Constitutive inequalities for isotropic elastic solids under finite strain

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(Received 16 June 1969)

Possible restrictions on isotropic constitutive laws for finitely deformed elastic solids are examined from the standpoint of Hill (1968). This introduced the notion of conjugate pairs of stress and strain measures, whereby families of contending inequalities can be generated. A typical member inequality stipulates that the scalar product of the rates of change of certain conjugate variables is positive in all circumstances. Interrelations between the various inequalities are explored, and some statical implications are established. The discussion depends on several ancillary theorems which are apparently new; these have, in addition, an intrinsic interest in the broad field of basic stress-strain analysis.

1. METHODS AND OBJECTIVES

In the continuing absence of adequate experimental data it remains problematical how to delimit a worthwhile class of elastic solids for theoretical study. The matter was raised originally by Truesdell (1956) and subsequently examined by Truesdell & Toupin (1963) and Truesdell & Noll (1965). They tentatively concluded that a constitutive inequality introduced by Coleman & Noll (1959) might be a rational basis. But this, as remarked by Hill (1968), excludes even the idealized neo-Hookean solid, which is universally regarded as a valid prototype of a group of rubberlike materials. In fact the inequality is altogether incompatible with elastic incompressibility, which has been the *sine qua non* of many finite strain analyses. Although plainly undesirable, this shortcoming is not decisive since actual materials are all compressible in some degree. On the other hand, the merits seen in the Coleman-Noll postulate by Truesdell & Toupin (1963, §9) seem insubstantial and have failed to secure its general adoption.

A needed perspective can be brought to the problem by pursuing an approach initiated by Hill (1968). In part, this considers a one-parameter family of contending inequalities defined via objective stress-rates of type

$$\boldsymbol{s} = \rho \frac{\mathscr{D}}{\mathscr{D}t} \left(\frac{\boldsymbol{\sigma}}{\rho} \right) - m(\boldsymbol{\sigma} \times \boldsymbol{\epsilon} + \boldsymbol{\epsilon} \times \boldsymbol{\sigma}), \tag{1}$$

where *m* is any real scalar. Here σ and ϵ are symbolic notations for the tensors of Cauchy stress and Eulerian strain-rate; a cross denotes their inner product; ρ is the present density, and $\mathscr{D}/\mathscr{D}t$ is the Jaumann (or rigid-body or co-rotational) flux. Well known stress-rates belonging to this family are given by m = 0 and ± 1 .

Each associated member inequality stipulates a positive scalar product of *every* corresponding stress-rate and strain-rate pair:

$$s \epsilon > 0$$
 at any strain. (2)

In particular, a strengthened version of the Coleman-Noll postulate is obtained with $m = \frac{1}{2}$ (cf. Coleman & Noll 1964, equation (6.19) or, less clearly, Truesdell & Toupin 1963, equation (6.10)).

In this paper some consequences of (2) are established for isothermal deformations of materials that are *isotropic* relative to some 'ground state'. The materials are required to be Cauchy-elastic merely, and not necessarily Green-elastic.[†] The main result is that (2) implies the generally weaker inequality

$$\dot{t}\dot{e} > 0$$
 at any strain, (3)

where t and e are certain symmetric stress and strain tensors based on the ground state, and 'conjugate' in the sense that the Pfaffian

$$\frac{1}{\rho_0} t \,\mathrm{d}e \tag{4}$$

always represents the work of incremental deformation per unit mass (ρ_0 being the density in the ground state). Specifically, for any considered *m*, the tensor *e* is coaxial with the Lagrangian strain ellipsoid and has principal values

$$e_i = \begin{cases} (a_i^{2m} - 1)/2m & (m \neq 0), \\ \ln a_i & (m = 0), \end{cases}$$
(5)

where a_i (i = 1, 2, 3) are the principal stretches relative to the ground state ('stretch' being the ratio of final to initial lengths of a linear element of material). Such strain measures have occasionally been mooted in other contexts (see, for example, Doyle & Ericksen 1956, p. 65).

The family (3) was also proposed by Hill (1968, §4 (ii)) for independent evaluation, since it is essentially distinct from (2). Exceptionally, the identity

$$\frac{1}{\rho_0} \dot{i} \dot{e} \equiv \frac{1}{\rho} s \epsilon \quad \text{when} \quad m = \pm 1 \tag{6}$$

will be established, together with the two-way implication

$$(2) \leftrightarrow (3) \quad \text{when} \quad m = 0. \tag{7}$$

Inequality (3) asserts that the stress-strain law t(e) is strongly convex in its tensor components. A well known consequence is that

$$\Delta t \Delta e > 0, \tag{8}$$

provided $\Delta e \neq 0$, where prefix Δ indicates an ordered difference of the variables in

[†] A case against the conventional thermostatics of Green-elasticity has been argued eloquently by Truesdell (e.g. 1964, p. 198).

any two (t, e) pairs; conversely (8) implies (3), apart perhaps in isolated strain configurations where the inequality is not strict. On the other hand, it does not seem to be known, and it will be proved, that (8) for isotropic materials is no more than a statement of a convex relation between the collective *principal* values t_i and e_i :

$$\sum_{i} \Delta t_i \, \Delta e_i > 0, \tag{9}$$

provided not every $\Delta e_i = 0$. That (8) entails (9) is self-evident by choosing any two coaxial strains. But the converse is non-obvious and was previously shown by Hill (1968, §4 (iii)) for Green-elastic solids where

$$\boldsymbol{t} = \partial \phi / \partial \boldsymbol{e},\tag{10}$$

 ϕ being a strain potential interpretable from (4) as an energy density. In this case ϕ is required by (8) and (9) respectively to be a convex function of the tensor components of the strain measure or of its principal values.

If a satisfactory solution of the opening problem is to be found at all among any of these inequalities, it seems likely to be of type (3), being less exclusive than (2). Moreover, the search need not be confined to the strain measures (5) but can be extended to f(x) = f(x) = f(x) (11)

$$e_i = f(a_i), \text{ with } f(1) = 0, f'(1) = 1,$$
 (11)

and f(a) any suitably smooth monotone function. Such an extension calls for a systematic way of generating conjugate stresses and their rates of change. This construction is made possible by the artifice of resolving all tensors on the axes of the Lagrangian ellipsoid (whose spin can be calculated in terms of the Eulerian strain-rate). The results of this auxiliary investigation appear to be new, and have an intrinsic interest in the wider field of basic stress-strain analysis.[†]

The eventual outcome for compressible solids is, briefly, in favour of inequality (3) in conjunction with the *logarithmic* measure of strain. Significantly, this is also the only inequality of its kind which does not exclude elastic incompressibility; further, it admits the Mooney-Rivlin solid. We have already noted that (3) implies, and is implied by, (2) when m = 0; that is,

$$(\mathscr{D}\boldsymbol{\tau}/\mathscr{D}t)\boldsymbol{\varepsilon} > 0 \tag{12}$$

for arbitrary strain-rates and configurations, where

$$\boldsymbol{\tau} = (\rho_0 | \rho) \, \boldsymbol{\sigma} \tag{13}$$

is the Kirchhoff stress based on the ground state. The equivalent hypothesis in the form (9) reduces to $\sum A = A \ln c = 0$

$$\sum_{i} \Delta \tau_i \Delta \ln a_i > 0,$$

where (τ_1, τ_2, τ_3) are the principal values of Kirchhoff stress. Inter alia this includes

[†] In the past the use even of the tensor logarithm has been thought to involve intractable analytic difficulties (e.g. Truesdell 1952, § 16), and practical measures of finite strain have been considered to be limited to a few members of the class (5), such as $m = \pm \frac{1}{2}, \pm 1$.

the ordering of the principal Cauchy stresses in the same algebraic sequence as the corresponding strains (whereas under the Coleman–Noll inequality the principal forces are so ordered). Proofs of these and other consequences follow in the paper.

2. Spins of the strain ellipsoids

As a necessary preliminary we derive some little-known formulae relating to the kinematics of finite deformation.

We consider first the rotation of the Eulerian strain ellipsoid, whose axes are the finally orthogonal triad of embedded line elements which were orthogonal and equal in the ground state. Suppose a pure infinitesimal strain is superimposed on the existing stretches (a_1, a_2, a_3) and let it be specified by tensor components $\delta \eta_{ij}$ on the present axes of the ellipsoid. Clearly, only the off-diagonal components $\delta \eta_{23}$, $\delta \eta_{31}$ and $\delta \eta_{12}$ produce further rotations. To the first order of infinitesimals their contributions are independent, say $\delta \theta_1$, $\delta \theta_2$ and $\delta \theta_3$ right-handedly about the 1, 2 and 3 axes.

To determine $\delta\theta_1$, for example, we observe that embedded line elements in the present 2 and 3 directions turn through angles $\delta\eta_{23}$ and $-\delta\eta_{23}$ respectively about the axis 1 during the additional strain $\delta\eta_{23}$ alone. These elements are then inclined at angles

$$(\delta \eta_{23} - \delta \theta_1)$$
 and $-(\delta \eta_{23} + \delta \theta_1)$

to the axes 2 and 3 respectively of the *new* ellipsoid. Since the line elements were perpendicular in the ground state, they form a pair of conjugate directions with respect to the new ellipsoid. By elementary geometry the condition for conjugacy is

$$a_{2}^{2}(\delta\eta_{23} - \delta\theta_{1}) + a_{3}^{2}(\delta\eta_{23} + \delta\theta_{1}) = 0$$

to first order. Thus, if $a_2 \neq a_3$, $\delta \theta_1 = \frac{a_2^2 + a_3^2}{a_2^2 - a_3^2} \delta \eta_{23}$

and similarly for the contributions from the other shear components. In terms of the Eulerian strain-rate e_{ij} , therefore, the spin of the Eulerian strain ellipsoid has components

$$\frac{a_2^2 + a_3^2}{a_2^2 - a_3^2} \epsilon_{23}, \quad \frac{a_3^2 + a_1^2}{a_3^2 - a_1^2} \epsilon_{31}, \quad \frac{a_1^2 + a_2^2}{a_1^2 - a_2^2} \epsilon_{12} \tag{14}$$

on its own axes, when the principal stretches are all different. This same derivation has been given elsewhere by the writer (1969), independently of an earlier and somewhat different one by Biot (1965, p. 92). When the incremental deformation is not a pure strain, (14) must naturally be augmented by the body spin (i.e. the rate of rotation of embedded directions momentarily coincident with the principal axes of ϵ_{ij}).

We consider next the rotation of the Lagrangian strain ellipsoid, whose axes are in the ground-state directions of the embedded triad that momentarily defines the Eulerian ellipsoid. Let rotations $\delta\psi_1$, $\delta\psi_2$ and $\delta\psi_3$ about the 1, 2 and 3 axes of the Lagrangian ellipsoid be the respective independent contributions from $\delta\eta_{23}$, $\delta\eta_{31}$ and $\delta\eta_{12}$. By definition $\delta\psi_1$ is the angle between the positions of the axis 2 before

and after the incremental strain $\delta\eta_{23}$ alone. The embedded pair of line elements occupying these neighbouring positions in the ground state are inclined at $a_3 \delta \psi_1/a_2$ after the stretch (a_1, a_2, a_3) . During the incremental strain both turn through an angle $\delta\eta_{23}$ about the axis 1 of the Eulerian ellipsoid. Thus

$$\begin{split} \delta\theta_1 &= \frac{a_3}{a_2} \delta\psi_1 + \delta\eta_{23} \\ \delta\psi_1 &= \frac{2a_2a_3}{a_2^2 - a_3^2} \delta\eta_{23} \end{split}$$

and so

from (14) if $a_2 \neq a_3$. Accordingly, when the principal stretches are all different, the rate of rotation of the Lagrangian strain ellipsoid has components $(\omega_1, \omega_2, \omega_3)$, say, equal to $2a_1a_2 = 2a_2a_3$

$$\frac{2a_2a_3}{a_2^2 - a_3^2} \epsilon_{23}, \quad \frac{2a_3a_1}{a_3^2 - a_1^2} \epsilon_{31}, \quad \frac{2a_1a_2}{a_1^2 - a_2^2} \epsilon_{12} \tag{15}$$

on its own axes (and is of course unaffected by body spin). This formula was obtained by the writer (1969); it could not be found in the standard literature on strain geometry.

3. REDUCTION OF INEQUALITIES OF THE FIRST KIND

The immediate task is to reduce (2) to its simplest explicit form for isotropic materials. In these the principal directions of Cauchy stress coincide with the axes of the Eulerian ellipsoid. To express the stress-strain law it is best here to use the Kirchhoff stress (13) and regard its principal values (τ_1, τ_2, τ_3) as functions of the stretches (a_1, a_2, a_3) with symmetries appropriate to the isotropy:

$$\tau_1(a_1, a_2, a_3) \equiv \tau_1(a_1, a_3, a_2), \tau_2(a_1, a_2, a_3) \equiv \tau_1(a_2, a_3, a_1), \tau_3(a_1, a_2, a_3) \equiv \tau_1(a_2, a_1, a_2).$$

$$(16)$$

From (1) and (2)

$$\frac{\rho_0}{\rho} s \epsilon = \frac{\mathscr{D} \tau}{\mathscr{D} t} \epsilon - 2m \sum_{ijk} \tau_{ij} \epsilon_{jk} \epsilon_{ki}.$$
(17)

Now, in tensor components on the Eulerian ellipsoid axes,

$$\frac{\mathscr{D}\tau_{11}}{\mathscr{D}t} = \dot{\tau}_1 = a_1 \frac{\partial \tau_1}{\partial a_1} e_{11} + a_2 \frac{\partial \tau_1}{\partial a_2} e_{22} + a_3 \frac{\partial \tau_1}{\partial a_3} e_{33}, \quad \text{etc.}$$

by the assumed isotropy and since the normal components of strain-rate are just $e_{11} = \dot{a}_1/a_1$, etc., while

$$\frac{\mathscr{D}\tau_{12}}{\mathscr{D}t} = \frac{a_1^2 + a_2^2}{a_1^2 - a_2^2} \left(\tau_1 - \tau_2\right) \epsilon_{12}, \quad \text{etc.},$$

from (14). (In passing, it is noted that the 'instantaneous moduli' of shear are positive if and only if the stresses and stretches are in the same algebraic sequence.)

Therefore, from (17),

$$\frac{\rho_{0}}{\rho} s \epsilon = \left(a_{1} \frac{\partial \tau_{1}}{\partial a_{1}} - 2m\tau_{1}\right) \epsilon_{11}^{2} + \left(a_{2} \frac{\partial \tau_{1}}{\partial a_{2}} + a_{1} \frac{\partial \tau_{2}}{\partial a_{1}}\right) \epsilon_{11} \epsilon_{22} + \dots + 2\left[\frac{a_{1}^{2} + a_{2}^{2}}{a_{1}^{2} - a_{2}^{2}}(\tau_{1} - \tau_{2}) - m(\tau_{1} + \tau_{2})\right] \epsilon_{12}^{2} + \dots\right)$$
(18)

when no two stretches are equal. This is a quadratic in the Eulerian components of strain-rate and is required by (2) to be always positive.

To begin with, since the shear components are uncoupled, their coefficients must be separately positive, so that

$$\frac{\tau_1 - \tau_2}{a_1^2 - a_2^2} > m \frac{\tau_1 + \tau_2}{a_1^2 - a_2^2}$$

with two similar inequalities obtained by cyclic permutation. Taking $a_1 > a_2 > a_3$ we can re-write these as

$$\begin{bmatrix} (1-m)a_1^2 + (1+m)a_2^2] \tau_1 > [(1+m)a_1^2 + (1-m)a_2^2] \tau_2, \\ [(1-m)a_2^2 + (1+m)a_3^2] \tau_2 > [(1+m)a_2^2 + (1-m)a_3^2] \tau_3, \\ [(1-m)a_3^2 + (1+m)a_1^2] \tau_3 < [(1+m)a_3^2 + (1-m)a_1^2] \tau_1. \end{bmatrix}$$

$$(19)$$

Suppose, however, that two stretches are equal, say $a_1 = a_2 = a$. Then the $\widehat{12}$ shear modulus is

$$\frac{1}{2}a \partial(\tau_1 - \tau_2)/\partial a_1,$$

either by a limiting procedure or directly from the incremental relations

$$\delta \tau_1 = \alpha \, \delta a_1 + \beta \, \delta a_2, \quad \delta \tau_2 = \beta \, \delta a_1 + \alpha \, \delta a_2,$$

$$\alpha = \frac{\partial \tau_1}{\partial a_1} = \frac{\partial \tau_2}{\partial a_2} \quad \text{and} \quad \beta = \frac{\partial \tau_1}{\partial a_2} = \frac{\partial \tau_2}{\partial a_1}.$$

From these, by analogy with the ordinary Hooke's law, the modulus for shearing in the $\widehat{12}$ plane can be read off as $\frac{1}{2}(\alpha - \beta)a$. The associated inequality is then

$$a \partial(\tau_1 - \tau_2)/\partial a_1 > 2m\tau,$$

where τ is written for $\tau_1 = \tau_2$.

The remaining terms in the normal components of strain-rate must be jointly positive. They can be put compactly as

$$\sum_{i} \dot{t}_{i} \dot{e}_{i} = \sum_{ij} \frac{\partial t_{i}}{\partial e_{j}} \dot{e}_{i} \dot{e}_{j} > 0,$$

where e_i are the strain measures defined in (5), so that

$$\dot{e}_i = a_i^{2m-1}\dot{a}_i = a_i^{2m}\epsilon_{ii}, \quad t_i = a_i^{-2m}\tau_i,$$

where i = 1, 2 or 3 and is not summed. Thus, at all strains the Jacobian matrix

 $(\partial t_i/\partial e_j)$ is positive-definite, (20)

or equivalently its symmetric part is. When the solid is Green-elastic, as in (10), the Jacobian equals $(\partial^2 \phi / \partial e_i \partial e_j)$ which is itself symmetric.

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Equations (19) with (20) are the required conditions necessary and sufficient for the constitutive inequality (2). From (20) it follows that the stress-strain law is invertible when expressed in the conjugate variables and further, as will be shown, that (t_1, t_2, t_3) are ordered algebraically like (e_1, e_2, e_3) and hence like (a_1, a_2, a_3) . That is,

$$a_1^{-2m}\tau_1 > a_2^{-2m}\tau_2 > a_3^{-2m}\tau_3, \tag{21}$$

but of course the converse does not hold. In particular when m = 0 the Kirchhoff and Cauchy stresses are ordered like the stretches, as proposed by Baker & Ericksen (1954).

When $m = \frac{1}{2}$ the quadratic (18) agrees with an expression obtained, at greater length and in other variables, by Truesdell & Toupin (1963, equation (6.12)) from the Coleman-Noll hypothesis. Truesdell & Toupin drew the inference (19), with $m = \frac{1}{2}$, together with the associated specialization of (20) asserting a strongly convex relation between the principal stretches and forces (which are $\tau_1/a_1 = a_2 a_3 \sigma_1$, etc.). But they were unable to prove the mutual ordering of forces and strains, † and this was done later by Truesdell & Noll (1965, p. 167).

When m = 0 or ± 1 it is apparent that (19) and (21) coincide. In that event (20) implies (19) and is by itself necessary and sufficient for (2). With any other value of m, however, there is always some range of strain in which (19) imposes a restriction additional to (20). When $m = \frac{1}{2}$ this was noted by Truesdell & Toupin (1963, p. 22), while Bragg & Coleman (1963) constructed a counter-example in which a particular ϕ satisfies (20) are all strains but not (2) (see also Truesdell & Noll 1965, p. 323).

4. CONJUGATE VARIABLES

At this stage in the argument we intercalate some needed theorems on general measures of stress and strain.

(a) We begin with a formula for the rate of change of the strain tensor e whose principal directions are the axes of the Lagrangian ellipsoid and whose principal values are defined by (11). The rate of change can be specified most conveniently by its components in a fixed frame of reference with which the Lagrangian axes momentarily coincide. In computing the rate we have to resolve on this frame the new tensor after an infinitesimal time (during which the Lagrangian axes may rotate). Then the normal components of \dot{e} are just

$$\dot{e}_i = a_i f'(a_i) \epsilon_{ii} \quad (i = 1, 2, 3),$$
(22)

while by (15) the shear components are

$$\begin{array}{c} (e_1 - e_2)\omega_3, \ (e_2 - e_3)\omega_1, \ (e_3 - e_1)\omega_2 \equiv \frac{2a_i a_j}{a_i^2 - a_j^2} (e_i - e_j)\epsilon_{ij} \quad \text{when} \quad a_i \neq a_j, \\ \\ a_i f'(a_i)\epsilon_{ij} \qquad \text{when} \quad a_i = a_j, \end{array}$$

[†] As remarked on p. 16 of their paper. There are consequent redundancies in some equations, for instance (6.18).

or

by a limiting process. It is recalled that e_{ij} here are the components of Eulerian strain-rate on the axes of the Eulerian ellipsoid.

When applied to the Green and Almansi measures (5), with m = 1 and -1 respectively, the preceding equations reproduce some familiar results. When m = 1 the rate of strain has Lagrangian components $a_i a_j e_{ij}$, numerically equal to the covariant components of e_{ij} on deforming coordinates initially embedded along the Lagrangian axes. Similarly, when m = -1 the rate of strain has Lagrangian components $e_{ij}/a_i a_j$, numerically equal to the contravariant components of e_{ij} on the same deforming coordinates.

With the logarithmic measure the rate of strain has normal components ϵ_{ii} (i = 1, 2, 3) and shear components

$$\frac{2a_ia_j}{a_i^2-a_j^2}\ln\left(\!\frac{a_i}{a_j}\!\right)\epsilon_{ij}.$$

When the total distortion is small, though not necessarily the dilatation, the last expression can be expanded as

$$\left\{1-\frac{1}{6}\left(\frac{a_i}{a_j}-1\right)^2+\ldots\right\}\epsilon_{ij},$$

which makes precise an order-of-magnitude formula given by Hill (1968, appendix).

(b) Any conjugate stress can now be calculated readily from the definition (4). Let its components on the axes of the Lagrangian ellipsoid be denoted by t_{ij} . The rate of working is then

$$t\dot{e} = \sum_{i} a_i f'(a_i) t_{ii} \epsilon_{ii} + \sum_{i \neq j} rac{2a_i a_j}{a_i^2 - a_j^2} (e_i - e_j) t_{ij} \epsilon_{ij}$$

per unit ground-state volume. But this rate is also given by the elementary formula $(\rho_0/\rho) \sigma \epsilon$, or $\tau \epsilon$ by (13). A comparison of coefficients of the arbitrary components of strain-rate gives $a_i f'(a_i) t_{ii} = \tau_{ii}$ (i = 1, 2, 3)

and, if
$$i \neq j$$
,
$$\frac{2a_i a_j}{a_i^2 - a_j^2} (e_i - e_j) t_{ij} = \tau_{ij} \quad \text{when} \quad a_i \neq a_j,$$
or
$$a_i f'(a_i) t_{ij} = \tau_{ij} \quad \text{when} \quad a_i = a_j,$$
(24)

where τ_{ij} are the components of Kirchhoff stress on the axes of the Eulerian ellipsoid. These expressions completely solve the problem in a formal sense. However, the statical interpretation of a particular conjugate stress may not be obvious; some of the simpler cases have been discussed by Hill (1968, § 2 and appendix) and McVean (1968) from various points of view. Here we need only note that corresponding to m = 1 and -1 in (5) one has $t_{ij} = \tau_{ij}/a_i a_j$ and $a_i a_j \tau_{ij}$, which are respectively the contravariant and covariant components of Kirchhoff stress on deforming coordinates initially cartesian and embedded along the Lagrangian axes.

Of course, stress tensors can be defined which are not conjugate to any strain measure in the present sense. One such is Cauchy stress for a compressible solid, as can be recognized at once from the incompatible normal components in (24).

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(c) Finally, the rate of change of any conjugate stress is determined for isotropic elastic solids. This is an easy matter since by (24) the Lagrangian ellipsoid axes are then always principal for t, just as the Eulerian ellipsoid axes are always principal for τ . One can therefore write

$$a_i f'(a_i) t_i = \tau_i \quad (i = 1, 2, 3) \tag{25}$$

and in particular $t_i = a_i^{-2m} \tau_i$ as already defined in connexion with (5). Analogously to (22) and (23) the normal components of \mathbf{i} on a fixed frame coinciding with the Lagrangian axes are

$$\dot{t}_i = \sum_j \frac{\partial t_i}{\partial e_j} \dot{e}_j.$$
⁽²⁶⁾

In terms of the spin (15) the shear components are

$$\begin{array}{l}
\left\{ (t_{1}-t_{2})\omega_{3}, (t_{2}-t_{3})\omega_{1}, (t_{3}-t_{1})\omega_{2}, \\
\equiv \frac{2a_{i}a_{j}}{a_{i}^{2}-a_{j}^{2}}(t_{i}-t_{j})\epsilon_{ij} & \text{when} \quad a_{i} \neq a_{j}, \\
a_{i}\frac{\partial}{\partial a_{i}}(t_{i}-t_{j})\epsilon_{ij} & \text{when} \quad a_{i} = a_{j}.
\end{array} \right\} (27)$$

There is, of course, no difficulty in finding the components of stress-rate when the material is not isotropic, but such formulae are not required here.

5. REDUCTION OF INEQUALITIES OF THE SECOND KIND

We are now in a position to analyse constitutive inequalities in the family (3) for isotropic solids.

(a) For this purpose the identity

$$\dot{t}\dot{e} = \sum_{i} \dot{t}_{i}\dot{e}_{i} + \sum_{ijk} (t_{i} - t_{j}) (e_{i} - e_{j})\omega_{k}^{2} \quad (k \neq i, j)$$

$$(28)$$

is fundamental. It follows at once by combining (22) with (26) and the first of (23) with the first of (27), when the stretches are all different. If two are equal, say $a_1 = a_2 = a$, the ω_3^2 terms are replaced by

$$2 \, \frac{\partial}{\partial e_1} \, (t_1 - t_2) \, \{af'(a) \, e_{12}\}^2.$$

Expression (28) is positive if and only if its independent parts are; namely,

$$\sum_{i} \dot{t}_{i} \dot{e}_{i} = \sum_{ij} \frac{\partial t_{i}}{\partial e_{j}} \dot{e}_{i} \dot{e}_{j} > 0, \qquad (29)$$

with

or

$$\begin{aligned} (t_i - t_j) \left(e_i - e_j \right) &> 0 \quad \text{when} \quad e_i \neq e_j, \\ \partial(t_i - t_j) / \partial e_i &> 0 \quad \text{when} \quad e_i = e_i. \end{aligned}$$

$$(30)$$

But (29) holds in every configuration and therefore in itself implies (30), which states that the conjugate variables have corresponding principal values in the same algebraic sequence. Accordingly, (29) alone is necessary and sufficient that (3) should hold universally. Put otherwise, it is enough to postulate (3) for strain-rates coaxial with the Eulerian ellipsoid under any strain relative to the ground state; it then holds automatically for all other strain-rates.

To deduce from (29) the mutual ordering of conjugate principal values we can proceed as follows. Let Δt_i and Δe_i (i = 1, 2, 3) be the varying differences of initial and final values generated during an additional deformation coaxial with the prestrain and executed at a *constant rate* with components \dot{e}_i . Then by (29) the quantity

$$\sum_{i} \dot{e}_i \Delta t_i$$

increases monotonically throughout this 'linear path' in (e_1, e_2, e_3) space and, being zero at the start, is subsequently positive. Whence

$$\sum_{i} \Delta t_i \Delta e_i > 0 \tag{31}$$

if some $\Delta e_i \neq 0$, since the ratios $\dot{e}_i / \Delta e_i$ are always equal and positive. Suppose, in particular, that the additional deformation takes (e_1, e_2, e_3) to (e_2, e_1, e_3) where $e_1 \neq e_2$, and likewise (t_1, t_2, t_3) to (t_2, t_1, t_3) by virtue of the isotropy. Then the preceding sum reduces to $(t_1 - t_2) (e_1 - e_2) > 0,$

as was to be proved.

In an exactly analogous manner (Hill 1968, p. 237) it can be shown by considering the product $\dot{e}\Delta t$ that $\dot{t}\dot{e} > 0 \rightarrow \Delta t\Delta e > 0$, (32)

if $\Delta e \neq 0$, where now the strain pair need not be coaxial nor the material isotropic. A corollary is that to any given stress tensor there corresponds just one strain tensor, since $\Delta t \equiv 0$ is incompatible with $\Delta e \neq 0$. Another obvious corollary is that te > 0 when the constitutive law admits a stress-free state, relative to which the strain is reckoned.

(b) In 1 it was stated that (9) implies (8). We give two proofs of this elementary but important theorem.

As is well known, by allowing the differences in (9) to be infinitesimals, one obtains (29) in the limit (except possibly at isolated strains where the inequality is not strict). But we have just shown that (29) implies (3), and this in turn implies (8) according to (32).

Alternatively, one can begin with the identity

$$\Delta t \Delta e \equiv \sum_{i} (t'_{i}e'_{i} + t_{i}e_{i}) - (t'e + te'),$$

where $\Delta t = t' - t$ and $\Delta e = e' - e$ explicitly. But, by an extremal property of tensor products (Hill 1968, p. 238),

$$t'e \leq \sum_{i} t'_{i}e_{i}$$
 and $te' \leq \sum_{i} t_{i}e'_{i}$,

where it is to be understood that the principal values of e and e' are arbitrarily

numbered in the same algebraic order, which is also automatically that of the principal values of t and t' by hypothesis (9). We conclude that

$$\Delta t \Delta e \ge \sum_{i} \left(t_{i}' e_{i}' + t_{i} e_{i} - t_{i}' e_{i} - t_{i} e_{i}' \right) \equiv \sum_{i} \Delta t_{i} \Delta e_{i}$$

and if the right side is positive so is the left.

(c) An explicit variant of (28) can be obtained with the help of (11), (15), (22) and (25):

$$\begin{split} \dot{t}\dot{e} &= \left\{ a_1 \frac{\partial \tau_1}{\partial a_1} - \tau_1 g(a_1) \right\} e_{11}^2 + \left(a_2 \frac{\partial \tau_1}{\partial a_2} + a_1 \frac{\partial \tau_2}{\partial a_1} \right) e_{11} e_{22} + \dots \\ &+ 2 \left(\frac{2a_1 a_2}{a_1^2 - a_2^2} \right)^2 (t_1 - t_2) \left(e_1 - e_2 \right) e_{12}^2 + \dots, \quad (33) \end{split}$$
it the notation
$$g(a_i) &= \frac{\mathrm{d}}{\mathrm{d}e_i} \left(a_i \frac{\mathrm{d}e_i}{\mathrm{d}a_i} \right) = 1 + a_i f''(a_i) / f'(a_i).$$

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This quadratic in the Eulerian components of strain-rate is to be compared with (18). If two stretches are equal, say $a_1 = a_2 = a$, the shear term is replaced by

$$2\left\{a\,\frac{\partial}{\partial a_1}\,(\tau_1-\tau_2)-\tau g(a)\right\}\,e_{12}^2.$$

With any strain measure (5) the function q reduces to a constant, 2m. The quadratics (18) and (33) then have identical terms in the normal components of strain-rate (and in a shear component when two stretches are equal).[†] These terms are alternatively expressible in the concise form (29) which, if it holds for all strains, was proved sufficient for (3). Thus, for such measures, (3) is generally weaker than (2), as stated in §1.

However, when m = 0 the coefficients of the shear terms in the two quadratics are

$$\begin{pmatrix} a_i^2 + a_j^2 \\ \overline{a_i^2 - a_j^2} \end{pmatrix} (\tau_i - \tau_j) \quad \text{and} \quad \left(\frac{2a_i a_j}{a_i^2 - a_j^2} \right)^2 (\tau_i - \tau_j) \ln \left(\frac{a_i}{a_j} \right)$$

respectively, and both have the sign of $(\tau_i - \tau_i)(a_i - a_i)$. In this case, therefore, (2) and (3) imply each other, even in a single configuration.

When $m = \pm 1$ the two quadratics are completely identical, as may easily be verified. Indeed (6) holds also when the material is not isotropic. For, from $\S 4(b)$, the components of the conjugate stresses for m = 1 or -1 on any rectangular coordinates in a fixed reference state are respectively equal to the contravariant or covariant components of Kirchhoff stress on these coordinates when embedded and deformed. Consequently, the rates of change of the conjugate stresses are equal to the convected derivatives of these components. And such derivatives are well known to be equal to the contravariant or covariant components of $\rho_0 s/\rho$ in (1) with m = 1

[†] From a broader standpoint this illustrates the fact that, under strain-rates coaxial with the Eulerian ellipsoid, $i\dot{e}/\rho_0$ is invariant with respect to choice of reference configuration for a measure (5). Two possible configurations have been singled out here: the ground state and the current state.

or -1 respectively. Correspondingly, as noted in §4(*a*), the rates of change of the reference components of the Green and Almansi strain measures are equal to the covariant and contravariant components of Eulerian strain-rate on the deformed coordinates.

6. INCOMPRESSIBLE SOLIDS

In the present context a direct treatment of incompressibility is preferable to extracting asymptotic formulae from the previous analysis. To avoid complicating the notation, (τ_1, τ_2, τ_3) in (16) are no longer regarded as principal values of Kirchhoff stress but merely as formal functions of the stretches, defined for all values though physically realizable only when $a_1a_2a_3 = 1$. Actual Kirchhoff (\equiv Cauchy) stresses capable of producing a considered deformation are obtained by superimposing on these functions an arbitrary hydrostatic pressure p (which may be positive or negative, and which does not alter the strain). Correspondingly, the principal stresses conjugate to a general strain measure are now

$$t_i - p/a_i f'(a_i)$$
 $(i = 1, 2, 3),$

where the quantities (t_1, t_2, t_3) are defined by (25).

Constitutive inequalities of the first kind then reduce to expressions formally similar to (17) or (18) augmented by

$$2mp\sum_{ij}e_{ij}^2$$
.

In fact p does not enter the moduli in the Jaumann flux, while the contribution from its time derivative vanishes in product with solenoidal strain-rates. When $m \neq 0$ the extra term is a definite quadratic and so (2) is violated in any strain configuration and for any strain-rate provided p has the appropriate sign and a sufficient magnitude. In particular the Coleman-Noll hypothesis fails. That (2) is admissible only in conjuction with the logarithmic measure was remarked by Hill (1968, § 3 (iii)), taking the neo-Hookean solid as an example.

Constitutive inequalities of the second kind give rise to an expression similar to (28) plus

$$p \sum_{i} g(a_{i}) e_{ii}^{2} + p \sum_{ijk} \left[\frac{1}{a_{j}f'(a_{j})} - \frac{1}{a_{i}f'(a_{i})} \right] (e_{i} - e_{j}) \omega_{k}^{2} \quad (k \neq i, j),$$

where the function g is as defined in (33). Both terms vanish identically only when the measure is logarithmic. Otherwise, (3) can be violated in every strain configuration by suitably choosing p for any admissible strain-rate. The inadequacy of (3) with measures (5), except when m = 0, was previously proved for Green-elastic solids by Hill (1968; cf. equation (30) there, which applies when $g \equiv 2m$ and the strain-rates are coaxial with the Eulerian ellipsoid).

When m = 0 the consequences of both kinds of inequality are the same, namely equations (29) and (30) in the variables $t_i = \tau_i$ and $e_i = \ln a_i$. Of course, the incompressibility restriction $e_1 + e_2 + e_3 = 0$ has to be observed. But still, as in §5(a), we can deduce first convexity and then the mutual ordering of the variables. To adapt the proof it is only necessary to observe that no incremental changes of volume occur along any linear path in the 'incompressibility plane' in (e_1, e_2, e_3) space, so that (29) can be applied at every stage. Whence, in analogy to (9),

$$\sum_{i} \Delta \tau_i \Delta \ln a_i > 0$$

for any pair of volume-preserving deformations.

For a Green-elastic solid with $\tau_i = a_i \partial \phi / \partial a_i$ the discriminating quadratic (29) with m = 0 becomes

$$\left(a_1^2 \frac{\partial^2 \phi}{\partial a_1^2} + a_1 \frac{\partial \phi}{\partial a_1}\right) x_1^2 + \ldots + 2a_1 a_2 \frac{\partial^2 \phi}{\partial a_1 \partial a_2} x_1 x_2 + \ldots > 0, \tag{34}$$

$$\phi = \frac{1}{2}\mu(\frac{1}{2} + \delta)\left(a_1^2 + a_2^2 + a_3^2\right) + \frac{1}{2}\mu(\frac{1}{2} - \delta)\left(a_1^{-2} + a_2^{-2} + a_3^{-2}\right)$$

$$2\mu \sum_{i} \{(\frac{1}{2} + \delta)a_{i}^{2} + (\frac{1}{2} - \delta)a_{i}^{-2}\}x_{i}^{2}$$

with m = 0 becomes $\left(a_1^2 \frac{\partial^2 \phi}{\partial a_1^2} + a_1 \frac{\partial \phi}{\partial a_1}\right) x_1^2 + \ldots + 2a_1 a_2 \frac{\partial^2 \phi}{\partial a_1 \partial a_2} x_1 x_2 + \ldots > 0, \quad (34)$ whenever x_1, x_2 and x_3 have a zero sum but do not all vanish. Before applying this test the functional form of ϕ can be modified, as one wishes, by using the connexion $a_1 a_2 a_3 = 1$. Take, for example, the Mooney–Rivlin material the potential of which may be put conveniently as $\phi = \frac{1}{2}\mu(\frac{1}{2} + \delta)(a_1^2 + a_2^2 + a_3^2) + \frac{1}{2}\mu(\frac{1}{2} - \delta)(a_1^{-2} + a_2^{-2} + a_3^{-2})$ where μ is the initial shear modulus and δ is a dimensionless constant. The quadratic (34) is here simply $2\mu \sum_i \{(\frac{1}{2} + \delta)a_i^2 + (\frac{1}{2} - \delta)a_i^{-2}\}x_i^2$ and is positive in all configurations if and only if $\mu > 0$ and $|\delta| \leq \frac{1}{2}$. In accordance with a general consequence of (3) for any Green-elastic solid (Hill 1968, §4) these restrictions ensure that net work is always *expended* in deformation from a stress-free ground state, ϕ being least there (the customary assumption). 7. STATICAL IMPLICATIONS AND SELECTION CRITERIA Consequences of inequalities of the second kind are now examined in greater detail for compressible solids. We recall that the objective is to find a strain measure for which the associated inequality is fully consistent with material behaviour, actual and conjectural. (a) We begin by considering configurations where the stress is purely *hydrostatic*. It may be tensile or compressive, as the constitutive law allows, and may perhaps never vanish. Any such configuration of course serves equally well as the ground state for an isotropic solid. The all-round stretch and Kirchhoff stress relative to this state are denoted by a and τ as usual. this state are denoted by a and τ as usual.

With any conjugate variables the rate equations in the considered deformed state are always isotropic-Hookean in character. By analogy the matrix of coefficients is positive-definite if and only if the associated bulk and rigidity moduli are positive. Whence by an easy calculation, or via (33), it is found that the net implications of (3) are

$$egin{aligned} &3a^3\kappa\equiv a\,\partial(au_1+2 au_2)/\partial a_1\ &2a^3\mu\equiv a\,\partial(au_1- au_2)/\partial a_1\ \end{pmatrix}> au g(a), \end{aligned}$$

where the derivatives (or their equivalents with other indices) are evaluated at $a_1 = a_2 = a_3 = a$. The conventional modulus of rigidity is here denoted by μ , and the bulk modulus with respect to Kirchhoff stress based on the *deformed* state by

$$\kappa = \rho \,\partial p / \partial \rho - p,$$

where $p = -\tau/a^3$ is the all-round pressure $(\rho \partial p/\partial \rho$ is the conventional bulk modulus). Thus |pg(z) - pg(z)| = |pg(z)| (25)

$$\kappa > -\frac{1}{3}pg(a), \quad \mu > -\frac{1}{2}pg(a). \tag{35}$$

With the measures (5), where $g(a) \equiv 2m$, these limitations on the moduli agree with ones obtained by Hill (1968, equations (15) and (16)) via inequality (2), which are here indistinguishable from (3) as noted in § 5 (c). Whatever the measure the conventional moduli are both positive in a stress-free state (if attainable), as is customarily assumed in infinitesimal elasticity theory.

For constructional metals the inequality for κ is comfortably satisfied with any reasonable choice of strain measure under practically attainable pressures. Beyond this range theoretical estimates of the dependence of κ on p seem insufficiently reliable for the present purpose. As for non-metals little of relevance is known about their compressibilities. At present, therefore, this particular inequality is undiscriminating. Turning to the other, we may presume from the nature of atomic or molecular bonds that μ is positive and varies fairly slowly with p (at least in materials which are only moderately compressible). If this is so, then for any function $g(a) \neq 0$ the inequality is likely to be violated beyond some critical dilatation or contraction, depending on the sign of g. This could not happen when the measure is logarithmic, since g then vanishes identically.

(b) In arbitrary configurations immediate consequences of (3) or of its variant (29) are $\partial t / \partial e_i > 0$ (i = 1, 2, 3)

$$\partial t_i / \partial e_i > 0 \quad (i = 1, 2, 3).$$

These correspond to incremental stretching in each principal direction in turn, while further strain is prevented in the other two. If such uniaxial deformation is continued, then by (25) $\sigma_i/f'(a_i)$ should increase monotonically throughout. More particularly, when the material is Green-elastic,

$$\frac{\partial \phi}{\partial a_i} \int f'(a_i)$$

should increase with a_i when the other stretches are held fixed.

If uniaxial deformation starts from the ground state, the active load (and hence that component of Cauchy stress) can be expected to rise steadily in all materials at first. There is a possibility that the load may in some cases attain a maximum and subsequently fall. However, no data is available and it is clear that qualitative speculation does not, in this instance, go far to restrict the choice of strain measure. For example, any $m < \frac{1}{2}$ in (5) is consistent with a tension that *perpetually* increases with extension, while any $m \ge \frac{1}{2}$ necessarily entails this behaviour (as does the Coleman-Noll hypothesis also).

(c) It is more fruitful to discuss the stresses needed for a finite simple shear relative to the ground state. The principal stretches are

$$a_1 = a, \quad a_2 = 1, \quad a_3 = a^{-1},$$

where $a \ge 1$, and the major axis of the Eulerian ellipse is inclined to the direction of shearing at an angle $\alpha \le \frac{1}{4}\pi$ given by

$$\cot 2\alpha = \frac{1}{2}(a_1 - a_3) = \gamma, \quad \text{say.}$$

Because of the assumed isotropy the stress is coaxial with the ellipse and its active shear component in the direction of shearing is

 $\sigma = \frac{1}{2}(\sigma_1 - \sigma_3)\sin 2\alpha.$

In all materials the senses of σ and γ can be expected to agree, irrespective of the other components of stress (which are merely passive). Consequently

$$\sigma_1 > \sigma_3 \tag{36}$$

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for the stated stretches.

With this inequality we can compare the prediction

$$\frac{\sigma_1}{a_1 f'(a_1)} > \frac{\sigma_3}{a_3 f'(a_3)}$$

from (3), in view of (25) and (30), the volume being preserved. That is,

$$\sigma_1 > \frac{a^2 f'(a)}{f'(a^{-1})} \sigma_3. \tag{37}$$

The coefficient on the right is always positive since f is supposed monotonic, and in particular is equal to a^{4m} with the choice (5).

Since inequalities (36) and (37) are in the same direction they are not necessarily in conflict, no matter what the function f. However, both exactly coincide only when the strain measure is logarithmic. With other measures, for example (5) with $m \neq 0$, the possible inferences depend on the sign of σ_3 . Suppose it to be positive: then in some actual materials (36) *might* hold without (37) when m > 0 but not when m < 0. On the other hand, if the sign is negative (as is more likely), (36) definitely entails (37) only when m > 0.

When the measure is logarithmic we can obtain further information from (12) itself by particularizing the strain-rate there to be that at a generic stage during the finite simple shearing. A short calculation leads to

$$(\dot{\sigma} + 2\sigma\gamma\dot{\gamma})\dot{\gamma} > 0$$

after using the isotropy condition to eliminate the difference of normal components of stress parallel and transverse to the direction of shearing. We conclude that $\sigma \exp \gamma^2$ should increase monotonically throughout. This requirement might be met even if σ attains a maximum and then falls.

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