Γ-convergence for a geometrically exact Cosserat shell-model of defective elastic crystals.

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Abstract I consider the Γ-limit to a three-dimensional Cosserat model as the aspect ratio $h > 0$ of a flat domain tends to zero. The bulk model involves already exact rotations as a second independent field intended to describe the rotations of the lattice in defective elastic crystals. The Γ-limit based on the natural scaling consists of a membrane like energy and a transverse shear energy both scaling with h, augmented by a curvature energy due to the Cosserat bulk, also scaling with h. A technical difficulty is to establish equi-coercivity of the sequence of functionals as the aspect ratio h tends to zero. Usually, equi-coercivity follows from a local coerciveness assumption. While the three-dimensional problem is well-posed for the Cosserat couple modulus $\mu_c \geq 0$, equi-coercivity needs a strictly positive $\mu_c > 0$. Then the Γ-limit model determines the midsurface deformation $m \in H^{1,2}(\omega, \mathbb{R}^3)$. For the true defective crystal case, however, $\mu_c = 0$ is appropriate. Without equi-coercivity, we obtain first an estimate of the $\Gamma - \liminf$ and $\Gamma - \limsup$ which can be strengthened to the Γ-convergence result. The Reissner-Mindlin model is "almost" the linearization of the Γ-limit for $\mu_c = 0$.

1 Introduction

1.1 Aspects of shell theory

The dimensional reduction of a given continuum-mechanical model is already an old subject has seen many "solutions". One possible way to proceed is the so called **d**erivation approach, i.e., reducing a given three-dimensional model via physically reasonable constitutive assumptions on the kinematics to a two-dimensional model. This is opposed to either the **i**ntrinsic approach which views the shell from the onset as a two-dimensional surface and invokes concepts from differential geometry or the **a**symptotic

methods which try to establish two-dimensional equations by formal expansion of the three-dimensional solution in power series in terms of a small nondimensional thickness parameter, the aspect ratio h. The intrinsic approach is closely related to the **d**irect approach which takes the shell to be a two-dimensional medium with additional **e**xtrinsic directors in the sense of a **r**estricted Cosserat surface Cosserat and Cosserat (1909).[1] There, two-dimensional equilibrium in appropriate new resultant stress and strain variables is postulated ab-initio more or less independent of three-dimensional considerations, cf. Antman (1995); Green et al. (1965); Ericksen and Truesdell (1958); Cohen and DeSilva (1966a,b); Cohen and Wang (1989); Rubin (2000).

A comprehensive presentation of the different approaches in classical shell theories can be found in the monograph Naghdi (1972). A thorough mathematical analysis of linear, infinitesimal-displacement shell theory, based on asymptotic methods is to be found in Ciarlet (1998) and the extensive references therein, see also Ciarlet (1997, 1999); Antman (1995); Destuynder and Salaun (1996); Dikmen (1982); Genevey (2000); Anzellotti et al. (1994). Excellent reviews of the modelling and finite element implementation may be found in Sansour and Bufler (1992); Sansour (1995); Sansour and Bocko (1998); Gruttmann et al. (1989); Gruttmann and Taylor (1992); Wriggers and Gruttmann (1993); Betsch et al. (1996); Büchter and Ramm (1992) and in the series of papers Simo and Fox (1989); Simo et al. (1989, 1990a,b); Simo and Kennedy (1992); Simo and Fox (1992). Properly invariant, geometrically exact, elastic plate theories are derived by formal asymptotic methods in Fox et al. (1993). This formal derivation is extended to curvilinear shells in Miara (1998); Lods and Miara (1998). Apart from the pure bending case Friesecke et al. (2002a, 2003), which is justified as the Γ-limit of the three-dimensional model for $h \to 0$ and which can be shown to be intrinsically well-posed, the obtained finite-strain models have not yet been shown to be well-posed. Indeed, the membrane energy contribution is notoriously not Legendre-Hadamard elliptic. The different membrane model formally justified in Le Dret and Raoult (1996) by Γ-convergence is geometrically exact and automatically quasiconvex/elliptic but unfortunately does not coincide upon linearization with the otherwise well-established infinitesimal-displacement membrane model. Moreover, this model does not describe the detailed geometry of deformation in compression but reduces to a tension-field theory Steigmann (1990). The quasiconvexification

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[1]Restricted, since no material length scale enters the direct approach, only the nondimensional **a**spect ratio h appears in the model. In terminology it is useful to distinguish between a "true" Cosserat model operating on SO(3) and theories with any number of directors.

step in Le Dret and Raoult (1996) appears since the membrane energy takes then into account the energy reducing effect of possible fine scale oscillations (wrinkles). The development of Le Dret and Raoult (1996) has been generalized to Young-measures in Freddi and Paroni (2004). A hierarchy of limiting theories based on Γ-convergence, distinguished by different scaling-exponents of the energy as a function of the aspect ratio h is developed in Friesecke et al. (2006). There the different scaling exponents can be controlled by scaling assumptions on the applied forces.

It is possible to include interfacial energy (here a second derivative term $\kappa \|D^2\varphi\|^2$ in the bulk energy) in the description of the material. The Γ-limit for constant κ has been investigated in Bhattacharya and James (1999) in an application to thin martensitic films. As a result, no quasiconvexification step is necessary (the higher derivative excludes arbitrary fine scale wrinkles) and in the limit one independent "Cosserat-director" appears. If simultaneously $\kappa \to 0$ faster than $h \to 0$, then the Γ-limit coincides (Shuh, 2000, Rem.5) with that of Le Dret and Raoult (1996). In our context (see below), including such an interfacial energy is tantamount to setting $\mu_c = \infty$ in the Cosserat bulk model, i.e. the Cosserat bulk model would degenerate into a second gradient model.

There are numerous proposals in the engineering literature for a finite-strain, geometrically exact plate formulation, see e.g. Fox and Simo (1992); Sansour and Bufler (1992); Sansour and Bednarczyk (1995); Sansour and Bocko (1998); Wriggers and Gruttmann (1993); Betsch et al. (1996); Büchter and Ramm (1992). These models are based on the Reissner-Mindlin kinematical assumption which is a variant of the direct approach; usually one independent director vector appears in the model. In many cases the need has been felt to devote attention to rotations $R \in \mathrm{SO}(3)$, since rotations are the dominant deformation mode of a thin flexible structure. This has led to the drill-rotation formulation which means that proper rotations either appear in the formulation as independent fields (leading to a restricted Cosserat surface) or they are an intermediary ingredient in the numerical treatment (constraint Cosserat surface, only continuum rotations matter finally). While the computational merit of this approach is well documented, such models lacked any asymptotic basis.

1.2   Outline of this contribution

In Neff (2004b) the author has proposed a Cosserat shell model for materials with rotational microstructure. In the underlying Cosserat-bulk model the Cosserat-rotation $\overline{R}$ and the gradients of $\overline{R}$ enter into the measure of deformation of the body. In fact, variation of $\overline{R}$ leads to a balance of

substructural interaction Capriz (1989). These gradients account therefore for the presence of interfaces between substructural units in a smeared sense. One may think of, e.g., liquid crystals, defective single crystals or metallic foams Neff (2006c); Neff and Forest (2007).

Assuming a strict principle of scale separation rules out the possibility of a direct comparison between macroscopic quantities (the usual deformation) and the microscopic ones (for example the lattice vectors in a defective crystal) and makes it plausible to assume that they behave independently of each other. For definiteness, we may view the Cosserat-rotations $\overline{R}$ as averaged lattice rotations, independent of the macroscopic rotation.[2] It can be shown that the Cosserat-rotation follows closely the macroscopic rotation in the bulk model provided that a constitutive parameter, the Cosserat couple modulus μ_c, is strictly positive. Therefore, the interesting case with independent microstructure is represented by $\mu_c = 0$. In this case, the amount of incompatibility of the lattice rotations, measured through $\mathrm{Curl}\,\overline{R}$, decisively influences the elastic response of the material and elastic coercivity can only be established for a reasonably smooth distribution of incompatibilities and defects. Every real pure single crystal contains still a massive amount of defects and incompatibilities. Thus, giving up the idealization of a defect free single crystal adds to the physical realism of the model. Let us henceforth refer to $\mu_c = 0$ as defective elastic crystal case.

The above mentioned shell model is shown to be well-posed in Neff (2004b) for the case $\mu_c > 0$ and in Neff (2007) for the case $\mu_c = 0$. Apart for technical details, this Cosserat shell model includes the generalized drill-rotation formulations alluded to above. Notably for $\mu_c = 0$, the in-plane drill-energy is absent in conformity with the classical Reissner-Mindlin model.

The formal derivation of the new shell model Neff (2004b), based on an asymptotic ansatz for a Cosserat bulk model with kinematical and physical assumptions appropriate for thin structures, however, still gives rise to questions as far as the asymptotic correctness and convergence is concerned. In this paper we address this point by showing, that the Γ-limit of the Cosserat bulk model for $h \to 0$ (under natural scaling assumptions) is, after descaling, given by the corresponding formal derivation, provided the energy contributions scaling with h are retained and the coefficient of the transverse shear energy is slightly modified. Given that the information provided by the Γ-limit hinges also on scaling assumptions, we think that

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[2] Compare with Zaafarani et al. (2006), where it is observed that lattice rotations are, in fact, independent of the macroscopic rotations in nano-indent single crystal copper experiments.

this result is a justification of the formal derivation in Neff (2004b) and the employed kinematical ansatz.

Central to our development is therefore the notion of Γ-convergence, a powerful theory originally initiated by De Giorgi Giorgi (1975, 1977) and especially suited for a variational framework on which in turn the numerical treatment with finite elements is based. This approach has thus far provided the only known convergence theorems for justifying lower dimensional nonlinear, frame-indifferent theories of elastic bodies.

Now, after presenting the notation, we recall in Section 2 the underlying "parent" three-dimensional finite-strain frame-indifferent Cosserat model with rotational substructure embodied by the **C**osserat rotations $\overline{R}$, i.e., a **t**riad of rigid directors $(\overline{R}_1|\overline{R}_2|\overline{R}_3) = \overline{R} \in \mathrm{SO}(3)$ and provide the existence results for this bulk model. Then we perform in Section 3 the transformation of the bulk model in physical space to a nondimensional thin domain and introduce the further scaling to a fixed reference domain Ω_1 with constant thickness on which the Γ-convergence procedure is finally based.

In Section 4 we recapitulate some points from Γ-convergence theory and introduce the Γ-limit for the rescaled formulation with respect to the two independent fields $(\varphi, \overline{R})$ of deformations and microrotations in Section 5. Two limit cases, $\mu_c = 0$ and $\mu_c = \infty$ deserve additional attention. Following, we provide the proof for the Γ-convergence results. First, for the simple case $\mu_c > 0$ in Section 6 similar to the development in Le Dret and Raoult (1996) and then for the case of defective elastic crystals $\mu_c = 0$ in Section 7. The case $\mu_c = \infty$ will be dealt with rigorously in a separate contribution. Our geometrically exact results have been first announced in Neff and Chełmiński (10/2004); Neff (2005, 2006b). In the meantime, the geometrically linear case for $\mu_c > 0$ has been treated by Aganovic et al. (2007a,b).

1.3 Notation

Notation for bulk material Let $\Omega \subset \mathbb{R}^3$ always be a bounded open domain with Lipschitz boundary $\partial\Omega$ and let Γ be a smooth subset of $\partial\Omega$ with non-vanishing 2-dimensional Hausdorff measure. For $a, b \in \mathbb{R}^3$ we let $\langle a, b \rangle_{\mathbb{R}^3}$ denote the scalar product on $\mathbb{R}^3$ with associated vector norm $\|a\|_{\mathbb{R}^3}^2 = \langle a, a \rangle_{\mathbb{R}^3}$. We denote by $\mathbb{M}^{3 \times 3}$ the set of real 3×3 second order tensors, written with capital letters. The standard Euclidean scalar product on $\mathbb{M}^{3 \times 3}$ is given by $\langle X, Y \rangle_{\mathbb{M}^{3 \times 3}} = \mathrm{tr}\left[XY^T \right]$, and the Frobenius tensor norm is $\|X\|^2 = \langle X, X \rangle_{\mathbb{M}^{3 \times 3}}$. In the following we omit the index $\mathbb{R}^3, \mathbb{M}^{3 \times 3}$. The identity tensor on $\mathbb{M}^{3 \times 3}$ will be denoted by $\mathbb{1}$, so that $\mathrm{tr}\left[X \right] = \langle X, \mathbb{1} \rangle$ and $\mathrm{tr}\left[X \right]^2 = \langle X, \mathbb{1} \rangle^2$. We let Sym and PSym denote the symmetric and

positive definite symmetric tensors respectively. We adopt the usual abbreviations of Lie-group theory, i.e., $\mathrm{GL}(3) := \{X \in \mathbb{M}^{3\times3} \,|\det[X] \neq 0\}$ the general linear group, $\mathrm{O}(3) := \{X \in \mathrm{GL}(3) \mid X^TX = \mathbb{1}\}$, $\mathrm{SO}(3) := \{X \in \mathrm{GL}(3) \,|X^TX = \mathbb{1},\ \det[X] = 1\}$ with corresponding Lie-algebra $\mathfrak{so}(3) := \{X \in \mathbb{M}^{3\times3} \,|X^T = -X\}$ of skew symmetric tensors. With $\mathrm{Adj}\,X$ we denote the tensor of transposed cofactors $\mathrm{Cof}(X)$ such that $\mathrm{Adj}\,X = \det[X]\,X^{-1} = \mathrm{Cof}(X)^T$ if $X \in \mathrm{GL}(3)$. We set $\mathrm{sym}(X) = \frac{1}{2}(X^T + X)$ and $\mathrm{skew}(X) = \frac{1}{2}(X - X^T)$ such that $X = \mathrm{sym}(X) + \mathrm{skew}(X)$. For vectors $\xi, \eta \in \mathbb{R}^n$ we have the tensor product $(\xi \otimes \eta)_{ij} = \xi_i\,\eta_j$. We write the polar decomposition in the form $F = RU = \mathrm{polar}(F)\,U$ with $R = \mathrm{polar}(F)$ the orthogonal part of F. For a second order tensor X we define the third order tensor $\mathfrak{h} = \mathrm{D}_x X(x) = (\nabla(X(x).e_1), \nabla(X(x).e_2), \nabla(X(x).e_3)) = (\mathfrak{h}^1, \mathfrak{h}^2, \mathfrak{h}^3) \in \mathbb{M}^{3\times3} \times \mathbb{M}^{3\times3} \times \mathbb{M}^{3\times3} \cong \mathfrak{T}(3)$. For third order tensors $\mathfrak{h} \in \mathfrak{T}(3)$ we set $\|\mathfrak{h}\|^2 = \sum_{i=1}^3 \|\mathfrak{h}^i\|^2$ together with $\mathrm{sym}(\mathfrak{h}) := (\mathrm{sym}\,\mathfrak{h}^1, \mathrm{sym}\,\mathfrak{h}^2, \mathrm{sym}\,\mathfrak{h}^3)$ and $\mathrm{tr}\,[\mathfrak{h}] := (\mathrm{tr}\,[\mathfrak{h}^1], \mathrm{tr}\,[\mathfrak{h}^2], \mathrm{tr}\,[\mathfrak{h}^3]) \in \mathbb{R}^3$. Moreover, for any second order tensor X we define $X \cdot \mathfrak{h} := (X\mathfrak{h}^1, X\mathfrak{h}^2, X\mathfrak{h}^3)$ and $\mathfrak{h} \cdot X$, correspondingly. Quantities with a bar, e.g. the micropolar rotation $\overline{R}$, represent the micropolar replacement of the corresponding classical continuum rotation R. For the deformation $\varphi \in C^1(\overline{\Omega}, \mathbb{R}^3)$ we have the deformation gradient $F = \nabla\varphi \in C(\overline{\Omega}, \mathbb{M}^{3\times3})$. $S_1(F) = D_F W(F)$ and $S_2(F) = F^{-1} D_F W(F)$ denote the first and second Piola Kirchhoff stress tensors. The first and second differential of a scalar valued function $W(F)$ are written $D_F W(F).H$ and $D_F^2 W(F).(H, H)$. We employ the standard notation of Sobolev spaces, i.e. $L^2(\Omega), H^{1,2}(\Omega), H_\circ^{1,2}(\Omega), W^{1,q}(\Omega)$, which we use indifferently for scalar-valued functions as well as for vector-valued and tensor-valued functions. The set $W^{1,q}(\Omega, \mathrm{SO}(3))$ denotes orthogonal tensors whose components are in $W^{1,q}(\Omega)$. Moreover, we set $\|X\|_\infty = \sup_{x\in\Omega} \|X(x)\|$. By $C_0^\infty(\Omega)$ we denote infinitely differentiable functions with compact support in Ω. We use capital letters to denote possibly large positive constants, e.g. C^+, K and lower case letters to denote possibly small positive constants, e.g. c^+, d^+.

Notation for plates and shells   Let $\omega \subset \mathbb{R}^2$ always be a bounded open domain with Lipschitz boundary $\partial\omega$ and let γ_0 be a smooth subset of $\partial\omega$ with non-vanishing 1-dimensional Hausdorff measure. The aspect ratio of the plate is $h > 0$. We denote by $\mathbb{M}^{m\times n}$ the set of matrices mapping $\mathbb{R}^n \mapsto \mathbb{R}^m$. For $H \in \mathbb{M}^{3\times2}$ and $\xi \in \mathbb{R}^3$ we write $(H|\xi) \in \mathbb{M}^{3\times3}$ for the matrix composed of H and the column ξ. Likewise $(v|\xi|\eta)$ is the matrix composed of the columns v, ξ, η. This allows us to write for $\varphi \in C^1(\mathbb{R}^3, \mathbb{R}^3)$: $\nabla\varphi = (\varphi_x|\varphi_y|\varphi_z) = (\partial_x\varphi|\partial_y\varphi|\partial_z\varphi)$. The identity tensor on $\mathbb{M}^{2\times2}$ is $\mathbb{1}_2$. The mapping $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ is the deformation of the midsurface, ∇m is the corresponding deformation gradient and $\vec{n}_m$ is the outer unit normal on

m. A matrix $X \in \mathbb{M}^{3 \times 3}$ can now be written as $X = (X.e_2 | X.e_2 | X.e_3) = (X_1 | X_2 | X_3)$. We write $v : \mathbb{R}^2 \mapsto \mathbb{R}^3$ for the displacement of the midsurface, such that $m(x,y) = (x,y,0)^T + v(x,y)$. The standard volume element is $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \mathrm{d}V = \mathrm{d}\omega\,\mathrm{d}z$.

2   The underlying three-dimensional Cosserat model

2.1   Problem statement in variational form

In Neff (2006a) a finite-strain, fully frame-indifferent, three-dimensional Cosserat micropolar model is introduced. The two-field problem has been posed in a variational setting. The task is to find a pair $(\varphi, \overline{R}) : \Omega \subset \mathbb{E}^3 \mapsto \mathbb{E}^3 \times \mathrm{SO}(3)$ of deformation φ and independent Cosserat-rotation $\overline{R} \in \mathrm{SO}(3)$, defined on the ambient physical space $\mathbb{E}^3$, minimizing the energy functional I,

$$I(\varphi, \overline{R}) = \int_{\Omega} W_{\mathrm{mp}}(\overline{R}^T \nabla \varphi) + W_{\mathrm{curv}}(\overline{R}^T \mathrm{D}_x \overline{R}) - \Pi_f(\varphi) - \Pi_M(\overline{R})\,\mathrm{d}V \qquad (1)$$

$$- \int_{\Gamma_S} \Pi_N(\varphi)\,\mathrm{d}S - \int_{\Gamma_C} \Pi_{M_c}(\overline{R})\,\mathrm{d}S \mapsto \min. \text{ w.r.t. } (\varphi, \overline{R}),$$

together with the Dirichlet boundary condition of place for the deformation φ on Γ: $\varphi_{|\Gamma} = g_{\mathrm{d}}$ and three possible **alternative** boundary conditions for the microrotations $\overline{R}$ on Γ,

$$\overline{R}_{|\Gamma} = \begin{cases} \overline{R}_{\mathrm{d}}, & \text{the case of \textbf{rigid} prescription}, \\ \mathrm{polar}(\nabla \varphi), & \text{the case of \textbf{strong} consistent coupling}, \\ \text{no condition for } \overline{R} \text{ on } \Gamma, & \textbf{N}\text{eumann-type relations for } \overline{R} \text{ on } \Gamma. \end{cases} \qquad (2)$$

The constitutive assumptions on the densities are

$$W_{\mathrm{mp}}(\overline{U}) = \mu \| \mathrm{sym}(\overline{U} - \mathbb{1}) \|^2 + \mu_c \| \mathrm{skew}(\overline{U}) \|^2 + \frac{\lambda}{2} \mathrm{tr} \left[\mathrm{sym}(\overline{U} - \mathbb{1}) \right]^2,$$

$$\overline{U} = \overline{R}^T F, \; F = \nabla \varphi, \qquad (3)$$

$$W_{\mathrm{curv}}(\mathfrak{K}) = \mu \frac{L_c^{1+p}}{12} \left(1 + \alpha_4 L_c^q \| \mathfrak{K} \|^q \right)$$

$$\left(\alpha_5 \| \mathrm{sym}\,\mathfrak{K} \|^2 + \alpha_6 \| \mathrm{skew}\,\mathfrak{K} \|^2 + \alpha_7 \mathrm{tr} \left[\mathfrak{K} \right]^2 \right)^{\frac{1+p}{2}},$$

$$\mathfrak{K} = \overline{R}^T \mathrm{D}_x \overline{R} := \left(\overline{R}^T \nabla(\overline{R}.e_1), \overline{R}^T \nabla(\overline{R}.e_2), \overline{R}^T \nabla(\overline{R}.e_3) \right),$$

the third order **c**urvature tensor,

under the minimal requirement $p \geq 1$, $q \geq 0$. The total elastically stored energy $W = W_{\mathrm{mp}} + W_{\mathrm{curv}}$ is quadratic in the stretch $\overline{U}$ and possibly super-quadratic in the curvature $\mathfrak{K}$. The strain energy W_{mp} depends on the deformation gradient $F = \nabla\varphi$ and the microrotations $\overline{R} \in \mathrm{SO}(3)$, which do not necessarily coincide with the **c**ontinuum rotations $R = \mathrm{polar}(F)$. The curvature energy W_{curv} depends moreover on the space derivatives $\mathrm{D}_x\overline{R}$ which describe the self-interaction of the microstructure.[3] In general, the **m**icropolar stretch tensor $\overline{U}$ is **n**ot symmetric and does not coincide with the **s**ymmetric continuum stretch tensor $U = R^T F = \sqrt{F^T F}$. By abuse of notation we set $\| \mathrm{sym}\,\mathfrak{K} \|^2 := \sum_{i=1}^{3} \| \mathrm{sym}\,\mathfrak{K}^i \|^2$ for third order tensors $\mathfrak{K}$, cf.(1.3).

Here $\Gamma \subset \partial\Omega$ is that part of the boundary, where Dirichlet conditions $g_{\mathrm{d}}, \overline{R}_{\mathrm{d}}$ for deformations and microrotations or coupling conditions for microrotations, are prescribed. $\Gamma_S \subset \partial\Omega$ is a part of the boundary, where traction boundary conditions in the form of the potential of applied surface forces Π_N are given with $\Gamma \cap \Gamma_S = \emptyset$. In addition, $\Gamma_C \subset \partial\Omega$ is the part of the boundary where the potential of external surface couples Π_{M_c} are applied with $\Gamma \cap \Gamma_C = \emptyset$. On the free boundary $\partial\Omega \setminus \{\Gamma \cup \Gamma_S \cup \Gamma_C\}$ corresponding natural boundary conditions for $(\varphi, \overline{R})$ apply. The potential of the external applied volume force is Π_f and Π_M takes on the role of the potential of applied external volume couples. For simplicity we assume

$$\begin{aligned}
\Pi_f(\varphi) &= \langle f, \varphi \rangle\,, \quad \Pi_M(\overline{R}) = \langle M, \overline{R} \rangle\,, \\
\Pi_N(\varphi) &= \langle N, \varphi \rangle\,, \quad \Pi_{M_c}(\overline{R}) = \langle M_c, \overline{R} \rangle\,,
\end{aligned} \tag{4}$$

for the potentials of applied loads with given functions $f \in L^2(\Omega, \mathbb{R}^3)$, $M \in L^2(\Omega, \mathbb{M}^{3\times 3})$, $N \in L^2(\Gamma_S, \mathbb{R}^3)$, $M_c \in L^2(\Gamma_C, \mathbb{M}^{3\times 3})$.

The parameters $\mu, \lambda > 0$ are the Lamé constants of classical isotropic elasticity, the additional parameter $\mu_c \geq 0$ is called the **C**osserat couple modulus. For $\mu_c > 0$ the elastic strain energy density $W_{\mathrm{mp}}(\overline{U})$ is **u**niformly convex in $\overline{U}$ and satisfies the **s**tandard growth assumption $\forall\, F \in \mathrm{GL}^+(3)$:

$$\begin{aligned}
W_{\mathrm{mp}}(\overline{U}) = W_{\mathrm{mp}}(\overline{R}^T F) &\geq \min(\mu, \mu_c)\, \|\overline{R}^T F - \mathbb{1}\|^2 = \min(\mu, \mu_c)\, \|F - \overline{R}\|^2 \\
&\geq \min(\mu, \mu_c) \inf_{R \in \mathrm{O}(3)} \|F - R\|^2 = \min(\mu, \mu_c)\, \mathrm{dist}^2(F, \mathrm{O}(3)) \\
&= \min(\mu, \mu_c)\, \mathrm{dist}^2(F, \mathrm{SO}(3)) = \min(\mu, \mu_c)\, \|F - \mathrm{polar}(F)\|^2 \\
&= \min(\mu, \mu_c)\, \|U - \mathbb{1}\|^2\,.
\end{aligned} \tag{5}$$

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[3]Observe that $\overline{R}^T \nabla(\overline{R}.e_i) \neq \overline{R}^T \partial_{x_i}\overline{R} \in \mathfrak{so}(3)$.

In contrast, for the interesting limit case of defective elastic crystals $\mu_c = 0$, where the Cosserat rotations $\overline{R}$ are viewed as the lattice rotations, the strain energy density is only convex w.r.t. F and does not satisfy (5).[4]

The parameter $L_c > 0$ (with dimension length) introduces an internal length which is characteristic for the material, e.g., related to the interaction length of the lattices in a defective single crystal. The internal length $L_c > 0$ is responsible for size effects in the sense that smaller samples are relatively stiffer than larger samples. We assume throughout that $\alpha_4, \alpha_5, \alpha_6 > 0, \alpha_7 \geq 0$. This implies the coercivity of curvature

$$\exists\, c^+ > 0 \quad \forall\, \mathfrak{K} \in \mathfrak{T}(3): \quad W_{\mathrm{curv}}(\mathfrak{K}) \geq c^+ \,\|\mathfrak{K}\|^{1+p+q}\,, \tag{6}$$

which is a basic ingredient of the mathematical analysis. Note that every subsequent result can also be obtained for a true lattice incompatibility measure W_{defect} replacing W_{curv} with

$$W_{\mathrm{defect}} = \mu\, L_c^{1+p+q} \,\|\overline{R}^T \,\mathrm{Curl}\,\overline{R}\|^{1+p+q}\,, \tag{7}$$

see Neff and Münch (2008). W_{defect} accounts for interfacial energy between adjacent regions of lattice orientations.

The non-standard boundary condition of strong consistent coupling ensures that no unwanted non-classical, polar effects may occur at the Dirichlet boundary Γ. It implies for the micropolar stretch that $\overline{U}_{|\Gamma} \in \mathrm{Sym}$ and for the second Piola-Kirchhoff stress tensor $S_2 := F^{-1} D_F W_{\mathrm{mp}}(\overline{U}) \in \mathrm{Sym}$ on Γ as in the classical, non-polar case. We refer to the weaker boundary condition $\overline{U}_{|\Gamma} \in \mathrm{Sym}$ as weak consistent coupling.

It is of prime importance to realize that a linearization of this Cosserat bulk model in the case of defective elastic crystals $\mu_c = 0$ for small displacement and small microrotations completely decouples the two fields of deformation φ and Cosserat-lattice rotations $\overline{R}$ and leads to the classical linear elasticity problem for the deformation.[5] For more details on the modelling of the three-dimensional Cosserat model we refer the reader to Neff (2006a).

2.2 Mathematical results for the Cosserat bulk problem

We recall the obtained results for the case without external loads. It can be shown Neff (2006c, 2004a):

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[4] The condition $F \in \mathrm{GL}^+(3)$ is necessary, otherwise $\|F - \mathrm{polar}(F)\|^2 = \mathrm{dist}^2(F, \mathrm{O}(3)) < \mathrm{dist}^2(F, \mathrm{SO}(3))$, as can be easily seen for the reflection $F = \mathrm{diag}(1, -1, 1)$.

[5] Thinking in the context of an infinitesimal-displacement Cosserat theory one might believe that $\mu_c > 0$ is necessary also for a "true" finite-strain Cosserat theory.

Theorem 2.2.1 (Existence for 3D-finite-strain elastic Cosserat model with $\mu_c > 0$). Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}} \in H^1(\Omega, \mathbb{R}^3)$ and $\overline{R}_{\mathrm{d}} \in W^{1,1+p}(\Omega, \mathrm{SO}(3))$. Then (1) with $\mu_c > 0$, $\alpha_4 \geq 0$, $p \geq 1$, $q \geq 0$ and either free or rigid prescription for $\overline{R}$ on Γ admits at least one minimizing solution pair $(\varphi, \overline{R}) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,1+p}(\Omega, \mathrm{SO}(3))$.   □

In the case of defective elastic crystals a more stringent control of the lattice incompatibility (higher curvature exponent) is necessary. Using the extended Korn's inequality Neff (2002); Pompe (2003), the following has been shown in Neff (2006c):

Theorem 2.2.2 (Existence for 3D-finite-strain elastic Cosserat model with $\mu_c = 0$). Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}} \in H^1(\Omega, \mathbb{R}^3)$ and $\overline{R}_{\mathrm{d}} \in W^{1,1+p+q}(\Omega, \mathrm{SO}(3))$. Then (1) with $\mu_c = 0$, $\alpha_4 > 0$, $p \geq 1$, $q > 1$ and either free or rigid prescription for $\overline{R}$ on Γ admits at least one minimizing solution pair $(\varphi, \overline{R}) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,1+p+q}(\Omega, \mathrm{SO}(3))$.   □

3   Dimensional reduction of the Cosserat bulk model

3.1   The three-dimensional Cosserat problem on a thin domain

The basic task of any shell theory is a consistent reduction of some presumably "exact" 3D-theory to 2D. The three-dimensional problem (1) defined on the physical space $\mathbb{E}^3$ will now be adapted to a shell-like theory. Let us therefore assume that the problem is already transformed in **n**ondimensional form. This means we are given a three-dimensional (nondimensional) **t**hin domain $\Omega_h \subset \mathbb{R}^3$

$$\Omega_h := \omega \times [-\frac{h}{2}, \frac{h}{2}], \quad \omega \subset \mathbb{R}^2 \,, \tag{1}$$

with **t**ransverse boundary $\partial\Omega_h^{\mathrm{trans}} = \omega \times \{-\frac{h}{2}, \frac{h}{2}\}$ and **l**ateral boundary $\partial\Omega_h^{\mathrm{lat}} = \partial\omega \times [-\frac{h}{2}, \frac{h}{2}]$, where ω is a bounded open domain[6] in $\mathbb{R}^2$ with smooth boundary $\partial\omega$ and $h > 0$ is the **n**ondimensional relative characteristic thickness (aspect ratio), $h \ll 1$. Moreover, assume we are given a deformation φ and microrotation $\overline{R}$,

$$\varphi: \ \Omega_h \subset \mathbb{R}^3 \mapsto \mathbb{R}^3, \qquad \overline{R}: \Omega_h \subset \mathbb{R}^3 \mapsto \mathrm{SO}(3) \,, \tag{2}$$

---

[6]For definiteness, one can think of $\omega = [0,1]] \times [0,1]$.

solving the following two-field minimization problem on the thin domain Ω_h:

$$
\begin{aligned}
I(\varphi, \overline{R}) &= \int_{\Omega_h} W_{\mathrm{mp}}(\overline{U}) + W_{\mathrm{curv}}(\mathfrak{K}) - \langle f, \varphi \rangle \, \mathrm{d}V \\
&\quad - \int_{\partial \Omega_h^{\mathrm{trans}} \cup \{\gamma_s \times [-\frac{h}{2}, \frac{h}{2}]\}} \langle N, \varphi \rangle \, \mathrm{d}S \mapsto \min. \text{ w.r.t. } (\varphi, \overline{R}) , \\
\overline{U} &= \overline{R}^T F, \quad \varphi_{|_{\Gamma_0^h}} = g_{\mathrm{d}}(x, y, z) , \\
\Gamma_0^h &= \gamma_0 \times [-\frac{h}{2}, \frac{h}{2}], \quad \gamma_0 \subset \partial \omega, \quad \gamma_s \cap \gamma_0 = \emptyset , \\
\overline{U}_{|_{\Gamma_0^h}} &= \overline{R}^T \nabla \varphi_{|_{\Gamma_0^h}} \in \mathrm{Sym}(3) ,
\end{aligned}
\tag{3}
$$

weak consistent coupling boundary condition **or**

$\overline{R}:$ free on Γ_0^h, alternative Neumann-type boundary condition ,

where $\widehat{L}_c = \frac{L_c}{L}$ is a nondimensional ratio. Without loss of mathematical generality we assume that $M, M_c \equiv 0$ in (4), i.e. that no external volume or surface couples are present in the bulk problem. We want to find a reasonable approximation $(\varphi_s, \overline{R}_s)$ of $(\varphi, \overline{R})$ involving only two-dimensional quantities.

3.2 Transformation on a fixed domain

In order to apply standard techniques of Γ-convergence, we transform the problem onto a **f**ixed domain Ω_1, independent of the aspect ratio $h > 0$. Define therefore

$$
\Omega_1 = \omega \times [-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R}^3, \quad \omega \subset \mathbb{R}^2 .
\tag{4}
$$

The scaling transformation

$$
\begin{aligned}
\zeta : \eta \in \Omega_1 \subset \mathbb{R}^3 \mapsto \mathbb{R}^3, \quad \zeta(\eta_1, \eta_2, \eta_3) &:= (\eta_1, \eta_2, h \cdot \eta_3) , \\
\zeta^{-1} : \xi \in \Omega_h \subset \mathbb{R}^3 \mapsto \mathbb{R}^3, \quad \zeta^{-1}(\xi_1, \xi_2, \xi_3) &:= (\xi_1, \xi_2, \xi_3/h) ,
\end{aligned}
\tag{5}
$$

maps Ω_1 into Ω_h and $\zeta(\Omega_1) = \Omega_h$. We consider the correspondingly scaled function (subsequently, scaled functions defined on Ω_1 will be indicated with a superscript $\sharp$) $\varphi^\sharp : \Omega_1 \to \mathbb{R}^3$, defined by

$$
\varphi(\xi_1, \xi_2, \xi_3) = \varphi^\sharp(\zeta^{-1}(\xi_1, \xi_2, \xi_3)) \quad \forall \, \xi \in \Omega_h ; \quad \varphi^\sharp(\eta) = \varphi(\zeta(\eta)) \quad \forall \, \eta \in \Omega_1 ,
$$

$$
\nabla \varphi(\xi_1, \xi_2, \xi_3) = \left(\partial_{\eta_1} \varphi^\sharp(\eta_1, \eta_2, \eta_3) | \partial_{\eta_2} \varphi^\sharp(\eta_1, \eta_2, \eta_3) | \frac{1}{h} \partial_{\eta_3} \varphi^\sharp(\eta_1, \eta_2, \eta_3) \right)
$$

$$
=: \nabla_\eta^h \varphi^\sharp = F_h^\sharp .
\tag{6}
$$

Similarly, we define a scaled rotation tensor $\overline{R}^{\sharp} : \Omega_1 \subset \mathbb{R}^3 \mapsto \mathrm{SO}(3)$ by

$$\overline{R}(\xi_1, \xi_2, \xi_3) = \overline{R}^{\sharp}(\zeta^{-1}(\xi_1, \xi_2, \xi_3)) \, \forall \xi \in \Omega_h \,;$$

$$\overline{R}^{\sharp}(\eta) = \overline{R}(\zeta(\eta)) \quad \forall \, \eta \in \Omega_1 \,,$$

$$\nabla_{\xi}[\overline{R}(\xi_1, \xi_2, \xi_3).e_i] = \left(\partial_{\eta_1}[\overline{R}^{\sharp}(\eta).e_i] | \partial_{\eta_2}[\overline{R}^{\sharp}(\eta).e_i] | \frac{1}{h}\partial_{\eta_3}[\overline{R}^{\sharp}(\eta).e_i] \right) \tag{7}$$

$$=: \nabla_{\eta}^{h}[\overline{R}^{\sharp}(\eta).e_i] \in \mathbb{M}^{3\times 3} \,,$$

$$\mathrm{D}_{\eta}^{h}\overline{R}_h^{3d,\sharp}(\eta) := \left(\nabla_{\eta}^{h}[\overline{R}^{\sharp}(\eta).e_1], \nabla_{\eta}^{h}[\overline{R}^{\sharp}(\eta).e_2], \nabla_{\eta}^{h}[\overline{R}^{\sharp}(\eta).e_3] \right) \in \mathfrak{T}(3) \,.$$

This allows us to define scaled nonsymmetric stretches $\overline{U}_h^{\sharp} = \overline{R}^{\sharp,T} F_h^{\sharp}$ and the scaled third order curvature tensor $\mathfrak{K}_h^{\sharp} : \Omega_1 \mapsto \mathfrak{T}(3)$

$$\mathfrak{K}_h^{\sharp}(\eta) = \left(\overline{R}^{\sharp,T}(\eta) \left(\partial_{\eta_1}[\overline{R}^{\sharp}(\eta).e_1] | \partial_{\eta_2}[\overline{R}^{\sharp}(\eta).e_1] | \frac{1}{h}\partial_{\eta_3}[\overline{R}^{\sharp}(\eta).e_1] \right) \,, \right.$$

$$\overline{R}^{\sharp,T}(\eta) \left(\partial_{\eta_1}[\overline{R}^{\sharp}(\eta).e_2] | \partial_{\eta_2}[\overline{R}^{\sharp}(\eta).e_2] | \frac{1}{h}\partial_{\eta_3}[\overline{R}^{\sharp}(\eta).e_2] \right) \,, \tag{8}$$

$$\left. \overline{R}^{\sharp,T}(\eta) \left(\partial_{\eta_1}[\overline{R}^{\sharp}(\eta).e_3] | \partial_{\eta_2}[\overline{R}^{\sharp}(\eta).e_3] | \frac{1}{h}\partial_{\eta_3}[\overline{R}^{\sharp}(\eta).e_3] \right) \right)$$

$$= \left(\overline{R}^{\sharp,T}(\eta)\nabla_{\eta}^{h}[\overline{R}^{\sharp}(\eta).e_1], \overline{R}^{\sharp,T}(\eta)\nabla_{\eta}^{h}[\overline{R}^{\sharp}(\eta).e_2], \overline{R}^{\sharp,T}(\eta)\nabla_{\eta}^{h}[\overline{R}^{\sharp}(\eta).e_3] \right)$$

$$= \overline{R}^{\sharp,T} \mathrm{D}_{\eta}^{h}\overline{R}^{\sharp}(\eta) \,.$$

Moreover, we define similarly scaled functions by setting

$$f^{\sharp}(\eta) := f(\zeta(\eta)), \qquad g_{\mathrm{d}}^{\sharp}(\eta) = g_{\mathrm{d}}(\zeta(\eta)), \qquad N^{\sharp}(\eta) := N(\zeta(\eta)) \,. \tag{9}$$

In terms of the introduced scaled deformations and rotations

$$\varphi^{\sharp} : \quad \Omega_1 \subset \mathbb{R}^3 \mapsto \mathbb{R}^3 \,, \qquad \overline{R}^{\sharp} : \Omega_1 \subset \mathbb{R}^3 \mapsto \mathrm{SO}(3) \,, \tag{10}$$

the scaled problem solves the following two-field minimization problem on the fixed domain Ω_1:

$$
\begin{aligned}
I^\sharp(\varphi^\sharp, &\nabla_\eta^h \varphi^\sharp, \overline{R}^\sharp, D_\eta^h \overline{R}^\sharp) \\
&= \int_{\eta \in \Omega_1} \left[W_{\mathrm{mp}}(\overline{U}_h^\sharp) + W_{\mathrm{curv}}(\mathfrak{K}_h^\sharp) - \langle f^\sharp, \varphi^\sharp \rangle \right] \det[\nabla \zeta(\eta)] \, dV_\eta \\
&\quad - \int_{\partial \Omega_1^{\mathrm{trans}} \cup \{\gamma_s \times [-\frac{1}{2}, \frac{1}{2}]\}} \langle N^\sharp, \varphi^\sharp \rangle \, \| \operatorname{Cof} \nabla \zeta(\eta).e_3 \| \, dS_\eta, \\
&= h \int_{\eta \in \Omega_1} W_{\mathrm{mp}}(\overline{U}_h^\sharp) + W_{\mathrm{curv}}(\mathfrak{K}_h^\sharp) - \langle f^\sharp, \varphi^\sharp \rangle \, dV_\eta \\
&\quad - \int_{\Omega_1^{\mathrm{trans}}} \langle N^\sharp, \varphi^\sharp \rangle \, 1 \, dS_\eta \\
&\quad - \int_{\gamma_s \times [-\frac{1}{2}, \frac{1}{2}]} \langle N^\sharp, \varphi^\sharp \rangle \, h \, dS_\eta \mapsto \min. \text{ w.r.t. } (\varphi^\sharp, \overline{R}^\sharp). \quad (11)
\end{aligned}
$$

3.3   The rescaled variational Cosserat bulk problem

Since the energy $\frac{1}{h} I^\sharp$ would not be finite for $h \to 0$ if tractions $N^\sharp$ on the transverse boundary were present, the investigations are in principle restricted to the case of $N^\sharp = 0$ on $\partial \Omega_1^{\mathrm{trans}}$.[7] For conciseness we investigate the following simplified and rescaled ($N^\sharp, f^\sharp = 0$, $g_{\mathrm{d}}(\xi_1, \xi_2, \xi_3) := g_{\mathrm{d}}(\xi_1, \xi_2)$) two-field minimization problem on Ω_1 with respect to Γ-convergence (without the factor $h > 0$ now), i.e. we are interested in the limiting behaviour of the energy per unit aspect ratio h:

$$
I_h^\sharp(\varphi^\sharp, \nabla_\eta^h \varphi^\sharp, \overline{R}^\sharp, D_\eta^h \overline{R}^\sharp) = \int_{\eta \in \Omega_1} W_{\mathrm{mp}}(\overline{U}_h^\sharp) + W_{\mathrm{curv}}(\mathfrak{K}_h^\sharp) \, dV_\eta,
$$

$$
\overline{U}_h^\sharp = \overline{R}^{\sharp,T} F_h^\sharp, \quad \varphi^\sharp\big|_{\Gamma_0^1}(\eta) = g_{\mathrm{d}}^\sharp(\eta) = g_{\mathrm{d}}(\zeta(\eta)) = g_{\mathrm{d}}(\eta_1, \eta_2, h\eta_3) = g_{\mathrm{d}}(\eta_1, \eta_2, 0),
$$

$$
\Gamma_0^1 = \gamma_0 \times [-\frac{1}{2}, \frac{1}{2}], \quad \gamma_0 \subset \partial \omega,
$$

$$
\overline{R}^\sharp : \quad \text{free on } \Gamma_0^1, \text{ Neumann-type boundary condition}, \quad (12)
$$

$$
\mathfrak{K}_h^\sharp = \overline{R}^{\sharp,T} D_\eta^h \overline{R}^\sharp(\eta).
$$

Here we assume that the boundary condition g_{d} is already independent of the transverse variable. We restrict attention to the weakest response, the **Neumann** boundary conditions on the Cosserat rotations $\overline{R}^\sharp$ in line with the

---

[7]The thin plate limit $h \to 0$ obviously cannot support non-vanishing transverse surface loads.

difficulty to experimentally influence the lattice rotations at the Dirichlet-boundary.[8] Moreover, we assume

$$p \geq 1 \,, \quad q > 1 \,, \tag{13}$$

such that both cases $\mu_c > 0$ and $\mu_c = 0$ can be considered simultaneously. External loads of various sort can be treated by Remark 4.0.5.

Within the rescaled formulation (12) we want to investigate the possible limit behaviour for $h \to 0$ and **f**ixed relative internal length $\widehat{L}_c > 0$. This amounts to considering sequences of plates with constant physical thickness d, increasing in plane-length L and accordingly increasing curvature strength of the microstructure, similar to letting $\kappa = const$ in Bhattacharya and James (1999).

3.4   On the choice of the scaling

The Γ-limit, if it exists, is unique. The only choice, which influences the final form of the Γ-limit is given by the initial scaling assumptions made on the unknowns, in order to relate them to the fixed domain Ω_1 and the assumption on the scaling of the energies, here the membrane scaling $\frac{1}{h} I^\sharp < \infty$. Our scaling ansatz is consistent with the one proposed in Le Dret and Raoult (1995); Friesecke et al. (2002b), but not consistent with the one taken in Ciarlet (1997), which scales transverse components of the displacement different in order to extract more information from the Γ-limit. Since we deal with a "two-field" model it is not possible to scale the fields differently. The general inadequacy of the scaling of linear elasticity adopted in Ciarlet (1997) in a geometrically exact context has been pointed out in Fonseca and Francfort (2001). The motivation for our choice is given by the apparent consistency of the results with formal developments and its linearization stability. Here we see that the energy scaling assumptions also introduce an ambiguity in the development. For example, starting from classical nonlinear elasticity, considering the present scaling for the unknowns and assuming $\frac{1}{h^5} I^\sharp < \infty$, a nonlinear von Kármán plate can be rigourously justified by Γ-convergence Friesecke et al. (2002b). These results have been extended to a hierarchy of models in Friesecke et al. (2006).

---

[8]We could as well treat the rigid case, i.e. $\overline{R}^\sharp_{|\Gamma_0^1} = \overline{R}_{\mathrm{d}}$. The case of weak consistent coupling would need additional provisions, the three-dimensional existence result already needs additional control in order to define the then necessary boundary terms.

4   Some facts from Γ-convergence

Let us briefly recapitulate the notions involved by using Γ-convergence. For a detailed treatment we refer to Dal Maso (1992); Braides (2002). We start by defining the lower and upper Γ-limit. In the following, X will always denote a metric space such that sequential compactness and compactness coincide. Moreover, we set $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. We consider a sequence of energy functionals $I_{h_j} : X \mapsto \overline{\mathbb{R}}, h_j \to 0$.

Definition 4.0.1 (Lower and upper Γ-limit). Let X be a metric space and let $I_{h_j} : X \mapsto \overline{\mathbb{R}}, h_j \to 0$ be a sequence of functionals. For $x \in X$ we define

$$\Gamma - \liminf_{h_j} I_{h_j} : X \mapsto \overline{\mathbb{R}},$$

$$\Gamma - \liminf_{h_j} I_{h_j}(x) := \inf \left\{ \liminf_{h_j} I_{h_j}(x_{h_j}), \quad x_{h_j} \to x \right\},$$

$$\Gamma - \limsup_{h_j} I_{h_j} : X \mapsto \overline{\mathbb{R}},$$

$$\Gamma - \limsup_{h_j} I_{h_j}(x) := \inf \left\{ \limsup_{h_j} I_{h_j}(x_{h_j}), \quad x_{h_j} \to x \right\}. \quad \square$$

It is clear that $\Gamma - \liminf_{h_j} I_{h_j}$ and $\Gamma - \limsup_{h_j} I_{h_j} : X \to \mathbb{R}$ always exist and are uniquely determined.

Definition 4.0.2 (Γ-convergence). Let X be a metric space. We say that a sequence of functionals $I_{h_j} : X \mapsto \overline{\mathbb{R}}$ Γ-converges in X to the limit functional $I_0 : X \mapsto \overline{\mathbb{R}}$, if for all $x \in X$ we have

$$\forall\, x \in X : \ \forall\, x_{h_j} \to x : \quad I_0(x) \leq \liminf_{h_j \to 0} I_{h_j}(x_{h_j}), \qquad \text{(lim inf-inequality)}$$

$$\forall\, x \in X : \ \exists\, x_{h_i} \to x : \quad I_0(x) \geq \limsup_{h_i \to 0} I_{h_i}(x_{h_i}), \qquad \text{(recovery sequence)} \ .$$

Corollary 4.0.3. Let X be a metric space. The sequence of functionals $I_{h_j} : X \mapsto \overline{\mathbb{R}}$ Γ-converges in X to $I_0 : X \mapsto \mathbb{R}$ if and only if

$$\Gamma - \liminf_{h_j} I_{h_j} = \Gamma - \limsup_{h_j} I_{h_j} = I_0 \,. \quad \square$$

Remark 4.0.4 (Lower semicontinuity of the Γ-limit). The lower and upper Γ-limits are always lower semicontinuous, hence the Γ-limit is a lower semicontinuous functional. Moreover, if the Γ-limit exists, it is unique.

Remark 4.0.5 (Stability under continuous perturbations). Assume that $I_{h_j} : X \mapsto \overline{\mathbb{R}}$ Γ-converges in X to $I_0 : X \mapsto \overline{\mathbb{R}}$ and let $\Pi : X \mapsto \mathbb{R}$,

independent of h_j, be continuous. Then $I_{h_j} + \Pi$ is Γ-convergent and it holds

$$(\Gamma - \lim_{h_j}[I_{h_j} + \Pi])(x) = (\Gamma - \lim_{h_j} I_{h_j})(x) + \Pi(x) = I_0(x) + \Pi(x), \quad (14)$$

see (Dal Maso, 1992, Prop. 6.21). Recall that when the functional Π, independent of h_j, is not continuous it can influence whether or not Γ-convergence takes place (Dal Maso, 1992, Ex. 6.23). $\qquad\square$

Let us also recapitulate the important **e**qui-coerciveness property. First we recall **c**oerciveness of a functional.[9]

Definition 4.0.6 (Coerciveness)**.** The functional $I : X \mapsto \overline{\mathbb{R}}$ is **c**oercive w.r.t. X, if for each fixed $C > 0$ the closure of the set $\{x \in X \,|\, I(x) \le C\}$ is compact in X, i.e. I has compact sub-levels. $\qquad\square$

Following (Dal Maso, 1992, p.70) we introduce

Definition 4.0.7 (Equi-coerciveness)**.** The sequence of functionals $I_{h_j} : X \mapsto \overline{\mathbb{R}}$ is **e**qui-coercive, if for each fixed $C > 0$ there exists a compact set $K_C \subset X$ such that $\{x \in X \,|\, I_{h_j}(x) \le C\} \subset K_C$, **i**ndependent of $h_j > 0$. $\quad\square$

Hence, if we know that I_{h_j} is equi-coercive over X and that along a sequence $\varphi_j \in X$ it holds that $I_{h_j}(\varphi_j) \le C$, then we can extract a subsequence, φ_{j_k} converging in the topology of X to some limit element $\varphi \in X$.

Theorem 4.0.8 (Characterization of equi-coerciveness)**.** The sequence of functionals $I_{h_j} : X \mapsto \mathbb{R}$ is **e**qui-coercive if and only if there exists a lower semicontinuous coercive function $\Psi : X \mapsto \overline{\mathbb{R}}$ such that $I_{h_j} \ge \Psi$ on X for every $h_j > 0$.

Proof. (Dal Maso, 1992, Prop. 7.7). $\qquad\square$

The following theorem concerns the convergence of the minimum values of an equi-coercive sequence of functions.

---

[9] Typically, coerciveness is given for $X = L^p(\Omega, \mathbb{R}^3)\,, 1 < p < \infty$ with Ω a bounded domain with smooth boundary and

$$I(\varphi) = \begin{cases} \int_\Omega W(\nabla\varphi)\,\mathrm{d}V & \text{if } \varphi \in W^{1,p}(\Omega, \mathbb{R}^3)\,, \quad \varphi_{|\partial\Omega} = 0\,, \\ +\infty & \text{else}\,, \end{cases} \quad (15)$$

with the **l**ocal coercivity assumption $W(F) \ge c_1^+ \|F\|^p - c_2^+$. Coerciveness follows by Poincaré's inequality and Rellich's compact embedding $W^{1,p}(\Omega, \mathbb{R}^3) \subset L^p(\Omega, \mathbb{R}^3)$. Recall that linear elasticity does not satisfy a local coercivity condition. This is the cause for some technical problems of the theory.

Theorem 4.0.9 (Coerciveness of the Γ-limit). Suppose that the sequence of functionals $I_{h_j} : X \mapsto \mathbb{R}$ is equi-coercive. Then the upper and lower Γ-limit are both coercive and

$$\min_{x \in X} \left(\Gamma - \liminf_{h_j} I_{h_j} \right)(x) = \liminf_{h_j} \inf_{x \in X} I_{h_j}(x). \tag{16}$$

If, in addition, the sequence of integral functionals $I_{h_j} : X \mapsto \overline{\mathbb{R}}$ Γ-converges to a functional $I_0 : X \mapsto \overline{\mathbb{R}}$, then I_0 itself is coercive and

$$\min_{x \in X} I_0(x) = \lim_{h_j} \inf_{x \in X} I_{h_j}(x). \tag{17}$$

   Proof. (Dal Maso, 1992, Theo. 7.8). $\qquad\qquad\qquad\qquad\qquad$ □
   Note that equi-coercivity is an additional feature in the development of Γ-convergence arguments, which allows to simplify proofs considerably through compactness arguments. As far as Γ-convergence is concerned, it may be useful to recall (Braides, 2002, p.19) that **m**inimizers of the Γ-limit variational problem may not be a limit of minimizers, so that Γ-convergence can be interpreted as a choice criterion. In addition, the Γ-limit of a constant sequence of functionals J, which is not lower semicontinuous, does not coincide with the constant functional J, instead one has $(\Gamma - \lim J)(x) < J(x)$. In this case, $(\Gamma - \lim J)(x) = QJ(x)$, where QJ is the lower semicontinuous envelope of J. In the case of non lower semicontinuous functionals, the Γ-limit is therefore introducing a different physical setting. We are dealing with lower-semicontinuous functionals.

5   The "two-field" Cosserat Γ-limit

5.1   The spaces and admissible sets

   Now let us proceed to the investigation of the Γ-limit for the rescaled problem (12). We do not use $I_{h_j}^\sharp$ directly in our investigation of Γ-convergence, since this would imply working with the weak topology of $H^{1,2}(\Omega_1, \mathbb{R}^3) \times W^{1,1+p+q}(\Omega_1, SO(3))$, which does not give rise to a metric space. Instead, we define suitable "bulk" spaces X, X' and suitable "two-dimensional" spaces X_ω, X'_ω. First, for $p \geq 1$, $q > 1$ we define the number $r > 1$ by

$$\frac{1}{1+p+q} + \frac{1}{r} = \frac{1}{2} \quad \Rightarrow \quad r = \frac{2(1+p+q)}{(1+p+q)-2}, \tag{18}$$

such that $L^{1+p+q} \cdot L^r \subset L^2$. Note that for $1 + p + q > 3$ it holds that $r < 6$ which implies the compact embedding $H^{1,2}(\Omega_1, \mathbb{R}^3) \subset L^r(\Omega_1, \mathbb{R}^3)$.

Now define the spaces

$$
\begin{aligned}
X &:= \{(\varphi, \overline{R}) \in L^r(\Omega_1, \mathbb{R}^3) \times L^{1+p+q}(\Omega_1, \mathrm{SO}(3))\}\,, \\
X' &:= \{(\varphi, \overline{R}) \in H^{1,2}(\Omega_1, \mathbb{R}^3) \times W^{1,1+p+q}(\Omega_1, \mathrm{SO}(3))\}\,, \qquad (19) \\
X_\omega &:= \{(\varphi, \overline{R}) \in L^r(\omega, \mathbb{R}^3) \times L^{1+p+q}(\omega, \mathrm{SO}(3))\}\,, \\
X'_\omega &:= \{(\varphi, \overline{R}) \in H^{1,2}(\omega, \mathbb{R}^3) \times W^{1,1+p+q}(\omega, \mathrm{SO}(3))\}\,,
\end{aligned}
$$

and the admissible sets

$$
\begin{aligned}
\mathcal{A}' &:= \{(\varphi, \overline{R}) \in H^{1,2}(\Omega_1, \mathbb{R}^3) \times W^{1,1+p+q}(\Omega_1, \mathrm{SO}(3))\,, \\
&\qquad\qquad \varphi_{|_{\Gamma_0^1}}(\eta) = g_{\mathrm{d}}^\sharp(\eta) \quad \}\,, \qquad\qquad (20) \\
\mathcal{A}'_\omega &:= \{(\varphi, \overline{R}) \in H^{1,2}(\omega, \mathbb{R}^3) \times W^{1,1+p+q}(\omega, \mathrm{SO}(3))\,, \\
&\qquad\qquad \varphi_{|_{\gamma_0}}(\eta_1, \eta_2) = g_{\mathrm{d}}^\sharp(\eta_1, \eta_2, 0) \quad \}\,, \\
\mathcal{A}'_{\Omega_1,\omega} &:= \{(\varphi, \overline{R}) \in H^{1,2}(\Omega_1, \mathbb{R}^3) \times W^{1,1+p+q}(\omega, \mathrm{SO}(3))\,, \\
&\qquad\qquad \varphi_{|_{\Gamma_0^1}}(\eta) = g_{\mathrm{d}}^\sharp(\eta) \quad \}\,,
\end{aligned}
$$

We note the compact embedding $X' \subset X$ and the natural inclusions $X_\omega \subset X$ and $X'_\omega \subset X'$. Now we extend the rescaled energies to the space X through redefining

$$
I_h^\sharp(\varphi^\sharp, \nabla_\eta^h \varphi^\sharp, \overline{R}^\sharp, \mathrm{D}_\eta^h \overline{R}^\sharp) = \begin{cases} I_h^\sharp(\varphi^\sharp, \nabla_\eta^h \varphi^\sharp, \overline{R}^\sharp, \mathrm{D}_\eta^h \overline{R}^\sharp) & \text{if } (\varphi^\sharp, \overline{R}^\sharp) \in \mathcal{A}' \\ +\infty & \text{else in } X\,, \end{cases}
$$
$$(21)$$

by abuse of notation. This is a classical trick used in applications of Γ-convergence. It has the virtue of incorporating the boundary conditions already in the energy functional. In the following, Γ-convergence results will be shown with respect to the encompassing metric space X.[10]

Definition 5.1.1 (The transverse averaging operator). For $\varphi \in L^2(\Omega_1, \mathbb{R}^3)$ define the averaging operator over the transverse (thickness) variable η_3

$$
\mathrm{Av} : L^2(\Omega_1, \mathbb{R}^3) \mapsto L^2(\omega, \mathbb{R}^3)\,, \quad \mathrm{Av}.\varphi(\eta_1, \eta_2) := \int_{-1/2}^{1/2} \varphi(\eta_1, \eta_2, \eta_3)\, \mathrm{d}\eta_3\,.
$$
$$(22)$$

---

[10]Of course, X, X' as such are not vector spaces, since one cannot add two rotations. Nevertheless, $L^r(\Omega_1, \mathrm{SO}(3)) \subset L^r(\Omega_1, \mathbb{M}^{3\times3})$ and this space is a Banach space.

It is clear that averaging with respect to the transverse variable η_3 commutes with differentiation w.r.t. the planar variables η_1, η_2, i.e.

$$[\text{Av} . \nabla_{(\eta_1, \eta_2)} \varphi(\eta_1, \eta_2, \eta_3)](\eta_1, \eta_2) = \nabla_{(\eta_1, \eta_2)} [\text{Av} . \varphi(\eta_1 \eta_1 \eta_1)](\eta_1, \eta_2) , \qquad (23)$$

for suitable regular functions φ. Note in passing that for a convex function $f : \mathbb{M}^{3 \times 2} \mapsto \mathbb{R}$ Jensen's inequality implies

$$\begin{aligned}
\int_\omega f(\nabla_{(\eta_1 \eta_2)} [\text{Av} . \varphi](\eta_1, \eta_2)) \, \mathrm{d}\omega &= \int_\omega f([\text{Av} . \nabla_{(\eta_1 \eta_2)} \varphi](\eta_1, \eta_2)) \, \mathrm{d}\omega \\
&\leq \int_\omega \int_{-1/2}^{1/2} f(\nabla_{(\eta_1 \eta_2)} \varphi(\eta_1, \eta_2, \eta_3)) \, \mathrm{d}\eta_3 \, \mathrm{d}\omega \\
&= \int_{\Omega_1} f(\nabla_{(\eta_1 \eta_2)} \varphi(\eta_1, \eta_2, \eta_3)) \, \mathrm{d}V_\eta . \qquad (24)
\end{aligned}$$

5.2   The Γ-limit variational "membrane" problem

Our first result is

Theorem 5.2.1 (Γ-limit for $\mu_c > 0$). For strictly positive Cosserat couple modulus $\mu_c > 0$ the Γ-limit for problem (12) in the setting of (21) is given by the limit energy functional $I_0^\sharp : X \mapsto \overline{\mathbb{R}}$,

$$I_0^\sharp(\varphi, \overline{R}) := \begin{cases} \int\limits_{\Omega_1} W_{\mathrm{mp}}^{\mathrm{hom}}(\nabla \text{Av} . \varphi, \overline{R}) + W_{\mathrm{curv}}^{\mathrm{hom}}(\mathfrak{K}_s) \, \mathrm{d}\omega \\ \qquad -\Pi(\text{Av} . \varphi, \overline{R}_3) & (\varphi, \overline{R}) \in \mathcal{A}'_{\Omega_1, \omega} \quad (25) \\ +\infty & \text{else in } X , \end{cases}$$

with $W_{\mathrm{mp}}^{\mathrm{hom}}$ and $W_{\mathrm{curv}}^{\mathrm{hom}}$ defined below.

The proof of this statement will be given in Section 6.

If we identify the thickness averaged deformation $\text{Av} . \varphi$ with the deformation of the midsurface $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$, this problem determines in fact a purely two-dimensional minimization problem for the deformation of the midsurface $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ and the microrotation of the plate (shell) $\overline{R} : \omega \subset \mathbb{R}^2 \mapsto \mathrm{SO}(3)$ on ω:

$$\begin{aligned}
I_0^\sharp(m, \overline{R}) = \int_\omega W_{\mathrm{mp}}^{\mathrm{hom}}(\nabla m, \overline{R}) + W_{\mathrm{curv}}^{\mathrm{hom}}(\mathfrak{K}_s) \, \mathrm{d}\omega \\
- \Pi(m, \overline{R}_3) \mapsto \min . \text{ w.r.t. } (m, \overline{R}) , \qquad (26)
\end{aligned}$$

and the boundary conditions of place for the midsurface deformation m on the Dirichlet part of the lateral boundary $\gamma_0 \subset \partial \omega$,

$$m_{|\gamma_0} = g_{\mathrm{d}}(x, y, 0) = \text{Av} . g_{\mathrm{d}}(x, y, 0) , \qquad \text{simply supported (fixed, welded)} . \qquad (27)$$

The boundary conditions for the microrotations $\overline{R}$ are automatically determined in the variational process. The dimensionally homogenized local density is[11] [12]

$$
W_{\mathrm{mp}}^{\mathrm{hom}}(\nabla m, \overline{R}) := \mu \underbrace{\| \operatorname{sym}((\overline{R}_1|\overline{R}_2)^T \nabla m - 1\!\!1_2) \|^2}_{\text{"intrinsic" shear-stretch energy}}
$$

$$
+ \mu_c \underbrace{\| \operatorname{skew}((\overline{R}_1|\overline{R}_2)^T \nabla m) \|^2}_{\text{"intrinsic" first order drill energy}} \tag{28}
$$

$$
+ 2\mu \frac{\mu_c}{\mu + \mu_c} \underbrace{\left(\langle \overline{R}_3, m_x \rangle^2 + \langle \overline{R}_3, m_y \rangle^2 \right)}_{\text{homogenized transverse shear energy}}
$$

$$
+ \frac{\mu\lambda}{2\mu + \lambda} \underbrace{\operatorname{tr} \left[\operatorname{sym}((\overline{R}_1|\overline{R}_2)^T \nabla m - 1\!\!1_2) \right]^2}_{\text{homogenized elongational stretch energy}} .
$$

The dimensionally homogenized curvature density is given by

$$
W_{\mathrm{curv}}^{\mathrm{hom}}(\mathfrak{K}_s) := \inf_{A \in \mathfrak{so}(3)} W_{\mathrm{curv}}^*(\overline{R}^T \partial_{\eta_1} \overline{R}, \overline{R}^T \partial_{\eta_2} \overline{R}, A) ,
$$

$$
\mathfrak{K}_s = \left(\overline{R}^T(\nabla(\overline{R}.e_1)|0), \overline{R}^T(\nabla(\overline{R}.e_2)|0), \overline{R}^T(\nabla(\overline{R}.e_3)|0) \right)
$$

$$
= \overline{R}^T(x,y) \mathrm{D}_x \overline{R}(x,y) , \tag{29}
$$

$$
\mathfrak{K}_s = (\mathfrak{K}_s^1, \mathfrak{K}_s^2, \mathfrak{K}_s^3) \in \mathfrak{T}(3) , \quad \text{the reduced } \mathbf{curvature \ tensor} ,
$$

where W_{curv}^* is an equivalent representation of the bulk curvature energy in terms of skew-symmetric arguments

$$
W_{\mathrm{curv}}(\mathfrak{K}) = W_{\mathrm{curv}}^*(\overline{R}^T \partial_{\eta_1} \overline{R}, \overline{R}^T \partial_{\eta_2} \overline{R}, \overline{R}^T \partial_{\eta_3} \overline{R}) ,
$$

$$
W_{\mathrm{curv}}^* : \mathfrak{so}(3) \times \mathfrak{so}(3) \times \mathfrak{so}(3) \mapsto \mathbb{R}^+ , \tag{30}
$$

with $\overline{R}^T \partial_{\eta_i} \overline{R} \in \mathfrak{so}(3)$ since $\partial_{\eta_i}[\overline{R}^T \overline{R}] = \partial_{\eta_i} 1\!\!1 = 0$. We note that W_{curv}^* remains a convex function in its argument and so is $W_{\mathrm{curv}}^{\mathrm{hom}}(\mathfrak{K}_s)$. Moreover, $W_{\mathrm{curv}}^{\mathrm{hom}}(\mathfrak{K}_s) = W_{\mathrm{curv}}(\mathfrak{K}_s)$ for $W_{\mathrm{curv}}(\mathfrak{K}) = \widehat{W}(\|\mathfrak{K}\|)$.

---

[11] $\| \operatorname{skew}((\overline{R}_1|\overline{R}_2)^T \nabla m) \|^2 = (\langle \overline{R}_1, m_y \rangle - \langle \overline{R}_2, m_x \rangle)^2$. Note that $\| \operatorname{skew}((\overline{R}_1|\overline{R}_2)^T \nabla m) \| = 0$ does not imply that $\overline{R}_3 = \vec{n}_m$.

[12] In the following, "intrinsic" refers to classical surface geometry, where intrinsic quantities are those which depend only on the first fundamental form $I_m = \nabla m^T \nabla m \in \mathbb{M}^{2 \times 2}$ of the surface. Then "intrinsic" in our terminology are terms, which reduce to such a dependence in the continuum limit $\overline{R} = \operatorname{polar}(\nabla m|\vec{n})$. For example $(\overline{R}_1|\overline{R}_2)^T \nabla m = \sqrt{\nabla m^T \nabla m}$, in this case.

In (26) Π denotes a general external loading functional, continuous in the topology of X, cf. Remark 4.0.5. It is clear that the limit functional $I_0^\sharp$ is weakly lower semicontinuous in the topology of $X' = H^{1,2}(\Omega, \mathbb{R}^3) \times W^{1,1+p+q}(\Omega, \mathrm{SO}(3))$ by simple convexity arguments. We note the twofold appearance of the **h**armonic mean $\mathcal{H}$,[13]

$$\frac{1}{2}\mathcal{H}(\mu, \frac{\lambda}{2}) = \frac{\mu\lambda}{2\mu + \lambda}, \qquad \mathcal{H}(\mu, \mu_c) = 2\mu \frac{\mu_c}{\mu + \mu_c}. \qquad (31)$$

An advantage of this formulation is that the dimensionally homogenized formulation remains **f**rame-indifferent. Note that the limit functional $I_0^\sharp$ is consistent with the following **p**lane stress requirement (c.f. (50))

$$\forall \eta_3 \in [-\frac{1}{2}, \frac{1}{2}] : \quad S_1(\eta_1, \eta_2, \eta_3).e_3 = 0, \qquad (32)$$

i.e. a vanishing normal stress over the entire thickness of the plate, while for any given thickness $h > 0$ from 3D-equilibrium one can only infer **z**ero normal stress at the upper and lower faces

$$\langle \overline{R}^T(\eta_1, \eta_2, \pm 1/2) S_1(\eta_1, \eta_2, \pm 1/2).e_3, e_3 \rangle = 0. \qquad (33)$$

In this sense, **t**he Cosserat "membrane" Γ-limit underestimates the real stresses, notably the transverse shear stresses, as noted in (Friesecke et al., 2006, 9.3) with respect to the membrane scaling.

5.3   The defective elastic crystal limit case $\mu_c = 0$

Since it is not possible to establish equi-coercivity for the defective crystal case $\mu_c = 0$, one cannot infer a Γ-limit result for $\mu_c = 0$ as a consequence of the result for $\mu_c > 0$. However, since the energy functional $I_{h_j}^\sharp$ for $\mu_c > 0$ is strictly bigger than the same functional for $\mu_c = 0$, independent of $h_j > 0$, it is easy to see (Dal Maso, 1992, Prop. 6.7) that on X we have the inequalities

$$\Gamma - \liminf I_{h_j}^\sharp{}_{|\mu_c=0} \leq \Gamma - \limsup I_{h_j}^\sharp{}_{|\mu_c=0} \leq \lim_{\mu_c \to 0} \left(\Gamma - \lim I_{h_j}^\sharp{}_{|\mu_c>0} \right) =: I_0^{\sharp,0},$$
$$(34)$$

---

[13] For $a, b > 0$ the harmonic, arithmetic and geometric mean are defined as $\mathcal{H}(a,b) := \frac{2}{\frac{1}{a} + \frac{1}{b}}$, $\quad \mathcal{A}(a,b) = \frac{a+b}{2}$, $\quad \mathcal{G}(a,b) = \sqrt{a\,b}$, respectively and one has the chain of inequalities $\mathcal{H}(a,b) \leq \mathcal{G}(a,b) \leq \mathcal{A}(a,b)$.

where

$$
I_0^{\sharp,0}(\varphi,\overline{R}) = \begin{cases} \int_\omega W_{\mathrm{mp}}^{\mathrm{hom},0}(\nabla\,\mathrm{Av}\,.\varphi,\overline{R}) + W_{\mathrm{curv}}^{\mathrm{hom}}(\mathfrak{K}_s)\,\mathrm{d}\omega \\ \quad -\Pi(\mathrm{Av}\,.\varphi,\overline{R}_3) & (\varphi,\overline{R}) \in \mathcal{A}_0^{\mathrm{mem}} \\ +\infty & \text{else in } X\,, \end{cases}
$$
(35)

with $\mathcal{A}_0^{\mathrm{mem}}$ defined as

$$
\mathcal{A}_0^{\mathrm{mem}} := \{(\varphi,\overline{R}) \in X \mid \mathrm{sym}\,(\overline{R}_1|\overline{R}_2)^T \nabla_{(\eta_1,\eta_2)}\,\mathrm{Av}\,.\varphi \in L^2(\Omega_1,\mathbb{M}^{2\times 2}),
$$
$$
\overline{R} \in W^{1,1+p+q}(\omega,\mathrm{SO}(3))\,,
$$
$$
\varphi^{\sharp}_{|_{\Gamma_0^1}}(\eta) = g_{\mathrm{d}}^{\sharp}(\eta) = g_{\mathrm{d}}(\eta_1,\eta_2,0)\,\}\,,
$$
(36)

and the understanding of $\nabla_{(\eta_1,\eta_2)}\,\mathrm{Av}\,.\varphi$ as **d**istributional derivative for $\varphi \in L^r(\Omega_1,\mathbb{R}^3)$. The corresponding local energy density in terms of $m = \mathrm{Av}\,.\varphi$ is

$$
W_{\mathrm{mp}}^{\mathrm{hom},0}(\nabla m,\overline{R}) := \mu\,\underbrace{\|\,\mathrm{sym}(\overline{R}_1|\overline{R}_2)^T\nabla m - \mathbb{1}_2\|^2}_{\text{"intrinsic" shear-stretch energy}}
$$
$$
+ \frac{\mu\lambda}{2\mu+\lambda}\,\underbrace{\mathrm{tr}\left[\mathrm{sym}((\overline{R}_1|\overline{R}_2)^T\nabla m - \mathbb{1}_2)\right]^2}_{\text{homogenized elongational stretch energy}}\,.
$$
(37)

Observe that the upper bound $I_0^{\sharp,0}$ for the $\Gamma-\limsup$ energy functional is **n**ot coercive w.r.t. $H^{1,2}(\omega,\mathbb{R}^3)$ due to the now missing transverse shear contribution, while it retains lower-semicontinuity. This degeneration remains true for whatever form the Γ-limit for $\mu_c = 0$ has, should it exist. Our main result is

Theorem 5.3.1 (Γ-limit for defective elastic crystals $\mu_c = 0$)**.** The Γ-limit of (12) for $\mu_c = 0$ in the setting of (21) exists and is given by (35). $\qquad\Box$

The proof of this statement is deferred to Section 7.

The loss of coercivity for $\mu_c = 0$ is primarily a loss of control for the "transverse" components $\langle m_x,\overline{R}_3\rangle, \langle m_y,\overline{R}_3\rangle$, while w.r.t. the remaining "in-plane" components compactness for minimizing sequences, whose midsurface deformations are supposed to be already bounded in $L^r(\omega)$, can be established (appropriate use of an extended Korn's second inequality, c.f. (101)). That homogenization may lead to a loss of (strict) rank-one convexity has been observed in Geymonat et al. (1993) for nonlinearly elastic composites, whose constituents are strictly rank-one convex.

For linearization consistency, it is easy to show that the linearization for $\mu_c = 0$ of the frame-indifferent Γ-limit $I_0^{\sharp,0}$ w.r.t. small midsurface displacement $v : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ and small curvature decouples the fields of infinitesimal midsurface displacement and infinitesimal microrotations: after de-scaling we are left with the classical infinitesimal "membrane" plate problem for $v : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$

$$\int_\omega h \left(\mu \| \operatorname{sym} \nabla(v_1, v_2) \|^2 + \frac{\mu\lambda}{2\mu+\lambda} \operatorname{tr} \left[\operatorname{sym} \nabla(v_1, v_2) \right]^2 \right) d\omega$$

$$- \langle f, \langle v, e_1 \rangle \cdot e_1 + \langle v, e_2 \rangle \cdot e_2 \rangle \mapsto \min. \text{ w.r.t. } v, \quad (38)$$

$$\langle v, e_i \rangle_{|\gamma_0} = \langle u^{\mathrm{d}}(x, y, 0), e_i \rangle, \quad i = 1, 2$$

$$\text{simply supported (horizontal components only)},$$

which leaves the vertical midsurface displacement v_3 undetermined due to the non-resistance of a linear "membrane" plate to vertical deflections. This problem coincides with a linearization[14] of the nonlinear membrane plate problem proposed in (Fox et al., 1993, par. 4.3), based on purely formal asymptotic methods applied to the St.Venant-Kirchhoff energy. The variational problem (38) is as well the Γ-limit of the classical linear elasticity bulk problem (if corresponding scaling assumptions are made, compare with (Anzellotti et al., 1994, Th.4.2), Bourquin et al. (1992) or (Ciarlet, 1997, Th.1.11.2). The classical linear bulk model in turn can be obtained as linearization for $\mu_c = 0$ of the Cosserat bulk problem. Hence, **o**nly in the defective elastic crystal case $\mu_c = 0$, linearization and taking the Γ-limit commute with the Γ-limit of classical linear elasticity.[15]

5.4   The formal limit $\mu_c = \infty$

This case is interesting, because the formal Γ-limit for $\mu_c \to \infty$ exists and still gives rise to an independent field of microrotations $\overline{R}$, while the Cosserat bulk problem for $\mu_c = \infty$ degenerates into a constraint theory (a so called indeterminate couple-stress model or second gradient model), where the microrotations $\overline{R}$ coincide necessarily with the continuum rotations $\operatorname{polar}(F)$ from the polar decomposition.

The formal Γ-limit problem is: find the deformation of the midsurface $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ and the microrotation of the plate (shell) $\overline{R} : \omega \subset \mathbb{R}^2 \mapsto$

---

[14] Expansion of the first fundamental form I_m of the midsurface m w.r.t. planar initial configuration yields $I_m - 1\!1_2 = \nabla m^T \nabla m - 1\!1_2 \approx \operatorname{sym} \nabla_{(x,y)}(v_1, v_2) + O(\|\nabla v\|^2)$. Hence control on vertical deflections v_3 is lost during linearization.

[15] As is well known (Ciarlet, 1999, p.464) this is not the case with the membrane Γ-limit, see Le Dret and Raoult (1995), based on the non-elliptic St.Venant-Kirchhoff energy.

SO(3) on ω such that for $I_0^{\sharp,\infty} : X \mapsto \overline{\mathbb{R}}$ in terms of the averaged deformation $m = \text{Av}.\varphi$,

$$I_0^{\sharp,\infty}(m,\overline{R}) \mapsto \min. \text{ w.r.t. } (m,\overline{R}), \tag{39}$$

with

$$I_0^{\sharp,\infty}(m,\overline{R}) = \begin{cases} \int_\omega W_{\text{mp}}^{\text{hom},\infty}(\nabla m,\overline{R}) + W_{\text{curv}}^{\text{hom}}(\mathfrak{K}_s)\,\mathrm{d}\omega \\ \quad -\Pi(m,\overline{R}_3) & (m,\overline{R}) \in \mathcal{A}_\omega^{\prime;\infty} \\ +\infty & \text{else in } X, \end{cases} \tag{40}$$

and the admissible set

$$\begin{aligned} \mathcal{A}_\omega^{\prime,\infty} := \{ (m,\overline{R}) &\in H^{1,2}(\omega,\mathbb{R}^3) \times W^{1,1+p+q}(\omega,\text{SO}(3)), \\ & m_{|\gamma_0}(\eta_1,\eta_2) = g_{\mathrm{d}}^{\sharp}(\eta_1,\eta_2,0), \\ & \langle \overline{R}_1, m_y \rangle = \langle \overline{R}_2, m_x \rangle \quad \}. \end{aligned} \tag{41}$$

The formal local energy density reads

$$W_{\text{mp}}^{\text{hom},\infty}(\nabla m,\overline{R}) := \mu \underbrace{\|(\overline{R}_1|\overline{R}_2)^T \nabla m - \mathbb{1}_2\|^2}_{\text{"intrinsic" shear-stretch energy}}$$

$$+ \underbrace{2\mu \left(\langle \overline{R}_3, m_x \rangle^2 + \langle \overline{R}_3, m_y \rangle^2 \right)}_{\text{homogenized transverse shear energy}}$$

$$+ \frac{\mu\lambda}{2\mu+\lambda} \underbrace{\text{tr}\left[\text{sym}((\overline{R}_1|\overline{R}_2)^T \nabla m - \mathbb{1}_2)\right]^2}_{\text{homogenized elongational stretch energy}}. \tag{42}$$

Note that $\mu_c = \infty$ rules out in-plane drill rotations Hughes and Brezzi (1989); Fox and Simo (1992), the transverse shear energy is doubled, but transverse shear is still possible since $\overline{R}_3$ need not coincide with the normal on m. In this sense, the resulting homogenized transverse shear modulus excludes what could be called "transverse shear locking" in accordance with the "Poisson thickness locking" which occurs, if the correct homogenized volumetric modulus is not taken.[16] In a future contribution we will discuss whether the formal limit (39) is the rigourous Γ-limit of the constraint Cosserat bulk problem. Note that in Bhattacharya and James (1999) it has been shown that the Γ-limit of a second gradient bulk model gives rise to one independent "Cosserat"-director, which here would correspond to $\overline{R}_3$.

---

[16]$\lim_{\lambda\to\infty} \frac{1}{2}\mathcal{H}(\mu,\frac{\lambda}{2}) = \mu < \infty$ but $\lim_{\lambda\to\infty} \frac{1}{2}\mathcal{A}(\mu,\frac{\lambda}{2}) = \infty$.

6 Proof for positive Cosserat couple modulus $\mu_c > 0$

We continue by proving Theorem 25, i.e. the claim on the form of the Γ-limit for strictly positive Cosserat couple modulus $\mu_c > 0$. The proof is split into several steps.

6.1 Equi-coercivity of $I_{h_j}^\sharp$, compactness and dimensional reduction

Theorem 6.1.1 (Equi-coercivity of $I_{h_j}^\sharp$). For positive Cosserat couple modulus $\mu_c > 0$ the sequence of rescaled energy functionals $I_{h_j}^\sharp$ defined in (12) is equi-coercive on the space X.

Proof. It is clear that for given $h > 0$ the problem (12) admits a minimizing pair $(\varphi_h^\sharp, \overline{R}_h^\sharp) \in H^{1,2}(\Omega_1, \mathbb{R}^3) \times W^{1,1+p+q}(\Omega_1, \mathrm{SO}(3))$ by the obvious scaling transformation of the minimizing solution of the bulk problem for values of $p \geq 1, q > 1$ and for both $\mu_c > 0$ and $\mu_c = 0$.[17] This is especially true for Neumann boundary conditions on the microrotations, since for exact rotations, $\|\overline{R}\| = \sqrt{3}$. This leads to a control of microrotations in $W^{1,1+p+q}(\Omega_1, \mathrm{SO}(3))$ already without specification of Dirichlet boundary data on the microrotations.

Consider now a sequence $h_j \to 0$ for $j \to \infty$. By inspection of the existence proof for the Cosserat bulk problem, it will become clear that for corresponding sequences $(\varphi_{h_j}^\sharp, \overline{R}_{h_j}^\sharp) \in H^{1,2}(\Omega_1, \mathbb{R}^3) \times W^{1,1+p+q}(\Omega_1, \mathrm{SO}(3)) = X'$ with $I_{h_j}^\sharp(\varphi_{h_j}^\sharp, \overline{R}_{h_j}^\sharp) < \infty$ bounded independent of h_j (not necessarily minimizers) we obtain a bound on the sequence $(\varphi_{h_j}^\sharp, \overline{R}_{h_j}^\sharp)$ in X', independent of h_j. To see this, note that for $\mu_c > 0$, it is immediate that $\nabla_\eta^h \varphi^\sharp = F_h^\sharp$ is bounded in $L^2(\Omega_1, \mathbb{M}^{3\times3})$, independent of $\overline{R}_{h_j}^\sharp$ on account of the local coercivity condition

$$W_{\mathrm{mp}}(\overline{R}_{h_j}^{\sharp,T} F_{h_j}^\sharp) \geq \min(\mu_c, \mu) \|\overline{R}_{h_j}^{\sharp,T} F_{h_j}^\sharp - \mathbb{1}\|^2$$
$$= \min(\mu_c, \mu) \left(\|F_{h_j}^\sharp\|^2 - 2\langle \overline{R}_{h_j}^{\sharp,T} F_{h_j}^\sharp, \mathbb{1} \rangle + 3 \right)$$
$$\geq \min(\mu_c, \mu) \left(\|F_{h_j}^\sharp\|^2 - 2\sqrt{3}\|F_{h_j}^\sharp\| + 3 \right), \qquad (43)$$

---

[17] In contrast to Γ-convergence arguments based on the finite-strain St.Venant-Kirchhoff energy Le Dret and Raoult (1995), which might not admit minimizers for any given $h > 0$.

and after integration

$$\infty > I_{h_j}^\sharp(\varphi_{h_j}^\sharp, \overline{R}_{h_j}^\sharp) > \int_{\Omega_1} W_{\mathrm{mp}}(\overline{U}_{h_j}^\sharp) + W_{\mathrm{curv}}(\mathfrak{K}_{h_j}^\sharp) \, dV_\eta$$

$$\geq \int_{\Omega_1} W_{\mathrm{mp}}(\overline{U}_{h_j}^\sharp) \, dV_\eta$$

$$\geq \int_{\Omega_1} \min(\mu_c, \mu) \left(\|F_{h_j}^\sharp\|^2 - 2\sqrt{3}\|F_{h_j}^\sharp\| + 3 \right) \, dV_\eta \qquad (44)$$

$$\geq \min(\mu_c, \mu) \int_{\Omega_1} \left(\left[\|\partial_{\eta_1}\varphi^\sharp\|^2 + \|\partial_{\eta_2}\varphi^\sharp\|^2 + \frac{1}{h_j{}^2}\|\partial_{\eta_3}\varphi^\sharp\|^2 \right] \right.$$
$$\left. - 2\sqrt{3}\left[\|\partial_{\eta_1}\varphi^\sharp\| + \|\partial_{\eta_2}\varphi^\sharp\| + \frac{1}{h_j}\|\partial_{\eta_3}\varphi^\sharp\| \right] + 3 \right) \, dV_\eta \, .$$

This implies a bound, independent of h_j, for the gradient $\nabla\varphi_{h_j}^\sharp$ in $L^2(\Omega_1, \mathbb{R}^3)$. The Dirichlet boundary conditions for $\varphi_{h_j}^\sharp$ together with Poincaré's inequality yield the boundedness of $\varphi_{h_j}^\sharp$ in $H^{1,2}(\Omega_1, \mathbb{R}^3)$.[18] With a similar argument, based on the local coercivity of curvature, the bound on $\overline{R}_{h_j}^\sharp$ can be obtained: we need only to observe that for a constant $c^+ > 0$, depending on the positivity of $\alpha_4, \alpha_5, \alpha_6, \alpha_7$, but independent of h_j,

$$\infty > I_{h_j}^\sharp(\varphi_{h_j}^\sharp, \overline{R}_{h_j}^\sharp) > \int_{\Omega_1} W_{\mathrm{mp}}(\overline{U}_{h_j}^\sharp) + W_{\mathrm{curv}}(\mathfrak{K}_{h_j}^\sharp) \, dV_\eta$$

$$\geq \int_{\Omega_1} W_{\mathrm{curv}}(\mathfrak{K}_{h_j}^\sharp) \, dV_\eta$$

$$\geq \int_{\Omega_1} c^+ \|\mathfrak{K}_{h_j}^\sharp\|^{1+p+q} \, dV_\eta = c^+ \int_{\Omega_1} \|\overline{R}_{h_j}^{\sharp,T} \mathrm{D}_\eta^{h_j} \overline{R}_{h_j}^\sharp\|^{1+p+q} \, dV_\eta$$

$$= c^+ \int_{\Omega_1} \|\mathrm{D}_\eta^{h_j} \overline{R}_{h_j}^\sharp\|^{1+p+q} \, dV_\eta \, , \qquad (45)$$

which establishes a bound on the gradient of rotations $\nabla_\eta^{h_j}[\overline{R}_{h_j}^\sharp(\eta).e_i]$, $i = 1, 2, 3$, independent of h_j. Moreover, $\|\overline{R}_{h_j}^\sharp\| = \sqrt{3}$, establishing the $W^{1,1+p+q}(\Omega_1, \mathrm{SO}(3))$ bound on $\overline{R}_{h_j}^\sharp$. Thus we may obtain a subsequence, not relabelled, such that

$$\varphi_{h_j}^\sharp \rightharpoonup \varphi_0^\sharp \quad \text{in } H^{1,2}(\Omega_1, \mathbb{R}^3), \quad \overline{R}_{h_j}^\sharp \rightharpoonup \overline{R}_0^\sharp \quad \text{in } W^{1,1+p+q}(\Omega_1, \mathrm{SO}(3)). \quad (46)$$

---

[18]This argument fails for the limit case $\mu_c = 0$ since local coercivity does not hold, which is realistic for defective elastic crystals.

Both weak limits $(\varphi_0^{\sharp}, \overline{R}_0^{\sharp})$ must be independent of the transverse coordinate η_3, otherwise the energy $I_{h_j}^{\sharp}$ could not remain finite for $h_j \to 0$, see (44) and compare with the definition of $D_{\eta}^{h_j}$ in (7). Hence the solution must be found in terms of functions defined on the two-dimensional domain ω. In this sense the domain of the limit problem is two-dimensional and the corresponding space is X_{ω}. Since the embedding $X' \subset X$ is compact, it is shown that the sequence of energy functionals $I_{h_j}^{\sharp}$ is equi-coercive w.r.t. X. $\square$

6.2   Lower bound-the $\liminf$-condition

If $I_0^{\sharp}$ is the Γ-limit of the sequence of energy functionals $I_{h_j}^{\sharp}$ then we must have ($\liminf$-inequality) that

$$I_0^{\sharp}(\varphi_0, \overline{R}_0) \leq \liminf_{h_j} I_{h_j}^{\sharp}(\varphi_{h_j}^{\sharp}, \overline{R}_{h_j}^{\sharp}), \qquad (47)$$

whenever

$$\varphi_{h_j}^{\sharp} \to \varphi_0^{\sharp} \quad \text{in } L^r(\Omega_1, \mathbb{R}^3), \quad \overline{R}_{h_j}^{\sharp} \to \overline{R}_0^{\sharp} \quad \text{in } L^{1+p+q}(\Omega_1, \mathrm{SO}(3)), \qquad (48)$$

for arbitrary $(\varphi_0^{\sharp}, \overline{R}_0^{\sharp}) \in X$. Observe that we can restrict attention to sequences $(\varphi_{h_j}^{\sharp}, \overline{R}_{h_j}^{\sharp}) \in X$ such that $I_{h_j}^{\sharp}(\varphi_{h_j}^{\sharp}, \overline{R}_{h_j}^{\sharp}) < \infty$ since otherwise the statement is true anyway. Sequences with $I_{h_j}^{\sharp}(\varphi_{h_j}^{\sharp}, \overline{R}_{h_j}^{\sharp}) < \infty$ are uniformly bounded in the space X', as seen previously. This implies weak convergence of a subsequence in X'. But we know already that the original sequences converge strongly in X to the limit $(\varphi_0^{\sharp}, \overline{R}_0^{\sharp}) \in X$. Hence we must have as well weak convergence to $\varphi_0^{\sharp} \in H^{1,2}(\omega, \mathbb{R}^3)$ and $\overline{R}_0^{\sharp} \in W^{1,1+p+q}(\omega, \mathrm{SO}(3))$, independent of the transverse variable η_3.

In a first step we consider now the local energy contribution: along sequences $(\varphi_{h_j}^{\sharp}, \overline{R}_{h_j}^{\sharp}) \in X$ with finite energy $I_{h_j}^{\sharp}$, the third column of the deformation gradient $\nabla_{\eta}^{h_j} \varphi_{h_j}^{\sharp}$ remains bounded but otherwise indetermined. Therefore, a trivial lower bound is obtained by minimizing the effect of the derivative in this direction in the local energy W_{mp}. To continue our development, we need some calculations: For smooth $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$, $\overline{R} : \omega \subset \mathbb{R}^2 \mapsto \mathrm{SO}(3)$ define the "director"-vector $b^* \in \mathbb{R}^3$ formally through

$$W_{\mathrm{mp}}^{\mathrm{hom}}(\nabla m, \overline{R}) = W_{\mathrm{mp}}(\overline{R}^T(\nabla m | b^*))$$
$$= \inf_{b \in \mathbb{R}^3} W_{\mathrm{mp}}(\overline{R}^T(\nabla m | b)). \qquad (49)$$

The vector b^*, which realizes this infimum, can be explicitly determined. Set $\tilde{F} := (\nabla m | b^*)$. The corresponding local optimality condition reads

$$
\begin{aligned}
\forall\, \delta b^* \in \mathbb{R}^3: \quad &\langle DW_{\mathrm{mp}}(\overline{R}^T(\nabla m | b^*)), \overline{R}^T(0|0|\delta b^*)\rangle = 0 \Rightarrow \\
&\langle \overline{R}\, DW_{\mathrm{mp}}(\overline{R}^T(\nabla m | b^*)), (0|0|\delta b^*)\rangle = 0 \Rightarrow \\
&\overline{R}\, DW_{\mathrm{mp}}(\overline{R}^T(\nabla m | b^*)).e_3 = 0 \Rightarrow \\
&D_{\tilde{F}} W_{\mathrm{mp}}(\overline{R}^T(\nabla m | b^*)).e_3 = 0 \Rightarrow \\
&S_1((\nabla m | b^*), \overline{R}).e_3 = 0\,.
\end{aligned}
\tag{50}
$$

Since

$$
\begin{aligned}
S_1(F, \overline{R}) = \overline{R}\, \Big(\mu \left(F^T \overline{R} + \overline{R}^T F - 2\,\mathbb{1} \right) \\
+ 2\mu_c\, \mathrm{skew}(\overline{R}^T F) + \lambda \operatorname{tr}\left[\overline{R}^T F - \mathbb{1} \right] \mathbb{1} \Big)
\end{aligned}
\tag{51}
$$

and

$$
\overline{R}^T \tilde{F} = \begin{pmatrix} \langle \overline{R}_1, m_x\rangle & \langle \overline{R}_1, m_y\rangle & \langle \overline{R}_1, b^*\rangle \\ \langle \overline{R}_2, m_x\rangle & \langle \overline{R}_2, m_y\rangle & \langle \overline{R}_2, b^*\rangle \\ \langle \overline{R}_3, m_x\rangle & \langle \overline{R}_3, m_y\rangle & \langle \overline{R}_3, b^*\rangle \end{pmatrix}, \quad \tilde{F}^T \overline{R} + \overline{R}^T \tilde{F} - 2\,\mathbb{1} = \tag{52}
$$

$$
\begin{pmatrix} 2[\langle \overline{R}_1, m_x\rangle - 1] & \langle \overline{R}_1, m_y\rangle + \langle \overline{R}_2, m_x\rangle & \langle \overline{R}_1, b^*\rangle + \langle \overline{R}_3, m_x\rangle \\ \langle \overline{R}_2, m_x\rangle + \langle \overline{R}_1, m_y\rangle & 2[\langle \overline{R}_2, m_y\rangle - 1] & \langle \overline{R}_2, b^*\rangle + \langle \overline{R}_3, m_y\rangle \\ \langle \overline{R}_3, m_x\rangle + \langle \overline{R}_1, b^*\rangle & \langle \overline{R}_3, m_y\rangle + \langle \overline{R}_2, b^*\rangle & 2[\langle \overline{R}_3, b^*\rangle - 1] \end{pmatrix},
$$

$$
\mathrm{skew}(\overline{R}^T \tilde{F}) = \begin{pmatrix} 0 & \frac{1}{2}\left(\langle \overline{R}_1, m_y\rangle - \langle \overline{R}_2, m_x\rangle\right) & \frac{1}{2}\left(\langle \overline{R}_1, b^*\rangle - \langle \overline{R}_3, m_x\rangle\right) \\ * & 0 & \frac{1}{2}\left(\langle \overline{R}_2, b^*\rangle - \langle \overline{R}_3, m_y\rangle\right) \\ * & * & 0 \end{pmatrix},
$$

the (plane-stress) requirement $S_1.e_3 = 0$ (50) implies

$$
\mu \begin{pmatrix} \langle \overline{R}_1, b^*\rangle + \langle \overline{R}_3, m_x\rangle \\ \langle \overline{R}_2, b^*\rangle + \langle \overline{R}_3, m_y\rangle \\ 2[\langle \overline{R}_3, b^*\rangle - 1] \end{pmatrix} + \mu_c \begin{pmatrix} \langle \overline{R}_1, b^*\rangle - \langle \overline{R}_3, m_x\rangle \\ \langle \overline{R}_2, b^*\rangle - \langle \overline{R}_3, m_y\rangle \\ 0 \end{pmatrix}
$$

$$
+ \lambda \left(\langle \overline{R}_1, m_x\rangle + \langle \overline{R}_2, m_y\rangle + \langle \overline{R}_3, b^*\rangle - 3 \right) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{53}
$$

The solution of the last system can conveniently be expressed in the

orthonormal triad $(\overline{R}_1, \overline{R}_2, \overline{R}_3)$ as

$$b^* = \frac{\mu_c - \mu}{\mu + \mu_c} \langle \overline{R}_3, m_x \rangle \, \overline{R}_1 + \frac{\mu_c - \mu}{\mu + \mu_c} \langle \overline{R}_3, m_y \rangle \, \overline{R}_2 + \varrho_m^* \, \overline{R}_3 \,,$$

$$\varrho_m^* = 1 - \frac{\lambda}{2\mu + \lambda} \left[\langle (\nabla m | 0), \overline{R} \rangle - 2 \right] . \tag{54}$$

Note that for $\overline{R} \in \mathrm{SO}(3)$ and $\nabla m \in L^2(\Omega_1, \mathbb{R}^3)$ it follows that $b^* \in L^2(\Omega_1, \mathbb{R}^3)$. Reinserting the solution b^* we have

$$\overline{R}^T \tilde{F} = \begin{pmatrix} \langle \overline{R}_1, m_x \rangle & \langle \overline{R}_1, m_y \rangle & \frac{\mu_c - \mu}{\mu + \mu_c} \langle \overline{R}_3, m_x \rangle \\ \langle \overline{R}_2, m_x \rangle & \langle \overline{R}_2, m_y \rangle & \frac{\mu_c - \mu}{\mu + \mu_c} \langle \overline{R}_3, m_y \rangle \\ \langle \overline{R}_3, m_x \rangle & \langle \overline{R}_3, m_y \rangle & \varrho_m^* \end{pmatrix}, \quad \tilde{F}^T \overline{R} + \overline{R}^T \tilde{F} - 2\,\mathbb{1} =$$

$$\begin{pmatrix} 2[\langle \overline{R}_1, m_x \rangle - 1] & \langle \overline{R}_1, m_y \rangle + \langle \overline{R}_2, m_x \rangle & \left(1 + \frac{\mu_c - \mu}{\mu + \mu_c} \right) \langle \overline{R}_3, m_x \rangle \\ \langle \overline{R}_2, m_x \rangle + \langle \overline{R}_1, m_y \rangle & 2[\langle \overline{R}_2, m_y \rangle - 1] & \left(1 + \frac{\mu_c - \mu}{\mu + \mu_c} \right) \langle \overline{R}_3, m_y \rangle \\ \left(1 + \frac{\mu_c - \mu}{\mu + \mu_c} \right) \langle \overline{R}_3, m_x \rangle & \left(1 + \frac{\mu_c - \mu}{\mu + \mu_c} \right) \langle \overline{R}_3, m_y \rangle & 2[\varrho_m^* - 1] \end{pmatrix},$$

$$\mathrm{skew}(\overline{R}^T \tilde{F}) = \begin{pmatrix} 0 & \frac{1}{2} \left(\langle \overline{R}_1, m_y \rangle - \langle \overline{R}_2, m_x \rangle \right) & \frac{1}{2} \left(\left(\frac{\mu_c - \mu}{\mu + \mu_c} - 1 \right) \langle \overline{R}_3, m_x \rangle \right) \\ * & 0 & \frac{1}{2} \left(\left(\frac{\mu_c - \mu}{\mu + \mu_c} - 1 \right) \langle \overline{R}_3, m_y \rangle \right) \\ * & * & 0 \end{pmatrix},$$

$$1 + \frac{\mu_c - \mu}{\mu + \mu_c} = \frac{2\mu_c}{\mu + \mu_c} \,, \qquad \frac{\mu_c - \mu}{\mu + \mu_c} - 1 = \frac{-2\mu}{\mu + \mu_c} \,. \tag{55}$$

We obtain finally for $W_{\mathrm{mp}}^{\mathrm{hom}}(\nabla m, \overline{R}) := W_{\mathrm{mp}}(\overline{R}^T (\nabla m | b^*))$ with $\tilde{\overline{U}} = \overline{R}^T (\nabla m | b^*) = \overline{R}^T \tilde{F}$ after a lengthy but otherwise straightforward computation

$$W_{\mathrm{mp}}^{\mathrm{hom}}(\nabla m, \overline{R}) := W_{\mathrm{mp}}(\tilde{\overline{U}})$$

$$= \mu \, \| \mathrm{sym}(\tilde{\overline{U}} - \mathbb{1}) \|^2 + \mu_c \, \| \mathrm{skew}(\tilde{\overline{U}}) \|^2 + \frac{\lambda}{2} \, \mathrm{tr} \left[\mathrm{sym}(\tilde{\overline{U}} - \mathbb{1}) \right]^2$$

$$= \mu \, \| \mathrm{sym}((\overline{R}_1 | \overline{R}_2)^T \nabla m - \mathbb{1}_2) \|^2 + \mu_c \, \| \mathrm{skew}((\overline{R}_1 | \overline{R}_2)^T \nabla m) \|^2 \tag{56}$$

$$+ 2\mu \, \frac{\mu_c}{\mu + \mu_c} \left(\langle \overline{R}_3, m_x \rangle^2 + \langle \overline{R}_3, m_y \rangle^2 \right)$$

$$+ \frac{\mu \lambda}{2\mu + \lambda} \, \mathrm{tr} \left[\mathrm{sym}((\overline{R}_1 | \overline{R}_2)^T \nabla m - \mathbb{1}_2) \right]^2 .$$

Along the sequence $(\varphi_{h_j}^{\sharp}, \overline{R}_{h_j}^{\sharp})$ we have by construction,

$$W_{\mathrm{mp}}(\overline{R}_{h_j}^{\sharp,T} \nabla_\eta^{h_j} \varphi_{h_j}^{\sharp}) = W_{\mathrm{mp}}(\overline{R}_{h_j}^{\sharp,T} (\nabla_{(\eta_1\eta_2)} \varphi_{h_j}^{\sharp} | \frac{1}{h_j} \partial_{\eta_3} \varphi_{h_j}^{\sharp}))$$

$$\geq W_{\mathrm{mp}}^{\mathrm{hom}}(\nabla_{(\eta_1\eta_2)} \varphi_{h_j}^{\sharp}, \overline{R}_{h_j}^{\sharp}). \tag{57}$$

Hence, integrating and taking the lim inf also

$$\liminf_{h_j} \int_{\Omega_1} W_{\mathrm{mp}}(\overline{R}_{h_j}^{\sharp,T} \nabla_\eta^{h_j} \varphi_{h_j}^{\sharp}) \, \mathrm{d}V_\eta$$

$$\geq \liminf_{h_j} \int_{\Omega_1} W_{\mathrm{mp}}^{\mathrm{hom}}(\nabla_{(\eta_1\eta_2)} \varphi_{h_j}^{\sharp}, \overline{R}_{h_j}^{\sharp}) \, \mathrm{d}V_\eta. \tag{58}$$

Now we use weak convergence of $\varphi_{h_j}^{\sharp}$ and strong convergence of $\overline{R}_{h_j}^{\sharp}$, to-gether with the convexity w.r.t. ∇m and continuity w.r.t. $\overline{R}$ of $\int_{\Omega_1} W_{\mathrm{mp}}^{\mathrm{hom}}(\nabla m, \overline{R}) \, \mathrm{d}V_\eta$ to get lower semi-continuity of the right hand side in (58) and to obtain altogether

$$\liminf_{h_j} \int_{\Omega_1} W_{\mathrm{mp}}(\overline{R}_{h_j}^{\sharp,T} \nabla_\eta^{h_j} \varphi_{h_j}^{\sharp}) \, \mathrm{d}V_\eta \geq \int_{\Omega_1} W_{\mathrm{mp}}^{\mathrm{hom}}(\nabla_{(\eta_1\eta_2)} \varphi_0^{\sharp}, \overline{R}_0^{\sharp}) \, \mathrm{d}V_\eta. \tag{59}$$

Next we are concerned with the **c**urvature contribution: it is always possi-ble to uniquely rewrite the curvature energy expression in terms of skew-symmetric quantities

$$W_{\mathrm{curv}}^* : \mathfrak{so}(3) \times \mathfrak{so}(3) \times \mathfrak{so}(3) \mapsto \mathbb{R}^+,$$
$$W_{\mathrm{curv}}^*(\overline{R}^T \partial_{\eta_1} \overline{R}, \overline{R}^T \partial_{\eta_2} \overline{R}, \overline{R}^T \partial_{\eta_3} \overline{R}) := W_{\mathrm{curv}}(\mathfrak{K}), \tag{60}$$

where $\overline{R}^T \partial_{\eta_i} \overline{R} \in \mathfrak{so}(3)$ since $\partial_{\eta_i}[\overline{R}^T \overline{R}] = 0$. We note that W_{curv}^* remains a convex function in its argument since $\mathfrak{K} \in \mathfrak{T}(3)$ can be obtained by a linear mapping from $(\overline{R}^T \partial_{\eta_1} \overline{R}, \overline{R}^T \partial_{\eta_2} \overline{R}, \overline{R}^T \partial_{\eta_3} \overline{R}) \in \mathfrak{so}(3) \times \mathfrak{so}(3) \times \mathfrak{so}(3)$. We define the "homogenized" (relaxed) curvature energy through

$$W_{\mathrm{curv}}^{*,\mathrm{hom}}(\overline{R}^T \partial_{\eta_1} \overline{R}, \overline{R}^T \partial_{\eta_2} \overline{R}) := W_{\mathrm{curv}}^*(\overline{R}^T \partial_{\eta_1} \overline{R}, \overline{R}^T \partial_{\eta_2} \overline{R}, A^*)$$

$$= \inf_{A \in \mathfrak{so}(3)} W_{\mathrm{curv}}^*(\overline{R}^T \partial_{\eta_1} \overline{R}, \overline{R}^T \partial_{\eta_2} \overline{R}, A), \tag{61}$$

and set accordingly

$$W_{\mathrm{curv}}^{\mathrm{hom}}(\mathfrak{K}_s) := W_{\mathrm{curv}}^{*,\mathrm{hom}}(\overline{R}^T \partial_{\eta_1} \overline{R}, \overline{R}^T \partial_{\eta_2} \overline{R}),$$
$$\mathfrak{K}_s = \left(\overline{R}^T (\nabla(\overline{R}.e_1)|0), \overline{R}^T (\nabla(\overline{R}.e_2)|0), \overline{R}^T (\nabla(\overline{R}.e_3)|0) \right), \tag{62}$$

in terms of the reduced curvature tensor $\mathfrak{K}_s \in \mathfrak{T}(3)$.

Similarly to (50) the infinitesimal rotation $A^* \in \mathfrak{so}(3)$, which realizes the infimum in (61), can be explicitly determined. We refrain from giving the explicit result. Suffice it to note that $W_{\text{curv}}^{\text{hom}}$ is uniquely defined, remains convex in its argument and has the same growth as W_{curv}. Then

$$W_{\text{curv}}(\overline{R}_{h_j}^{\sharp,T} D_\eta^{h_j} \overline{R}_{h_j}^\sharp) = W_{\text{curv}}^*(\overline{R}_{h_j}^{\sharp,T} \partial_{\eta_1} \overline{R}_{h_j}^\sharp, \overline{R}_{h_j}^{\sharp,T} \partial_{\eta_2} \overline{R}_{h_j}^\sharp, \frac{1}{h_j} \overline{R}_{h_j}^{\sharp,T} \partial_{\eta_3} \overline{R}_{h_j}^\sharp)$$

$$\geq W_{\text{curv}}^{*,\text{hom}}(\overline{R}_{h_j}^{\sharp,T} \partial_{\eta_1} \overline{R}_{h_j}^\sharp, \overline{R}_{h_j}^{\sharp,T} \partial_{\eta_2} \overline{R}_{h_j}^\sharp). \tag{63}$$

Integrating the last inequality, taking the lim inf on both sides and using that $W_{\text{curv}}^{*,\text{hom}}$ is convex in its argument, together with weak convergence of the two in-plane components of the curvature tensor, i.e.

$$(\overline{R}_{h_j}^{\sharp,T} \partial_{\eta_1} \overline{R}_{h_j}^\sharp, \overline{R}_{h_j}^{\sharp,T} \partial_{\eta_2} \overline{R}_{h_j}^\sharp, 0) \rightharpoonup (\overline{R}_0^{\sharp,T} \partial_{\eta_1} \overline{R}_0^\sharp, \overline{R}_0^{\sharp,T} \partial_{\eta_2} \overline{R}_0^\sharp, 0)$$
$$\text{in } L^{1+p+q}(\Omega_1, \mathfrak{T}(3)), \tag{64}$$

we obtain

$$\liminf_{h_j} \int_{\Omega_1} W_{\text{curv}}(\overline{R}_{h_j}^{\sharp,T} D_\eta^{h_j} \overline{R}_{h_j}^\sharp) \, dV_\eta$$

$$\geq \int_{\Omega_1} W_{\text{curv}}^{*,\text{hom}}(\overline{R}_{h_j}^{\sharp,T} \partial_{\eta_1} \overline{R}_{h_j}^\sharp, \overline{R}_{h_j}^{\sharp,T} \partial_{\eta_2} \overline{R}_{h_j}^\sharp) \, dV_\eta$$

$$= \int_{\Omega_1} W_{\text{curv}}^{\text{hom}}(\overline{R}_0^{\sharp,T} D\overline{R}_0^\sharp) \, dV_\eta. \tag{65}$$

Then, because $W_{\text{curv}}, W_{\text{mp}} \geq 0$,

$$\liminf_{h_j} \int_{\Omega_1} W_{\text{mp}}(\overline{R}_{h_j}^{\sharp,T} \nabla_\eta^{h_j} \varphi_{h_j}^\sharp) + W_{\text{curv}}(\overline{R}_{h_j}^{\sharp,T} D_\eta^{h_j} \overline{R}_{h_j}^\sharp) \, dV_\eta$$

$$\geq \liminf_{h_j} \int_{\Omega_1} W_{\text{mp}}(\overline{R}_{h_j}^{\sharp,T} \nabla_\eta^{h_j} \varphi_{h_j}^\sharp) \, dV_\eta + \liminf_{h_j} \int_{\Omega_1} W_{\text{curv}}(\overline{R}_{h_j}^{\sharp,T} D_\eta^{h_j} \overline{R}_{h_j}^\sharp) \, dV_\eta$$

$$\geq \int_{\Omega_1} W_{\text{mp}}^{\text{hom}}(\nabla_{(\eta_1,\eta_2)} \varphi_0^\sharp, \overline{R}_0^\sharp) \, dV_\eta + \int_{\Omega_1} W_{\text{curv}}^{\text{hom}}(\overline{R}_0^{\sharp,T} D\overline{R}_0^\sharp) \, dV_\eta \tag{66}$$

$$= \int_{\Omega_1} W_{\text{mp}}^{\text{hom}}(\nabla_{(\eta_1,\eta_2)} \varphi_0^\sharp, \overline{R}_0^\sharp) + W_{\text{curv}}^{\text{hom}}(\overline{R}_0^{\sharp,T} D\overline{R}_0^\sharp) \, dV_\eta,$$

where we used (59) and (65). Now we use that $\varphi_0^\sharp$ is independent of the transverse variable η_3, which allows us to insert the averaging operator

without any change to see that

$$\int_{\Omega_1} W_{\mathrm{mp}}^{\mathrm{hom}}(\nabla_{(\eta_1 \eta_2)}\varphi_0^\sharp, \overline{R}_0^\sharp) = \int_{\Omega_1} W_{\mathrm{mp}}^{\mathrm{hom}}(\nabla_{(\eta_1 \eta_2)}\,\mathrm{Av}\,.\varphi_0^\sharp, \overline{R}_0^\sharp)\,\mathrm{d}V_\eta$$

$$= \int_\omega W_{\mathrm{mp}}^{\mathrm{hom}}(\nabla_{(\eta_1 \eta_2)}\,\mathrm{Av}\,.\varphi_0^\sharp, \overline{R}_0^\sharp)\,\mathrm{d}\omega\,, \qquad (67)$$

since $\overline{R}_0^\sharp$ is also independent of the transverse variable. Hence we obtain altogether the desired $\lim\inf$-inequality

$$I_0^\sharp(\varphi_0^\sharp, \overline{R}_0^\sharp) \le \liminf_{h_j} I_{h_j}^\sharp(\varphi_{h_j}^\sharp, \overline{R}_{h_j}^\sharp) \qquad (68)$$

for

$$I_0^\sharp(\varphi_0, \overline{R}_0) := \int_{\Omega_1} W_{\mathrm{mp}}^{\mathrm{hom}}(\nabla_{(\eta_1 \eta_2)}\,\mathrm{Av}\,.\varphi_0, \overline{R}_0) + W_{\mathrm{curv}}^{\mathrm{hom}}(\overline{R}_0^T D\overline{R}_0)\,\mathrm{d}V_\eta$$

$$= \int_\omega W_{\mathrm{mp}}^{\mathrm{hom}}(\nabla_{(\eta_1 \eta_2)}\,\mathrm{Av}\,.\varphi_0, \overline{R}_0) + W_{\mathrm{curv}}^{\mathrm{hom}}(\overline{R}_0^T D\overline{R}_0)\,\mathrm{d}\omega\,. \quad \square$$

6.3   Upper bound-the recovery sequence

Now we show that the lower bound will actually be reached. A sufficient requirement for the recovery sequence is that

$$\forall\,(\varphi_0, \overline{R}_0) \in X = L^r(\Omega_1, \mathbb{R}^3) \times L^{1+p+q}(\Omega_1, \mathrm{SO}(3))$$

$$\exists\,\varphi_{h_j}^\sharp \to \varphi_0 \quad \text{in } L^r(\Omega_1, \mathbb{R}^3)\,, \quad \overline{R}_{h_j}^\sharp \to \overline{R}_0 \quad \text{in } L^{1+p+q}(\Omega_1, \mathrm{SO}(3)):$$

$$\limsup I_{h_j}^\sharp(\varphi_{h_j}^\sharp, \overline{R}_{h_j}^\sharp) \le I_0^\sharp(\varphi_0, \overline{R}_0)\,. \qquad (69)$$

Observe that this is now only a condition if $I_0^\sharp(\varphi_0, \overline{R}_0) < \infty$. In this case the uniform coercivity of $I_{h_j}^\sharp(\varphi_{h_j}^\sharp, \overline{R}_{h_j}^\sharp)$ over $X' = H^{1,2}(\Omega_1, \mathbb{R}^3) \times W^{1,1+p+q}(\Omega_1, \mathrm{SO}(3))$ implies that we can restrict attention to sequences $(\varphi_{h_j}^\sharp, \overline{R}_{h_j}^\sharp)$ converging weakly to some
$(\varphi_0, \overline{R}_0) \in H^{1,2}(\omega, \mathbb{R}^3) \times W^{1,1+p+q}(\omega, \mathrm{SO}(3)) = X'_\omega$, defined over the two-dimensional domain ω only. Note, however, that finally it is strong convergence in X which matters.

The natural candidate for the recovery sequence for the bulk deformation is given by the "reconstruction"

$$\varphi_{h_j}^\sharp(\eta_1, \eta_2, \eta_3) := m(\eta_1, \eta_2) + h_j\,\eta_3\,b^*(\eta_1, \eta_2) = \varphi_0(\eta_1, \eta_2) + h_j\,\eta_3\,b^*(\eta_1, \eta_2)\,,$$
$$(70)$$

where, with the abbreviation $m = \varphi_0 = \text{Av}.\varphi_0$ at places,

$$b^*(\eta_1, \eta_2) := \frac{\mu_c - \mu}{\mu + \mu_c} \langle \overline{R}_{0,3}, m_x \rangle \overline{R}_{0,1} + \frac{\mu_c - \mu}{\mu + \mu_c} \langle \overline{R}_{0,3}, m_y \rangle \overline{R}_{0,2} + \varrho_m^* \overline{R}_{0,3},$$

$$\varrho_m^* = 1 - \frac{\lambda}{2\mu + \lambda} \left[\langle (\nabla m | 0), \overline{R}_0 \rangle - 2 \right]. \tag{71}$$

Observe that $b^* \in L^2(\omega, \mathbb{R}^3)$. Convergence of $\varphi_{h_j}^\sharp$ in $L^r(\Omega_1, \mathbb{R}^3)$ to the limit φ_0 as $h_j \to 0$ is obvious.

The reconstruction for the rotation $\overline{R}_0$ is, however, not obvious since on the one hand we have to maintain the rotation constraint along the sequence and on the other hand we must approach the lower bound, which excludes the simple reconstruction $\overline{R}_{h_j}^\sharp(\eta_1, \eta_2, \eta_3) = \overline{R}_0(\eta_1, \eta_2)$. In order to meet both requirements we consider therefore

$$\overline{R}_{h_j}^\sharp(\eta_1, \eta_2, \eta_3) := \overline{R}_0(\eta_1, \eta_2) \cdot \exp\left(h_j \, \eta_3 \, A^*(\eta_1, \eta_2)\right), \tag{72}$$

where $A^* \in \mathfrak{so}(3)$ is the term obtained in (61), depending on the given $\overline{R}_0$ and we note that $A^* \in L^{1+p+q}(\omega, \mathfrak{so}(3))$ by the coercivity of W_{curv}^*. It is clear that $\overline{R}_{h_j}^\sharp \in \text{SO}(3)$, since $\exp : \mathfrak{so}(3) \mapsto \text{SO}(3)$ and we have the convergence $\overline{R}_{h_j}^\sharp \to \overline{R}_0$ in $L^{1+p+q}(\Omega_1, \text{SO}(3))$ for $h_j \to 0$.

Since neither b^* nor A^* need be differentiable, we have to consider slightly modified recovery sequences, however. With fixed $\varepsilon > 0$ choose $b_\varepsilon \in W^{1,2}(\omega, \mathbb{R}^3)$ such that $\|b_\varepsilon - b^*\|_{L^2(\omega, \mathbb{R}^3)} < \varepsilon$ and similarly for A^* choose $A_\varepsilon \in W^{1,1+p+q}(\omega, \mathfrak{so}(3))$ such that $\|A_\varepsilon - A^*\|_{L^{1+p+q}(\omega, \mathfrak{so}(3))} < \varepsilon$. This allows us to present finally our **recovery sequence**

$$\varphi_{h_j, \varepsilon}^\sharp(\eta_1, \eta_2, \eta_3) := \varphi_0(\eta_1, \eta_2) + h_j \, \eta_3 \, b_\varepsilon(\eta_1, \eta_2),$$

$$\overline{R}_{h_j, \varepsilon}^\sharp(\eta_1, \eta_2, \eta_3) := \overline{R}_0(\eta_1, \eta_2) \cdot \exp\left(h_j \, \eta_3 \, A_\varepsilon(\eta_1, \eta_2)\right). \tag{73}$$

This definition implies

$$\nabla\varphi^\sharp_{h_j,\varepsilon}(\eta_1,\eta_2,\eta_3) = (\nabla\varphi_0(\eta_1,\eta_2)|h_j\, b_\varepsilon(\eta_1,\eta_2)) + h_j\,\eta_3\,(\nabla b_\varepsilon(\eta_1,\eta_2)|0)\,,$$

$$\overline{R}^{\sharp,T}_{h_j,\varepsilon}\partial_{\eta_1}\overline{R}^\sharp_{h_j,\varepsilon} = \exp\left(h_j\,\eta_3\,A_\varepsilon\right)^T\overline{R}^T_0$$
$$\left[\partial_{\eta_1}\overline{R}_0\,\exp\left(h_j\,\eta_3\,A_\varepsilon\right) + \overline{R}_0\,D\exp\left(h_j\,\eta_3\,A_\varepsilon\right).[h_j\,\eta_3\partial_{\eta_1}A_\varepsilon]\right]\,,$$

$$\overline{R}^{\sharp,T}_{h_j,\varepsilon}\partial_{\eta_2}\overline{R}^\sharp_{h_j,\varepsilon} = \exp\left(h_j\,\eta_3\,A_\varepsilon\right)^T\overline{R}^T_0$$
$$\left[\partial_{\eta_2}\overline{R}_0\,\exp\left(h_j\,\eta_3\,A_\varepsilon\right) + \overline{R}_0\,D\exp\left(h_j\,\eta_3\,A_\varepsilon\right).[h_j\,\eta_3\partial_{\eta_2}A_\varepsilon]\right]\,,$$

$$\overline{R}^{\sharp,T}_{h_j,\varepsilon}\partial_{\eta_3}\overline{R}^\sharp_{h_j,\varepsilon} = \exp\left(h_j\,\eta_3\,A_\varepsilon\right)^T\overline{R}^T_0$$
$$\left[\partial_{\eta_3}\overline{R}_0\,\exp\left(h_j\,\eta_3\,A_\varepsilon\right) + \overline{R}_0\,D\exp\left(h_j\,\eta_3\,A_\varepsilon\right).[h_j\,A_\varepsilon]\right]$$
$$= h_j\,\exp\left(h_j\,\eta_3\,A_\varepsilon\right)^T D\exp\left(h_j\,\eta_3\,A_\varepsilon\right).[A_\varepsilon]\,, \tag{74}$$

with $\partial_{\eta_i}A_\varepsilon \in \mathfrak{so}(3)$. In view of the prominent appearance of the exponential in these expressions it is useful to briefly recapitulate the basic features of the matrix exponential exp acting on $\mathfrak{so}(3)$. We note

$$\exp:\ \mathfrak{so}(3)\mapsto \mathrm{SO}(3)\quad\text{is infinitely differentiable}\,,$$
$$\forall\,A\in\mathfrak{so}(3):\quad \|\exp(A)\| = \sqrt{3}\ \Rightarrow$$
$$\exp:L^{1+p+q}(\Omega_1,\mathfrak{so}(3))\mapsto L^{1+p+q}(\Omega_1,\mathrm{SO}(3))\quad\text{is continuous}\,,$$
$$D\exp:\mathfrak{so}(3)\mapsto\mathrm{Lin}(\mathfrak{so}(3),\mathbb{M}^{3\times3})\quad\text{is locally continuous}\,,$$
$$\forall\,H\in\mathfrak{so}(3):\quad D\exp(0).H = H\,,$$
$$\forall\,A,H\in\mathfrak{so}(3):\exp(A)^T\cdot D\exp(A).H \in \mathfrak{so}(3)\,. \tag{75}$$

Note that by appropriately choosing $h_j,\varepsilon > 0$ we can arrange that strong convergence of (74) to the correct limit still obtains by using $(75)_3$. Now abbreviate

$$\widetilde{\overline{U}} := \overline{R}^T_0\,(\nabla\varphi_0(\eta_1,\eta_2)|b^*) \in \mathbb{M}^{3\times3}\,,$$
$$\widetilde{\overline{V}}^\varepsilon_{h_j} := \overline{R}^{\sharp,T}_{h_j,\varepsilon}\,[(\nabla\varphi_0(\eta_1,\eta_2)|b_\varepsilon(\eta_1,\eta_2)) + h_j\,\eta_3\,(\nabla b_\varepsilon(\eta_1,\eta_2)|0)] \in \mathbb{M}^{3\times3},$$
$$\widetilde{\overline{V}}^\varepsilon_0 := \overline{R}^T_0\,(\nabla\varphi_0(\eta_1,\eta_2)|b_\varepsilon(\eta_1,\eta_2)) \in \mathbb{M}^{3\times3}\,, \tag{76}$$
$$\mathfrak{k}^{\sharp,i}_{h_j,\varepsilon} := \overline{R}^{\sharp,T}_{h_j,\varepsilon}\partial_{\eta_i}\overline{R}^\sharp_{h_j,\varepsilon} \in \mathfrak{so}(3),\quad i = 1,2,3\,,$$
$$\mathfrak{k}^i_0 := \overline{R}^T_0\,\partial_{\eta_i}\overline{R}_0 \in \mathfrak{so}(3),\quad i = 1,2\,,$$
$$\tilde{A}_{h_j,\varepsilon} := \exp\left(h_j\,\eta_3\,A_\varepsilon(\eta_1,\eta_2)\right)^T D\exp\left(h_j\,\eta_3\,A_\varepsilon(\eta_1,\eta_2)\right).[A_\varepsilon] \in \mathfrak{so}(3)\,,$$
$$\mathfrak{K}^\sharp_{h_j,\varepsilon} := \overline{R}^{\sharp,T}_{h_j,\varepsilon}\mathrm{D}^{h_j}_\eta\overline{R}^\sharp_{h_j,\varepsilon}(\eta_1,\eta_2,\eta_3) \in \mathfrak{T}(3)$$
$$\mathfrak{K}_0(\eta_1,\eta_2) = \overline{R}^T_0\,D\overline{R}_0(\eta_1,\eta_2) \in \mathfrak{T}(3)\,.$$

We note that by the smoothness of $A_\varepsilon \in W^{1,1+p+q}(\omega_1, \mathfrak{so}(3))$

$$
\begin{aligned}
\|\tilde{A}_{h_j,\varepsilon} - A_\varepsilon\|_{L^{1+p+q}(\Omega_1, \mathfrak{so}(3))} &\to 0 \quad \text{if} \quad h_j \to 0, \\
\|\mathfrak{k}^{\sharp,i}_{h_j,\varepsilon} - \mathfrak{k}^i_0\|_{L^{1+p+q}(\Omega_1, \mathfrak{so}(3))} &\to 0 \quad \text{if} \quad h_j \to 0, \\
\|\widetilde{\overline{V}}^\varepsilon_{h_j} - \widetilde{\overline{V}}^\varepsilon_0\|_{L^2(\Omega_1, \mathbb{M}^{3\times3})} &\to 0 \quad \text{if} \quad h_j \to 0, \\
\|\widetilde{\overline{V}}^\varepsilon_{h_j} - \widetilde{\overline{U}}\|_{L^2(\Omega_1, \mathbb{M}^{3\times3})} &\to 0 \quad \text{if} \quad h_j, \varepsilon \to 0.
\end{aligned}
\tag{77}
$$

The abbreviations in (76) imply

$$
\begin{aligned}
I^\sharp_{h_j}(\varphi^\sharp_{h_j,\varepsilon}, \overline{R}^\sharp_{h_j,\varepsilon}) &= \int_{\Omega_1} W_{\mathrm{mp}}(\widetilde{\overline{V}}^\varepsilon_{h_j}) + W^*_{\mathrm{curv}}(\mathfrak{k}^{\sharp,1}_{h_j,\varepsilon}, \mathfrak{k}^{\sharp,2}_{h_j,\varepsilon}, \frac{1}{h_j}\overline{R}^{\sharp,T}_{h_j,\varepsilon}\partial_{\eta_3}\overline{R}^\sharp_{h_j,\varepsilon}) \, \mathrm{d}V_\eta \\
&= \int_{\Omega_1} W_{\mathrm{mp}}(\widetilde{\overline{V}}^\varepsilon_{h_j}) + W^*_{\mathrm{curv}}(\mathfrak{k}^{\sharp,1}_{h_j,\varepsilon}, \mathfrak{k}^{\sharp,2}_{h_j,\varepsilon}, \tilde{A}_{h_j,\varepsilon}) \, \mathrm{d}V_\eta,
\end{aligned}
\tag{78}
$$

where we used that $h_j \cdot b_\varepsilon$ in the definition of the recovery deformation gradient $(74)_1$ is cancelled by the factor $\frac{1}{h_j}$ in the definition of $I^\sharp_{h_j}$. Whence, adding and subtracting $W_{\mathrm{mp}}(\widetilde{\overline{U}})$

$$
\begin{aligned}
&I^\sharp_{h_j}(\varphi^\sharp_{h_j,\varepsilon}, \overline{R}^\sharp_{h_j,\varepsilon}) \\
&= \int_{\Omega_1} W_{\mathrm{mp}}(\widetilde{\overline{U}}) + W_{\mathrm{mp}}(\widetilde{\overline{V}}^\varepsilon_{h_j}) - W_{\mathrm{mp}}(\widetilde{\overline{U}}) + W^*_{\mathrm{curv}}(\mathfrak{k}^{\sharp,1}_{h_j,\varepsilon}, \mathfrak{k}^{\sharp,2}_{h_j,\varepsilon}, \tilde{A}_{h_j,\varepsilon}) \, \mathrm{d}V_\eta \\
&= \int_{\Omega_1} W_{\mathrm{mp}}(\widetilde{\overline{U}}) + W_{\mathrm{mp}}(\widetilde{\overline{U}} + \widetilde{\overline{V}}^\varepsilon_{h_j} - \widetilde{\overline{U}}) - W_{\mathrm{mp}}(\widetilde{\overline{U}}) + W_{\mathrm{curv}}(\mathfrak{K}_{h_j}) \, \mathrm{d}V_\eta
\end{aligned}
$$

since W_{mp} and W_{curv} are both positive, we get from the triangle inequality

$$
\begin{aligned}
&\leq \int_{\Omega_1} W_{\mathrm{mp}}(\widetilde{\overline{U}}) + |W_{\mathrm{mp}}(\widetilde{\overline{U}} + \widetilde{\overline{V}}^\varepsilon_{h_j} - \widetilde{\overline{U}}) - W_{\mathrm{mp}}(\widetilde{\overline{U}})| \\
&\quad + W^*_{\mathrm{curv}}(\mathfrak{k}^{\sharp,1}_{h_j,\varepsilon}, \mathfrak{k}^{\sharp,2}_{h_j,\varepsilon}, \tilde{A}_{h_j,\varepsilon}) \, \mathrm{d}V_\eta
\end{aligned}
$$

Expanding the quadratic energy W_{mp} we obtain

$$
\begin{aligned}
&= \int_{\Omega_1} W_{\mathrm{mp}}(\widetilde{\overline{U}}) + |W_{\mathrm{mp}}(\widetilde{\overline{U}}) + \langle DW_{\mathrm{mp}}(\widetilde{\overline{U}}), \widetilde{\overline{V}}^{\varepsilon}_{h_j} - \widetilde{\overline{U}} \rangle \\
&\qquad + D^2 W_{\mathrm{mp}}(\widetilde{\overline{U}}).(\widetilde{\overline{V}}^{\varepsilon}_{h_j} - \widetilde{\overline{U}}, \widetilde{\overline{V}}^{\varepsilon}_{h_j} - \widetilde{\overline{U}})| + W^*_{\mathrm{curv}}(\mathfrak{k}^{\sharp,1}_{h_j,\varepsilon}, \mathfrak{k}^{\sharp,2}_{h_j,\varepsilon}, \tilde{A}_{h_j,\varepsilon}) \, dV_\eta
\end{aligned}
\tag{79}
$$

$$
\begin{aligned}
&\leq \int_{\Omega_1} W_{\mathrm{mp}}(\widetilde{\overline{U}}) + \|DW_{\mathrm{mp}}(\widetilde{\overline{U}})\| \, \|\widetilde{\overline{V}}^{\varepsilon}_{h_j} - \widetilde{\overline{U}}\| + C \, \|\widetilde{\overline{V}}^{\varepsilon}_{h_j} - \widetilde{\overline{U}}\|^2 \\
&\qquad + W^*_{\mathrm{curv}}(\mathfrak{k}^{\sharp,1}_{h_j,\varepsilon}, \mathfrak{k}^{\sharp,2}_{h_j,\varepsilon}, \tilde{A}_{h_j,\varepsilon}) \, dV_\eta
\end{aligned}
$$

for $\|\widetilde{\overline{V}}^{\varepsilon}_{h_j} - \widetilde{\overline{U}}\| \leq 1$ we have

$$
\begin{aligned}
&\leq \int_{\Omega_1} W_{\mathrm{mp}}(\widetilde{\overline{U}}) + \left(C + \|DW_{\mathrm{mp}}(\widetilde{\overline{U}})\|\right) \|\widetilde{\overline{V}}^{\varepsilon}_{h_j} - \widetilde{\overline{U}}\| \\
&\qquad + W^*_{\mathrm{curv}}(\mathfrak{k}^{\sharp,1}_{h_j,\varepsilon}, \mathfrak{k}^{\sharp,2}_{h_j,\varepsilon}, \tilde{A}_{h_j,\varepsilon}) \, dV_\eta
\end{aligned}
$$

since $\|DW_{\mathrm{mp}}(\widetilde{\overline{U}})\| \leq C_2 \|\widetilde{\overline{U}}\|$ we obtain

$$
\leq \int_{\Omega_1} W_{\mathrm{mp}}(\widetilde{\overline{U}}) + \left(C + \|\widetilde{\overline{U}}\|\right) \|\widetilde{\overline{V}}^{\varepsilon}_{h_j} - \widetilde{\overline{U}}\| + W^*_{\mathrm{curv}}(\mathfrak{k}^{\sharp,1}_{h_j,\varepsilon}, \mathfrak{k}^{\sharp,2}_{h_j,\varepsilon}, \tilde{A}_{h_j,\varepsilon}) \, dV_\eta
$$

and by Hölder's inequality we get

$$
\begin{aligned}
&\leq \int_{\Omega_1} W_{\mathrm{mp}}(\widetilde{\overline{U}}) + W^*_{\mathrm{curv}}(\mathfrak{k}^{\sharp,1}_{h_j,\varepsilon}, \mathfrak{k}^{\sharp,2}_{h_j,\varepsilon}, \tilde{A}_{h_j,\varepsilon}) \, dV_\eta \\
&\qquad + \left(C + \|\widetilde{\overline{U}}\|_{L^2(\Omega_1)}\right) \|\widetilde{\overline{V}}^{\varepsilon}_{h_j} - \widetilde{\overline{U}}\|_{L^2(\Omega_1)}.
\end{aligned}
$$

Continuing the estimate with regard to $W^*_{\mathrm{curv}}(\mathfrak{k}^{\sharp,1}_{h_j,\varepsilon}, \mathfrak{k}^{\sharp,2}_{h_j,\varepsilon}, \tilde{A}_{h_j,\varepsilon})$ and adding and subtracting $\widetilde{\overline{V}}^{\varepsilon}_0$ we may obtain

$$
\begin{aligned}
&I^{\sharp}_{h_j}(\varphi^{\sharp}_{h_j,\varepsilon}, \overline{R}^{\sharp}_{h_j,\varepsilon}) \\
&\quad \leq \int_{\Omega_1} W_{\mathrm{mp}}(\widetilde{\overline{U}}) + W^*_{\mathrm{curv}}(\mathfrak{k}^1_0, \mathfrak{k}^2_0, A^*) + W^*_{\mathrm{curv}}(\mathfrak{k}^{\sharp,1}_{h_j,\varepsilon}, \mathfrak{k}^{\sharp,2}_{h_j,\varepsilon}, \tilde{A}_{h_j,\varepsilon}) \\
&\qquad - W^*_{\mathrm{curv}}(\mathfrak{k}^1_0, \mathfrak{k}^2_0, A^*) \, dV_\eta \\
&\qquad + \left(C + \|\widetilde{\overline{U}}\|_{L^2(\Omega_1)}\right) \|\widetilde{\overline{V}}^{\varepsilon}_{h_j} - \widetilde{\overline{V}}^{\varepsilon}_0 + \widetilde{\overline{V}}^{\varepsilon}_0 - \widetilde{\overline{U}}\|_{L^2(\Omega_1)}
\end{aligned}
$$

Thus

$$
\begin{aligned}
I_{h_j}^{\sharp}(\varphi_{h_j,\varepsilon}^{\sharp}, &\overline{R}_{h_j,\varepsilon}^{\sharp}) \\
&\leq \int_{\Omega_1} W_{\mathrm{mp}}(\widetilde{\overline{U}}) + W_{\mathrm{curv}}^*(\mathfrak{k}_0^1, \mathfrak{k}_0^2, A^*) \, \mathrm{d}V_\eta \\
&\quad + \|W_{\mathrm{curv}}^*(\mathfrak{k}_{h_j,\varepsilon}^{\sharp,1}, \mathfrak{k}_{h_j,\varepsilon}^{\sharp,2}, \tilde{A}_{h_j,\varepsilon}) - W_{\mathrm{curv}}^*(\mathfrak{k}_0^1, \mathfrak{k}_0^2, A_\varepsilon)\|_{L^1(\Omega_1)} \quad\quad (80) \\
&\quad + \|W_{\mathrm{curv}}^*(\mathfrak{k}_0^1, \mathfrak{k}_0^2, A_\varepsilon) - W_{\mathrm{curv}}^*(\mathfrak{k}_0^1, \mathfrak{k}_0^2, A^*)\|_{L^1(\Omega_1)} \\
&\quad + \left(C + \|\widetilde{\overline{U}}\|_{L^2(\Omega_1)}\right) \left(\|\widetilde{V}_{h_j}^\varepsilon - \widetilde{V}_0^\varepsilon\|_{L^2(\Omega_1)} + \|\widetilde{V}_0^\varepsilon - \widetilde{\overline{U}}\|_{L^2(\Omega_1)}\right).
\end{aligned}
$$

Now take $h_j \to 0$ to obtain by the continuity of W_{curv}^* in its first two arguments and $(77)_3$

$$
\begin{aligned}
\limsup_{h_j \to 0} I_{h_j}^{\sharp}(\varphi_{h_j,\varepsilon}^{\sharp}, \overline{R}_{h_j,\varepsilon}^{\sharp}) &\leq \int_{\Omega_1} W_{\mathrm{mp}}(\widetilde{\overline{U}}) + W_{\mathrm{curv}}^*(\mathfrak{k}_0^1, \mathfrak{k}_0^2, A^*) \, \mathrm{d}V_\eta \\
&\quad + \|W_{\mathrm{curv}}^*(\mathfrak{k}_0^1, \mathfrak{k}_0^2, A_\varepsilon) - W_{\mathrm{curv}}^*(\mathfrak{k}_0^1, \mathfrak{k}_0^2, A^*)\|_{L^1(\Omega_1)} \\
&\quad + \left(C + \|\widetilde{\overline{U}}\|_{L^2(\Omega_1)}\right) \|\widetilde{V}_0^\varepsilon - \widetilde{\overline{U}}\|_{L^2(\Omega_1)}. \quad (81)
\end{aligned}
$$

Since

$$
\begin{aligned}
\|\widetilde{V}_0^\varepsilon - \widetilde{\overline{U}}\|^2 &= \|\overline{R}_0^T (\nabla\varphi_0(\eta_1,\eta_2)|b_\varepsilon) - \overline{R}_0^T (\nabla\varphi_0(\eta_1,\eta_2)|b^*)\|^2 \\
&= \|\overline{R}_0^T ((\nabla\varphi_0(\eta_1,\eta_2)|b_\varepsilon) - (\nabla\varphi_0(\eta_1,\eta_2)|b^*))\|^2 \\
&= \|(\nabla\varphi_0(\eta_1,\eta_2)|b_\varepsilon) - (\nabla\varphi_0(\eta_1,\eta_2)|b^*)\|^2 = \|b_\varepsilon - b^*\|^2, \quad (82)
\end{aligned}
$$

we get, by letting $\varepsilon \to 0$ and using now the continuity of W_{curv}^* in its last argument together with $\|A_\varepsilon - A^*\|_{L^{1+p+q}(\omega,\mathfrak{so}(3))} < \varepsilon$, the bound

$$
\begin{aligned}
\limsup_{h_j \to 0} I_{h_j}^{\sharp}(\varphi_{h_j,\varepsilon}^{\sharp}, \overline{R}_{h_j,\varepsilon}^{\sharp}) &\leq \int_{\Omega_1} W_{\mathrm{mp}}(\widetilde{\overline{U}}) + W_{\mathrm{curv}}^*(\mathfrak{k}_0^1, \mathfrak{k}_0^2, A^*) \, \mathrm{d}V_\eta \\
&= \int_{\Omega_1} W_{\mathrm{mp}}(\widetilde{\overline{U}}) + W_{\mathrm{curv}}^{*,\mathrm{hom}}(\mathfrak{k}_0^1, \mathfrak{k}_0^2) \, \mathrm{d}V_\eta \\
&= \int_{\Omega_1} W_{\mathrm{mp}}^{\mathrm{hom}}(\nabla\varphi_0, \overline{R}_0) + W_{\mathrm{curv}}^{\mathrm{hom}}(\mathfrak{K}_0) \, \mathrm{d}V_\eta. \quad (83)
\end{aligned}
$$

Since $\varphi_0, \overline{R}_0$ are two-dimensional (independent of the transverse variable),

we may write as well

$$\limsup_{h_j \to 0} I_{h_j}^{\sharp}(\varphi_{h_j,\varepsilon}^{\sharp}, \overline{R}_{h_j,\varepsilon}^{\sharp}) \leq \int_{\Omega_1} W_{\mathrm{mp}}^{\hom}(\nabla_{(\eta_1,\eta_2)} \mathrm{Av}.\varphi_0, \overline{R}_0) + W_{\mathrm{curv}}^{\hom}(\mathfrak{K}_0)\, dV_\eta$$
$$= \int_\omega W_{\mathrm{mp}}^{\hom}(\nabla_{(\eta_1,\eta_2)} \mathrm{Av}.\varphi_0, \overline{R}_0) + W_{\mathrm{curv}}^{\hom}(\mathfrak{K}_0)\, d\omega$$
$$= I_0^{\sharp}(\varphi_0, \overline{R}_0)\,, \tag{84}$$

which shows the desired upper bound. Note that the appearance of the averaging operator Av is not strictly necessary since the limit problem for $\mu_c > 0$ is independent of the transverse variable anyhow. This finishes the proof of Theorem 25.   □

7 Proof for zero Cosserat couple modulus $\mu_c = 0$

Now we supply the proof for Theorem 5.3.1, i.e. we show that the formal limit of $\mu_c \to 0$ of the Γ-limit for $\mu_c > 0$ is in fact the Γ-limit for $\mu_c = 0$. This result cannot be inferred from the case with $\mu_c > 0$ since equi-coercivity is lost.

Remark 7.0.1 (Loss of equi-coercivity)**.** If we consider Γ-convergence in the weak topology of $W^{1,2}(\Omega, \mathbb{R}^3)$ for the deformations φ instead of working with the strong topology $L^r(\Omega, \mathbb{R}^3)$, i.e. assuming for minimizing sequences a priori that $\|\nabla\varphi_{h_j}\|_{L^2(\Omega)}$ is bounded, then the problem related to a loss of equi-coercivity does not appear and the Γ-limit result for $\mu_c = 0$ is an easy consequence of the case for $\mu_c > 0$.

For $\mu_c > 0$ equi-coercivity is enough to provide for the uniform bound on the deformation gradients in the minimization process. The crucial question is whether to obtain a uniform bound on the deformation gradients in the minimization process also for $\mu_c = 0$. For thickness $h \to 0$ the deformations of the thin structure might develop high oszillations (wrinkles) which exclude such a bound on the gradients but the sequence of deformations could still converge strongly in $L^r(\Omega, \mathbb{R}^3)$. Therefore, the strong topology of $L^r(\Omega, \mathbb{R}^3)$ is the convenient framework for Γ-convergence results.

In order to circumvent the loss of equi-coercivity we investigate first a lower bound of the rescaled three-dimensional formulation for the limit case $\mu_c = 0$.

7.1   The "membrane" lower bound for $\mu_c = 0$

We introduce a new family of functionals $I_h^{\sharp,\mathrm{mem}} : X' \mapsto \overline{\mathbb{R}}$, where all transverse shear terms have been omitted, more precisely

$$
\begin{aligned}
I_h^{\sharp,\mathrm{mem}}&(\varphi^\sharp, \nabla_\eta^h \varphi^\sharp, \overline{R}^\sharp, \mathrm{D}_\eta^h \overline{R}^\sharp) \\
&= \int_{\eta \in \Omega_1} W_{\mathrm{mp}}(\overline{U}_h^{\sharp,\mathrm{mem}}) + W_{\mathrm{curv}}(\mathfrak{K}_h^\sharp) \, \mathrm{d}V_\eta \mapsto \min. \text{ w.r.t. } (\varphi^\sharp, \overline{R}^\sharp),
\end{aligned}
$$

$$
\varphi^\sharp_{|_{\Gamma_0^1}}(\eta) = g_{\mathrm{d}}^\sharp(\eta) = g_{\mathrm{d}}(\zeta(\eta)) = g_{\mathrm{d}}(\eta_1, \eta_2, h \cdot \eta_3) = g_{\mathrm{d}}(\eta_1, \eta_2, 0),
$$

$$
\overline{U}_h^\sharp = \overline{R}^{\sharp,T} F_h^\sharp,
$$

$$
\overline{U}_h^{\sharp,\mathrm{mem}} = \begin{pmatrix} \overline{U}_{h,11}^\sharp & \overline{U}_{h,12}^\sharp & 0 \\ \overline{U}_{h,21}^\sharp & \overline{U}_{h,22}^\sharp & 0 \\ 0 & 0 & \overline{U}_{h,33}^\sharp \end{pmatrix}
$$

$$
= \begin{pmatrix} \langle \overline{R}_1^{3d,\sharp}, \partial_{\eta_1} \varphi^\sharp \rangle & \langle \overline{R}_1^{3d,\sharp}, \partial_{\eta_2} \varphi^\sharp \rangle & 0 \\ \langle \overline{R}_2^{3d,\sharp}, \partial_{\eta_1} \varphi^\sharp \rangle & \langle \overline{R}_2^{3d,\sharp}, \partial_{\eta_2} \varphi^\sharp \rangle & 0 \\ 0 & 0 & \frac{1}{h} \langle \overline{R}_3^{3d,\sharp}, \partial_{\eta_3} \varphi^\sharp \rangle \end{pmatrix},
$$

$$
\Gamma_0^1 = \gamma_0 \times [-\frac{1}{2}, \frac{1}{2}], \quad \gamma_0 \subset \partial\omega,
$$

$$
\overline{R}^\sharp : \quad \text{free on } \Gamma_0^1, \text{ Neumann-type boundary condition}, \tag{85}
$$

$$
W_{\mathrm{mp}}(\overline{U}_h^{\sharp,\mathrm{mem}}) = \mu \, \| \mathrm{sym}(\overline{U}_h^{\sharp,\mathrm{mem}} - \mathbb{1}) \|^2 + \frac{\lambda}{2} \, \mathrm{tr} \left[\mathrm{sym}(\overline{U}_h^{\sharp,\mathrm{mem}} - \mathbb{1}) \right]^2
$$

$$
=: W_{\mathrm{mp}}^{\mathrm{mem}}(\nabla \varphi_h^\sharp, \overline{R}^\sharp),
$$

$$
W_{\mathrm{curv}}(\mathfrak{K}_h^\sharp) = \mu \frac{\widehat{L}_c^{1+p}}{12} \left(1 + \alpha_4 \, \widehat{L}_c^q \, \|\mathfrak{K}_h^\sharp\|^q \right)
$$

$$
\left(\alpha_5 \, \| \mathrm{sym}\, \mathfrak{K}_h^\sharp \|^2 + \alpha_6 \, \| \mathrm{skew}\, \mathfrak{K}_h^\sharp \|^2 + \alpha_7 \, \mathrm{tr} \left[\mathfrak{K}_h^\sharp \right]^2 \right)^{\frac{1+p}{2}},
$$

$$
\mathfrak{K}_h^\sharp = \overline{R}^{\sharp,T} \, \mathrm{D}_\eta^h \overline{R}^\sharp(\eta).
$$

Note that for $(\varphi^\sharp, \overline{R}^\sharp) \in X$ the product $\overline{U}_h^\sharp$ does not have a classical meaning if $\nabla \varphi^\sharp \notin L^2(\Omega_1, \mathbb{M}^{3\times3})$. However, the product $\overline{U}_h^\sharp$ does already have a distributional meaning because $\overline{R}^\sharp \in W^{1,1+p+q}(\Omega_1, \mathrm{SO}(3))$ and $\nabla \varphi^\sharp \in$

$W^{-1,r}(\Omega_1, \mathbb{M}^{3\times3})$. Accordingly, we define the admissible set

$$\mathcal{A}_h^{\mathrm{mem}} := \{(\varphi, \overline{R}) \in X \mid \mathrm{sym}\, \overline{U}_h^{\sharp,\mathrm{mem}} \in L^2(\Omega_1, \mathbb{M}^{3\times3}),$$
$$\overline{R} \in W^{1,1+p+q}(\Omega_1, \mathrm{SO}(3)),$$
$$\varphi_{|_{\Gamma_0^1}}(\eta) = g_{\mathrm{d}}^{\sharp}(\eta) = g_{\mathrm{d}}(\eta_1, \eta_2, 0)\}, \qquad (86)$$

where the distribution $\overline{U}_h^{\sharp,\mathrm{mem}}$ is regular and belongs to $L^2(\Omega_1, \mathbb{M}^{3\times3})$. As in (21) we extend the rescaled energies to the larger space X through redefining

$$I_h^{\sharp,\mathrm{mem}}(\varphi^{\sharp}, \nabla_{\eta}^h \varphi^{\sharp}, \quad \overline{R}^{\sharp}, \mathrm{D}_{\eta}^h \overline{R}^{\sharp})$$
$$= \begin{cases} I_h^{\sharp,\mathrm{mem}}(\varphi^{\sharp}, \nabla_{\eta}^h \varphi^{\sharp}, \overline{R}^{\sharp}, \mathrm{D}_{\eta}^h \overline{R}^{\sharp}) & \text{if } (\varphi^{\sharp}, \overline{R}^{\sharp}) \in \mathcal{A}_h^{\mathrm{mem}} \\ +\infty & \text{else in } X\,. \end{cases} \qquad (87)$$

Observe that

$$\forall\, h > 0: \quad I_{h|_{\mu_c=0}}^{\sharp}(\varphi^{\sharp}, \nabla_{\eta}^h \varphi^{\sharp}, \overline{R}^{\sharp}, \mathrm{D}_{\eta}^h \overline{R}^{\sharp}) \geq I_h^{\sharp,\mathrm{mem}}(\varphi^{\sharp}, \nabla_{\eta}^h \varphi^{\sharp}, \overline{R}^{\sharp}, \mathrm{D}_{\eta}^h \overline{R}^{\sharp})\,, \qquad (88)$$

which implies (Dal Maso, 1992, Prop. 6.7) that

$$\Gamma - \liminf_h I_{h|_{\mu_c=0}}^{\sharp} \geq \Gamma - \liminf_h I_h^{\sharp,\mathrm{mem}}\,. \qquad (89)$$

Hence $\Gamma - \liminf I_h^{\sharp,\mathrm{mem}}$ provides a lower bound for $\Gamma - \liminf I_{h|_{\mu_c=0}}^{\sharp}$. Putting inequalities (34) and (89) together, we obtain the natural chain of inequalities on X

$$\Gamma - \liminf I_h^{\sharp,\mathrm{mem}} \leq \Gamma - \liminf I_{h|_{\mu_c=0}}^{\sharp} \qquad (90)$$
$$\leq \Gamma - \limsup I_{h|_{\mu_c=0}}^{\sharp} \leq \lim_{\mu_c \to 0} \left(\Gamma - \lim I_{h|_{\mu_c>0}}^{\sharp}\right) =: I_0^{\sharp,0}\,.$$

7.2 A lower bound for the "membrane" lower bound

Let us consider the following energy functional $I_0^{\sharp,\mathrm{mem}} : X \mapsto \overline{\mathbb{R}}$,

$$I_0^{\sharp,\mathrm{mem}}(\varphi, \overline{R}) :=$$
$$\begin{cases} \int\limits_{\omega} W_{\mathrm{mp}}^{\mathrm{hom},0}(\nabla_{(\eta_1,\eta_2)}\, \mathrm{Av}\,.\varphi(\eta_1,\eta_2,\eta_3), \overline{R}) + W_{\mathrm{curv}}^{\mathrm{hom}}(\mathfrak{K}_s)\, \mathrm{d}\omega & (\varphi, \overline{R}) \in \mathcal{A}_0^{\mathrm{mem}} \\ +\infty & \text{else in } X\,, \end{cases}$$
$$\qquad (91)$$

where $W_{\mathrm{mp}}^{\mathrm{hom},0}$ is defined in (37) and the admissible set is now

$$
\begin{aligned}
\mathcal{A}_0^{\mathrm{mem}} := \{ (\varphi, \overline{R}) \in X \mid \ & \mathrm{sym}\,(\overline{R_1}|\overline{R_2})^T \nabla_{(\eta_1 \eta_2)}\,\mathrm{Av}\,.\varphi \in L^2(\Omega_1, \mathbb{M}^{2 \times 2}), \\
& \overline{R} \in W^{1,1+p+q}(\omega, \mathrm{SO}(3)), \\
& \varphi_{|\Gamma_0^1}^\sharp(\eta) = g_{\mathrm{d}}^\sharp(\eta) = g_{\mathrm{d}}(\eta_1, \eta_2, 0) \},
\end{aligned}
\tag{92}
$$

with a distributional meaning for $(\overline{R_1}|\overline{R_2})^T \nabla_{(\eta_1 \eta_2)}\,\mathrm{Av}\,.\varphi$. Note that $I_0^{\sharp,\mathrm{mem}} = I_0^{\sharp,0}$. We show next the

Lemma 7.2.1 (Membrane lower bound). For arbitrary $(\varphi_0^\sharp, \overline{R}_0^\sharp) \in X$ it holds that

$$
I_0^{\sharp,\mathrm{mem}}(\varphi_0, \overline{R}_0) \leq \liminf_{h_j} I_{h_j}^{\sharp,\mathrm{mem}}(\varphi_{h_j}^\sharp, \overline{R}_{h_j}^\sharp),
$$

whenever

$$
\varphi_{h_j}^\sharp \to \varphi_0^\sharp \quad \text{in } L^r(\Omega_1, \mathbb{R}^3), \quad \overline{R}_{h_j}^\sharp \to \overline{R}_0^\sharp \quad \text{in } L^{1+p+q}(\Omega_1, \mathrm{SO}(3)).
$$

Proof. Observe that we can restrict attention to sequences $(\varphi_{h_j}^\sharp, \overline{R}_{h_j}^\sharp) \in X$ such that $I_{h_j}^{\sharp,\mathrm{mem}}(\varphi_{h_j}^\sharp, \overline{R}_{h_j}^\sharp) < \infty$ since otherwise the statement is true anyway. If $I_{h_j}^{\sharp,\mathrm{mem}}(\varphi_{h_j}^\sharp, \overline{R}_{h_j}^\sharp) < \infty$, then equicoercivity w.r.t. rotations remains untouched by a change from W_{mp} to $W_{\mathrm{mp}}^{\mathrm{mem}}$ in the local energy. Hence, as usual by now, we can restrict attention to sequences of rotations $\overline{R}_{h_j}^\sharp$ converging weakly to some $\overline{R}_0 \in W^{1,1+p+q}(\omega, \mathrm{SO}(3))$, defined over the two-dimensional domain ω only. However, we cannot conclude that φ_0 is independent of the transverse variable, contrary to the case with $\mu_c > 0$.

Along sequences $(\varphi_{h_j}^\sharp, \overline{R}_{h_j}^\sharp) \in X$ with finite energy the product $\frac{1}{h_j}\langle \overline{R}_{h_j,3}, \partial_{\eta_3}\varphi_{h_j}^\sharp \rangle$ remains bounded but otherwise indeterminate. Therefore, a trivial lower bound is obtained by minimizing the effect in the 33-component in the local energy $W_{\mathrm{mp}}^{\mathrm{mem}}$. To do this, we need some calculations: for smooth $\varphi : \Omega_1 \mapsto \mathbb{R}^3$, $\overline{R} : \omega \subset \mathbb{R}^2 \mapsto \mathrm{SO}(3)$ define the "director"-vector $b^* = (0, 0, \varrho^*)^T \in \mathbb{R}^3$ with $b(\varrho) = (0, 0, \varrho)^T \in \mathbb{R}^3$ formally through

$$
\begin{aligned}
W_{\mathrm{mp}}^{\mathrm{hom},0}(\nabla_{(\eta_1 \eta_2)}\varphi, \overline{R}) &= W_{\mathrm{mp}}^{\mathrm{mem}}(\overline{R}^T (\nabla_{(\eta_1 \eta_2)}\varphi|b^*)) \\
&:= \inf_{\varrho \in \mathbb{R}} W_{\mathrm{mp}}^{\mathrm{mem}}(\overline{R}^T (\nabla_{(\eta_1 \eta_2)}\varphi|b(\varrho))).
\end{aligned}
\tag{93}
$$

The real number ϱ^*, which realizes this infimum, can be explicitly determined. Without giving the calculation, which follows as in (50), we obtain

$$\varrho^* = 1 - \frac{\lambda}{2\mu + \lambda} \left[\langle (\nabla_{(\eta_1 \eta_2)} \varphi | 0), \overline{R} \rangle - 2 \right]$$

$$= 1 - \frac{\lambda}{2\mu + \lambda} \operatorname{tr} \left[\operatorname{sym}((\overline{R}_1 | \overline{R}_2)^T \nabla_{(\eta_1 \eta_2)} \varphi - \mathbb{1}_2) \right]. \tag{94}$$

Note that if $\overline{R} \in SO(3)$ and $\operatorname{sym}((\overline{R}_1 | \overline{R}_2)^T \nabla_{(\eta_1 \eta_2)} \varphi - \mathbb{1}_2) \in L^2(\Omega_1, \mathbb{R}^3)$ one has $\varrho^* \in L^2(\Omega_1, \mathbb{R}^3)$.

We obtain for $W_{\mathrm{mp}}^{\mathrm{hom},0}(\nabla_{(\eta_1 \eta_2)} \varphi, \overline{R}) := W_{\mathrm{mp}}(\overline{R}^T (\nabla_{(\eta_1 \eta_2)} \varphi | b^*))$ after a lengthy but straightforward computation

$$W_{\mathrm{mp}}^{\mathrm{hom},0}(\nabla_{(\eta_1 \eta_2)} \varphi, \overline{R}) := \mu \| \operatorname{sym}((\overline{R}_1 | \overline{R}_2)^T \nabla_{(\eta_1 \eta_2)} \varphi - \mathbb{1}_2) \|^2 \tag{95}$$

$$+ \frac{\mu\lambda}{2\mu + \lambda} \operatorname{tr} \left[\operatorname{sym}((\overline{R}_1 | \overline{R}_2)^T \nabla_{(\eta_1 \eta_2)} \varphi - \mathbb{1}_2) \right]^2.$$

Along the sequence $(\varphi_{h_j}^\sharp, \overline{R}_{h_j}^\sharp)$ we have therefore by construction,

$$W_{\mathrm{mp}}^{\mathrm{mem}}(\overline{R}_{h_j}^{\sharp,T} \nabla_\eta^{h_j} \varphi_{h_j}^\sharp) = W_{\mathrm{mp}}^{\mathrm{mem}}(\overline{R}_{h_j}^{\sharp,T} (\nabla_{(\eta_1 \eta_2)} \varphi_{h_j}^\sharp | \frac{1}{h_j} \partial_{\eta_3} \varphi_{h_j}^\sharp))$$

$$\geq W_{\mathrm{mp}}^{\mathrm{hom},0}(\nabla_{(\eta_1 \eta_2)} \varphi_{h_j}^\sharp, \overline{R}_{h_j}^\sharp). \tag{96}$$

Hence, integrating and taking the lim inf also

$$\liminf_{h_j} \int_{\Omega_1} W_{\mathrm{mp}}(\overline{R}_{h_j}^{\sharp,T} \nabla_\eta^{h_j} \varphi_{h_j}^\sharp) \, dV_\eta \geq \liminf_{h_j} \int_{\Omega_1} W_{\mathrm{mp}}^{\mathrm{hom},0}(\nabla_{(\eta_1 \eta_2)} \varphi_{h_j}^\sharp, \overline{R}_{h_j}^\sharp) \, dV_\eta. \tag{97}$$

As in (59) (and subsequently) the proof of statement (7.2.1) would be finished, if we could show weak convergence of $\nabla_{(\eta_1 \eta_2)} \varphi_{h_j}^\sharp$ in $L^2(\Omega_1, \mathbb{M}^{3\times 3})$ whenever $\varphi_{h_j}^\sharp \to \varphi_0^\sharp$ strong in $L^r(\Omega_1, \mathbb{R}^3)$ and $I_{h_j}^{\sharp,\mathrm{mem}}(\varphi_{h_j}^\sharp, \overline{R}_{h_j}^\sharp) < \infty$. Boundedness and weak convergence of the sequence $\nabla_{(\eta_1 \eta_2)} \varphi_{h_j}^\sharp$ in $L^2(\Omega_1, \mathbb{M}^{3\times 3})$ is, however, not clear at all, since we now basically control only the "symmetric intrinsic" term $\| \operatorname{sym}((\overline{R}_1 | \overline{R}_2)^T \nabla_{(\eta_1 \eta_2)} \varphi - \mathbb{1}_2) \|^2$ in the integrand. Instead, we will prove a weaker statement, namely that

$$(\overline{R}_{1,h_j}^\sharp | \overline{R}_{2,h_j}^\sharp)^T \nabla_{(\eta_1 \eta_2)} \varphi_{h_j}^\sharp \rightharpoonup (\overline{R}_{1,0}^\sharp | \overline{R}_{2,0}^\sharp)^T \nabla_{(\eta_1 \eta_2)} \varphi_0^\sharp \in L^2(\Omega_1, \mathbb{M}^{2\times 2}), \tag{98}$$

after showing, that the above expressions have a well-defined distributional meaning along the sequence, since $\nabla_{(\eta_1 \eta_2)} \varphi_{h_j}^\sharp$ has no classical meaning if we know only that $\varphi_{h_j}^\sharp \in L^r(\Omega_1, \mathbb{R}^3)$.

In order to give a precise distributional meaning to the expression in (98) along the sequence we define first for smooth $\phi \in C^\infty(\overline{\Omega}_1, \mathbb{R}^3)$ and $\overline{R} \in W^{1,1+p+q}(\Omega_1, \mathrm{SO}(3))$ an intermediate function Ψ,

$$\Psi : \Omega_1 \mapsto \mathbb{R}^2, \quad \Psi(\eta_1, \eta_2, \eta_3) := \begin{pmatrix} \langle \overline{R}_1, \phi \rangle \\ \langle \overline{R}_2, \phi \rangle \end{pmatrix}. \tag{99}$$

This implies that $\Psi \in W^{1,1+p+q}(\Omega_1, \mathbb{R}^2)$. It holds

$$(\overline{R}_1 | \overline{R}_2)^T \nabla_{(\eta_1, \eta_2)} \phi = \begin{pmatrix} \langle \overline{R}_1, \partial_{\eta_1} \phi \rangle & \langle \overline{R}_1, \partial_{\eta_2} \phi \rangle \\ \langle \overline{R}_2, \partial_{\eta_1} \phi \rangle & \langle \overline{R}_2, \partial_{\eta_2} \phi \rangle \end{pmatrix},$$

$$D(\overline{R}_1 | \overline{R}_2).\phi := \begin{pmatrix} \langle \partial_{\eta_1} \overline{R}_1, \phi \rangle & \langle \partial_{\eta_2} \overline{R}_1, \phi \rangle \\ \langle \partial_{\eta_1} \overline{R}_2, \phi \rangle & \langle \partial_{\eta_2} \overline{R}_2, \phi \rangle \end{pmatrix},$$

$$\nabla \Psi = \begin{pmatrix} \partial_{\eta_1} \langle \overline{R}_1, \phi \rangle & \partial_{\eta_2} \langle \overline{R}_1, \phi \rangle \\ \partial_{\eta_1} \langle \overline{R}_2, \phi \rangle & \partial_{\eta_2} \langle \overline{R}_2, \phi \rangle \end{pmatrix}$$

$$= (\overline{R}_1 | \overline{R}_2)^T \nabla_{(\eta_1, \eta_2)} \phi + D(\overline{R}_1 | \overline{R}_2).\phi. \tag{100}$$

The last equality shows

$$(\overline{R}_1 | \overline{R}_2)^T \nabla_{(\eta_1, \eta_2)} \phi := \nabla_{(\eta_1, \eta_2)} \Psi - D(\overline{R}_1 | \overline{R}_2).\phi. \tag{101}$$

We note the local estimate

$$\begin{aligned} \| \operatorname{sym} \nabla_{(\eta_1, \eta_2)} \Psi \|^2 &= \| \operatorname{sym}((\overline{R}_1 | \overline{R}_2)^T \nabla_{(\eta_1, \eta_2)} \phi) + \operatorname{sym}(D(\overline{R}_1 | \overline{R}_2).\phi) \|^2 \\ &\leq 2\| \operatorname{sym}((\overline{R}_1 | \overline{R}_2)^T \nabla_{(\eta_1, \eta_2)} \phi) \|^2 + 2\| \operatorname{sym}(D(\overline{R}_1 | \overline{R}_2).\phi) \|^2 \\ &\leq 2\| \operatorname{sym}((\overline{R}_1 | \overline{R}_2)^T \nabla_{(\eta_1, \eta_2)} \phi) \|^2 + 2\| D(\overline{R}_1 | \overline{R}_2).\phi \|^2 \quad (102) \\ &\leq 2\| \operatorname{sym}((\overline{R}_1 | \overline{R}_2)^T \nabla_{(\eta_1, \eta_2)} \phi) \|^2 + 2\| D(\overline{R}_1 | \overline{R}_2) \|^2 \cdot \| \phi \|^2. \end{aligned}$$

The last inequality implies after integration and Hölder's inequality (reminder $r = \frac{2(1+p+q)}{(1+p+q)-2}$, c.f. (18))

$$\int_{\Omega_1} \| \operatorname{sym} \nabla_{(\eta_1, \eta_2)} \Psi \|^2 \, dV_\eta$$

$$\leq 2 \int_{\Omega_1} \| \operatorname{sym}((\overline{R}_1 | \overline{R}_2)^T \nabla_{(\eta_1, \eta_2)} \phi) \|^2 \, dV_\eta + 2\| \overline{R} \|^2_{W^{1,1+p+q}(\Omega_1)} \| \phi \|^2_{L^r(\Omega_1, \mathbb{R}^3)}. \tag{103}$$

Moreover,

$$\int_{\Omega_1} \| \operatorname{sym} \nabla_{(\eta_1, \eta_2)} \Psi \|^2 + \| \Psi \|^2 \, dV_\eta \leq 2 \int_{\Omega_1} \| \operatorname{sym}((\overline{R}_1 | \overline{R}_2)^T \nabla_{(\eta_1, \eta_2)} \phi) \|^2 \, dV_\eta$$

$$+ 2\| \overline{R} \|^2_{W^{1,1+p+q}(\Omega_1)} \| \phi \|^2_{L^r(\Omega_1, \mathbb{R}^3)} + 2\| \phi \|^2_{L^2(\Omega_1, \mathbb{R}^3)}, \tag{104}$$

since $\|\Psi\|^2 = \langle \overline{R}_1, \phi \rangle^2 + \langle \overline{R}_2, \phi \rangle^2 \leq \|\overline{R}_1\|^2 \|\phi\|^2 + \|\overline{R}_2\|^2 \|\phi\|^2 = 2\|\phi\|^2$.
Furthermore, adding and subtracting $\mathbb{1}_2$

$$\int_{\Omega_1} \|\operatorname{sym} \nabla_{(\eta_1,\eta_2)} \Psi\|^2 + \|\Psi\|^2 \, dV_\eta$$

$$\leq 2 \int_{\Omega_1} \|\operatorname{sym}((\overline{R}_1|\overline{R}_2)^T \nabla_{(\eta_1,\eta_2)} \phi)\|^2 \, dV_\eta$$
$$+ 2\|\overline{R}\|_{W^{1,1+p+q}(\Omega_1)}^2 \|\phi\|_{L^r(\Omega_1,\mathbb{R}^3)}^2 + 2\|\phi\|_{L^2(\Omega_1,\mathbb{R}^3)}^2$$

$$= 2 \int_{\Omega_1} \|\operatorname{sym}((\overline{R}_1|\overline{R}_2)^T \nabla_{(\eta_1,\eta_2)} \phi - \mathbb{1}_2 + \mathbb{1}_2)\|^2 \, dV_\eta$$
$$+ 2\|\overline{R}\|_{W^{1,1+p+q}(\Omega_1)}^2 \|\phi\|_{L^r(\Omega_1,\mathbb{R}^3)}^2 + 2\|\phi\|_{L^2(\Omega_1,\mathbb{R}^3)}^2$$

$$\leq \int_{\Omega_1} 4\|\operatorname{sym}((\overline{R}_1|\overline{R}_2)^T \nabla_{(\eta_1,\eta_2)} \phi - \mathbb{1}_2)\|^2 + 4\|\mathbb{1}_2\|^2 \, dV_\eta \qquad (105)$$
$$+ 2\|\overline{R}\|_{W^{1,1+p+q}(\Omega_1)}^2 \|\phi\|_{L^r(\Omega_1,\mathbb{R}^3)}^2 + 2\|\phi\|_{L^2(\Omega_1,\mathbb{R}^3)}^2 \,.$$

Hence, considering $\varphi_{h_j}^\sharp$ instead of ϕ we obtain along the sequence $(\varphi_{h_j}^\sharp, \overline{R}_{h_j}^\sharp) \in X$ with

$$I_{h_j}^{\sharp,\mathrm{mem}}(\varphi_{h_j}^\sharp, \overline{R}_{h_j}^\sharp) < \infty \,, \qquad (106)$$

and the distributional meaning of the gradient on $\varphi_{h_j}^\sharp$ the additional uniform bound

$$\int_{\Omega_1} \|\operatorname{sym} \nabla_{(\eta_1,\eta_2)} \Psi\|^2 + \|\Psi\|^2 \, dV_\eta$$
$$\leq \frac{4}{\mu} I_{h_j}^{\sharp,\mathrm{mem}}(\varphi_{h_j}^\sharp, \overline{R}_{h_j}^\sharp) + \int_{\Omega_1} 4\|\mathbb{1}_2\|^2 \, dV_\eta \qquad (107)$$
$$+ 2\|\overline{R}_{h_j}^\sharp\|_{W^{1,1+p+q}(\Omega_1)}^2 \|\varphi_{h_j}^\sharp\|_{L^r(\Omega_1,\mathbb{R}^3)}^2 + 2\|\varphi_{h_j}^\sharp\|_{L^2(\Omega_1,\mathbb{R}^3)}^2 < \infty \,.$$

The classical Korn's second inequality without boundary conditions on a Lipschitz domain (Temam, 1985, Prop.1.1) implies therefore that

$$\infty > \int_{\Omega_1} \|\operatorname{sym} \nabla_{(\eta_1,\eta_2)} \Psi_{h_j}\|^2 + \|\Psi_{h_j}\|^2 \, dV_\eta \qquad (108)$$

$$= \int_{-1/2}^{1/2} \left[\int_\omega \|\operatorname{sym} \nabla_{(\eta_1,\eta_2)} \Psi_{h_j}(\eta_1,\eta_2,\eta_3)\|^2 + \|\Psi_{h_j}(\eta_1,\eta_2,\eta_3)\|^2 \, d\omega \right] d\eta_3$$

$$\geq \int_{-1/2}^{1/2} \left[c_K^+ \int_\omega \|\nabla_{(\eta_1,\eta_2)} \Psi_{h_j}(\eta_1,\eta_2,\eta_3)\|^2 + \|\Psi_{h_j}(\eta_1,\eta_2,\eta_3)\|^2 \, d\omega \right] d\eta_3 \,,$$

which allows us to conclude the boundedness of $\nabla_{(\eta_1,\eta_2)}\Psi_{h_j}$ in $L^2(\Omega_1,\mathbb{R}^2)$ and weak convergence of this sequence of gradients to a limit. By construction we know already that $\Psi_{h_j} \to \Psi_0 \in L^2(\Omega_1,\mathbb{R}^2)$ (assumed strong convergence of $\overline{R}_{h_j}$ and $\varphi_{h_j}^\sharp$). Hence $\nabla_{(\eta_1,\eta_2)}\Psi_{h_j}$ converges weakly to $\nabla_{(\eta_1,\eta_2)}\Psi_0$. Since we know as well that $\partial_{\eta_i}\overline{R}_{h_j}^\sharp \rightharpoonup \partial_{\eta_i}\overline{R}_0^\sharp$ in $L^{1+p+q}(\Omega_1,\mathbb{M}^{3\times 3})$, $i = 1,2$ and $\varphi_{h_j}^\sharp \to \varphi_0^\sharp$ in $L^r(\Omega_1,\mathbb{R}^3)$ we obtain

$$D(\overline{R}_{1,h_j}^\sharp|\overline{R}_{2,h_j}^\sharp).\phi_{h_j}^\sharp \rightharpoonup D(\overline{R}_{1,0}^\sharp|\overline{R}_{2,0}^\sharp).\phi_0^\sharp \in L^2(\Omega_1,\mathbb{M}^{2\times 2}). \tag{109}$$

Looking now back at (101) shows that

$$(\overline{R}_{1,h_j}^\sharp|\overline{R}_{2,h_j}^\sharp)^T \nabla_{(\eta_1,\eta_2)}\varphi_{h_j}^\sharp \in L^2(\Omega_1,\mathbb{M}^{2\times 2}), \tag{110}$$

is a well defined expression with distributional meaning of $\nabla_{(\eta_1,\eta_2)}\varphi_{h_j}^\sharp$ for which (98) holds. Due to the convexity of $W_{\mathrm{mp}}^{\mathrm{hom},0}$ in the argument $\mathrm{sym}((\overline{R}_1|\overline{R}_2)^T \nabla_{(\eta_1,\eta_2)}\varphi$, we may pass to the limit in (97) to obtain

$$\liminf_{h_j} \int_{\Omega_1} W_{\mathrm{mp}}(\overline{R}_{h_j}^{\sharp,T}\nabla_\eta^{h_j}\varphi_{h_j}^\sharp)\,\mathrm{d}V_\eta \geq \int_{\Omega_1} W_{\mathrm{mp}}^{\mathrm{hom},0}(\nabla_{(\eta_1,\eta_2)}\varphi_0^\sharp,\overline{R}_0^\sharp)\,\mathrm{d}V_\eta. \tag{111}$$

The convexity of $W_{\mathrm{mp}}^{\mathrm{hom},0}$ and Jensen's inequality (24) show then

$$\int_\omega W_{\mathrm{mp}}^{\mathrm{hom},0}(\nabla_{(\eta_1,\eta_2)}\,\mathrm{Av}\,.\varphi(\eta_1,\eta_2),\overline{R})\,\mathrm{d}\omega$$
$$\leq \int_\omega \int_{-1/2}^{1/2} W_{\mathrm{mp}}^{\mathrm{hom},0}(\nabla_{(\eta_1,\eta_2)}\varphi(\eta_1,\eta_2,\eta_3),\overline{R})\,\mathrm{d}\eta_3\,\mathrm{d}\omega$$
$$= \int_{\Omega_1} W_{\mathrm{mp}}^{\mathrm{hom},0}(\nabla_{(\eta_1,\eta_2)}\varphi(\eta_1,\eta_2,\eta_3),\overline{R})\,\mathrm{d}V_\eta. \tag{112}$$

Combining (112) with (111) gives

$$\liminf_{h_j} \int_{\Omega_1} W_{\mathrm{mp}}(\overline{R}_{h_j}^{\sharp,T}\nabla_\eta^{h_j}\varphi_{h_j}^\sharp)\,\mathrm{d}V_\eta \geq \int_\omega W_{\mathrm{mp}}^{\mathrm{hom},0}(\nabla_{(\eta_1,\eta_2)}\,\mathrm{Av}\,.\varphi(\eta_1,\eta_2),\overline{R})\,\mathrm{d}\omega. \tag{113}$$

The proof of Lemma 7.2.1 is finished along the lines of (59). Note that (110) does definitely not yield control of $\nabla_{(\eta_1,\eta_2)}\varphi_{h_j}^\sharp$ in $L^2(\Omega_1,\mathbb{M}^{3\times 2})$. □
Proof of Theorem 5.3.1: To finish the proof of Γ-convergence for zero Cosserat couple modulus Theorem 5.3.1 we observe first that Lemma 7.2.1

implies that

$$I_0^{\sharp,\mathrm{mem}} \leq \Gamma - \liminf_{h_j} I_{h_j}^{\sharp,\mathrm{mem}}, \tag{114}$$

which is "almost" a liminf result for $I_h^{\sharp,\mathrm{mem}}$ since $I_0^{\sharp,\mathrm{mem}}$ could be strictly smaller. We combine this result with the chain of inequalities (115) which yields that on $X = L^r(\Omega_1, \mathbb{R}^3) \times L^{1+p+q}(\Omega_1, \mathrm{SO}(3))$

$$I_0^{\sharp,\mathrm{mem}} \leq \Gamma - \liminf I_h^{\sharp,\mathrm{mem}} \leq \Gamma - \liminf I_{h|_{\mu_c=0}}^{\sharp}$$
$$\leq \Gamma - \limsup I_{h|_{\mu_c=0}}^{\sharp} \leq \lim_{\mu_c \to 0} \left(\Gamma - \lim I_{h|_{\mu_c>0}}^{\sharp} \right) =: I_0^{\sharp,0}. \tag{115}$$

Since, however, $I_0^{\sharp,\mathrm{mem}} = I_0^{\sharp,0}$, the last inequality is in fact an equality, which shows that

$$\Gamma - \lim I_{h_j|_{\mu_c=0}}^{\sharp} = I_0^{\sharp,0}. \tag{116}$$

This gives us complete information on the behaviour of sequences of minimizing problems for $\mu_c = 0$, should such sequences exist and converge to a limit in the encompassing space X. ☐

8 Comparison with the formal finite-strain Cosserat thin plate model with size effects

Statement of the formal Cosserat plate model

A formal "rational" of dimensional descend has led us in Neff (2004b, 2007) to **p**ostulate the following two-dimensional minimization problem for the deformation of the midsurface $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ and the microrotation of the plate (shell) $\overline{R} : \omega \subset \mathbb{R}^2 \mapsto \mathrm{SO}(3)$ on ω:

$$I(m,\overline{R}) = \int_\omega h\, W_{\mathrm{mp}}(\overline{U}) + h\, W_{\mathrm{curv}}(\mathfrak{K}_s) + \frac{h^3}{12}\, W_{\mathrm{bend}}(\mathfrak{K}_b)\, d\omega$$
$$- \Pi(m, \overline{R}_3) \mapsto \min. \text{ w.r.t. } (m,\overline{R}), \tag{1}$$

under the constraints

$$\overline{U} = \overline{R}^T \widehat{F}, \qquad \widehat{F} = (\nabla m | \overline{R}_3) \in \mathbb{M}^{3\times 3}, \tag{2}$$

$$\mathfrak{K}_s = \left(\overline{R}^T (\nabla(\overline{R}.e_1)|0), \overline{R}^T(\nabla(\overline{R}.e_2)|0), \overline{R}^T(\nabla(\overline{R}.e_3)|0) \right) \in \mathfrak{T}(3), \quad \mathfrak{K}_b = \mathfrak{K}_s^3,$$

and the boundary conditions of place for the midsurface deformation m on the Dirichlet part of the lateral boundary γ_0,

$$m_{|\gamma_0} = g_{\mathrm{d}}(x, y, 0)\,, \qquad \text{simply supported (fixed, welded)}\,. \qquad (3)$$

The basically three possible **a**lternative boundary conditions (consistent coupling, free, rigid) for the microrotations $\overline{R}$ on γ_0 are

$$\overline{R}_{|\gamma_0} = \mathrm{polar}((\nabla m | \nabla g_{\mathrm{d}}(x, y, 0).e_3))_{|\gamma_0}\,,$$

strong form of reduced consistent coupling,

$$\forall\, A \in C_0^\infty(\gamma_0, \mathfrak{so}(3)): \qquad\qquad\qquad\qquad\qquad\qquad (4)$$

$$\int_{\gamma_0} \langle \overline{R}^T\, (\nabla m(x, y) | \nabla g_{\mathrm{d}}(x, y, 0).e_3), A(x, y) \rangle \, \mathrm{d}s = 0\,,$$

very **w**eak consistent coupling,

$$\overline{R}_{|\gamma_0}: \qquad\quad \textbf{f}\text{ree on } \gamma_0,\ \text{Neumann-type boundary condition}\,,$$

$$\overline{R}_{3_{|\gamma_0}} = \frac{\nabla g_{\mathrm{d}}(x, y, 0).e_3}{\|\nabla g_{\mathrm{d}}(x, y, 0).e_3\|}\,, \qquad \textbf{r}\text{igid director prescription}\,.$$

The constitutive assumptions on the reduced densities are for the strain energy[19]

$$
\begin{aligned}
W_{\mathrm{mp}}(\overline{U}) &= \mu\,\|\operatorname{sym}(\overline{U} - \mathbb{1})\|^2 + \mu_c\,\|\operatorname{skew}(\overline{U})\|^2 \\
&\quad + \frac{\mu\lambda}{2\mu + \lambda}\,\operatorname{tr}\left[\operatorname{sym}(\overline{U} - \mathbb{1})\right]^2 \\
&= \mu\,\underbrace{\|\operatorname{sym}((\overline{R}_1|\overline{R}_2)^T \nabla m - \mathbb{1}_2)\|^2}_{\text{shear-stretch energy}} + \mu_c\,\underbrace{\|\operatorname{skew}((\overline{R}_1|\overline{R}_2)^T \nabla m)\|^2}_{\text{first order drill energy}} \\
&\quad + \underbrace{\frac{\kappa(\mu + \mu_c)}{2}\left(\langle \overline{R}_3, m_x\rangle^2 + \langle \overline{R}_3, m_y\rangle^2\right)}_{\text{classical transverse shear energy}} \\
&\quad + \frac{\mu\lambda}{2\mu + \lambda}\,\underbrace{\operatorname{tr}\left[\operatorname{sym}((\overline{R}_1|\overline{R}_2)^T \nabla m - \mathbb{1}_2)\right]^2}_{\text{elongational stretch energy}}\,,
\end{aligned}
$$

$$(5)$$

---

[19] $\|\operatorname{skew}((\overline{R}_1|\overline{R}_2)^T \nabla m)\|^2 = (\langle \overline{R}_1, m_y\rangle - \langle \overline{R}_2, m_x\rangle)^2.$

and for the curvature energy

$$W_{\mathrm{curv}}(\mathfrak{K}_s) = \mu \frac{L_c^{1+p}}{12} \left(1 + \alpha_4 L_c^q \, \|\mathfrak{K}_s\|^q\right)$$

$$\left(\alpha_5 \, \| \operatorname{sym} \mathfrak{K}_s\|^2 + \alpha_6 \, \| \operatorname{skew} \mathfrak{K}_s\|^2 + \alpha_7 \operatorname{tr}\left[\mathfrak{K}_s\right]^2\right)^{\frac{1+p}{2}} ,$$

$$\mathfrak{K}_s = \left(\overline{R}^T(\nabla(\overline{R}.e_1)|0), \overline{R}^T(\nabla(\overline{R}.e_2)|0), \overline{R}^T(\nabla(\overline{R}.e_3)|0)\right) ,$$

$$\mathfrak{K}_s = (\mathfrak{K}_s^1, \mathfrak{K}_s^2, \mathfrak{K}_s^3) \in \mathfrak{T}(3) ,$$

the reduced third order **c**urvature tensor ,

$$W_{bend}(\mathfrak{K}_b) = \mu \, \| \operatorname{sym}(\mathfrak{K}_b)\|^2 + \mu_c \, \| \operatorname{skew}(\mathfrak{K}_b)\|^2 + \frac{\mu\lambda}{2\mu + \lambda} \operatorname{tr}\left[\operatorname{sym}(\mathfrak{K}_b)\right]^2 ,$$

$$\mathfrak{K}_b = \overline{R}^T(\nabla\overline{R}_3|0) = \mathfrak{K}_s^3 ,$$

the second order **n**on-symmetric bending tensor .

The (relative) thickness of the plate (shell) is $h > 0$. The total elastically stored energy density due to **m**embrane-strain, total **p**late-curvature and specific **p**late-bending

$$W = \underbrace{h \, W_{\mathrm{mp}}}_{\text{membrane}} + \underbrace{h \, W_{\mathrm{curv}}}_{\text{curvature}} + \underbrace{\frac{h^3}{12} W_{bend}}_{\text{bending}} , \qquad (6)$$

depends on the midsurface deformation gradient ∇m and microrotations $\overline{R}$ together with their space derivatives only through the frame-indifferent measures $\overline{U}$ and $\mathfrak{K}_s$. The **m**icropolar stretch tensor $\overline{U}$ of the plate is in general **n**on-symmetric, neither is the **m**icropolar reduced third order curvature tensor $\mathfrak{K}_s$. The three-dimensional plate deformation is supposed to be reconstructed as

$$\varphi_s(x, y, z) = m(x, y) + \left(z \, \varrho_m(x, y) + \frac{z^2}{2} \, \varrho_b(x, y)\right) \overline{R}(x, y).e_3 , \qquad (7)$$

where

$$\varrho_m = 1 - \frac{\lambda}{2\mu + \lambda}\left[\langle(\nabla m|0), \overline{R}\rangle - 2\right] + \frac{\langle N_{\text{diff}}, \overline{R}_3\rangle}{(2\mu + \lambda)}$$

$$= \underbrace{1 - \frac{\lambda}{2\mu + \lambda}\text{tr}\left[\overline{U} - \mathbb{1}\right] + \frac{\langle N_{\text{diff}}, \overline{R}_3\rangle}{(2\mu + \lambda)}}_{\text{first order thickness change due to elongational stretch}},$$

$$\varrho_b = \underbrace{-\frac{\lambda}{2\mu + \lambda}\langle(\nabla \overline{R}_3|0), \overline{R}\rangle + \frac{\langle N_{\text{res}}, \overline{R}_3\rangle}{(2\mu + \lambda)\,h}}_{\text{non-symmetric shift of the midsurface due to bending}}$$

$$= -\frac{\lambda}{2\mu + \lambda}\,\text{tr}\left[\mathfrak{K}_b\right] + \frac{\langle N_{\text{res}}, \overline{R}_3\rangle}{(2\mu + \lambda)\,h} \tag{8}$$

and N_{diff}, N_{res} given by

$$N_{\text{res}} := \left[N(x, y, \frac{h}{2}) + N(x, y, -\frac{h}{2})\right],$$

$$N_{\text{diff}} := \frac{1}{2}\left[N(x, y, \frac{h}{2}) - N(x, y, -\frac{h}{2})\right]. \tag{9}$$

To first order, the reconstructed deformationgradient is given by $F_s = (\nabla m | \varrho_m \, \overline{R}_3)$. Here $\omega \subset \mathbb{R}^2$ is a domain with boundary $\partial\omega$ and $\gamma_0 \subset \partial\omega$ is that part of the boundary, where Dirichlet conditions g_d for deformations and microrotations and/or consistent coupling conditions for microrotations, respectively, are prescribed. The reduced external loading functional $\Pi(m, \overline{R}_3)$ is a linear form in $(m, \overline{R}_3)$ in terms of the underlying three-dimensional loads. The parameters $\mu, \lambda > 0$ are the Lamé constants of classical elasticity, $\mu_c \geq 0$ is called the Cosserat couple modulus and $L_c > 0$ introduces the internal length. We assume throughout that $\alpha_5 > 0, \alpha_6 > 0, \alpha_7 \geq 0$. We have included the so called **s**hear correction factor κ ($0 < \kappa \leq 1$) to keep in line with classical infinitesimal-displacement plate models. In our formal derivation, however, we obtain $\kappa = 1$. The reduced model (1) is fully **f**rame-indifferent, meaning that

$$\forall\, Q \in SO(3): \quad W_{\text{mp}}(Q\widehat{F}, Q\overline{R}) = W_{\text{mp}}(\widehat{F}, \overline{R}), \quad \mathfrak{K}_s(Q\overline{R}) = \mathfrak{K}_s(\overline{R}). \tag{10}$$

The non-invariant term ϱ_m is only needed to reconstruct the 3D-deformation, which depends on the non-invariant loading.[20] **S**train and **c**urvature parts

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[20]Of course, if the external tractions are rotated as well, we obtain invariance: $\langle Q.N_{\text{diff}}, Q.\overline{R}_3\rangle = \langle N_{\text{diff}}, \overline{R}_3\rangle$.

are **a**dditively decoupled, as in the underlying parent Cosserat bulk model
(1). We note the appearance of the **h**armonic mean $\mathcal{H}$ and **a**rithmetic mean
$\mathcal{A}$

$$\frac{1}{2}\mathcal{H}(\mu, \frac{\lambda}{2}) = \frac{\mu\lambda}{2\mu + \lambda}\,, \qquad \kappa\, \mathcal{A}(\mu, \mu_c) = \kappa\, \frac{\mu + \mu_c}{2}\,. \tag{11}$$

8.1 Mathematical results for the formal Cosserat thin plate model

For conciseness we state only the obtained results for the case without ex-
ternal loads. It can be shown directly, without recourse to three-dimensional
considerations Neff (2004b):

Theorem 8.1.1 (Existence for 2D-Cosserat thin plate with $\mu_c > 0$ and
$\kappa > 0$). Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and assume for the
boundary data $g_{\mathrm{d}} \in H^1(\omega, \mathbb{R}^3)$ and $\overline{R}_{\mathrm{d}} \in W^{1,1+p}(\omega, \mathrm{SO}(3))$. Then (1) with
$\mu_c > 0$, $\kappa > 0$, $\alpha_4 \geq 0$, $p \geq 1$, $q \geq 0$ and either free or rigid prescription for
$\overline{R}$ on γ_0 admits at least one minimizing solution pair $(m, \overline{R}) \in H^1(\omega, \mathbb{R}^3) \times$
$W^{1,1+p}(\omega, \mathrm{SO}(3))$. ◻

Using the extended Korn's inequality Neff (2002); Pompe (2003), the
following has been shown in Neff (2007):

Theorem 8.1.2 (Existence for 2D-Cosserat thin plate with $\mu_c = 0$ and
$\kappa > 0$). Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and assume for the
boundary data $g_{\mathrm{d}} \in H^1(\omega, \mathbb{R}^3)$ and $\overline{R}_{\mathrm{d}} \in W^{1,1+p+q}(\omega, \mathrm{SO}(3))$. Then (1)
with $\mu_c = 0$, $\kappa > 0$, $\alpha_4 > 0$, $p \geq 1$, $q > 0$ and either free or rigid pre-
scription for $\overline{R}$ on γ_0 admits at least one minimizing solution pair $(m, \overline{R}) \in$
$H^1(\omega, \mathbb{R}^3) \times W^{1,1+p+q}(\omega, \mathrm{SO}(3))$. ◻

9 The form of the transverse shear energy for non-vanishing thickness and the shear correction factor κ

Γ-convergence describes the thin shell limit, but misses of course the fact
that in actual computations of thin structures one wants to describe a ma-
terial with finite thickness, which certainly can sustain some amount of
transverse shear.

If we compare the two different limit models (26),(1) described herein,
we see that $\lim_{h_j \to 0} \frac{1}{h_j} I(m, \overline{R})$ in (1) coincides with the Γ-limit $I_0^\sharp$ in (26)
as far as the local energy contribution W_{mp} is concerned, apart from the
coefficient of the transverse shear energy. How then should the transverse

shear contribution a priori look like, starting from a three-dimensional view-point?[21]

There is a large number of papers concerned with the effective (homog-enized) coefficient of the transverse shear energy for isotropic linear elastic bulk material. The transverse shear deformation in the finite-strain Cosserat approach is proportional to $\left(\langle \overline{R}_3, m_x \rangle, \langle \overline{R}_3, m_y \rangle\right)$. The corresponding trans-verse shear energy is proportional to $\langle \overline{R}_3, m_x \rangle^2 + \langle \overline{R}_3, m_y \rangle^2$. If we assume **no** warping (transverse sections remain straight), i.e. an ansatz of the form $\varphi(x, y, z) = m(x, y) + \varrho^+(z)\, \overline{R}(x, y).e_3$ with $\varrho^+ : \mathbb{R} \mapsto \mathbb{R}^+$ and a constant director $\overline{R}.e_3$ over the thickness, the transverse shear energy is generally over-estimated. This ansatz leads to a linear distribution of the transverse shear-stresses in the plate.

From direct equilibrium considerations for the bulk it follows, however, that the director should be S-shaped over the thickness. Including this effect amounts to introduce warping. This corresponds to a "weaker" kinematical ansatz $\varphi(x, y, z) = m(x, y) + \varrho^+(z)\, Q(z)\, \overline{R}(x, y).e_3$ with an additional inde-pendent rotation field $Q \in \mathrm{SO}(3)$, depending only on the transverse variable z Wisniewski and Turska (2001, 2002). It leads to a quadratic distribution of the transverse shear stresses in thickness direction. In order to relieve the effect of not including warping in the simpler ansatz, the introduction of the **s**hear correction factor κ can be motivated.

For both presented models, the transverse shear energy in our notation can be written in the form

$$\mathrm{G}' \left(\langle \overline{R}_3, m_x \rangle^2 + \langle \overline{R}_3, m_y \rangle^2 \right) , \tag{12}$$

with a constitutive coefficient G′, the **t**ransverse shear modulus $[\mathrm{G}'] = [\mathrm{N/m}^2]$.[22] Summarizing, we have

$$\mathrm{G}' = \kappa\, \mathcal{A}(\mu, \mu_c) = \kappa\, \frac{\mu + \mu_c}{2} \qquad\qquad \text{formal reduction (1)},$$

$$\mathrm{G}' = \mathcal{H}(\mu, \mu_c) = 2\mu\, \frac{\mu_c}{\mu + \mu_c} \qquad\qquad \text{Γ-limit (26)}\ ,\ \ (13)$$

$$\mathrm{G}' = \kappa\, \mathcal{A}(\mu, 0) = \kappa\, \frac{\mu}{2} \qquad\qquad \text{classical linear Reissner-Mindlin} ,$$

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[21]The possible difference between W_{curv} and $W_{\mathrm{curv}}^{\mathrm{hom}}$ is not our concern, since the consti-tutive coefficients of W_{curv} are rather a matter of convenience at present, as long as coercivity of curvature is guaranteed.

[22]Mindlin's notation (Mindlin, 1951, eq.7).

with $\kappa \geq 0$, the so called shear correction factor.[23] There are various values for the shear correction factor κ proposed in the engineering literature, among them prominently

$$\kappa = \frac{\pi^2}{12} \approx 0.8225\,, \qquad \text{Mindlin's value Mindlin (1951)}\,,$$

$$\kappa = \frac{87}{100} = 0.8700\,, \qquad \text{Babuska's value for } \nu = 0.3\,,$$

$$\kappa = \frac{10}{12 - 2\nu} \approx 0.8772\,, \qquad \text{Zhilin's value for } \nu = 0.3$$
$$\text{Altenbach and Zhilin (2004)}\,,$$

$$\kappa = \frac{10}{12} \approx 0.8333\,, \qquad \text{Reissner's value Reissner (1945, 1985)}\,, \quad (14)$$

$$\kappa = \frac{10}{12 - 7\nu} \approx 1.01 \qquad \text{Rössle's value for } \nu = 0.3\,,$$

$$\frac{\pi^2}{12} \leq \kappa < 1\,, \;\; \text{Altenbach's estimate Altenbach and Zhilin (2004)}\,.$$

These values for κ are proposed in terms of best fitting of certain simple infinitesimal three-dimensional quasistatic or dynamic test cases. Mindlin's value $\kappa = \frac{\pi^2}{12}$ is obtained from a best fit of the first eigenfrequency of the linearized plate model as compared to the three-dimensional linear elasticity solution. Reissner's value appears through an additional assumption regarding the stress distribution through the thickness (Reissner, 1945, eq.10). Babuska's value Babuska and Li (1992) is based on numerical "experiments". By dimensional analysis it can be shown Altenbach and Zhilin (2004) that κ should depend on the Lamé constants only through the Poisson ratio $0 < \nu < \frac{1}{2}$. Another motivation for the introduction of κ is obtained by trying to optimize the rate of convergence of the linear Reissner-Mindlin model to the solution of the linear elasticity model as $h \to 0$. This is

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[23] "In the classical Reissner-Mindlin model, the shear stresses $\sigma_{13}, \sigma_{23} (= \langle \overline{R}_3, m_x \rangle, \langle \overline{R}_3, m_y \rangle)$ are constant through the thickness of the plate. However, three-dimensional traction free boundary conditions at the upper and lower face of the shell imply that at these faces, the stresses have to be zero, hence also the shear stresses have to be zero. An analysis of equilibrium for an elastic beam shows that the shear stress should be quadratic through the thickness and vanish at the faces. A constant shear stress distribution over the thickness overestimates therefore the shear energy. A correction factor, known as the shear correction factor is often used to reduce the energy associated with transverse shear and accurate estimates of this factor can be made for elastic beams and shells. For nonlinear materials, however, it is difficult to estimate a shear correction factor." (Belytschko et al., 2000, p.554).

the argument for Rössle's value, Rössle (1999). The fact that there κ might be bigger than one cannot easily be accepted from a purely engineering point of view.

For $0 \leq \kappa = \frac{4\mu\mu_c}{(\mu+\mu_c)^2} \leq 1$ it holds that $\kappa\,\mathcal{A}(\mu,\mu_c) = \mathcal{H}(\mu,\mu_c)$. Hence, in view of our deduction of the Γ-limit as compared to the formal reduction and the general inequality $\mathcal{H}(\mu,\mu_c) \leq \mathcal{A}(\mu,\mu_c)$ together with the linearization consistency of the Γ-limit (35) if $\mu_c = 0$ **it** is strongly suggested that $\kappa < 1$, in accordance with engineering practice, also in the finite strain case.

The question of the form of the homogenized transverse shear energy is as well related to the observation, that the Γ-limit energy functional for $\mu_c = 0$ will necessarily loose coercivity, which can directly be traced to the missing transverse shear contribution but this loss of coercivity is not due to the missing drill-energy. In this respect, note that $W_{\mathrm{mp}}(\overline{U})$ in (5) leads to a coercive formulation w.r.t. the midsurface deformation m also for $\mu_c = 0$. Moreover, in a linearized context, this energy is asymptotically correct for $\mu_c = 0$ and $\kappa = 1$.

For numerical calculations, the "homogenized" energy $I_0^{\sharp,0}$, which is indeed the Γ-limit energy functional for $\mu_c = 0$, can hardly be regarded as suitable in this case. From a more practical, computational viewpoint then, the introduction of a strictly positive shear correction factor $0 < \kappa < 1$ is fully justified and provides exactly that necessary minimal change of the local energy used in $I_0^{\sharp,0}$, in order to re-establish first strict Legendre-Hadamard ellipticity w.r.t. m (but not local strict convexity) and second coercivity for the midsurface in $H^{1,2}(\omega,\mathbb{R}^3)$. This underlines the salient features of the formal derivation together with $\mu_c = 0$ and $0 < \kappa \leq 1$.

10   Consequences for the Cosserat couple modulus μ_c

It is generally accepted in the engineering literature that really thin structures cannot support a non-vanishing transverse shear contribution. We introduce therefore the **p**ostulate

Postulate 10.0.3 (Vanishing transverse shear). Regardless of material constants, in the limit of arbitrarily thin, homogeneous isotropic structures, i.e. for $h \to 0$, transverse shear effects are altogether absent.   □

Since the Γ-limit faithfully describes the leading order term for vanishing thickness, this postulate implies that the Cosserat couple modulus μ_c must vanish as well, since otherwise one would have to deal with a remaining homogenized transverse shear contribution in the thin plate limit.

This statement has far reaching consequences: it has never been possible to unequivocally identify specific values for the Cosserat couple modulus
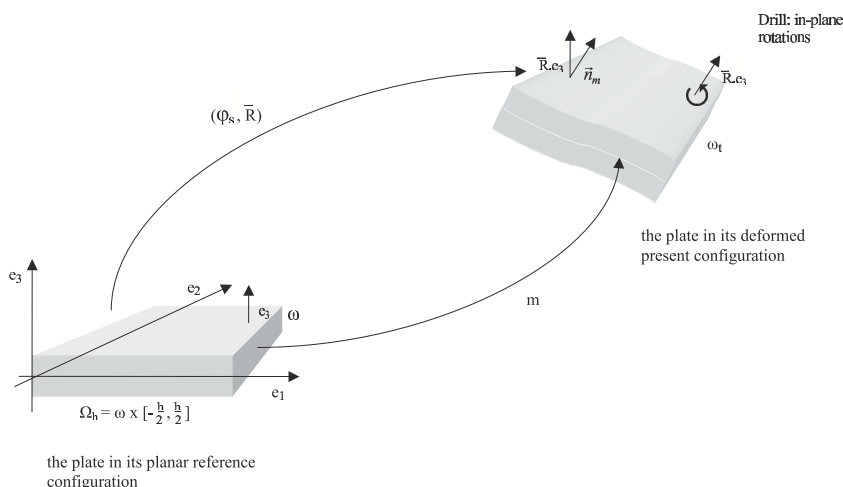
Figure 1. The assumed Cosserat plate kinematics incorporating transverse shear $(\overline{R}_3 \neq \vec{n}_m)$, thickness stretch $(\varrho_m \neq 1)$ and drill-rotations. Reconstructed three-dimensional deformation $\varphi_s : \Omega_h \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$, reconstructed microrotation $\overline{R}^{3d} : \Omega_h \subset \mathbb{R}^3 \mapsto \mathrm{SO}(3)$, $\overline{R}^{3d}(x,y,z) = \overline{R}(x,y)$, midsurface deformation $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ and microrotation of the plate $\overline{R} : \omega \subset \mathbb{R}^2 \mapsto \mathrm{SO}(3)$.

$\mu_c > 0$ in the experimentally oriented literature. In light of our development it is suggested to resolve the problem in the following way: $\mu_c > 0$ in the finite-strain Cosserat bulk model is a numerical tuning or penalty parameter but not a material constant. That μ_c should be zero as a material constant has been conjectured in Neff (2004b). The unexpected formal proof of this statement has been reached now by our Γ-convergence result.

A striking consequence of this reasoning is that **a** linear Cosserat bulk model describing faithfully the behaviour of a material body, does not exist, since for $\mu_c = 0$ the linearized fields of infinitesimal displacement and infinitesimal microrotation decouple, see Neff (2003). In summary Postulate 10.0.3 implies that the **i**nfinitesimal Cauchy stress tensor σ must always be symmetric.

11 Conclusion

We have justified the dimensional reduction of a geometrically exact Cosserat bulk model to its two-dimensional counterpart by use of Γ - convergence arguments. The underlying Cosserat bulk model features already independent rotations which may be identified with the averaged lattice rotations in defective elastic crystals if $\mu_c = 0$. Thus the appearance of an independent director field $\overline{R}_3$ is natural and not primarily due to the dimensional reduction/relaxation step. The argument is given for plates (flat reference configuration) only, but it is straightforward to extend the result to genuine shells with curvilinear reference configuration and it should be noted that the extension to shells is independent of geometrical features of this curvilinear reference configuration: the inclusion of transverse shear effects makes the distinction between elliptic, parabolic and hyperbolic surfaces in a certain sense obsolete. A welcome feature of the obtained Γ-limit for the defective crystal case $\mu_c = 0$ is its linearization consistency.

Apart for bending terms, the obtained Γ-limit is similar to the previously given formal development in Neff (2004b) and constitutes therefore a rigourous mathematical justification of Reissner-Mindlin type models. Future work will discuss the engineering implications of our results as far as the numerical value of the Cosserat couple modulus μ_c and its relation to the transverse shear modulus in classical Reissner-Mindlin type theories is concerned.

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