Correct traction boundary conditions in the indeterminate couple stress model

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Abstract

In this paper we consider the Grioli-Koiter-Mindlin-Toupin indeterminate couple stress model. The main aim is to show that the traction boundary conditions were not yet completely deduced. As it turns out, and to our own surprise, restricting the boundary condition framework from the strain gradient models to the couple stress model does not reduce to Mindlin's set of accepted boundary conditions. We present therefore, for the first time the complete, consistent set of traction boundary conditions.

Key words: generalized continua, strain gradient elasticity, modified couple stress model, consistent traction boundary conditions.

1 Introduction

Higher gradient elasticity models are nowadays increasingly used to describe mechanical structures at the micro- and nano-scale or to regularize certain ill-posed problems by means of higher gradient contributions [1, 2]. One of the very first among such models is the so called indeterminate couple stress model [3, 4, 5, 6] in which the higher gradient contributions only enter through gradients on the continuum rotation, i.e. the total elastic energy can be written as $W(\nabla u, \nabla(\nabla u)) = W_e(\text{sym}\nabla u) + W_{\text{curv}}(\nabla(\text{curl} u))$.

The question of boundary conditions in higher gradient elasticity models has been a subject of continuous attention. The crux of the matter in higher gradient models is the impossibility to vary the test function and its gradient independently. A suitable split into tangential and normal parts must always be considered. This is well known in general higher gradient models, see e.g. [7, 8]. The indeterminate couple stress model has been investigated in this respect as well. A first answer has been given by Mindlin and Tiersten [4] as well as Koiter [6] who established (correctly) that only 5 geometric and 5 traction boundary conditions can be prescribed due to the dependence of the curvature energy only on gradients of rotations. We agree that there are 5 traction boundary conditions in the indeterminate couple stress model which may be independently prescribed. However, we show in [9, 10] that the correct traction boundary conditions are not those proposed by Mindlin and Tiersten [4] and which are currently used in the literature. Since all papers dealing with the indeterminate couple stress model use this incomplete set of boundary conditions we will not refer further to any specific one.

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2 The indeterminate couple stress model

We consider a body which occupies a bounded open set Ω of the three-dimensional Euclidian space $\mathbb{R}^3$ and assume that its boundary $\partial\Omega$ is a piecewise smooth surface. An elastic material fills the domain $\Omega \subset \mathbb{R}^3$ and we refer the motion of the body to rectangular axes Ox_i, $i = 1, 2, 3$. For vector fields v with components $v_i \in \mathrm{H}^1(\Omega)$, $i = 1, 2, 3$, we define $\nabla v = \left((\nabla v_1)^T, (\nabla v_2)^T, (\nabla v_3)^T\right)^T$, while for tensor fields P with the rows $P_i \in \mathrm{H}(\mathrm{div}\,; \Omega)$, $i = 1, 2, 3$, we define $\mathrm{Div}\, P = (\mathrm{div}\, P_1, \mathrm{div}\, P_2, \mathrm{div}\, P_3)^T$. Equivalently, in index notation: $(\nabla v)_{ik} = v_{i,k}$ and $(\mathrm{Div}\, P)_i = P_{ij,j}$. In the remainder of the paper, $\mathrm{sym}\, X$ and $\mathrm{skew}\, X$ denote the symmetric and the skew symmetric part of the matrix X, respectively, $\mathrm{tr}(X)$ denotes the trace of the matrix X, $\|X\|$ is the Frobenius norm of the matrix X. The identity tensor on $\mathbb{R}^{3\times 3}$ will be denoted by $\mathbb{1}$. We also use the operator $\mathrm{anti}: \mathbb{R}^3 \to \mathfrak{so}(3)$, $\mathfrak{so}(3) := \{X \in \mathbb{R}^{3\times 3}\ |X^T = -X\}$, defined by $(\mathrm{anti}(v))_{ij} = -\varepsilon_{ijk} v_k, \forall\, v \in \mathbb{R}^3$, where ε_{ijk} is the totally antisymmetric third order permutation Levi-Civita tensor. We use the curl operator, $\mathrm{curl}\, v = \varepsilon_{ijk} v_{k,j}, \forall\, v \in \mathbb{R}^3$ and denote respectively by $\cdot$, $:$ and $\langle\cdot,\cdot\rangle$ a simple and double contraction and the scalar product between two tensors of any suitable order[1]. Everywhere we adopt the Einstein convention of sum over repeated indices if not differently specified.

The Grioli-Koiter-Mindlin-Toupin isotropic indeterminate couple stress model [3, 4, 5, 6] considers the curvature energy $W_{\mathrm{curv}}(\nabla(\mathrm{curl}\, u)) = \frac{\alpha_1}{4}\, \|\,\mathrm{sym}\, \nabla \mathrm{curl}\ u\|^2 + \frac{\alpha_2}{4}\, \|\,\mathrm{skew}\, \nabla \mathrm{curl}\ u\|^2$ and the classical elastic energy $W_e(\mathrm{sym}\, \nabla u) = \mu\, \|\mathrm{sym} \nabla u\|^2 + \frac{\lambda}{2}\, [\mathrm{tr}(\mathrm{sym} \nabla u)]^2$, where μ, λ, α_1 and α_2 are constitutive coefficients. The correct and accepted strong form of the Euler-Lagrange equations are

$$
\begin{aligned}
&\mathrm{Div}\ \ (\sigma - \tfrac{1}{2}\ \mathrm{anti}(\mathrm{Div}\, \widetilde{m})) + f = 0, && \text{equilibrium of forces} \\
&\sigma \ \ = D_{\mathrm{sym}\, \nabla u} W_e(\mathrm{sym}\, \nabla u) = 2\,\mu\ \mathrm{sym}\, \nabla u + \lambda\ \mathrm{tr}(\nabla u) \mathbb{1}, && \text{symmetric Cauchy-stress} \\
&\widetilde{m} \ \ = D_{\nabla \mathrm{curl}\, u} W_{\mathrm{curv}}(\nabla \mathrm{curl}\, u) = \alpha_1\ \mathrm{sym}(\nabla \mathrm{curl}\, u) + \alpha_2\ \mathrm{skew}(\nabla \mathrm{curl}\, u), && \text{couple stress tensor.}
\end{aligned}
\tag{2.1}
$$

Note that the couple stress tensor $\widetilde{m}$ is a second order and trace free tensor. Having the Euler-Lagrange equation, the question of which boundary conditions may be prescribed arises.

3 The incomplete boundary conditions considered in literature

We want to stress the fact that in the framework of a complete second gradient theory we can arbitrarily prescribe u and the normal derivative of the displacement $\nabla u \cdot n$ on the Dirichlet boundary Γ. This means that one has 6 independent geometric (or kinematical) boundary conditions that can be assigned on the boundary of the considered second gradient medium. Analogously, one can assign 6 traction (or natural) conditions on the force (in duality of u) and double force (in duality of $\nabla u \cdot n$), respectively, at $\partial\Omega \setminus \overline{\Gamma}$. The situation is slightly different in the indeterminate couple stress model since only a certain linear combination of second derivatives, i.e. $\nabla \mathrm{curl}\, u$, is controlled. Mindlin and Tiersten [4] concluded that the geometric boundary conditions on $\Gamma \subset \partial\Omega$ are the five independent conditions

$$
u\Big|_{\Gamma} \ = \ \widetilde{u}^0, \qquad (\mathbb{1} - n \otimes n) \cdot \mathrm{curl}\, u\Big|_{\Gamma} = (\mathbb{1} - n \otimes n) \cdot \mathrm{curl}\, \widetilde{u}^0, \tag{3.2}
$$

for a given vector function $\widetilde{u}^0$ at the boundary, where n is the unit normal vector on $\partial\Omega$ and $\otimes$ denotes the dyadic product of two vectors. The latter condition, in fact, prescribes only the tangential component of $\mathrm{curl}\, u$. Therefore, one may prescribe only 5 independent boundary conditions.

The possible traction boundary conditions on the remaining boundary $\partial\Omega \setminus \overline{\Gamma}$ given first by Mindlin and Tiersten [4] are

$$
\begin{aligned}
\left\{\left(\sigma - \tfrac{1}{2}\ \mathrm{anti}(\mathrm{Div}\, \widetilde{m})\right) \cdot\, n - \tfrac{1}{2} n \times \nabla[\langle n, (\mathrm{sym}\, \widetilde{m}) \cdot n\rangle]\right\}\Big|_{\partial\Omega\setminus\overline{\Gamma}} &= \widetilde{t}, \\
(\mathbb{1} - n \otimes n) \cdot \widetilde{m} \cdot n\Big|_{\partial\Omega\setminus\overline{\Gamma}} &= (\mathbb{1} - n \otimes n) \cdot \widetilde{g},
\end{aligned}
\tag{3.3}
$$

for prescribed vector functions $\widetilde{t}$ and $\widetilde{g}$ at the boundary, where $\langle\cdot,\cdot\rangle$ denotes the scalar product of two vectors. Mindlin and Tiersten [4] have correctly concluded that the maximal number of independent traction boundary conditions is also 5. The same conclusion has been arrived at by Koiter [6]. These traction boundary conditions (3.3) have been rederived again and again. However, they are erroneous.

[1] For example, $(A \cdot v)_i = A_{ij} v_j$, $(A \cdot B)_{ik} = A_{ij} B_{jk}$, $A : B = A_{ij} B_{ji}$, $(C \cdot B)_{ijk} = C_{ijp} B_{pk}$, $(C : B)_i = C_{ijp} B_{pj}$, $\langle v, w\rangle = v \cdot w = v_i w_i$, $\langle A, B\rangle = A_{ij} B_{ij}$ etc.

4 The correct boundary conditions in the indeterminate couple stress model

The prescribed traction boundary conditions (3.3) proposed by Mindlin and Tiersten [4] do not remain independent, in the sense that $\widetilde{g}$ leads to a further energetic conjugate, besides $\widetilde{t}$, of u. From this reason and looking back to the clear and correct boundary conditions considered in the more general second gradient elasticity model, in order to prescribe independent geometric boundary conditions and their corresponding completely independent energetic conjugate (traction boundary conditions), we have to prescribe u and $(\mathbb{1} - n \otimes n) \cdot (\nabla u \cdot n)$. Let us remark that prescribing $u\big|_\Gamma = \widetilde{u}^0$ and $(\mathbb{1} - n \otimes n) \cdot (\nabla u \cdot n)\big|_\Gamma = (\mathbb{1} - n \otimes n) \cdot \nabla \widetilde{u}^0 \cdot n$ is fully equivalent with prescribing $u\big|_\Gamma = \widetilde{u}^0$ and $(\mathbb{1} - n \otimes n) \cdot \operatorname{curl} u\big|_\Gamma = (\mathbb{1} - n \otimes n) \cdot \operatorname{curl} \widetilde{u}^0$, which is (3.2). However, in the formulation of the principle of virtual power, the energetic conjugate of $(\mathbb{1} - n \otimes n) \cdot \operatorname{curl} u$ is not equal to the energetic conjugate of $(\mathbb{1} - n \otimes n) \cdot (\nabla u \cdot n)$.

Using the principle of virtual power proposed by Mindlin and Tiersten [4, Eq. (5.13)], but now suitably applying the surface divergence theorem [9, 11, 12], we arrive at the following traction boundary conditions on $\partial\Omega \setminus \overline{\Gamma}$

$$\begin{aligned} \Big\{\Big(\sigma \quad -\tfrac{1}{2}\,\operatorname{anti}(\operatorname{Div}\widetilde{m})\Big)\cdot n - \tfrac{1}{2} n \times \nabla[\langle n, (\operatorname{sym}\widetilde{m})\cdot n\rangle] & \\ -\tfrac{1}{2}\nabla\,[\,\operatorname{anti}\,(\,(\mathbb{1} - n \otimes n)\cdot \widetilde{m}\cdot n\,)\cdot(\mathbb{1} - n\otimes n)\,] : (\mathbb{1} - n\otimes n)\Big\}\Big|_{\partial\Omega\setminus\overline{\Gamma}} &= \widetilde{t}, \\ (\mathbb{1} - n\otimes n)\cdot \operatorname{anti}[(\mathbb{1} - n\otimes n)\cdot\widetilde{m}\cdot n]\cdot n\Big|_{\partial\Omega\setminus\overline{\Gamma}} &= (\mathbb{1} - n\otimes n)\cdot\widetilde{g}, \end{aligned} \tag{4.4}$$

together with the traction boundary conditions on $\partial\Gamma$

$$\{([\operatorname{anti}[(\mathbb{1} - n\otimes n)\cdot\widetilde{m}\cdot n]]^+ - [\operatorname{anti}[(\mathbb{1} - n\otimes n)\cdot\widetilde{m}\cdot n]]^-)\cdot\nu\}\Big|_{\partial\Gamma} = \widetilde{\pi}, \tag{4.5}$$

where $\widetilde{t}$ and $\widetilde{g}$ are prescribed vector functions on $\partial\Omega \setminus \overline{\Gamma}$, while $\widetilde{\pi}$ is a prescribed vector function on $\partial\Gamma$. Here, ν is a vector tangential to the surface Γ and which is orthogonal to its boundary $\partial\Gamma$. The term $[\operatorname{anti}[(\mathbb{1} - n\otimes n)\cdot\widetilde{m}\cdot n]]^+ - [\operatorname{anti}[(\mathbb{1} - n\otimes n)\cdot\widetilde{m}\cdot n]]^-$ measures the discontinuity of $\operatorname{anti}[(\mathbb{1} - n\otimes n)\cdot\widetilde{m}\cdot n]$ across $\partial\Gamma$.

Comparing (3.3) and (4.4), we remark that in the Mindlin and Tiersten formulation $(3.3)_2$ it remains a missing boundary term $-\frac{1}{2}\nabla\,[\,\operatorname{anti}\,(\,(\mathbb{1} - n\otimes n)\cdot\widetilde{m}\cdot n\,)\cdot(\mathbb{1} - n\otimes n)\,] : (\mathbb{1} - n\otimes n)$ which also performs work against u. On the other hand, we show in [9] that when the higher gradient contributions only enter through gradients on the continuum rotation, i.e., $\nabla\operatorname{curl} u$, the independent traction boundary conditions which are coming from the representation in terms of a particular case of second gradient elasticity model written with third order moment tensors coincide with our novel traction boundary conditions (4.4) and (4.5), and not with the traction boundary conditions (3.3) proposed by Mindlin and Tiersten.

Our renewed interest in traction boundary conditions in the indeterminate couple stress model was triggered by the controversial papers [13, 14]. There, the authors have made far reaching claims on the possible anti-symmetric nature of the second order couple stress tensor $\widetilde{m}$. Their reasoning is based on physically plausible assumptions similar to a Cosserat or micromorphic theory [15] which led them to require a total split of the effect of force and moment tensors in (3.3). In (3.3), this can be achieved if and only if $\widetilde{m}$ is skew-symmetric and this constitutes the essence of their claim. However, since (3.3) is incomplete, their conclusion is misleading, see also [16]. The couple stress tensor in the indeterminate couple stress theory is not necessarily skew-symmetric! Quite to the contrary, the couple stress tensor may be chosen to be symmetric [17, 18].

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