of the following conditions is satisfied:*

(i) *proj-dim$(U/U') \leq 1$ for every submodule U' of U;*
(ii) *inj-dim$(U) \leq 1$ and every U-filtered R-module has injective dimension at most* 1.

For instance every R-module over any ring of global dimension at most 1 is a good R-module. Hence good R-modules exist in abundance.

THEOREM 1.8 ($V = L$). *Let T be any R-module such that ${}^{\perp}T$ is closed under pure submodules. Moreover, let $\kappa \geq |R| + |E(T)| + \aleph_0$ be an infinite cardinal and $\mathfrak{C}(T) \subseteq {}^{\perp}T$ be suitable. Furthermore, assume that there exists a good $(\kappa, \mathfrak{C}(T))$-universal R-module U for T. If $M \in \mathfrak{C}(T)$, then there exists $M' \in \mathfrak{C}(T)$ such that*

- *$M \subseteq M'$;*
- *M' has a filtration $M' = \bigcup_{\alpha<\lambda} M'_\alpha$ of submodules $M'_\alpha \in \mathfrak{C}(T)$;*
- *$M'_{\alpha+1}/M'_\alpha \cong U$ for all $\alpha < \lambda$.*

PROOF. Let T, $\mathfrak{C}(T)$ and U be given. If $M \in \mathfrak{C}(T)$, then Corollary 1.4 shows that there is a λ-filtration $M = \bigcup_{\alpha<\lambda} M_\alpha$ of M such that each quotient $M_{\alpha+1}/M_\alpha$ is of size at most κ and satisfies $M_{\alpha+1}/M_\alpha \in \mathfrak{C}(T)$ since $\mathfrak{C}(T)$ is suitable. Inductively we shall construct the R-module M' containing M which satisfies the conditions of the theorem. Assume α is an ordinal and M'_β has been constructed such that

(i) $M_\beta \subseteq M'_\beta$ for all $\beta < \alpha$;
(ii) $M'_\beta \in \mathfrak{C}(T)$ for all $\beta < \alpha$;
(iii) $M'_{\beta+1}/M'_\beta \cong U \in \mathfrak{C}(T)$ for all $\beta + 1 < \alpha$;
(iv) the injective dimension of M'_β is at most 1 for all $\beta < \alpha$ if U satisfies Definition 1.7 (ii).

If α is a limit ordinal we put $M'_\alpha = \bigcup_{\beta<\alpha} M'_\beta$ and clearly (i) to (iii) are satisfied for M'_α by Lemma 1.1 and since $\mathfrak{C}(T)$ is suitable. Moreover, M'_α has injective dimension at most 1 if U was assumed to satisfy Definition 1.7 (ii).
Thus assume that $\alpha = \beta + 1$. We consider the exact sequence

$$0 \to M_\beta \to M_\alpha \to M_\alpha/M_\beta \to 0.$$

Since $M_\beta \subseteq M'_\beta$ there exists a pushout R-module B such that $M_\alpha \subseteq B$ and the following sequence is exact:

$$0 \to M'_\beta \to B \to M_\alpha/M_\beta \to 0.$$

By Corollary 1.4 the quotient R-module M_α/M_β is of size at most κ and an element of $\mathfrak{C}(T)$ since $\mathfrak{C}(T)$ is suitable. Hence it embeds into the $(\kappa, \mathfrak{C}(T))$-universal R-module U. Applying $\mathrm{Hom}(-, M'_\beta)$ to the exact sequence

$$0 \to M_\alpha/M_\beta \to U \to U/(M_\alpha/M_\beta) \to 0$$

yields an epimorphism

$$\mathrm{Ext}^1_R(U, M'_\beta) \to \mathrm{Ext}^1_R(M_\alpha/M_\beta, M'_\beta) \to 0,$$

since either the injective dimension of the R-module M'_β is at most 1 by (iv) or proj-dim$(U/(M_\alpha/M_\beta))$ is at most 1. Thus there exists a R-module M'_α containing B (and hence M_α) such that the following sequence is exact:

$$0 \to M'_\beta \to M'_\alpha \to U \to 0.$$

Since M'_β and U are elements of $\mathfrak{C}(T)$ we conclude that also M'_α is in $\mathfrak{C}(T)$ by the closure under extensions. Moreover, $M'_\alpha/M'_\beta \cong U$ as required and M'_α has injective dimension at most 1 by Kaplansky's Lemma [**FuSa**, Lemma VI 2.4] if U satisfies Definition 1.7 (ii). Putting $M' = \bigcup_{\alpha<\kappa} M'_\alpha$ completes the proof using Lemma 1.1 and the closure of $\mathfrak{C}(T)$ under unions. □

For the next result recall that an R-module M is a *splitter* if M satisfies $\mathrm{Ext}^1_R(M, M) = 0$, i.e. has no self-extensions. M is a λ-*splitter* for some cardinal λ if $\mathrm{Ext}^1_R(U, U^{(\lambda)}) = 0$ and a *strong splitter* if it is a λ-splitter for all cardinals λ. The next result shows that splitters exist in abundance.

LEMMA 1.9. *For any R-module A there is a R-module $B \in A^\perp$ such that $\mathrm{Ext}^1_R(B, B) = 0$ and for some cardinal λ, $B = \bigcup_{\alpha<\lambda} B_\alpha$ such that*

$B_0 = 0$ and for all $\alpha + 1 < \lambda$, $B_{\alpha+1}/B_\alpha \cong A$. Moreover, B can be chosen to have cardinality 2^κ with $\kappa = |R| + |A|$.

PROOF. See [**EkMe**, Corollary XVI 1.3]. $\square$

Examples for strong splitters are for instance all (partial) tilting R-modules. However, in contrast to the above Lemma 1.9 strong splitters are in many cases of a particular form - at least in the constructible universe. Recall that the nucleus of a torsion-free abelian group U is the largest subring S of $\mathbb{Q}$ such that U is still a module over S.

LEMMA 1.10 (V=L). *Let U be a ω-splitter over $\mathbb{Z}$. Then U is free over its nucleus.*

PROOF. See [**EkMe**, Corollary XVI 1.11]. $\square$

We deduce an immediate corollary which is our main theorem. Note that the closure of ${}^\perp T$ under pure submodules is required since we apply Theorem 1.13. Recall that a submodule $N \subseteq M$ is *pure* in M if every finite system of equations over N which is solvable in M is also solvable in N (see [**FuSa**, Chapter I]).

THEOREM 1.11 (V=L). *Let T be any R-module such that ${}^\perp T$ is closed under pure submodules. Let $\kappa \geq |R| + |E(T)| + \aleph_0$ be a cardinal and $\mathfrak{C}(T) \subseteq {}^\perp T$ be a suitable class. Furthermore assume that there exists a good $(\kappa, \mathfrak{C}(T))$-universal R-module U for T which is a strong splitter. Then $\bigoplus\limits_{\lambda} U$ is $(\lambda, \mathfrak{C}(T))$-universal for T for every infinite cardinal λ. In particular, for $\lambda \geq |U|$ there is a $(\lambda, \lambda, \mathfrak{C}(T))$-universal R-module for T.*

PROOF. Let $M \in \mathfrak{C}(T)$ be of size less than or equal to λ. By Theorem 1.8 we can embed M into a R-module $M' \in \mathfrak{C}(T)$ such that M' has a filtration $M' = \bigcup\limits_{\alpha < \lambda'} M'_\alpha$ such that $M'_{\alpha+1}/M'_\alpha \cong U$ for every $\alpha + 1 < \lambda' \leq \lambda$. However, inductively we can show that the filtration of M' satisfies $M'_\alpha \cong \bigoplus\limits_{\beta<\alpha} U$ since U is a strong splitter. Hence $M' \cong \bigoplus\limits_{\alpha<\lambda'} U$ and therefore $\bigoplus\limits_{\alpha<\lambda} U$ is $(\lambda, \mathfrak{C}(T))$-universal for T. If $\lambda \geq |U|$, then $\bigoplus\limits_{\lambda} U$ is of size λ and hence a $(\lambda, \lambda, \mathfrak{C}(T))$-universal R-module for T. This completes the proof. $\square$

Note that if U is only a λ-splitter then the filtration from Theorem 1.8 has the property that $M'_\alpha \cong \bigoplus\limits_{\alpha} U$ for every $\alpha < \lambda^+$.

We now use the existence of splitters to construct universal R-modules for large cardinals λ.

LEMMA 1.12. *Let T be any R-module such that ${}^{\perp}T$ is closed under pure submodules. Let $\mathfrak{C}(T)$ be a subclass of ${}^{\perp}T$ which is suitable. For every infinite cardinal λ there exists a $(\lambda, \mathfrak{C}(T))$-universal R-module U for T which is a splitter such that $|U| \leq 2^{\delta}$ with $\delta = 2^{|R|\lambda}$.*

PROOF. Let $\mathfrak{C}(T)' = \{C \in \mathfrak{C}(T) : |C| \leq \lambda\}$ and put $A = \bigoplus\limits_{C \in \mathfrak{C}(T)'} C$. Then $A \in \mathfrak{C}(T)$ and $|A| \leq 2^{|R|\lambda}$. By Lemma 1.9 there exists a splitter U of size less than or equal to 2^{δ} with $\delta = 2^{|R|\lambda}$. Note that $\delta \geq |A| + |R|$. Moreover, $U = \bigcup\limits_{\alpha < \lambda'} U_{\alpha}$ such that $U_0 = 0$ and for all $\alpha < \lambda'$, $U_{\alpha+1}/U_{\alpha} \cong A$. Hence Lemma 1.1 yields that $\mathrm{Ext}^1_R(U, T) = 0$ and thus $U \in \mathfrak{C}(T)$ by the closure property of $\mathfrak{C}(T)$. Since every $C \in \mathfrak{C}(T)'$ embeds into $A = U_1$ and hence into U we obtain that U is $(\lambda, \mathfrak{C}(T))$-universal for T. □

We conclude this section with a theorem in which we restrict ourselves to rings of global dimension 1 for simplicity. In this case all R-modules have projective dimension at most 1, hence are good R-modules. However, we need to know the existence of strong splitters. Note that Lemma 1.12 just gives the existence of splitters.

THEOREM 1.13 (V=L). *Let R be of global dimension 1 and let T be any R-module such that ${}^{\perp}T$ is closed under pure submodules. Let $\mathfrak{C}(T) \subseteq {}^{\perp}T$ be suitable and put $\kappa = |E(T)| + |R| + \aleph_0$. Assume that there is a strong splitter which is $(\kappa, \kappa, \mathfrak{C}(T))$-universal for T. If $\lambda \geq \kappa$, then there exists a $(\lambda, \lambda, \mathfrak{C}(T))$-universal R-module for T.*

PROOF. By assumption there is a strong splitter for T which is $(\kappa, \kappa, \mathfrak{C}(T))$-universal. Hence Theorem 1.11 shows that for any $\lambda \geq \kappa$ there is a $(\lambda, \lambda, \mathfrak{C}(T))$-universal R-module for T. □

The author would like to mention that Theorem 1.8 is not provable in ZFC as follows from [**ShSt**] where Kulikov's questions was answered in the negative in a particular model of set-theory, hence universal $\mathbb{Z}$-modules do not exist.

REMARK 1.14. *In the proof of Theorem 1.8 we can strengthen the filtration of M from Corollary 1.4 if we assume R to be a hereditary domain. In this case, it is even possible to construct a filtration as stated but with successive factors $< \kappa$-generated. It then suffices for Theorem 1.13 to find a strong splitter in which every $< \kappa$-generated*

R-module from $\mathfrak{C}(T)$ embeds. We will use this fact in the last section on Dedekind domains (see Theorem 2.9).

2. Universal modules over Dedekind domains

In this section we shall apply our Main Theorem 1.13 to R-module classes over Dedekind domains. Let R be a Dedekind domain which is not a field, $Q = Q(R)$ its quotient field and $\mathrm{Spec}(R)$ the set of all prime ideals of R. Then, all non-zero prime ideals of R are maximal. Moreover, since R is a Prüfer domain, divisible R-modules are injective, torsion-free R-modules are flat and by [**Ma**, Theorem 18.8] we have

$$Q/R \cong \bigoplus_{0 \neq p \in \mathrm{Spec}(R)} E(R/p)$$

where $E(R/p)$ denotes the injective hull of R/p. For any prime ideal p, the localization R_p of R at p is a noetherian valuation domain and hence a principal ideal domain. The ideals of R_p are of the form $p^n R_p$ and pR_p is the unique maximal ideal of R_p. Note that Dedekind domains are exactly the hereditary domains in which every ideal is projective, hence R is of global dimension 1.
Following notation in [**TrWa**], for a set $P \subseteq \mathrm{Spec}(R)$, we put $R_P = \pi^{-1}(\bigoplus_{p \in P} E(R/p))$ where

$$0 \to R \to Q \xrightarrow{\pi} \bigoplus_{p \in \mathrm{Spec}(R)} E(R/p) \to 0$$

is the minimal injective resolution of R. Then $R_P = \bigcap_{q \in \mathrm{Spec}(R) \setminus P} R_q$ with R_q being the usual localization of R at the prime ideal q.

Lemma 2.1. *The following are true for $P \subseteq \mathrm{Spec}(R)$.*

(i) $R \subseteq R_P \subseteq Q$;
(ii) $R_P/R = \bigoplus_{p \in P} E(R/p)$;
(iii) $Q/R_P = \bigoplus_{q \in \mathrm{Spec}(R) \setminus P} E(R/q)$.

Proof. Left to the reader. □

In order to apply Theorem 1.13 we first show that for any subset $P \subseteq \mathrm{Spec}(R)$ the corresponding ring R_P is a strong splitter.

Lemma 2.2. *Let $P \subseteq \mathrm{Spec}(R)$. Then R_P is a strong splitter.*

PROOF. The lemm follows if it is noted that $\mathrm{Ext}^1_R(A,B) = \mathrm{Ext}^1_{R_P}(A,B)$ for all torsion-free R-modules A and B. However, we give an alternate proof. We have to prove that $\mathrm{Ext}^1_R(R_P, R_P^{(\kappa)}) = 0$ for all cardinals κ. Since

$$0 \to R \to R_P \to \bigoplus_{p \in P} E(R/p) \to 0$$

is short exact with $I_P = \bigoplus_{p \in P} E(R/p)$ injective, it suffices to prove that $\mathrm{Ext}^1_R(I_P, R_P^{(\kappa)}) = 0$. For each $p \in \mathrm{Spec}(R)$, the R-module $E(R/p)$ has an infinite composition series with successive factors isomorphic to R/p. Since R/p is finitely presented, we are left to prove that $\mathrm{Ext}^1_R(R/p, R_P) = 0$ for all $p \in P$. Since $\mathrm{Hom}(R/p, E(N)/N) \cong \mathrm{Ext}^1_R(R/p, N)$ for all R-modules N this follows from the fact that $\mathrm{Hom}_R(R/p, Q/R_P) = 0$ for all $p \in P$. □

Now let M be a torsion R-module. Then M decomposes into the direct sum of its p-primary components (see e.g. [**FuSa**, page 131])

$$M = \bigoplus_{p \in \mathrm{Spec}(R)} M \otimes R_p = \bigoplus_{p \in \mathrm{Spec}(R)} M_p.$$

Accordingly, we let

- $\delta(M) = \{p \in \mathrm{Spec}(R) : M_p \text{ is bounded}\}$,
- $\delta_0(M) = \{p \in \mathrm{Spec}(R) : M_p = 0\}$ and
- $\delta_b(M) = \delta(M) \backslash \delta_0(M)$.

Recall that an R-module T is *bounded* if there exists $r \in R$ such that $rT = 0$ and it is *p-divisible* for some $p \in \mathrm{Spec}(R)$ if $\mathrm{Ext}^1_R(R/p, T) = 0$. As usual, T is called *divisible* if it is p-divisible for all $p \in \mathrm{Spec}(R)$. Since divisible submodules are direct summands we may restrict ourselves in the following to reduced R-modules. We recall an easy lemma from [**TrWa**, Lemma 1].

LEMMA 2.3. *Let $0 \neq p \in \mathrm{Spec}(R)$ and M be a R-module. The following are equivalent:*

(i) *M is p-divisible;*
(ii) *the localization $M_p = M \otimes R_p$ is a divisible R_p-module;*
(iii) *$pM = M$.*

PROOF. See [**TrWa**, Lemma 1]. □

LEMMA 2.4. *Let M be a reduced torsion R-module with only finitely many non-trivial bounded primary components. Then it follows that* $\mathrm{Ext}^1_R(R_{\delta(M)}, M) = 0$.

PROOF. Let M be as stated, hence $\delta_b(M)$ is finite. Let $M = M_b \oplus M_1$ with $M_b = \bigoplus_{p\in\delta_b(M)} M_p$ and $M_1 = \bigoplus_{p\notin\delta_b(M)} M_p$. Then $\mathrm{Ext}^1_R(R_{\delta(M)}, M) = 0$ if and only if $\mathrm{Ext}^1_R(R_{\delta(M)}, M_b) = 0$ and $\mathrm{Ext}^1_R(R_{\delta(M)}, M_1) = 0$. Since $\delta_b(M)$ is finite, the R-module M_b is cotorsion and hence $\mathrm{Ext}^1_R(R_{\delta(M)}, M_b) = 0$ by the torsion freeness of $R_{\delta(M)}$. Thus we may assume without loss of generality that $\delta(M) = \delta_0(M)$ and $M = M_1$. The short exact sequence

$$0 \to R \to R_{\delta(M)} \to \bigoplus_{p\in\delta(M)} E(R/p) \to 0$$

induces the exactness of

$$\cdots \to \mathrm{Ext}^1_R(R_{\delta(M)}/R, M) \to \mathrm{Ext}^1_R(R_{\delta(M)}, M) \to \mathrm{Ext}^1_R(R, M) = 0.$$

Now, $\mathrm{Ext}^1_R(R_{\delta(M)}/R, M) \cong \prod_{p\in\delta(M)} \mathrm{Ext}^1_R(E(R/p), M)$. But the latter equals zero since $M_p = 0$ for every $p \in \delta(M) = \delta_0(M)$, hence M is p-divisible for all $p \in \delta(M)$ by Lemma 2.3. Thus $\mathrm{Ext}^1_R(R_{\delta(M)}/R, M) = 0$ and hence also $\mathrm{Ext}^1_R(R_{\delta(M)}, M) = 0$. $\square$

We now turn our attention to submodules N of the quotient field Q satisfying $\mathrm{Ext}^1_R(N, M) = 0$ for particular torsion R-modules M.

LEMMA 2.5. *Let $R \subseteq N \subseteq Q$ be a submodule of Q. If M is a reduced torsion R-module such that $\mathrm{Ext}^1_R(N, M) = 0$, then the following hold:*

(i) *If N is p-divisible for some $p \in \mathrm{Spec}(R)$, then M_p is bounded;*
(ii) *Let $P = \{p \in \mathrm{Spec}(R) : (N/R)_p \neq 0\}$. Then $M_p = 0$ for almost all $p \in P$.*

PROOF. We first prove (ii). The short exact sequence

$$0 \to R \to N \to N/R \to 0 \tag{2.1}$$

induces the exact sequence

$$\cdots \to \mathrm{Hom}(R, M) \to \mathrm{Ext}^1_R(N/R, M) \to \mathrm{Ext}^1_R(N, M) = 0.$$

Now, $\mathrm{Hom}(R, M) \cong M$ which is torsion by assumption. Therefore, $\mathrm{Ext}^1_R(N/R, M) \cong \prod_{p\in P} \mathrm{Ext}^1_R((N/R)_p, M)$ is torsion. But this can only happen if $\mathrm{Ext}^1_R((N/R)_p, M) = 0$ for almost all $p \in P$. However, $(N/R)_p \neq 0$ for $p \in P$ implies that $R/p \subseteq (N/R)_p$ and hence also $\mathrm{Ext}^1_R(R/p, M) = 0$. By Lemma 2.3 we obtain that M and hence M_p is $p-$divisible for almost all $p \in P$ which is a contradiction since M was assumed to be reduced. Thus $M_p = 0$ for almost all $p \in P$ as claimed and (ii) holds.

Now assume that N is p-divisible for some $p \in \mathrm{Spec}(R)$. Then $N \otimes R_p$

is a torsion-free divisible R_p-module of rank 1, hence isomorphic to Q. Thus the short exact sequence 2.1 induces the short exact sequence

$$0 \to R_p \to Q \to (N/R)_p \to 0$$

since R_p is flat. Applying $\mathrm{Hom}(-, M_p)$ we obtain

$$0 \to M_p \to \mathrm{Ext}^1_R((N/R)_p, M_p) \to \mathrm{Ext}^1_R(Q, M_p) \to 0$$

and by the above reasoning $\mathrm{Ext}^1_R(Q, M_p)$ must be torsion. This can only happen if $\mathrm{Ext}^1_R(Q, M_p) = 0$ and thus M_p is reduced torsion cotorsion and hence bounded. Therefore, (i) holds.

□

We now proceed to a first step towards finding universal torsion-free R-modules for a torsion R-module M.

THEOREM 2.6. *Let M be a reduced torsion R-module with only finitely many non-trivial bounded p-components. If N is torsion-free of finite rank such that* $\mathrm{Ext}^1_R(N, M) = 0$*, then N embeds into $\bigoplus_\omega R_{\delta(M)}$.*

PROOF. Since a bounded R-module is cotorsion we may assume as in the proof of Lemma 2.4 that $\delta(M) = \delta_0(M)$. First assume that N is torsion-free of rank 1, hence isomorphic to a submodule of Q containing R. By Lemma 2.5 the set $P = \{p \in \mathrm{Spec}(R) : (N/R)_p \neq 0$ and $M_p \neq 0\}$ is finite. Moreover, if $p \in P$, then N is not p-divisible. Therefore, the sum $\bigoplus_{p \in P} (N/R)_p$ is bounded and hence we may assume without loss of generality that $(N/R)_p = 0$ for all $M_p \neq 0$ by passing to a suitable isomorphic copy of N. Thus, $\{p \in \mathrm{Spec}(R) : (N/R)_p \neq 0\} \subseteq \delta(M)$ and the short exact sequence

$$0 \to R \to N \to \bigoplus_{p \in \delta(M)} E(R/p) \to 0$$

induces the sequence

$$0 = \mathrm{Hom}(\bigoplus_{p \in \delta(M)} E(R/p), R_{\delta(M)}) \to \mathrm{Hom}(N, R_{\delta(M)}) \to \mathrm{Hom}(R, R_{\delta(M)})$$

$$\to \mathrm{Ext}^1_R(\bigoplus_{p \in \delta(M)} E(R/p), R_{\delta(M)}) = 0$$

and hence $\mathrm{Hom}(S, R_{\delta(M)})$ is isomorphic to $\mathrm{Hom}(R, R_{\delta(M)}) \cong R_{\delta(M)}$. This shows that N embeds into $R_{\delta(M)}$ and hence into $\bigoplus_\omega R_{\delta(M)}$.

If N is of arbitrary finite rank, then the arguments above show that every rank 1 submodule of N can be embedded into $R_{\delta(M)}$. We consider the short exact sequence

$$0 \to R \to R_{\delta(M)} \to I = \bigoplus_{p\in\delta(M)} E(R/p) \to 0.$$

Tensoring by the flat R-module N we get

$$0 \to N \to N \otimes R_{\delta(M)} \to I \otimes N \to 0.$$

Applying $\operatorname{Ext}^1_R(-, M)$ we obtain

$$\cdots \to Ext^1_R(I \otimes N, M) \to \operatorname{Ext}^1_R(N \otimes R_{\delta(M)}, M) \to 0.$$

Now, $\operatorname{Ext}^1_R(I\otimes N, M) \cong \prod_{p\in\delta(M)} \operatorname{Ext}^1_R(E(R/p)\otimes N, M) = 0$ since $M_p = 0$ for all $p \in \delta(M)$. Thus $\operatorname{Ext}^1_R(N \otimes R_{\delta(M)}, M) = 0$ as well. By the above reasoning we have a short exact sequence

$$0 \to \bigoplus R_{\delta(M)} \to N \otimes R_{\delta(M)} \to \bigoplus_p I_p \to 0$$

with I_p being p-torsion. Thus we get

$$\operatorname{Hom}(\bigoplus R_{\delta(M)}, M) \to \operatorname{Ext}^1_R(\bigoplus_p I_p, M) \to 0$$

where $\operatorname{Hom}(\bigoplus R_{\delta(M)}, M) \cong \bigoplus \operatorname{Hom}(R_{\delta(M)}, M) \cong \bigoplus M$ is torsion since N is of finite rank. Therefore, also $\operatorname{Ext}^1_R(\bigoplus_p I_p, M) \cong \prod_p \operatorname{Ext}^1_R(I_p, M)$ is torsion which implies that $\operatorname{Ext}^1_R(I_p, M) = 0$ for almost all p. As in the proof of Theorem 2.5 we obtain that $\bigoplus_{p\notin\delta(M)} I_p$ is bounded and hence without loss of generality trivial, by passing to an isomorphic copy of $N \otimes R_{\delta(M)}$. Therefore, we have the following short exact sequence

$$0 \to \bigoplus R_{\delta(M)} \to N \otimes R_{\delta(M)} \to \bigoplus_{p\in\delta(M)} I_p \to 0.$$

Now, let M' be any reduced $R_{\delta(M)}$-module. Then $M'_p = 0$ for all $p \in \delta(M)$. Applying $\operatorname{Ext}^1_R(-, M')$ implies

$$\cdots \to \operatorname{Ext}^1_{R_{\delta(M)}}(\bigoplus_{p\in\delta(M)} I_p, M') \to \operatorname{Ext}^1_{R_{\delta(M)}}(N \otimes R_{\delta(M)}, M') \to 0$$

where the first term is zero since $M'_p = 0$ for all $p \in \delta(M)$. Consequently, $N \otimes R_{\delta(M)}$ is a Baer module of finite rank over the Prüfer domain $R_{\delta(M)}$ and thus a free $R_{\delta(M)}$-module by [**FuSa**, Theorem XVI 8.9]. Therefore, N embeds into $\bigoplus_\omega R_{\delta(M)}$. $\square$

In order to pass from finite rank torsion-free R-modules to R-modules of countable rank we need to restrict to Dedekind domains having a countably generated quotient field. In this case Pontryagin's well-known test criterion for projectivity holds.

LEMMA 2.7. *Assume Q is countably generated. Let N be a torsion-free R-module of countable rank. Then N is projective if and only if all finite rank submodules of N are projective.*

PROOF. See [**FuSa**, page 537]. $\square$

THEOREM 2.8. *Assume that Q is countably generated. Let M be a reduced torsion R-module with only finitely many non-trivial bounded primary components. If N is torsion-free of countable rank such that $\mathrm{Ext}^1_R(N, M) = 0$, then N embeds into $\bigoplus_{\omega} R_{\delta(M)}$.*

PROOF. Since Q is countably generated any torsion-free R-module ($R_{\delta(M)}$-module) is free if and only if all of its finite rank pure submodules are free by Lemma 2.7. We write N as the union of an ascending chain of pure submodules N_n of finite rank. Since $R_{\delta(M)}$ is flat we obtain that $N \otimes R_{\delta(M)}$ is the union of the ascending chain of pure submodules $N_n \otimes R_{\delta(M)}$. However, as in the proof of Theorem 2.6 we deduce that $N_n \otimes R_{\delta(M)}$ is a free $R_{\delta(M)}$-module of finite rank for all $n \in \omega$ and hence $N \otimes R_{\delta(M)}$ is free as an $R_{\delta(M)}$-module and therefore embeds into $\bigoplus_{\omega} R_{\delta(M)}$. $\square$

Finally, we can state our main theorem which uses the axioms of $(V = L)$. As before, for an R-module M, let $\mathfrak{C}(M)$ denote the class of all torsion-free R-modules N satisfying $\mathrm{Ext}^1_R(N, M) = 0$. Recall from [**TrWa**] that a Dedekind domain R is called *small* if $|R| \leq 2^{\aleph_0}$ and the spectrum of R is countable.

THEOREM 2.9 (V=L). *Let R be a small Dedekind domain and M be a torsion R-module with only finitely many non-trivial bounded primary components. Then $\bigoplus_{\lambda} R_{\delta(M)}$ is $(\lambda, \lambda, \mathfrak{C}(M))$-universal for M for every cardinal $\lambda \geq |R| + \omega$.*

PROOF. Let $\kappa = |R| + \aleph_0$. Since R is small, Q is countably generated by [**Ma**, Theorem 18.4]. Hence $|Q| = \kappa$ and for every R-module T, the injective envelope $E(T)$ has size at most $|T|\kappa$. Let M be as stated. Without loss of generality M is reduced and as before we may also assume that $\delta(M) = \delta_0(M)$. We claim that we even may assume M to be of cardinality κ as well. Since each R_p ($p \in \mathrm{Spec}(R)$) is a local principal ideal domain, every p-component M_p of M possesses a basic submodule B_p of the form

$$B_p = \bigoplus_{n\geq 1}(R_p/p^nR_p)^{(\alpha_n^p)} \oplus R_p^{(\beta^p)}$$

for uniquely determined cardinals α_n^p and β^p by [**EkMe**, Theorem V 1.7]. Recall that B_p is basic in M_p if B_p is pure in M_p and M_p/B_p is divisible. As for abelian groups, B_p is an epimorphic image of M_p (see [**Fu**, Theorem]). Since M_p is unbounded for $p \notin \delta(M)$, there exists an unbounded direct sum D_p of countably many cyclic R_p-modules which is a summand of B_p and hence itself an epimorphic image of M_p. Since $\mathrm{Spec}(R)$ is countable, we deduce that $D = \bigoplus\limits_{p\notin\delta(M)} D_p$ is an epimorphic image of M which is of cardinality at most κ and satisfies $\delta(M) = \delta(D)$. Since R is hereditary, any $N \in \mathfrak{C}(M)$ is also an element of $\mathfrak{C}(D)$. Moreover, $\bigoplus\limits_{\lambda} R_{\delta(M)} \in \mathfrak{C}(M) \subseteq \mathfrak{C}(D)$. Thus it suffices to prove that $\bigoplus\limits_{\lambda} R_{\delta(M)}$ is $(\lambda, \lambda, \mathfrak{C}(D))$-universal for D for all $\lambda \geq \kappa$. This shows that we may assume without loss of generality that M is of cardinality at most κ. Hence, $|E(M)| \leq \kappa$ and therefore $\kappa = |R| + \aleph_0 + |E(M)|$. Since we are assuming Gödel's constructible universe and R is small we deduce $|R| \leq 2^{\aleph_0} = \aleph_1$ and since $R_{\delta(M)}$ is a strong splitter it suffices to prove that $\bigoplus\limits_{\kappa} R_{\delta(M)}$ is $(\kappa, \kappa, \mathfrak{C}(M))$-universal for M by Theorem 1.13. Thus, let $N \in \mathfrak{C}(M)$ be of cardinality $\rho \leq \aleph_1$. We induct on the rank $\mathrm{rk}(N)$ of N to prove that $N \otimes R_{\delta(M)}$ is a free $R_{\delta(M)}$-module. If $rk(N) = \aleph_0$, then N embeds into $N \otimes R_{\delta(M)} = \bigoplus\limits_{\omega} R_{\delta(M)}$ by Theorem 2.8. Thus let $\mathrm{rk}(N) = \aleph_1$ and assume that the result is true for all R-modules of rank less than $\mathrm{rk}(N)$. Then we obtain $N' \otimes R_{\delta(M)} \in \mathfrak{C}(M)$ for every submodule N' of N as in the proof of Theorem 2.8. Since $\mathrm{rk}(N)$ is regular and R is a hereditary domain, N has a filtration of pure submodules with countably generated successive factors in $\mathfrak{C}(M)$ by [**EkMe**, Theorem XII 1.10]. As in the proof of Theorem 1.13 we use the fact that $\bigoplus\limits_{\omega} R_{\delta(M)}$ is a strong splitter in which every countably generated R-module from $\mathfrak{C}(M)$ embeds, to show that $N \otimes R_{\delta(M)}$ is a free $R_{\delta(M)}$-module (see also Remark 1.14). $\square$

3. Application to abelian groups

In this section we will use Theorem 1.13 and Theorem 2.9 to show the existence of universal abelian groups and, as a consequence, solve a problem by Kulikov in Gödel's universe for particular torsion groups T. The result is not new but has been obtained in [**St1**] using different techniques. Recall, that $R = \mathbb{Z}$ is a countable Dedekind domain and that for every infinite abelian group T, the injective envelope $E(T)$ satisfies $|T| = |E(T)|$. Moreover, ${}^{\perp}T$ is always closed under pure subgroups.

Let $\mathfrak{C}(T) = \{G \in {}^{\perp}T : G \textit{ is torsion-free}\}$. We first assume that T is a torsion abelian group. By Theorem 2.9 we have the following result which partially solves a problem by Kulikov from [**KN**]. This result was also proved in [**St1**].

THEOREM 3.1 (V=L). *Let T be a torsion group with only finitely many non-trivial bounded primary components. Then there exist $(\lambda, \lambda, \mathfrak{C}(T))$-universal groups for T for any cardinal $\lambda \geq \aleph_0$.*

PROOF. Follows from Theorem 2.9. □

We now aim for universal groups for finite rank torsion–free abelian groups. First we need to recall the definition of the outer type of a torsion-free group of finite rank.

DEFINITION 3.2. *Let G be a torsion–free abelian group of finite rank. Moreover, let $g_1, \cdots, g_n$ be a maximal linearly independent subset of G. Put $H_i = \langle g_1, \cdots, \hat{g}_i, g_{i+1}, \cdots, g_n\rangle_* \subseteq G$, where $\hat{g}$ means that the element is missing. Letting $S_i = G/H_i \subseteq \mathbb{Q}$ we define the* outer type *of G by $\mathrm{OT}(G) = \sup\{S_1, \cdots, S_n\}$. Similarly, the inner type $\mathrm{IT}(G)$ of G is defined as $\mathrm{IT}(G) = \inf\{\langle g_i\rangle_* : i \leq n\}$.*

Note, that for a pure subgroup H of G we have $\mathrm{OT}(G) \geq \mathrm{OT}(H)$. Warfield proved that the definition of inner and outer type does not depend on the chosen set of linearly independent elements. Moreover, we have $\mathrm{IT}(G) \leq \mathrm{OT}(G)$, $\mathrm{IT}(G) \leq R$ for all $R \in \mathrm{Tst}(G) = \{\langle g\rangle_* \subseteq G : x \in G\}$, and $R \leq \mathrm{OT}(G)$ for every R in the cotypeset of G, respectively. Obviously, $\mathrm{OT}(R) = R = \mathrm{IT}(R)$ holds for every rational group $R \subseteq \mathbb{Q}$. Using a result due to Warfield [**Wa**] we can explicitly determine the outer type and inner type for any torsion–free abelian group G of finite rank. Recall that two height sequences are equivalent if and only if they differ in finitely many finite entries only. The equivalence class $[(h_p)_{p\in\Pi}]$ of a height sequence $(h_p)_{p\in\Pi}$ determines uniquely the corresponding type.

LEMMA 3.3 (Warfield). *Let G be a torsion–free abelian group of finite rank and let F be a free subgroup of G such that G/F is torsion, i.e.*

$$G/F = \bigoplus_{p\in\Pi} T_p \textit{ with } T_p = \mathbb{Z}(p^{i_{p,1}}) \oplus \cdots \oplus \mathbb{Z}(p^{i_{p,n_p}})$$

and $0 \leq i_{p,1} \leq \cdots \leq i_{p,n_p} \leq \infty$. *Then* $\mathrm{OT}(G) = [(i_{p,n_p})_{p\in\Pi}]$ *and* $\mathrm{IT}(G) = [(i_{p,1})_{p\in\Pi}]$.

The key stone for obtaining universal groups for a torsion–free group of finite rank is given by the following result due to Goeters [**Goe**] and Wickless [**Wi**]. In the sequel we shall consider the outer type as an equivalence class of height sequences and also at the same time as an isomorphism class of rational groups.

Lemma 3.4 (Goeters and Wickless). *Let* X, Y *be torsion–free abelian groups of finite rank. Then* $\mathrm{Ext}_{\mathbb{Z}}(X,Y) = 0$ *if and only if for every height sequence* $h \in OT(X)$, *we have* $\sum_{p\in S} h(p) < \infty$ *where* $S = \{p : pY \neq Y\}$.

For an abelian group Y put $\sigma(Y) = \{p \in \Pi : pY \neq Y\}$ and define

$$R_Y = \langle 1/p^n : n \in \omega, p \notin \sigma(Y)\rangle$$

to be the subring of $\mathbb{Q}$ generated by all primes which are not in $\sigma(Y)$. Then R_Y is called the *nucleus* of Y which is the largest subring of $\mathbb{Q}$ over which Y is still an R-module. Thus we have

Corollary 3.5. *Let* Y *be a torsion-free abelian group of finite rank. Then a torsion-free finite rank group* X *satisfies* $\mathrm{Ext}_{\mathbb{Z}}(X,Y) = 0$ *if and only if* $\mathrm{OT}(X) \leq R_Y$.

Proof. Let X be torsion-free of finite rank. By Lemma 3.4 we have $\mathrm{Ext}_{\mathbb{Z}}(X,Y) = 0$ if and only if $\sum_{p\in\sigma(Y)} h(p) < \infty$ for every height sequence $h \in \mathrm{OT}(X)$. Clearly this is equivalent to $\mathrm{OT}(X) \leq R_Y$. $\square$

Theorem 3.6 (V=L). *Let* Y *be a torsion-free group of finite rank and* $R = R_Y$ *the nucleus of* Y. *Then* $\bigoplus_{\lambda} R$ *is* $(\lambda, \lambda, \mathfrak{C}(Y))$-*universal for* Y *for every infinite cardinal* λ.

Proof. Obviously, $U = \bigoplus_{\omega} R$ is a strong splitter and satisfies $U \in \mathfrak{C}(Y)$. Hence, it suffices to prove that U is $(\omega, \omega, \mathfrak{C}(Y))$-universal for Y by Theorem 1.13. Thus it remains to prove that any countable torsion-free group X satisfying $\mathrm{Ext}_{\mathbb{Z}}(X,Y) = 0$ embeds into U. If X is of rank 1, then Corollary 3.5 implies that $X \leq R$ as types, hence X embeds into R. If X is of finite rank, then the above implies that $X \otimes R$ is homogeneous of type R. Since

$$\mathrm{Ext}_{\mathbb{Z}}(X,R) = \mathrm{Ext}_{\mathbb{Z}}(X, \mathrm{Hom}(R,R)) = \mathrm{Ext}_{\mathbb{Z}}(X \otimes R, R)$$

we obtain by Corollary 3.5 that $\mathrm{Ext}_{\mathbb{Z}}(X \otimes R, R) = 0$. Using [**Ha**] it follows that $X \otimes R$ is completely decomposable of type R and therefore

$X \subseteq X \otimes R$ embeds into U. Finally, if X is of countable rank, then X posesses a filtration $\{X_n : n \in \omega\}$ of pure subgroups X_n of finite rank which implies a filtration $\{X_n \otimes R : n \in \omega\}$ of pure subgroups $X_n \otimes R$ of $X \otimes R$. By the above, every $X_n \otimes R$ is a free R-module and hence it is well-known that also $X \otimes R$ is completely decomposable of type R. Consequently, $X \subseteq X \otimes R$ embeds into U. $\square$

As an immediate corollary we obtain

THEOREM 3.7 (V=L). *Let M be a mixed abelian group such that its torsion part $T = t(M)$ has only finitely many non-trivial bounded primary components. Let $Y = M/T$ be of finite rank. If $R_Y = \mathbb{Z}_{\delta(T)}$, then $\bigoplus_\lambda R_Y$ is $(\lambda, \lambda, \mathfrak{C}(M))$-universal for M for all infinite cardinals λ.*

PROOF. By the assumptions we have $R_Y = \mathbb{Z}_{\delta(T)}$ and hence $\bigoplus_\lambda R_Y \in \mathfrak{C}(M)$ for all infinite cardinals λ since $\bigoplus_\lambda R_Y \in \mathfrak{C}(Y)$ and $\bigoplus_\lambda R_Y \in \mathfrak{C}(T)$ by Theorem 3.6 and Theorem 3.1. Since $\mathrm{Ext}_{\mathbb{Z}}(G, M) = 0$ implies $\mathrm{Ext}_{\mathbb{Z}}(G, Y) = 0$ for every torsion-free abelian group G we conclude that $\bigoplus_\lambda R_Y$ is $(\lambda, \lambda, \mathfrak{C}(M))$-universal for M for all infinite cardinals λ. $\square$

References

[Ar] **D.M. Arnold**, *Abelian Groups and Representations of Finite Partially Ordered Sets*, Lecture Notes in Math. **931**, CMS Books in Math., Springer Verlag Berlin-Heidelberg-New York (2000).

[Ba] **R. Baer**, *The subgroup of the elements of finite order of an abelian group*, Ann. of Math. **37** (1936), 766–781.

[EkMe] **P.C. Eklof and A. Mekler**, *Almost Free Modules; Set–Theoretic Methods*, North–Holland (1990).

[Fu] **L. Fuchs**, *Infinite Abelian Groups* **I, II**, Academic Press (1970, 1973).

[FuSa] **L. Fuchs and L. Salce**, *Modules over non-noetherian domains*,Mathematical Surveys and Monographs **84**, AMS (2001). Academic Press (1970, 1973).

[Goe] **P. Göters**, *Generating cotorsion theories and injective classes*, Acta Math. Hung. **51** (1988), 99–107.

[Ha] **J. Hausen**, *Automorphismen gesättigte Klassen abzählbarer abelscher Gruppen*, Studies on Abelian Groups, Springer (1968), 147–181.

[KN] **The Kourovka Notebook - unsolved questions in group theory** , Russian Academy of Science, **14th edition**(1999).

[Ma] **H. Matsumura**, *Commutative Ring Theory*, Cambridge Studies in Advanced Mathematics **8**, Cambridge University Press, Cambridge, (1989).

[Sa] **L. Salce**, *Cotorsion theories for abelian groups*, Symposia Mathematica **23** (1979), 11–32.

[SaTr] **J. Saroch and J. Trlifaj**, *Completeness of cotorsion pairs*, submitted.

[Sh] **S. Shelah**, *A compactness theorem for singular cardinals, free algebras, Whitehead problem and transversals*, Israel J. Math. **21**(1975), 319–349.

[ShSt] **S. Shelah and L. Strüngmann**, *Kulikov's problem on universal torsion-free abelian groups*, Journal of London Math. Soc. **67** (2003), 626–642.

[St1] **L. Strüngmann**, *On problems by Baer and Kulikov using $V = L$*, Illinois J. of Math. **46** (2002), 477–490.

[St2] **L. Strüngmann**, *Derived cotorsion theories*, Habilitationsschrift, Universität Duisburg-Essen (2004).

[St3] **L. Strüngmann**, *Baer cotorsion pairs*, to appear in Israel J. Math. (2005).

[St4] **L. Strüngmann**, $(\mathfrak{V}, \mathfrak{W})$-*cotorsion pairs*, to appear in Archiv der Math. (2005).

[StWa] **L. Strüngmann and S.L. Wallutis**, *On the torsion groups in cotorsion classes*, AGRAM 2000 Conference, Perth, Western Australia, Contemp. Math. **273** (2001), 269–283. Perth (2001).

[TrWa] **J. Trlifaj and S.L. Wallutis**, *Tilting modules over small Dedkind domains*, J. Pure and Applied Algebra **172** (2002), 109–117.

[Wa] **R.B. Warfield Jr.**, *Homomorphisms and duality for torsion–free groups*, Math. Z. **107** (1968), 189–200.

[Wi] **W. J. Wickless**, *Projective classes of torsion–free abelian groups II*, Acta Math. Hung. **44** (1984), 13–20.

Fachbereich Mathematik, University of Duisburg–Essen, 45117 Essen, Germany

E-mail address: `lutz.struengmann@uni-essen.de`

Current address: Department of Mathematics, University of Hawaii, 2565 McCarthy Mall, Honolulu, HI 96822-2273, USA

E-mail address: `lutz@math.hawaii.edu`