

# Construction of polyconvex energies for non-trivial anisotropy classes

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**Abstract** Hyperelastic material behavior can be preferably described by using polyconvex energies, since the existence of minimizers is then guaranteed, if, in addition, the coercivity condition is satisfied. We give an overview of the construction of polyconvex energies for the description of non-trivial anisotropy classes, namely the triclinic, monoclinic, rhombic, tetragonal, trigonal and cubic symmetry groups, as well as transverse isotropy. The anisotropy of the material is described by invariants in terms of the right Cauchy-Green tensor and a specific second-order and a fourth-order structural tensor, respectively. To show the capability of the proposed polyconvex energies to simulate real anisotropic material behavior we focus on fittings of fourth-order elasticity tensors near the reference state to experimental data of different anisotropic materials.

## 1 Introduction

The polyconvexity (Ball (1977)) of anisotropic free energy functions was first proved in Schröder and Neff (2001) for transverse isotropy; a general extension of this work to orthotropic polyconvex functions is discussed in Schröder and Neff (2003). There the concept of structural tensors, introduced in an attractive way by Boehler (1978, 1979, 1987), was taken into account. The main idea of this concept is to extend anisotropic functions to isotropic functions by introducing first- and second-order tensors characterizing the underlying material symmetry group, called structural tensors, as additional agencies. Thus, the known isotropic tensor function representations, as e.g. extensively developed by Wang (1969a,b, 1970a,b, 1971), Smith (1970, 1971), Boehler (1977), can be used to yield representations for anisotropic tensor functions, see Boehler (1979, 1987), Liu (1982), Spencer (1971, 1982) and Zheng (1994b). However, with regard to e.g. Zheng and Spencer (1993), transverse isotropy and orthotropy as well as the triclinic and monoclinic symmetry groups are the only symmetry groups, which can

be described by first- and second-order structural tensors. Therefore, only in these cases the tensor-representation theory yields a complete polynomial basis of principal and mixed invariants, which are suitable for the construction of specific energy functions. For the representation of eight mechanically important anisotropy types structural tensors of order higher than two are required: for the trigonal, tetragonal and cubic systems fourth-order and for the hexagonal systems sixth-order structural tensors are necessary, see Zheng and Spencer (1993), Zheng (1994b). In this context Zheng (1994b) pointed out some difficulties which

*depend on the much less well knowledge of complete and irreducible representations for isotropic tensor functions with tensors of orders higher than two.*

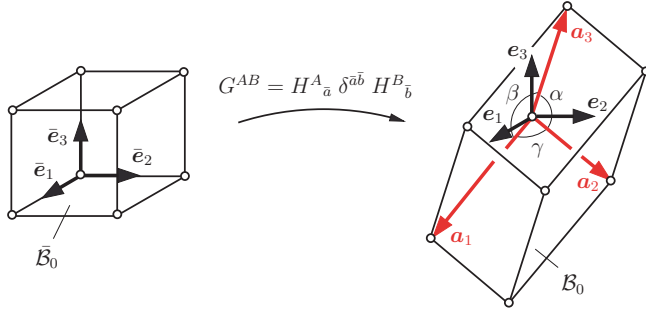
There are some particular cases in literature discussing this topic, e.g. Betten (1982, 1987, 1991, 1998), Betten and Helisch (1992, 1995, 1996), Xiao (1996, 1997), Zheng and Betten (1995) and Zheng (1994a). Xiao (1996, 1997) presented vector-valued and second-order tensor-valued tensor functions for all 32 crystal and non-crystal classes.

The construction of polyconvex triclinic, monoclinic, rhombic and transversely isotropic energy functions based on second-order, symmetric and positive-definite structural tensors is given in Schröder et al. (2008), see also Ebbing et al. (2009). A direct extension of the results presented in Schröder and Neff (2003) by introducing a single fourth-order structural tensor for a basis of a polyconvex cubic energy is given in Kambouchev et al. (2007). Furthermore coordinate-free representations of tetragonal, trigonal and cubic polyconvex energy functions, based on crystallographically motivated fourth-order structural tensors, can be found in Schröder et al. (2009).

In this contribution we give an overview of the development of polyconvex anisotropic energy functions for non-trivial anisotropy classes – the triclinic, monoclinic, rhombic, tetragonal, trigonal and cubic mechanically important symmetry classes – as well as transverse isotropy based on the works Schröder et al. (2009, 2008). These energies are formulated in terms of specific second-order and fourth-order structural tensors, respectively, which are constructed by taking the base system of the underlying crystal class into account. In order to show the applicability of this procedure, we present the fittings of fourth-order elasticity tensors near the reference state with the proposed polyconvex anisotropic energy and compare the results with experimental data taken from the literature.

## 2 Non-Trivial Anisotropy Groups

For the formulation of anisotropic energies the Neumann's Principle postulating relations between crystal symmetries and the crystal's physical properties, plays an important role, see Neumann (1885). This principle, which states that *symmetry elements associated with any crystal physical property must include the elements of the symmetry point group of the crystal* leads to several restrictions of the specific form of anisotropic energies and is therefore of high importance for the representations of tensor functions. The symmetry of crystals as well as their anisotropic physical properties are closely related to the crystal lattices. A so-called Bravais lattice is characterized by the lengths of the three base vectors  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ , given by  $a = \|\mathbf{a}_1\|, b = \|\mathbf{a}_2\|, c = \|\mathbf{a}_3\|$  and the orientations of the base vectors described by the axial angles  $\alpha, \beta, \gamma$  as can be seen in Fig. 1. Here we intro-



**Figure 1.** Push-forward operation  $\mathbf{G}$ .

duce a fictitious reference configuration  $\bar{\mathcal{B}}_0$ , in this context see also Menzel and Steinmann (2001). Let  $\mathbf{H}$  represent a linear tangent map, which maps cartesian base vectors  $\bar{\mathbf{e}}_i \in \bar{\mathcal{B}}_0$  onto crystallographically motivated base vectors  $\mathbf{a}_i \in \mathcal{B}_0$ , i.e.,

$$\mathbf{H} : \bar{\mathbf{e}}_i \mapsto \mathbf{a}_i \quad \rightarrow \quad \mathbf{H} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3] \quad \text{with} \quad \mathbf{a}_i = \mathbf{H} \bar{\mathbf{e}}_i. \quad (1)$$

Then the second-order symmetric and positive (semi-)definite tensor

$$\mathbf{G} = \mathbf{H} \mathbf{H}^T \quad (2)$$

can be interpreted as a push-forward of the cartesian metric of the fictitious configuration  $\bar{\mathcal{B}}_0$  to the real reference configuration  $\mathcal{B}_0$ . The triclinic, monoclinic, rhombic system as well as the transversely isotropic symmetry can be described by such specific second-order metric tensors. In all cases we

assume that

$$\mathbf{a}_1 \parallel \bar{\mathbf{e}}_1 \quad \text{and} \quad \mathbf{a}_2 \perp \bar{\mathbf{e}}_3. \quad (3)$$

Concerning the notation for the material symmetry groups we refer to Zheng and Spencer (1993). In the triclinic case, i.e. here for the symmetry group  $\mathcal{C}_1$ , no additional restrictions on the orientation or lengths of the base vectors have to be taken into account. The Voigt notation of the triclinic elasticity tensor reads

$$\mathcal{C}_1 : \mathbb{C} = \begin{bmatrix} \mathbb{C}_{1111} & \mathbb{C}_{1122} & \mathbb{C}_{1133} & \mathbb{C}_{1112} & \mathbb{C}_{1123} & \mathbb{C}_{1113} \\ & \mathbb{C}_{2222} & \mathbb{C}_{2233} & \mathbb{C}_{2212} & \mathbb{C}_{2223} & \mathbb{C}_{2213} \\ & & \mathbb{C}_{3333} & \mathbb{C}_{3312} & \mathbb{C}_{3323} & \mathbb{C}_{3313} \\ & & & \mathbb{C}_{1212} & \mathbb{C}_{1223} & \mathbb{C}_{1213} \\ & \text{sym.} & & & \mathbb{C}_{2323} & \mathbb{C}_{2313} \\ & & & & & \mathbb{C}_{1313} \end{bmatrix}. \quad (4)$$

In the monoclinic case ( $\mathcal{C}_2$ ) the base vector  $\mathbf{a}_1$  as well as  $\mathbf{a}_2$  are oriented perpendicularly to  $\mathbf{a}_3$ ; therefore the relation  $\alpha = \beta = 90^\circ$  must hold. The monoclinic elasticity tensor appears as

$$\mathcal{C}_2 : \mathbb{C} = \begin{bmatrix} \mathbb{C}_{1111} & \mathbb{C}_{1122} & \mathbb{C}_{1133} & \mathbb{C}_{1112} & 0 & 0 \\ & \mathbb{C}_{2222} & \mathbb{C}_{2233} & \mathbb{C}_{2212} & 0 & 0 \\ & & \mathbb{C}_{3333} & \mathbb{C}_{3312} & 0 & 0 \\ & & & \mathbb{C}_{1212} & 0 & 0 \\ & \text{sym.} & & & \mathbb{C}_{2323} & \mathbb{C}_{2313} \\ & & & & & \mathbb{C}_{1313} \end{bmatrix}. \quad (5)$$

The base vectors of the rhombic lattice ( $\mathcal{C}_3$ ) are aligned in the directions of the three mutually orthogonal two-fold axes, see Fig. 2. Here the coefficient scheme of the elasticity tensor is

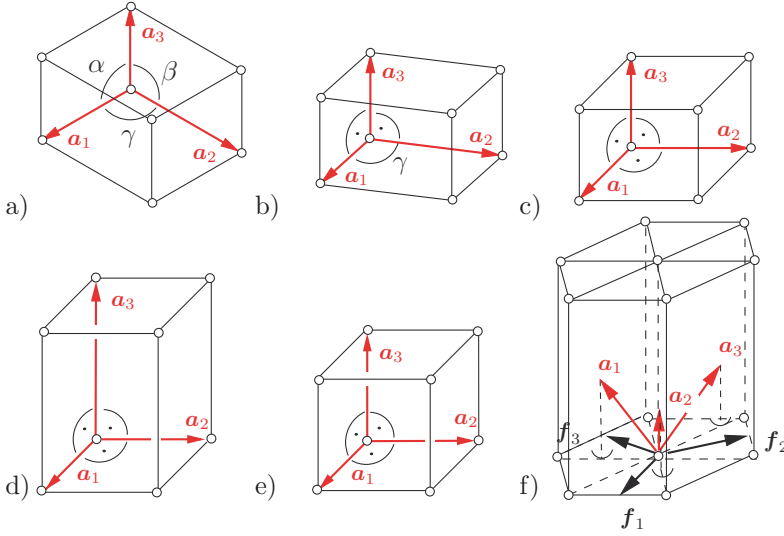
$$\mathcal{C}_3 : \mathbb{C} = \begin{bmatrix} \mathbb{C}_{1111} & \mathbb{C}_{1122} & \mathbb{C}_{1133} & 0 & 0 & 0 \\ & \mathbb{C}_{2222} & \mathbb{C}_{2233} & 0 & 0 & 0 \\ & & \mathbb{C}_{3333} & 0 & 0 & 0 \\ & & & \mathbb{C}_{1212} & 0 & 0 \\ & \text{sym.} & & & \mathbb{C}_{2323} & 0 \\ & & & & & \mathbb{C}_{1313} \end{bmatrix}. \quad (6)$$

The detailed metric tensor representations for these cases are presented in the Appendix. The parameters  $a, b, c, \tilde{a}, \tilde{b}, \tilde{c}, \bar{a}, \bar{b}, \bar{c}$ , which appear there, can be interpreted as additional material parameters and must satisfy the conditions for the positive (semi-)definiteness of the concerning metric tensor. In case of transverse isotropy  $\mathbf{G}$  is given by  $\mathbf{G} = \text{diag}[a, b, b]$  with

$a, b \geq 0$ , if the assumed preferred direction of the material  $\mathbf{a}_1$  lies parallel to the  $\mathbf{e}_1$ -axis. For the representation of the tetragonal, trigonal and cubic anisotropy fourth-order structural tensors are needed, see Zheng and Spencer (1993). Thus, for the description of these symmetry classes we constructed crystallographically motivated fourth-order “structural tensors” (Schröder et al. (2009)). These are decomposed in a sum of dyadic products, i.e.,

$$\mathbb{G} = \sum_{i=1}^3 \mathbf{a}_i \otimes \mathbf{a}_i \otimes \mathbf{a}_i \otimes \mathbf{a}_i, \quad (7)$$

where the vectors  $\mathbf{a}_i$ ,  $i = 1, 2, 3$  are the base vectors of an associated underlying crystallographic base system. The definition (7) leads to an elasticity tensor -type representation of the fourth-order structural tensor.



**Figure 2.** a) Triclinic ( $\mathcal{C}_1$ ), b) monoclinic ( $\mathcal{C}_2$ ), c) rhombic ( $\mathcal{C}_3$ ), d) tetragonal ( $\mathcal{C}_5$ ), e) cubic ( $\mathcal{C}_7$ ), f) trigonal ( $\mathcal{C}_8$ ) cell .

The base system of the tetragonal symmetry classes  $\mathcal{C}_4$  and  $\mathcal{C}_5$  is characterized by two orthogonal base vectors  $\mathbf{a}_1$  and  $\mathbf{a}_2$  of equal lengths which are perpendicular to the four-fold axis  $\mathbf{a}_3$ . Whereas for  $\mathcal{C}_4$   $\mathbf{a}_1$  and  $\mathbf{a}_2$  can be taken as any two orthogonal vectors perpendicular to  $\mathbf{a}_3$ , for  $\mathcal{C}_5$  they are aligned in directions of two mutually orthogonal two-fold axes of  $\mathcal{C}_5$ , e.g.,  $\mathbf{a}_1 = a \mathbf{e}_1$ ,  $\mathbf{a}_2 = a \mathbf{e}_2$ ,  $\mathbf{a}_3 = c \mathbf{e}_3$ , see Fig. 2d. The anisotropic elasticity tensors at reference state  $\mathbb{C} = 4\psi^2_{\mathbb{C}\mathbb{C}}|_{\mathcal{C}=1}$  for the crystal type  $\mathcal{C}_4$  and  $\mathcal{C}_5$  in

Voigt notation are given by the expressions

$$\mathcal{C}_4 : \mathbb{C} = \begin{bmatrix} \mathbb{C}_{1111} & \mathbb{C}_{1122} & \mathbb{C}_{1133} & \mathbb{C}_{1112} & 0 & 0 \\ & \mathbb{C}_{1111} & \mathbb{C}_{1133} & -\mathbb{C}_{1112} & 0 & 0 \\ & & \mathbb{C}_{3333} & 0 & 0 & 0 \\ & & & \mathbb{C}_{1212} & 0 & 0 \\ sym. & & & & \mathbb{C}_{2323} & 0 \\ & & & & & \mathbb{C}_{2323} \end{bmatrix}, \quad (8)$$

$$\mathcal{C}_5 : \mathbb{C} = \begin{bmatrix} \mathbb{C}_{1111} & \mathbb{C}_{1122} & \mathbb{C}_{1133} & 0 & 0 & 0 \\ & \mathbb{C}_{1111} & \mathbb{C}_{1133} & 0 & 0 & 0 \\ & & \mathbb{C}_{3333} & 0 & 0 & 0 \\ & & & \mathbb{C}_{1212} & 0 & 0 \\ sym. & & & & \mathbb{C}_{2323} & 0 \\ & & & & & \mathbb{C}_{2323} \end{bmatrix}. \quad (9)$$

In the trigonal case ( $\mathcal{C}_8$  and  $\mathcal{C}_9$ ) we consider the basis of the associated rhombohedral cell denoted by  $\mathbf{a}_1$ ,  $\mathbf{a}_2$ ,  $\mathbf{a}_3$  of equal lengths and which include the same angle with one another. The three-fold axis is given along the  $(\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3)$ -direction. Whereas for  $\mathcal{C}_8$  the base vectors  $\mathbf{a}_i$  can be taken as any of the described base vectors, for  $\mathcal{C}_9$  the base vectors  $\mathbf{a}_i$  are perpendicular to the three two-fold axes of  $\mathcal{C}_9$  denoted here by  $\mathbf{f}_i$ ,  $i = 1, 2, 3$ , see Fig. 2f, i.e.,  $\mathbf{a}_i \perp \mathbf{f}_i$ . The trigonal elasticity tensors of type  $\mathcal{C}_8$  and type  $\mathcal{C}_9$  appear in the forms

$$\mathcal{C}_8 : \mathbb{C} = \begin{bmatrix} \mathbb{C}_{1111} & \mathbb{C}_{1122} & \mathbb{C}_{1133} & 0 & \mathbb{C}_{1123} & \mathbb{C}_{1113} \\ & \mathbb{C}_{1111} & \mathbb{C}_{1133} & 0 & -\mathbb{C}_{1123} & -\mathbb{C}_{1113} \\ & & \mathbb{C}_{3333} & 0 & 0 & 0 \\ & & & \frac{1}{2}(\mathbb{C}_{1111} - \mathbb{C}_{1122}) & -\mathbb{C}_{1113} & \mathbb{C}_{1123} \\ sym. & & & & \mathbb{C}_{2323} & 0 \\ & & & & & \mathbb{C}_{2323} \end{bmatrix}, \quad (10)$$

$$\mathcal{C}_9 : \mathbb{C} = \begin{bmatrix} \mathbb{C}_{1111} & \mathbb{C}_{1122} & \mathbb{C}_{1133} & 0 & \mathbb{C}_{1123} & 0 \\ & \mathbb{C}_{1111} & \mathbb{C}_{1133} & 0 & -\mathbb{C}_{1123} & 0 \\ & & \mathbb{C}_{3333} & 0 & 0 & 0 \\ & & & \frac{1}{2}(\mathbb{C}_{1111} - \mathbb{C}_{1122}) & 0 & \mathbb{C}_{1123} \\ sym. & & & & \mathbb{C}_{2323} & 0 \\ & & & & & \mathbb{C}_{2323} \end{bmatrix}. \quad (11)$$

In the cubic case ( $\mathcal{C}_7$ ) the three mutually orthogonal base vectors  $\mathbf{a}_i$  of equal lengths coincide with the three four-fold symmetry axes of  $\mathcal{C}_7$ , i.e.,  $\mathbf{a}_i = a \mathbf{e}_i$ ,  $i = 1, 2, 3$ , which are depicted in Fig. 2e. The coefficient scheme

of the elasticity tensor is

$$\mathcal{C}_7 : \quad \mathbb{C} = \begin{bmatrix} \mathbb{C}_{1111} & \mathbb{C}_{1122} & \mathbb{C}_{1122} & 0 & 0 & 0 \\ & \mathbb{C}_{1111} & \mathbb{C}_{1122} & 0 & 0 & 0 \\ & & \mathbb{C}_{1111} & 0 & 0 & 0 \\ & & & \mathbb{C}_{1212} & 0 & 0 \\ \text{sym.} & & & & \mathbb{C}_{1212} & 0 \\ & & & & & \mathbb{C}_{1212} \end{bmatrix}. \quad (12)$$

The resulting fourth-order structural tensors are listed in the Appendix.

### 3 Construction of Anisotropic Polyconvex Energies

For the construction of energy functions the principle of objectivity must be fulfilled, which is here satisfied by using reduced constitutive equations in terms of the right Cauchy-Green tensor  $\mathbf{C}$ , i.e.,

$$\psi(\mathbf{C}) = W(\mathbf{F}) \quad \text{with} \quad \mathbf{C} = \mathbf{F}^T \mathbf{F}, \quad (13)$$

where  $\mathbf{F}$  denotes the deformation gradient. Furthermore, the principle of material symmetry enforces the invariance of the constitutive equations with respect to the transformations  $\mathbf{Q} \in \mathcal{G} \subset \text{O}(3)$  of the material symmetry group  $\mathcal{G} \subset \text{O}(3)$ . Thus, the invariance condition

$$\psi(\mathbf{C}) = \psi(\mathbf{Q} \mathbf{C} \mathbf{Q}^T) \quad \forall \mathbf{Q} \in \mathcal{G} \quad (14)$$

must hold. Under consideration of the concept of structural tensors, see Boehler (1979), and representation theorems of tensor functions, we insert structural tensors as additional arguments into the free energy function (14). Second-order structural tensors  $\mathbf{G}$  and fourth-order structural tensors  $\mathbb{G}$  characterize the underlying material symmetry group  $\mathcal{G} \subset \text{O}(3)$  if

$$\left. \begin{aligned} \mathbf{G} &= \mathbf{Q} \mathbf{G} \mathbf{Q}^T \\ \mathbb{G} &= \mathbf{Q} \boxtimes \mathbf{Q} : \mathbb{G} : \mathbf{Q}^T \boxtimes \mathbf{Q}^T \end{aligned} \right\} \quad \forall \mathbf{Q} \in \mathcal{G} \subset \text{O}(3), \quad (15)$$

wherein the Kronecker product is defined by  $(\mathbf{A} \boxtimes \mathbf{B}) : (\mathbf{a} \otimes \mathbf{b}) = (\mathbf{A} \mathbf{a}) \otimes (\mathbf{B} \mathbf{b})$  with  $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{3 \times 3}$  and  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ . The index representation of  $\mathbb{G}$  reads

$$\mathbb{G}^{ABCD} = Q^A_{\bar{A}} Q^B_{\bar{B}} Q^C_{\bar{C}} Q^D_{\bar{D}} \mathbb{G}^{\bar{A}\bar{B}\bar{C}\bar{D}}. \quad (16)$$

Finally, for the description of anisotropic materials we obtain the function

$$\psi(\mathbf{C}, \mathbf{G}, \mathbb{G}) = \psi(\mathbf{Q} \mathbf{C} \mathbf{Q}^T, \mathbf{Q} \mathbf{G} \mathbf{Q}^T, \mathbf{Q} \boxtimes \mathbf{Q} : \mathbb{G} : \mathbf{Q}^T \boxtimes \mathbf{Q}^T) \quad \forall \mathbf{Q} \in \text{O}(3). \quad (17)$$

Hence, the introduction of structural tensors as additional tensor agencies in the free energy function extends the  $\mathcal{G}$ -invariant functions (14) to isotropic tensor functions (17). For the construction of energy functions which satisfy this property we use polynomial bases in terms of the principal invariants

$$I_1 = \text{tr}\mathbf{C}, \quad I_2 = \text{tr}[\text{Cof}\mathbf{C}], \quad I_3 = \det\mathbf{C} \quad (18)$$

and mixed invariants in terms of the right Cauchy-Green tensor and the second- and fourth-order structural tensors, respectively, appearing in the Appendix. For the formulation of triclinic, monoclinic, rhombic as well as transversely isotropic energy functions we consider the polyconvex polynomial basis

$$\mathcal{P}^{a,m,o} := \{I_1, I_2, I_3, J_4, J_5\}, \quad (19)$$

where the polyconvex mixed invariants  $J_4$  and  $J_5$  are defined as

$$J_4 = \text{tr}[\mathbf{C}\mathbf{G}], \quad J_5 = \text{tr}[\text{Cof}[\mathbf{C}]\mathbf{G}]. \quad (20)$$

Trigonal, tetragonal and cubic energy functions are based on the elements of the polyconvex polynomial basis

$$\mathcal{P}^{ht,t,c} := \{I_1, I_2, I_3, J_6, J_7, J_8, J_9\}. \quad (21)$$

Here the polyconvex mixed invariants are formulated in terms of the underlying fourth-order structural tensor given in the Appendix and  $\mathbf{C}$ , i.e.,

$$J_6 = \mathbf{C} : \mathbb{G} : \mathbf{1}, \quad J_7 = \text{Cof}\mathbf{C} : \mathbb{G} : \mathbf{1}, \quad (22)$$

$$J_8 = \mathbf{C} : \mathbb{G} : \mathbf{C}, \quad J_9 = \text{Cof}\mathbf{C} : \mathbb{G} : \text{Cof}\mathbf{C}.$$

The proof of polyconvexity of the above mentioned invariants is given in detail in Schröder et al. (2008, 2009).

### 3.1 Specific Anisotropic Polyconvex Energies.

In general we consider an additive decomposition of the polyconvex anisotropic energy functions, i.e.,

$$\psi_i = \psi^{iso} + \psi_i^{aniso}, \quad i = 1, 2 \quad (23)$$

where as isotropic part the Mooney-Rivlin model

$$\psi^{iso} = \delta_1 I_1 + \delta_2 I_2 + \delta_3 I_3 - (2\delta_1 + 4\delta_2 + 2\delta_3) \ln \sqrt{I_3} \quad (24)$$



is taken into account. For the symmetry groups  $\mathcal{C}_1 - \mathcal{C}_3$  as well as for transverse isotropy we use the energy function

$$\psi_1 = \psi^{iso} + \psi_1^{aniso}, \quad (25)$$

where the anisotropic part is assumed to be of the type

$$\psi_1^{aniso} = \sum_{j=1}^m \left[ f_j(J_{4j}) + \tilde{f}_j(J_{5j}) + h_j(I_3) \right], \quad (26)$$

with the  $j$ -th invariants

$$J_{4j} = \text{tr}[\mathbf{C}\mathbf{G}_j], \quad J_{5j} = \text{tr}[\text{Cof}[\mathbf{C}]\mathbf{G}_j]. \quad (27)$$

As an example, in case of monoclinic anisotropy the  $j$ -th structural tensor has the form

$$\mathbf{G}_j^m = \begin{bmatrix} a_j & d_j & 0 \\ d_j & b_j & 0 \\ 0 & 0 & c_j \end{bmatrix}, \quad \text{with } a_j, c_j \geq 0, a_j b_j \geq d_j^2. \quad (28)$$

In detail, the polyconvex anisotropic functions in (26) are given by

$$\begin{aligned} f_j(J_{4j}) &= \sum_{r=1}^n \frac{\xi_{rj}}{(\alpha_{rj} + 1)g_j^{\alpha_{rj}}} J_{4j}^{(\alpha_{rj}+1)}, \\ \tilde{f}_j(J_{5j}) &= \sum_{r=1}^n \frac{\xi_{rj}}{(\beta_{rj} + 1)g_j^{\beta_{rj}}} J_{5j}^{(\beta_{rj}+1)}, \\ h_j(I_3) &= \sum_{r=1}^n \xi_{rj} \frac{g_j}{\gamma_{rj}} I_3^{-\gamma_{rj}}, \end{aligned} \quad (29)$$

with the polyconvexity conditions  $\alpha_{rj}, \beta_{rj}, \xi_{rj} \geq 0, \gamma_{rj} \geq -1/2$  and the relation  $g_j := \mathbf{G}_j : \mathbf{1}$ . The function (25) satisfies also without the isotropic part the coercivity condition; the proof is given in Schröder et al. (2008). The additional consideration of an isotropic function  $h_j(I_3)$ , see (29)<sub>3</sub>, was proposed in an analogous way in Itskov and Aksel (2004). This term allows for a simple fulfillment of a stress-free reference configuration, i.e.,

$$\begin{aligned} \mathbf{S}|_{C=1} = 2\partial_{\mathbf{C}}\psi_1|_{C=1} &= \sum_{r=1}^n \sum_{j=1}^m 2\xi_{rj} \left[ \left( -g_j + \frac{1}{(g_j)^{\beta_{rj}}} g_j^{\beta_{rj}+1} \right) \mathbf{1} + \right. \\ &\quad \left. \left( \frac{1}{(g_j)^{\alpha_{rj}}} g_j^{\alpha_{rj}} - \frac{1}{(g_j)^{\beta_{rj}}} g_j^{\beta_{rj}} \right) \mathbf{G}_j \right] = \mathbf{0}. \end{aligned} \quad (30)$$

For the description of the symmetry groups  $\mathcal{C}_4$ ,  $\mathcal{C}_5$  and  $\mathcal{C}_7$ ,  $\mathcal{C}_8$ ,  $\mathcal{C}_9$  we choose the second energy function of (23)

$$\psi_2 = \psi^{iso} + \psi_2^{aniso}, \quad (31)$$

which consists of the anisotropic part

$$\psi_2^{aniso} = \sum_{j=1}^m \left[ f_j(J_{6j}) + \tilde{f}_j(J_{7j}) + \bar{f}_j(J_{8j}) + \tilde{h}_j(I_3) \right], \quad (32)$$

with the polyconvex functions

$$\begin{aligned} f_j(J_{6j}) &= \sum_{r=1}^n \frac{\xi_{rj}}{(\alpha_{rj} + 1)m_j^{\alpha_{rj}}} J_{6j}^{(\alpha_{rj}+1)}, \\ \tilde{f}_j(J_{7j}) &= \sum_{r=1}^n \frac{3\xi_{rj}}{(\beta_{rj} + 1)m_j^{\beta_{rj}}} J_{7j}^{(\beta_{rj}+1)}, \\ \bar{f}_j(J_{8j}) &= \sum_{r=1}^n \frac{\xi_{rj}}{(\eta_{rj} + 1)m_j^{\eta_{rj}}} J_{8j}^{(\eta_{rj}+1)}, \\ \tilde{h}_j(I_3) &= \sum_{r=1}^n \xi_{rj} \frac{3m_j}{\gamma_{rj}} I_3^{-\gamma_{rj}}. \end{aligned} \quad (33)$$

Here the  $j$ -th invariants are given by

$$J_{6j} = \mathbf{C} : \mathbb{G}_j : \mathbf{1}, \quad J_{7j} = \text{Cof} \mathbf{C} : \mathbb{G}_j : \mathbf{1}, \quad J_{8j} = \mathbf{C} : \mathbb{G}_j : \mathbf{C}; \quad (34)$$

the abbreviation  $m_j := \mathbf{1} : \mathbb{G}_j : \mathbf{1}$  and the polyconvexity restrictions

$$\alpha_{rj}, \beta_{rj}, \eta_{rj}, \xi_{rj} \geq 0, \gamma_{rj} \geq -\frac{1}{2} \quad (35)$$

are taken into account. As an example, for the crystal class  $\mathcal{C}_5$  the  $j$ -th fourth-order structural tensor appears in the form

$$\mathbb{G}^{t2}_j = \text{diag}[a_j^4, a_j^4, c_j^4, 0, 0, 0], \quad \text{with } a_j, c_j \geq 0. \quad (36)$$

The second Piola-Kirchhoff stresses at natural state, which are here automatically equal to zero, are obtained by

$$\begin{aligned} \mathbf{S}|_{C=1} &= 2\partial_{\mathbf{C}}\psi_2|_{C=1} = \sum_{r=1}^n \sum_{j=1}^m 2\xi_{rj} \left[ (-3m_j + \frac{3}{(m_j)^{\beta_{rj}}} m_j^{\beta_{rj}+1}) \mathbf{1} \right. \\ &\quad \left. + (\frac{1}{(m_j)^{\alpha_{rj}}} m_j^{\alpha_{rj}} - \frac{3}{(m_j)^{\beta_{rj}}} m_j^{\beta_{rj}} + \frac{2}{(m_j)^{\eta_{rj}}} m_j^{\eta_{rj}}) (\mathbb{G}_j : \mathbf{1}) \right] = \mathbf{0}. \end{aligned} \quad (37)$$

## 4 Fitting to Referential Data

In order to show the capability of the polyconvex functions (23) to describe the phenomenological response of materials characterized by the presented symmetry groups, we adjust the underlying linearized fourth-order elasticity tensor near the reference state  $\mathbb{C}_0$ , with  $\mathbb{C}_0 := 4\partial_{\mathbf{C}}\psi|_{\mathbf{C}=\mathbf{1}}$ , to experimental measurements. The linearization of the stress response functions at a natural state, i.e.,

$$\text{Lin}[\mathbf{S}] = \mathbb{C}_0 : \text{Lin}[\mathbf{E}], \quad \text{with} \quad \mathbb{C}_0 := 2 \frac{\partial \mathbf{S}}{\partial \mathbf{C}} \Big|_{\mathbf{C}=\mathbf{1}}, \quad \mathbf{S}|_{\mathbf{C}=\mathbf{1}} = \mathbf{0} \quad (38)$$

and the Green-Lagrange strain tensor  $\mathbf{E} := \frac{1}{2}(\mathbf{C} - \mathbf{1})$ , reduces to

$$\boldsymbol{\sigma} = \mathbb{C}_0 : \boldsymbol{\varepsilon}, \quad (39)$$

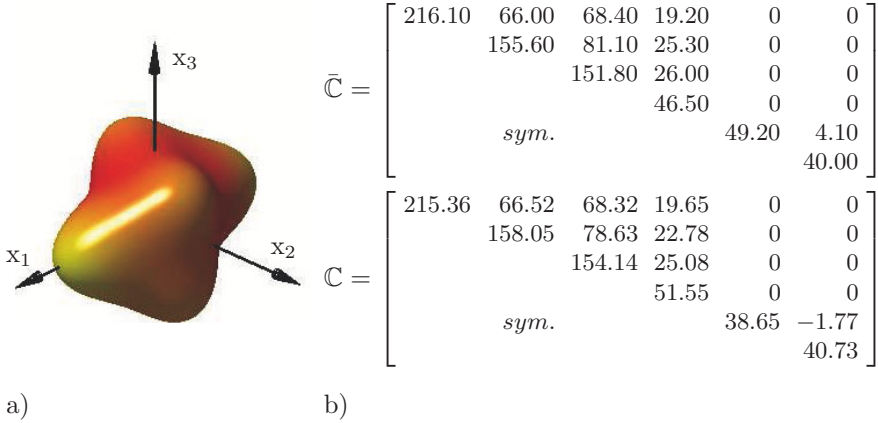
where the term  $\text{Lin}[\mathbf{S}]$  can be identified with the linear stress tensor  $\boldsymbol{\sigma}$  and the term  $\text{Lin}[\mathbf{E}]$  with the linear strain tensor  $\boldsymbol{\varepsilon}$  in the case of small strains; therefore, the linearized tangent moduli  $\mathbb{C}_0$  at the reference state is identical to the classical representation of the elasticity tensor in the small strain regime. For the approximations we use experimental data on monoclinic, rhombic, tetragonal, trigonal and cubic elasticities presented in Simmons and Wang (1971). In Schröder et al. (2008) and Ebbing et al. (2009) approximations of rhombic and monoclinic elasticity tensors using the polyconvex function (25) can be found. In Schröder et al. (2009) a trigonal elasticity tensor is adjusted by considering the polyconvex function (31). The adjustment of moduli is based on the minimization of the error function

$$e = \frac{\|\mathbb{C} - \bar{\mathbb{C}}\|}{\|\bar{\mathbb{C}}\|}. \quad (40)$$

$\mathbb{C} \in \mathbb{R}^{6 \times 6}$  denotes the computed tangent moduli  $\mathbb{C}_0$  in Voigt notation and  $\bar{\mathbb{C}} \in \mathbb{R}^{6 \times 6}$  is the associated coefficient scheme of experimental values. For the minimization of (40) we use the evolution strategy proposed by Schwefel (1996).

In the following the results of the fittings for materials of different anisotropy types are presented. Note that in all cases the adjustments are done by using the energy functions (23) with  $n = m = 3$ . In order to get a better understanding of the underlying anisotropic material behavior the characteristic surfaces of Young's moduli of the adjusted materials are depicted; in this context see Shuvalov (1988) and Böhlke and Brüggemann (2001). Furthermore, the explicit form of Young's Moduli and the errors of the adjustments as well as detailed results on the material parameters and the structural tensors are given.

**Monoclinic material, type  $\mathcal{C}_2$ :** First we approximate the elasticity tensor of the monoclinic material Aegirite-Augite. We consider the energy function (25) with  $n = m = 3$ , the anisotropic part (26) and  $\mathbf{G}_j^m$ , given by (28). The characteristic surface and representations of the experimentally determined and computed elasticity tensor can be found in Fig. 3. The



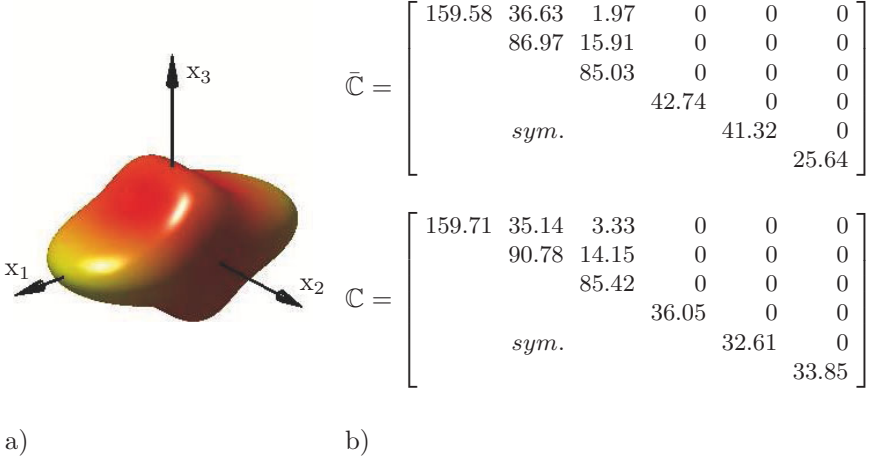
**Figure 3.** Aegirite-Augite: a) Characteristic surface of Young's moduli, b)  $\bar{\mathbb{C}}$  and  $\mathbb{C}$  in [GPa].

optimization of (40) with  $\mathbb{C} := 4\partial_{\mathbb{C}\mathbb{C}}\psi|_{\mathbb{C}=1}$  yields a relative error  $e$  of approximately 4.3 %. The optimized set of material parameters and resulting monoclinic structural tensors are shown in Table 1.

**Table 1.** Material parameter set of Aegirite-Augite

isotropic material parameters:						anisotropic material parameters:					
$\delta_1 = 2.08, \quad \delta_2 = 1.08, \quad \delta_3 = 0.23$						$r$	$j$	$\alpha_{rj}$	$\beta_{rj}$	$\gamma_{rj}$	$\xi_{rj}$
$\mathbf{G}_1 = \begin{bmatrix} 0.27 & 0.65 & 0 \\ 0.65 & 1.58 & 0 \\ 0 & 0 & 1.31 \end{bmatrix}$						1	1	0.89	0.17	-0.28	0.40
						2	1	0.90	0.09	-0.37	1.54
						3	1	0.87	0.73	-0.15	0.96
$\mathbf{G}_2 = \begin{bmatrix} 1.08 & -0.91 & 0 \\ -0.91 & 1.30 & 0 \\ 0 & 0 & 1.36 \end{bmatrix}$						1	2	0.00	2.95	-0.39	1.74
						2	2	-	-	-	-
						3	2	0.00	2.86	-0.40	1.20
$\mathbf{G}_3 = \begin{bmatrix} 2.46 & -0.04 & 0 \\ -0.04 & 0.001 & 0 \\ 0 & 0 & 0.000002 \end{bmatrix}$						1	3	2.95	0.29	-0.37	0.61
						2	3	2.45	0.32	-0.37	1.26
						3	3	2.44	0.66	-0.44	0.35

**Rhombic material, type  $\mathcal{C}_3$ :** For the approximation of the experimentally determined elasticities of Aragonite we use the energy function (25), where we set  $n = m = 3$  and take the anisotropic part (26) with the rhombic structural tensor  $\mathbf{G}_j^o$  given in the Appendix into account. The characteristic surface of Young's moduli reflects the typical rhombic symmetry properties described in section 2, see Fig. 4. Optimizing (40) leads here to an error



**Figure 4.** Aragonite: a) Characteristic surface of Young's moduli, b)  $\bar{\mathbb{C}}$  and  $\mathbb{C}$  in [GPa].

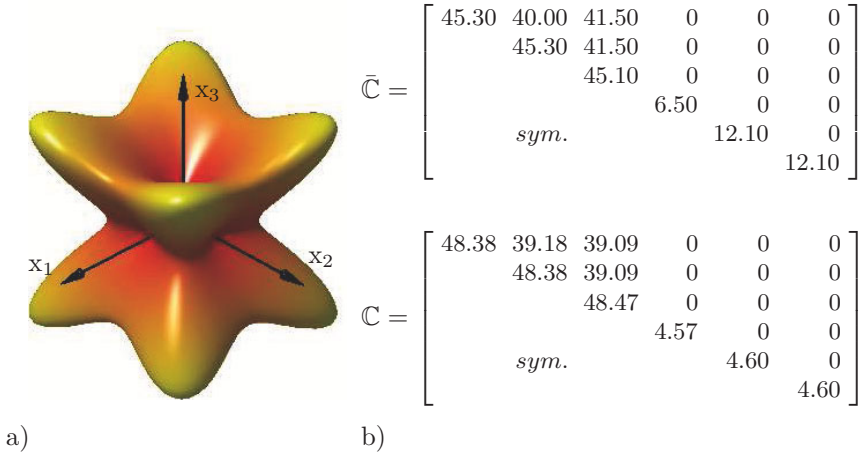
of  $e = 6.7\%$ . Detailed results on the computed elasticities and the material parameters can be found in Fig. 4 and in Table 2.

**Table 2.** Material parameter set of Aragonite

isotropic material parameters:				anisotropic material parameters:					
$\delta_1 = 2.80, \quad \delta_2 = 7 \cdot 10^{-4}, \quad \delta_3 = 0.03$				$r$	$j$	$\alpha_{rj}$	$\beta_{rj}$	$\gamma_{rj}$	$\xi_{rj}$
$\mathbf{G}_1 = \text{diag}(2.04, 0.83, 0.0000001)$				1	1	1.72	0.24	-0.47	0.20
				2	1	3.08	0.30	-0.50	1.64
				3	1	3.03	0.17	-0.50	2.04
$\mathbf{G}_2 = \text{diag}(0.000004, 0.97, 0.61)$				1	2	0.41	0.39	-0.50	0.04
				2	2	0.04	0.09	-0.50	0.75
				3	2	0.05	0.07	-0.44	0.01
$\mathbf{G}_3 = \text{diag}(0.0000002, 1.23, 2.13)$				1	3	1.58	1.40	-0.50	0.86
				2	3	0.71	1.22	-0.50	0.62
				3	3	0.95	0.65	-0.50	1.20



**Tetragonal material, type  $\mathcal{C}_5$ :** The elasticities of Indium are taken, see Fig. 6. For the computation of the elasticity tensor at the reference state  $\mathbb{C} = 4\psi^2 \bar{\mathbb{C}}|_{C=1}$  the energy function (31) with  $n = m = 3$ , the anisotropic part (32) and  $j$ -th tetragonal structural tensors  $\mathbb{G}_j^{t2}$  given in the Appendix are used. The characteristic surface of Young's moduli depicted



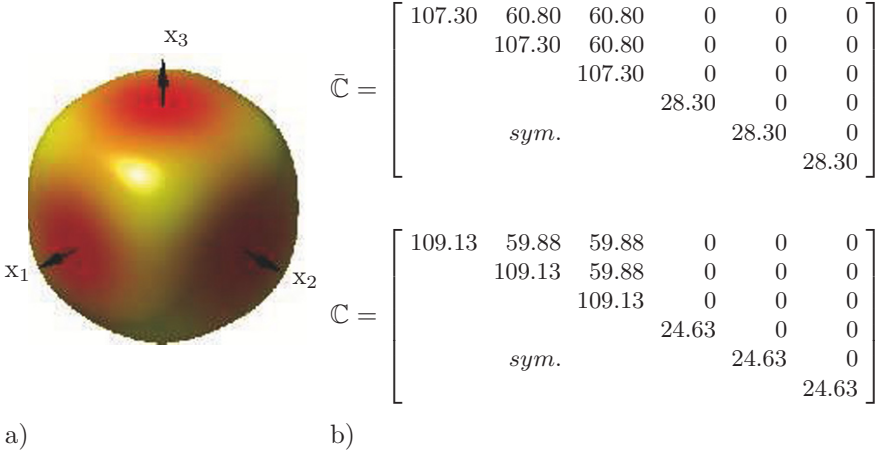
**Figure 6.** Indium: a) Characteristic surface of Young's moduli, b)  $\bar{\mathbb{C}}$  and  $\mathbb{C}$  in [GPa].

in Fig. 6 shows here the typical symmetry properties of a tetragonal material of  $\mathcal{C}_5$ -type. The optimization procedure yields the material parameters and three tetragonal structural tensors as listed in Table 4; the obtained error is  $e = 10.17\%$ .

**Table 4.** Material parameter set of Indium

isotropic material parameters:	anisotropic material parameters:						
$\delta_1 = 0.02, \quad \delta_2 = 2.23, \quad \delta_3 = 7.49$	$r$	$j$	$\alpha_{rj}$	$\beta_{rj}$	$\gamma_{rj}$	$\eta_{rj}$	$\xi_{rj}$
$\mathbb{G}_1 = \text{diag}(0.0002, 0.0002, 0.0001, 0, 0, 0)$	1	1	1.24	2.04	0.47	1.87	1.17
	2	1	1.61	3.55	0.89	0.98	0.28
	3	1	1.45	2.65	0.96	1.12	0.93
$\mathbb{G}_2 = \text{diag}(0.0006, 0.0006, 0.0024, 0, 0, 0)$	1	2	0.68	0.63	1.00	1.19	1.88
	2	2	2.82	2.03	0.18	0.30	1.11
	3	2	2.31	1.26	2.43	1.03	1.34
$\mathbb{G}_3 = \text{diag}(0.001, 0.001, 0.0002^2, 0, 0, 0)$	1	3	1.23	2.85	1.61	0.44	1.99
	2	3	2.13	2.48	0.74	2.13	0.36
	3	3	0.60	1.28	1.71	0.98	1.32

**Cubic material, type  $\mathcal{C}_7$ :** Fig. 7 depicts the experimental and optimized elasticity tensor of Aluminium as well as the characteristic surface of Young's moduli reflecting the cubic symmetry properties of Aluminium. For the approximation of the elasticity tensor the energy function (31) with  $n = m = 3$ , the anisotropic part (32) and the  $j$ -th cubic structural tensors given in the Appendix are taken into account. The corresponding cubic ma-



**Figure 7.** Aluminium: a) Characteristic surface of Young's moduli, b)  $\bar{\mathbb{C}}$  and  $\mathbb{C}$  in [GPa].

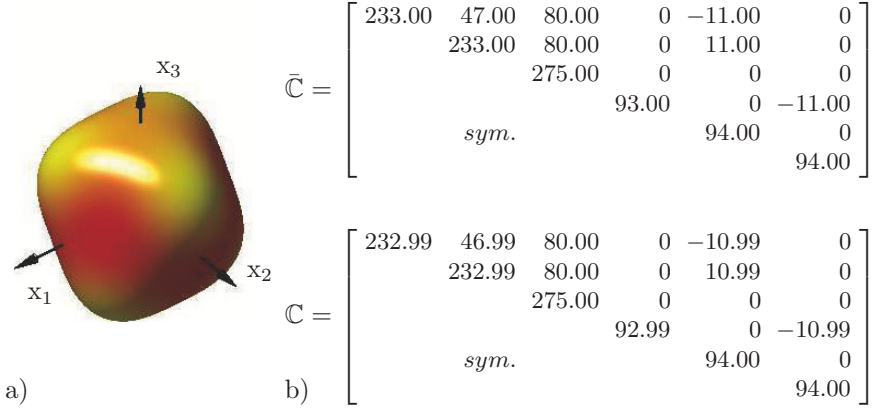
terial parameters and resulting three cubic structural tensors can be found in Table 5. From the optimization of (40) we obtain a relative error of approximately 3.07%.

**Table 5.** Material parameter set of Aluminium

isotropic material parameters:	anisotropic material parameters:						
$\delta_1 = 4.76, \delta_2 = 7.56, \delta_3 = 7.41$	$r$	$j$	$\alpha_{rj}$	$\beta_{rj}$	$\gamma_{rj}$	$\eta_{rj}$	$\xi_{rj}$
$\mathbb{G}_1 = \text{diag}(0.0065^4, 0.0065^4, 0.0065^4, 0, 0, 0)$	1	1	0.07	2.17	2.54	1.46	1.14
	2	1	1.26	1.55	0.50	1.52	1.00
	3	1	1.02	2.32	2.58	0.25	0.37
$\mathbb{G}_2 = \text{diag}(0.0024^4, 0.0024^4, 0.0024^4, 0, 0, 0)$	1	2	2.29	2.58	3.34	2.59	1.31
	2	2	1.78	3.51	1.17	0.91	0.16
	3	2	0.97	4.15	2.46	1.51	2.14
$\mathbb{G}_3 = \text{diag}(0.0012^4, 0.0012^4, 0.0012^4, 0, 0, 0)$	1	3	1.94	2.20	1.81	0.88	0.29
	2	3	2.43	1.65	2.59	1.94	0.59
	3	3	1.61	3.10	2.96	2.68	0.46



**Trigonal material, type  $\mathcal{C}_9$ :** The elasticity tensor at the reference state is here computed under consideration of the energy function (31) with the anisotropic part (32) and the trigonal fourth-order structural tensor  $\mathbb{G}^{ht2}$ . A comparison of the computed and experimentally determined elasticity



**Figure 8.** Lithium tantalate: a) Characteristic surface of Young's moduli, b)  $\bar{C}$  and  $C$  in [GPa].

tensors of Lithium tantalate, see Fig. 8, shows that this trigonal material can be described using the energy function (31), the error is 0.0021%. For more details on the material parameters see Table 6, with  $z = 10^{-2}$ .

**Table 6.** Material parameter set of Lithium tantalate

isotropic material parameters:						
$\delta_1 = 1.32, \quad \delta_2 = 0.14, \quad \delta_3 = 0.11$						
anisotropic material parameters:						
$r$	$j$	$\alpha_{rj}$	$\beta_{rj}$	$\gamma_{rj}$	$\eta_{rj}$	$\xi_{rj}$
1	1	0.12	0.59	-0.42	0.01	1.26
2	1	0.64	1.16	-0.48	0.13	0.15
3	1	0.00	0.74	-0.49	0.04	2.26
1	2	1.90	0.04	-0.23	1.66	1.79
2	2	0.81	0.57	-2.39	1.21	0.15
3	2	2.92	0.06	-0.37	1.71	0.91
1	3	2.18	0.05	-0.36	1.98	2.30
2	3	0.77	1.16	0.42	1.93	0.09
3	3	1.34	0.22	-0.24	1.98	0.90
relative error:						
$e = 0.0021 \%$						

$\mathbb{G}_1 =$	$\begin{bmatrix} 0.90 & 0.30 & 0.40 & 0 & -0.35 & 0 \\ & 0.90 & 0.40 & 0 & 0.35 & 0 \\ & & 0.27 & 0 & 0 & 0 \\ & & & 0.30 & 0 & -0.35 \\ sym. & & & & 0.40 & 0 \\ & & & & & 0.40 \end{bmatrix}$
$\mathbb{G}_2 =$	$\begin{bmatrix} 2z & 0.7z & 5.1z & 0 & 1.8z & 0 \\ & 2z & 5.1z & 0 & -1.8z & 0 \\ & & 20z & 0 & 0 & 0 \\ & & & 0.65z & 0 & 1.8z \\ sym. & & & & 5.1z & 0 \\ & & & & & 5.1z \end{bmatrix}$
$\mathbb{G}_3 =$	$\begin{bmatrix} 7z & 2.3z & 8.8z & 0 & -4.5z & 0 \\ & 7z & 8.8z & 0 & 4.5z & 0 \\ & & 16.7z & 0 & 0 & 0 \\ & & & 2.35z & 0 & -4.5z \\ sym. & & & & 8.8z & 0 \\ & & & & & 8.8z \end{bmatrix}$

## 5 Conclusion

In this contribution we have presented a framework for the construction of polyconvex energy functions suitable for the description of triclinic, monoclinic, rhombic, trigonal, tetragonal, cubic and transversely isotropic hyperelastic materials. For this we have used polyconvex polynomial invariants in terms of the right Cauchy-Green tensor and second-order and fourth-order structural tensors, respectively. For the construction of the structural tensor the base system of the underlying crystal class has been taken into account. Representative examples have been discussed in order to show the capability of the proposed energy function for the fitting of real anisotropic material behavior.

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## Appendix

Triclinic system, group $\mathcal{C}_1$ :	
$\mathbf{a}_1 = (a, 0, 0)^T, \quad \mathbf{a}_2 = (b \cos \gamma, b \sin \gamma, 0)^T, \quad \mathbf{a}_3 = (c \cos \beta, X, Y)^T$	
$\mathbf{G}^a = \begin{bmatrix} \tilde{a} & \tilde{d} & \tilde{e} \\ \tilde{d} & \tilde{b} & \tilde{f} \\ \tilde{e} & \tilde{f} & \tilde{c} \end{bmatrix},$	
<p>with <math>\tilde{a} = a^2 + b^2 \cos^2 \gamma + c^2 \cos^2 \beta</math></p>	
$\tilde{b} = b^2 \sin^2 \gamma + \frac{c^2 (\cos \alpha - \cos \beta \cos \gamma)^2}{\sin^2 \gamma}$	
$\tilde{c} = \frac{c^2 (1 + 2 \cos \alpha \cos \beta \cos \gamma - \cos^2 \alpha - \cos^2 \beta - \cos^2 \gamma)}{\sin^2 \gamma}$	
$\tilde{d} = b^2 \cos \gamma \sin \gamma + \frac{c^2 \cos \beta (\cos \alpha - \cos \beta \cos \gamma)}{\sin \gamma}$	
$\tilde{e} = \frac{c^2 \cos \beta (1 + 2 \cos \alpha \cos \beta \cos \gamma - \cos^2 \alpha - \cos^2 \beta - \cos^2 \gamma)^{1/2}}{\sin \gamma}$	
$\tilde{f} = \frac{c^2 (\cos \alpha - \cos \beta \cos \gamma)}{\sin \gamma}$	
$\cdot \frac{(1 + 2 \cos \alpha \cos \beta \cos \gamma - \cos^2 \alpha - \cos^2 \beta - \cos^2 \gamma)^{1/2}}{\sin \gamma}$	
$X = c (\cos \alpha - \cos \beta \cos \gamma) / \sin \gamma$	
$Y = c [1 + 2 \cos \alpha \cos \beta \cos \gamma - (\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma)]^{1/2} / \sin \gamma$	
Monoclinic system, group $\mathcal{C}_2$ :	
$\mathbf{a}_1 = (a, 0, 0)^T, \quad \mathbf{a}_2 = (b \cos \gamma, b \sin \gamma, 0)^T, \quad \mathbf{a}_3 = (0, 0, c)^T$	
$\mathbf{G}^m = \begin{bmatrix} a^2 + b^2 \cos^2 \gamma & b^2 \cos \gamma \sin \gamma & 0 \\ b^2 \cos \gamma \sin \gamma & b^2 \sin^2 \gamma & 0 \\ 0 & 0 & c^2 \end{bmatrix}$	

Rhombic system, group $\mathcal{C}_3$ :					
$\mathbf{a}_1 = (a, 0, 0)^T, \quad \mathbf{a}_2 = (0, b, 0)^T, \quad \mathbf{a}_3 = (0, 0, c)^T$ $\mathbf{G}^o = \begin{bmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{bmatrix}$					
Tetragonal system, group $\mathcal{C}_4$ :					
$\mathbf{a}_1 = (\bar{a}, \bar{b}, 0)^T, \quad \mathbf{a}_2 = (-\bar{b}, \bar{a}, 0)^T, \quad \mathbf{a}_3 = (0, 0, c)^T$ $\mathbb{G}^{t1} = \begin{bmatrix} \bar{a}^4 + \bar{b}^4 & 2\bar{a}^2\bar{b}^2 & 0 & \bar{a}^3\bar{b} - \bar{b}^3\bar{a} & 0 & 0 \\ & \bar{a}^4 + \bar{b}^4 & 0 & \bar{b}^3\bar{a} - \bar{a}^3\bar{b} & 0 & 0 \\ & & c^4 & 0 & 0 & 0 \\ & & & sym. & 2\bar{a}^2\bar{b}^2 & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{bmatrix}$					
Tetragonal system, group $\mathcal{C}_5$ :					
$\mathbf{a}_1 = (a, 0, 0)^T, \quad \mathbf{a}_2 = (0, a, 0)^T, \quad \mathbf{a}_3 = (0, 0, c)^T$ $\mathbb{G}^{t2} = \begin{bmatrix} a^4 & 0 & 0 & 0 & 0 & 0 \\ & a^4 & 0 & 0 & 0 & 0 \\ & & c^4 & 0 & 0 & 0 \\ & & & sym. & 0 & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{bmatrix}$					

Trigonal system, group $\mathcal{C}_8$ :					
$\mathbf{a}_1 = \begin{pmatrix} \frac{\bar{a}}{\sqrt{3}} \\ -\frac{\bar{b}}{\sqrt{3}} \\ \frac{\bar{c}}{3} \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} -\frac{\bar{a}}{2\sqrt{3}} + \frac{\bar{b}}{2} \\ \frac{\bar{b}}{2\sqrt{3}} + \frac{\bar{a}}{2} \\ \frac{\bar{c}}{3} \end{pmatrix}, \quad \mathbf{a}_3 = \begin{pmatrix} -\frac{\bar{a}}{2\sqrt{3}} - \frac{\bar{b}}{2} \\ \frac{\bar{b}}{2\sqrt{3}} - \frac{\bar{a}}{2} \\ \frac{\bar{c}}{3} \end{pmatrix}$					
$\mathbb{G}^{ht1} = \begin{bmatrix} \frac{1}{8} C^2 & \frac{1}{24} C^2 & \frac{1}{18} C \bar{c}^2 & 0 & A & B \\ & \frac{1}{8} C^2 & \frac{1}{18} C \bar{c}^2 & 0 & -A & -B \\ & & \frac{1}{27} \bar{c}^4 & 0 & 0 & 0 \\ & sym. & & \frac{1}{24} C^2 & -B & A \\ & & & & \frac{1}{18} C \bar{c}^2 & 0 \\ & & & & & \frac{1}{18} C \bar{c}^2 \end{bmatrix}$					
with $A = -\frac{1}{4\sqrt{3}}\bar{a}^2\bar{b}\bar{c} + \frac{1}{12\sqrt{3}}\bar{b}^3\bar{c}$ , $B = -\frac{1}{4\sqrt{3}}\bar{a}\bar{b}^2\bar{c} + \frac{1}{12\sqrt{3}}\bar{a}^3\bar{c}$ , $C = \bar{a}^2 + \bar{b}^2$					
Trigonal system, group $\mathcal{C}_9$ :					
$\mathbf{a}_1 = (0, -\bar{b}/\sqrt{3}, \bar{c}/3)^T, \quad \mathbf{a}_2 = (\bar{b}/2, \bar{b}/(2\sqrt{3}), \bar{c}/3)^T, \quad \mathbf{a}_3 = (-\bar{b}/2, \bar{b}/(2\sqrt{3}), \bar{c}/3)^T$					
$\mathbb{G}^{ht2} = \begin{bmatrix} \frac{1}{8} \bar{b}^4 & \frac{1}{24} \bar{b}^4 & \frac{1}{18} \bar{b}^2 \bar{c}^2 & 0 & \frac{1}{12\sqrt{3}} \bar{b}^3 \bar{c} & 0 \\ & \frac{1}{8} \bar{b}^4 & \frac{1}{18} \bar{b}^2 \bar{c}^2 & 0 & -\frac{1}{12\sqrt{3}} \bar{b}^3 \bar{c} & 0 \\ & & \frac{1}{27} \bar{c}^4 & 0 & 0 & 0 \\ & sym. & & \frac{1}{24} \bar{b}^4 & 0 & \frac{1}{12\sqrt{3}} \bar{b}^3 \bar{c} \\ & & & & \frac{1}{18} \bar{b}^2 \bar{c}^2 & 0 \\ & & & & & \frac{1}{18} \bar{b}^2 \bar{c}^2 \end{bmatrix}$					
Cubic system, group $\mathcal{C}_7$ :					
$\mathbf{a}_1 = (a, 0, 0)^T, \quad \mathbf{a}_2 = (0, a, 0)^T, \quad \mathbf{a}_3 = (0, 0, a)^T$					
$\mathbb{G}^c = \begin{bmatrix} a^4 & 0 & 0 & 0 & 0 & 0 \\ & a^4 & 0 & 0 & 0 & 0 \\ & & a^4 & 0 & 0 & 0 \\ & sym. & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{bmatrix}$					