# Some Results Concerning the Mathematical Treatment of Finite Plasticity.

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**Abstract**

The initial-boundary value problems arising in the context of finite elasto-plasticity models relying on the multiplicative split $F = F_e\, F_p$ are investigated. First, we present such a model based on the elastic Eshelby tensor. We highlight the behaviour of the system at frozen plastic flow. It is shown how the direct methods of variations can be applied to the resulting boundary value problem. Next the coupling with a viscoplastic flow rule is discussed. With stringent elastic stability assumptions and with a nonlocal extension in space local existence in time can be proved.

Subsequently, a new model is introduced suitable for small elastic strains. A key feature of the model is the introduction of an independent field of elastic rotations $R_e$. An evolution equation for $R_e$ is presented which relates $R_e$ to $F_e$. The equilibrium equations at frozen plastic flow are now linear elliptic leading to a local existence and uniqueness result without further stability assumptions or other modifications. An extended Korn's first inequality is used taking the plastic incompatibility of $F_p$ into account.

## 1 Notation

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth Lipschitz boundary $\partial\Omega$ and let $\Gamma$ be a smooth subset of $\partial\Omega$ with non-vanishing 2-dimensional Hausdorff measure. For $a, b \in \mathbb{R}^3$ we let $\langle a, b\rangle_{\mathbb{R}^3}$ denote the scalar product on $\mathbb{R}^3$ with associated vector norm $\|a\|^2_{\mathbb{R}^3} = \langle a, a\rangle_{\mathbb{R}^3}$. We denote by $\mathbb{M}^{3\times 3}$ the set of real $3\times 3$ tensors. The standard Euclidean scalar product on $\mathbb{M}^{3\times 3}$ is given by $\langle A, B\rangle_{\mathbb{M}^{3\times 3}} = \operatorname{tr}\left[AB^T\right]$ and thus the Frobenius tensor norm is $\|A\|^2 = \langle A, A\rangle_{\mathbb{M}^{3\times 3}}$. In the following we omit the index $\mathbb{R}^3, \mathbb{M}^{3\times 3}$. The identity tensor on $\mathbb{M}^{3\times 3}$ will be denoted by $1\!\!1$, so that $\operatorname{tr}[A] = \langle A, 1\!\!1\rangle$. We let $\mathbb{S}ym$ and $P\mathbb{S}ym$ denote the symmetric and positive definite symmetric tensors, respectively. We adopt the usual abbreviations of Lie-Group theory, i.e., $\mathrm{GL}(3,\mathbb{R}) := \{X \in \mathbb{M}^{3\times 3}\ |\det[X] \neq 0\}$ the general linear group, $\mathrm{SL}(3,\mathbb{R}) := \{X \in \mathrm{GL}(3,\mathbb{R})\ |\det[X] = 1\}$, $\mathrm{O}(3) := \{X \in$

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$\mathrm{GL}(3,\mathbb{R}) \mid X^T X = 1\!\!1\}$, $\mathrm{SO}(3,\mathbb{R}) := \{X \in \mathrm{GL}(3,\mathbb{R}) \,|X^T X = 1\!\!1,\ \det[X] = 1\}$ with corresponding Lie-Algebras $\mathfrak{so}(3) := \{X \in \mathbb{M}^{3\times3} \,|X^T = -X\}$ of skew symmetric tensors and $\mathfrak{sl}(3) := \{X \in \mathbb{M}^{3\times3} \,|\mathrm{tr}\,[X] = 0\}$ of traceless tensors. With $\mathrm{Adj}\,A$ we denote the tensor of transposed cofactors $\mathrm{Cof(A)}$ such that $\mathrm{Adj}\,A = \det[A]\,A^{-1} = \mathrm{Cof(A)}^{\mathrm{T}}$ if $A \in \mathrm{GL}(3,\mathbb{R})$. We set $\mathrm{sym}(A) = \frac{1}{2}(A^T + A)$ and $\mathrm{skew}(A) = \frac{1}{2}(A - A^T)$ such that $A = \mathrm{sym}(A) + \mathrm{skew}(A)$. For $X \in \mathbb{M}^{3\times3}$ we set $\mathrm{dev}\,X = X - \frac{1}{3}\,\mathrm{tr}\,[X]\,1\!\!1 \in \mathfrak{sl}(3)$ and for vectors $\xi, \eta \in \mathbb{R}^n$ we have $(\xi \otimes \eta)_{ij} = \xi_i\,\eta_j$. We write the polar decomposition in the form $F = R\,U = \mathrm{polar}(F)\,U$. In general we work in the context of nonlinear, finite elasticity. For the total deformation $\varphi \in C^1(\overline{\Omega}, \mathbb{R}^3)$ we have the deformation gradient $F = \nabla\varphi \in C(\overline{\Omega}, \mathbb{M}^{3\times3})$. Furthermore $S_1(F)$ and $S_2(F)$ denote the first and second Piola Kirchhoff stress tensors, respectively. Total time derivatives are written $\frac{d}{dt}A(t) = \dot{A}$. The first and second differential of a scalar valued function $W(F)$ are written $D_F W(F).H$ and $D_F^2 W(F).(H,H)$ respectively. $\partial\mathcal{X}$ is the (possibly set valued) subdifferential of the scalar valued function $\mathcal{X}$. We set $C = F^T F$, $C_p = F_p^T F_p$, $C_e = F_e^T F_e$, $E = \frac{1}{2}(C - 1\!\!1)$, $E_p = \frac{1}{2}(C_p - 1\!\!1)$, $E_e = \frac{1}{2}(C_e - 1\!\!1)$. We employ the standard notation of Sobolev spaces, i.e. $L^2(\Omega), H^{1,2}(\Omega), H_\circ^{1,2}(\Omega)$ which we use indifferently for scalar-valued functions as well as for vector-valued and tensor-valued functions. Moreover we set $\|A\|_\infty = \sup_{x\in\Omega}\|A(x)\|$. For $A \in C^1(\overline{\Omega}, \mathbb{M}^{3\times3})$ we define $\mathrm{Rot}\,A(x) = \mathrm{Curl}\,A(x)$ as the operation curl applied row wise. We define $H_\circ^{1,2}(\Omega,\Gamma) := \{\phi \in H^{1,2}(\Omega) \mid \phi_{|_\Gamma} = 0\}$, where $\phi_{|_\Gamma} = 0$ is to be understood in the sense of traces and by $C_0^\infty(\Omega)$ we denote infinitely differentiable functions with compact support in $\Omega$. We use capital letters to denote possibly large positive constants, e.g.,$C^+, K$ and lower case letters to denote possibly small positive constants, e.g., $c^+, d^+$. The smallest eigenvalue of a positive definite symmetric tensor $P$ is abbreviated by $\lambda_{\min}(P)$. Finally, w.r.t. abbreviates "with respect to".

# 2 Introduction

In the nonlinear theory of elasto-visco-plasticity at large deformation gradients it is often assumed that the deformation gradient $F = \nabla\varphi$ splits multiplicatively into an elastic and plastic part Lee [26], Mandel [30]

$$\nabla\varphi(x) = F(x) = F_e(x)\,F_p(x), \quad F_e, F_p \in \mathrm{GL}^+(3,\mathbb{R})\,, \tag{1}$$

where $F_e, F_p$ are explicitly understood to be incompatible configurations, i.e. $F_e, F_p \neq \nabla\Psi$ for any $\Psi : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$. Thus $F_p$ introduces in a natural way a **non-Riemannian manifold** structure. In our context we assume that this decomposition is uniquely defined only up to a **global** rigid rotation, since for arbitrary $\overline{Q} \in \mathrm{SO}(3)$ we have

$$\nabla\varphi(x) = F(x) = F_e(x)\,F_p(x) = F_e(x)\,\overline{Q}\,\overline{Q}^T\,F_p(x) = \tilde{F}_e(x)\,\tilde{F}_p(x)\,, \tag{2}$$

implying invariance under $F_p \mapsto \overline{Q}\,F_p$, $\forall\,\overline{Q} \in \mathrm{SO}(3)$. This multiplicative split, which has gained more or less permanent status in the literature, is micromechanically motivated by the kinematics of single crystals where dislocations move along fixed slip systems through the crystal lattice. The source for the incompatibility are those dislocations which did not completely traverse the crystal and consequently give rise to an inhomogeneous plastic deformation. Therefore, in the case of single crystal plasticity it is reasonable to introduce the deviation of the plastic intermediate configuration $F_p$ from compatibility as a kind of plastic **dislocation density**. This deviation should be related somehow to the quantity $\operatorname{Curl} F_p$ and indeed in NEFF [43] we see the important role played by $\operatorname{Curl} F_p$ in the existence theory related to models in this area.

The constitutive assumption (1) is incorporated into balance equations governing the elastic response of the material and supplemented by flow rules in the form of ordinary differential equations determining the evolution of the plastic part.

We refer the reader to BLOOM [6], KONDO [22], KRÖNER [23, 24] and MAUGIN [31], STEINMANN&STEIN [54], CERMELLI&GURTIN [9] for more details on the subject of dislocations and incompatibilities and to ORTIZ,REPETTO&STAINIER [47] and ORTIZ&STAINIER [48] for an account of the occurrence of the microstructure related to dislocations. A summary presentation of the theory for single crystals can be found in GURTIN [18]. For applications of the general theory of polycrystalline materials in the engineering field we refer to the non exhaustive list DAFALIAS [12, 13], MIEHE [35] and SIMO&HUGHES [51], SIMO [52], SIMO&ORTIZ [53]. An introduction to the theory of materials and inelastic deformations can be found in HAUPT [20], BESSELING&GIESSEN [5] and LEMAITRE&CHABOCHE [27]. Abstract mathematical treatments concerning the modelling of elasto-plasticity may be found in SILHAVY [50] and LUCCHESI&PODIO-GUIDUGLI [29].

# 3 The General Finite Elasto-Plastic Model

To begin with let us first introduce the considered finite compressible 3D-model. In most applications inertia effects can be safely neglected; one confines attention to the so called quasistatic case. Moreover, we restrict our considerations to the adiabatic problem without hardening. In general, hardening laws can be incorporated and will not affect the subsequent mathematical results. For simplicity the exposition is based on the phenomenological approach for isotropic polycrystals with associated flow rule, but the single crystal case as well as non-associated flow rules and general anisotropies can also be treated in the same spirit. We have opted to present a theory with elastic domain and yield function, but unified constitutive models cf. BODNER&PARTOM [7], SANSOUR&KOLLMANN [49] can also be considered. The inclusion of dead load body forces is standard and for brevity omitted.

In the quasi-static setting without body forces we are therefore led to study the following system of coupled partial differential and evolution equations for the deformation $\varphi : [0,T] \times \overline{\Omega} \mapsto \mathbb{R}^3$ and the plastic variable $F_p : [0,T] \times \overline{\Omega} \mapsto \mathrm{GL}^+(3,\mathbb{R})$:

$$
\begin{aligned}
&\int_\Omega W(F_e)\, \det[F_p]\, dx \mapsto \min. \quad \text{w.r.t. } \varphi \text{ at given } F_p\,, \\
&0 = \operatorname{Div}\, D_F\big[W(F_e)\, \det[F_p]\big] = \operatorname{Div}\,\big[S_1(F_e)\, \det[F_p]\big]\,, \\
&W(F_e) = \frac{\mu}{2}\, \|F_e\|^2 + \frac{\lambda}{4}\, \det[F_e]^2 - \frac{2\mu+\lambda}{2}\, \ln\det[F_e]\,, \qquad (3) \\
&F_e = \nabla\varphi\, F_p^{-1},\ \Sigma_E = F_e^T D_{F_e} W(F_e) \det[F_p] - W(F_e)\det[F_p]\, 1\!\!1\,,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{d}{dt}\left[F_p^{-1}\right](t) \in -F_p^{-1}(t)\, f(\Sigma_E)\,, \\
&\varphi_{|_\Gamma}(t,x) = g(t,x) \quad x \in \Gamma, \quad F_p^{-1}(0) = F_{p_0}^{-1}, \quad F_{p_0} \in \mathrm{GL}^+(3,\mathbb{R})\,,
\end{aligned}
$$

with the constitutive monotone multifunction $f : \mathbb{M}^{3\times 3} \mapsto \mathbb{M}^{3\times 3}$, that governs the plastic evolution and which is motivated by the principle of maximal dissipation relevant for the thermodynamical consistency of the model. Subsequently, $f$ will be obtained as $f = \partial\chi$, with a nonlinear flow potential $\chi : \mathbb{M}^{3\times 3} \mapsto \mathbb{R}$ (associated plasticity). $W(F_e)$ is the underlying elastic free energy which is already specified to be of Neo-Hooke type and $\mu, \lambda > 0$ are the Lam$\acute{e}$ constants of the material. Here $\Sigma_E$ denotes the elastic Eshelby energy momentum tensor which may be reduced to $\Sigma_M = F_e^T D_{F_e} W(F_e)$, the elastic Mandel stress tensor in case of a deviatoric flow rule according to isochoric plasticity. $F_{p_0}$ is the initial condition for the plastic variable. The inclusion sign $\in$ indicates that rate-independent, ideal plasticity is covered in this formulation.

The peculiar form of the elastic ansatz

$$
\int_\Omega W(\nabla\varphi\, F_p^{-1})\, \det[F_p]\, dx \mapsto \min. \quad \text{w.r.t. } \varphi \text{ at given } F_p\,, \tag{4}
$$

is motivated by the following observation. If $F_p(x) = \nabla\Psi_p(x)$ is compatible and $\Psi_p$ is a diffeomorphism, then the multiplicative decomposition (1) turns into

$$
\begin{aligned}
\nabla\varphi(x) &= \nabla_x\left[\Psi_e(\Psi_p(x))\right] = \nabla_\xi\Psi_e(\Psi_p(x))\, \nabla_x\Psi_p(x) = F_e(x)\, F_p(x)\,, \\
F_e &= \nabla\varphi(x)\, F_p^{-1} = \nabla_\xi\Psi_e(\Psi_p(x))
\end{aligned}
\tag{5}
$$

and (4) is nothing but the change of variables formula

$$
\begin{aligned}
\int_\Omega W(\nabla\varphi\, F_p^{-1})\, \det[F_p]\, dx &= \int_\Omega W(F_e)\, \det[\nabla\Psi_p]\, dx \\
&= \int_\Omega W(\nabla_\xi\Psi_e(\Psi_p(x)))\, \det[\nabla\Psi_p]\, dx = \int_{\xi\in\Psi_p(\Omega)} W(\nabla_\xi\Psi_e(\xi))\, d\xi.
\end{aligned}
\tag{6}
$$

If $DW(1\!\!1) = 0$, as is usually the case, we see that $\Psi_e(\xi) = \xi$ induces a globally stress free compatible new (intermediate) reference configuration $\Psi_p(\Omega)$ and the invariance requirement (2) preserves the compatibility of $\Psi_p$. Hence, locally, $F_p$ induces a change of coordinates to a stress free reference configuration.

# 4 Infinitesimal Model - Linearized Kinematics

If we identify $F = 1\!\!1 + \nabla u$ and $F_p = 1\!\!1 + p$, with both displacement gradient $\nabla u$ and plastic variable $p$ infinitesimally small, then the finite model (3) may be approximated by the reduced, partially linearized system

$$
\begin{aligned}
&\int_\Omega \tfrac{1}{2}\langle \mathcal{D}.\varepsilon_e, \varepsilon_e\rangle\, \left[1 + \mathrm{tr}\,[\varepsilon_p]\,\right]\, dx \mapsto \min. \quad \text{w.r.t. } u \text{ at given } \varepsilon_p\,, \\
&0 = \mathrm{Div}\ D_\varepsilon\Big[\tfrac{1}{2}\langle \mathcal{D}.\varepsilon_e, \varepsilon_e\rangle\, \left[1 + \mathrm{tr}\,[\varepsilon_p]\,\right]\Big] = \mathrm{Div}\,\Big[T\big[1 + \mathrm{tr}\,[\varepsilon_p]\,\big]\Big]\,,
\end{aligned}
$$

$$
\begin{aligned}
&\Psi(\varepsilon_e) = \tfrac{1}{2}\langle \mathcal{D}.\varepsilon_e, \varepsilon_e\rangle = \mu\, \|\varepsilon_e\|^2 + \frac{\lambda}{2}\, \mathrm{tr}\,[\varepsilon_e]^2, \quad T = \mathcal{D}.\varepsilon_e = \frac{\partial \Psi(\varepsilon_e)}{\partial \varepsilon}\,, \\
&\varepsilon_e = \varepsilon - \varepsilon_p, \quad \varepsilon(\nabla u(x)) = \tfrac{1}{2}(\nabla u^T + \nabla u), \quad \varepsilon_p = \tfrac{1}{2}(p^T + p)\,, \\
&\Psi^{thermo}(\varepsilon, \varepsilon_p) = \Psi(\varepsilon_e)\big[1 + \mathrm{tr}\,[\varepsilon_p]\,\big]\,, \\
&\dot{\varepsilon}_p(t) \in f(T_E)\,, \\
&T_E = -\partial_{\varepsilon_p}\Big[\Psi^{thermo}(\varepsilon, \varepsilon_p)\Big] = T\,\big[1 + \mathrm{tr}\,[\varepsilon_p]\,\big] - \tfrac{1}{2}\langle \mathcal{D}.\varepsilon_e, \varepsilon_e\rangle\, 1\!\!1\,, \\
&u_{|\Gamma}(t,x) = \tilde{g}(t,x) \quad x \in \Gamma, \quad \varepsilon_p(0) \in \mathbb{S}ym(3)\,,
\end{aligned}
\tag{7}
$$

where $\mathcal{D}$ is the 4th. order elasticity tensor, $T$ is the Cauchy stress tensor and the multiplicative decomposition (1) has been replaced by the additive decomposition of the infinitesimal strains into elastic and plastic parts

$$
\varepsilon = \varepsilon_e + \varepsilon_p\,. \tag{8}
$$

Here, $\Psi^{thermo}$ acts as thermodynamic potential for the plastic flow. The new reduced system (7) remains intrinsically thermodynamically correct. There is a rich mathematical literature successfully treating models based on (8) with $\mathrm{tr}\,[\varepsilon_p] = 0$, i.e., $\mathrm{tr}\,[f(T_E)] = 0$ or $T_E = T$, in which case $\Psi^{thermo} = \Psi(\varepsilon_e)$ and the model is of pre-monotone type in the sense of Alber. See, e.g., Alber [1], Han&Reddy [19], Ionescu&Sofonea [21], Chelminski [10, 11] and references therein.

A general mathematical treatment either of the reduced system (7) or of finite plasticity is, however, largely wanting. In the following we want to contribute some partial results in respect of finite plasticity.

**Remark 4.1 (Linearisation)**
*The reduced system* (7) *is not the exact formal linearisation of* (3). *However, the performed reduction yields a system of equations which is, where different from the formal linearisation, correct of higher order and remains intrinsically thermodynamically admissible. In addition it retains the Eshelbian like structure.*

# 5 Thermodynamically Consistent Plastic Flow Rules

In this part of the paper we would like to indicate how to obtain the **canonical** flow rules of finite multiplicative elasto-plasticity. In our context we use the term canonical in the sense of Miehe [34] meaning that fundamental dissipation principles together with tools from convex analysis are invoked to get an overall framework for multiplicative elasto-plasticity. In a more abstract setting this fits into the framework of a Thermodynamics with Internal Variables (TIV) as in Maugin [32]. The development is in principle well known but the use of the Eshelby tensor has only surfaced recently. We will see how the ansatz (4) naturally leads to the use of the Eshelby tensor. We include therefore the following for the presentation to be sufficiently self-contained.

A word of caution may be in order. Contrary to some other papers concerned with multiplicative elasto-plasticity we use as independent set of variables $F, F_p^{-1}$ leading to left-rate flow rules of the form $F_p\,\frac{d}{dt}\left[F_p^{-1}\right] =: L_{F_p^{-1}}$. The traditional approach would use as independent variables $F, F_p$ $(C, C_p)$ leading to flow rules of the form $\frac{d}{dt}F_p\, F_p^{-1} =: L_{F_p}$. However, there is no mathematical reason to prefer one representation over the other. The main point is that both types lead to similar mathematical structures. We employ throughout a material description, any quantity being defined with respect to the reference configuration; thus avoiding any discussion on consistent stress rates.

Let $W = W(F_e) = W(F\,F_p^{-1})$ be the given hyperelastic energy. The first Piola-Kirchhoff stress tensor $S_1$ is then $S_1(F, F_p^{-1}) = D_F W(FF_p^{-1}) = DW(FF_p^{-1})F_p^{-T}$. Using the objective Lie derivative and the principle of maximal dissipation one arrives at the canonical flow rule

$$-F_p\,\frac{d}{dt}\left[F_p^{-1}\right] \in \partial\chi(\Sigma_E)\,, \tag{9}$$

where $\partial\chi$ is the set valued subdifferential of the indicator function $\chi$ of a convex set $\mathcal{E}$ in the stress space related to $\Sigma_E$. Thus for rate-independent ideal plasticity

$$\chi(\Sigma_E) = \begin{cases} 0 & \Sigma_E \in \mathcal{E} \\ \infty & \Sigma_E \notin \mathcal{E}. \end{cases} \tag{10}$$

This flow rule can accommodate the assumption of isochoric plasticity, i.e., $\det[F_p] = 1$ (which replaces $\operatorname{tr}[\varepsilon_p] = 0$ in (8)) by defining the convex set $\mathcal{E}$ to be

$$\mathcal{E} := \{\Sigma_E \in \mathbb{M}^{3\times 3} \mid \quad \|\operatorname{dev}\operatorname{sym}(\Sigma_E)\| \le \sigma_{\mathrm{y}}\}\,, \tag{11}$$

where $\sigma_{\mathrm{y}}$ is the yield limit. Moreover with $0 = \frac{d}{dt}\left[F_p F_p^{-1}\right] = \dot{F}_p F_p^{-1} + F_p \frac{d}{dt}\left[F_p^{-1}\right]$ and

$$\begin{aligned} \dot{F}F^{-1} &= \frac{d}{dt}\left[F_e F_p\right](F_e F_p)^{-1} = \dot{F}_e F_e^{-1} + F_e \dot{F}_p F_p^{-1} F_e^{-1} \\ &= \dot{F}_e F_e^{-1} + F_e\left[-F_p \frac{d}{dt}\left[F_p^{-1}\right]\right] F_e^{-1} \end{aligned} \tag{12}$$

the spatial velocity gradient $\dot{F}F^{-1}$ may be additively decomposed into elastic and plastic parts as usual.

This is a straightforward generalization of the classical von Mises type $J_2$-plasticity for infinitesimal strains to finite strains. The macroscopic yield limit $\sigma_{\mathrm{y}}$ corresponds conceptually to the microscopic activation level of dislocation glide. Observe that the choice $\operatorname{sym}(\Sigma_E)$ instead of $\Sigma_E$ sets the so called plastic material spin Dafalias [14] to zero. But for isotropic $W$ the elastic Eshelby tensor is already symmetric, which has been noted previously Maugin&Epstein [33]. A shortcut way to see that this flow rule is thermodynamically admissible proceeds as follows: Let the deformation gradient $F$ be constant in time and consider

$$\begin{aligned} &\frac{d}{dt}\left[W(FF_p^{-1}(t))\det[F_p]\right] = \\ &= \langle DW(FF_p^{-1}(t)), F\frac{d}{dt}\left[F_p^{-1}\right]\rangle \det[F_p] + W(F_e)\,\langle \operatorname{Adj} F_p{}^T, \frac{d}{dt}\left[F_p\right]\rangle \\ &= \det[F_p]\left[\langle DW(FF_p^{-1}(t)), FF_p^{-1}F_p\frac{d}{dt}\left[F_p^{-1}\right]\rangle + W(F_e)\,\langle F_p^{-T}, \frac{d}{dt}\left[F_p\right]\rangle\right] \end{aligned}$$

$$= \det[F_p]\big[\langle F_p^{-T}F^T DW(FF_p^{-1}(t)), F_p\frac{d}{dt}\left[F_p^{-1}\right]\rangle + W(F_e)\,\langle 1\!\!1, \frac{d}{dt}[F_p]\,F_p^{-1}\rangle\big]$$

$$= \det[F_p]\big[\langle \underbrace{F_e^T DW(F_e(t))}_{\Sigma_M}, F_p\frac{d}{dt}\left[F_p^{-1}\right]\rangle - W(F_e)\,\langle 1\!\!1, F_p\frac{d}{dt}\left[F_p^{-1}\right]\rangle\big]$$

$$= \langle\underbrace{\det[F_p]\left(F_e^T DW(F_e(t)) - W(F_e)\,1\!\!1\right)}_{\Sigma_E}, F_p\frac{d}{dt}\left[F_p^{-1}\right]\rangle\,. \tag{13}$$

In the absence of thermal effects, classical continuum mechanics may be based on a second law in the form

$$\forall\, V\subset\mathbb{R}^3:\quad \frac{d}{dt}\int_V W\;dx \le \int_{\partial V}\langle S_1.n,\dot\varphi\rangle\;dS + \int_V\langle f,\dot\varphi\rangle\;dx\,, \tag{14}$$

where $W$ is the free energy, $n$ is the unit outward normal to the control volume $V$ and $f$ are the body forces Gurtin [17, p.41]. A sufficient condition for (14) to hold is the reduced dissipation inequality which is fulfilled whenever

$$\frac{d}{dt}\left[W(F\,F_p^{-1}(t))\det[F_p]\right]\le 0\,, \tag{15}$$

for arbitrary $F$ fixed in time. Thus, when choosing

$$-F_p\frac{d}{dt}\left[F_p^{-1}\right] = \lambda^+\,\partial\chi(\Sigma_E)\,, \tag{16}$$

with $\chi$ convex, the reduced dissipation inequality (15) is guaranteed, since from convex analysis $\langle\Sigma_E,\partial\chi(\Sigma_E)\rangle\ge 0$. Moreover, if

$$\partial\chi(\Sigma_E) = \lambda^+\;\mathrm{dev}(\mathrm{sym}(\Sigma_E))\,, \tag{17}$$

the right-hand side is traceless implying that $\det[F_p^{-1}] = 1 = \det[F_p]$.

Observe that the choice of the elastic Eshelby tensor $\Sigma_E$ as the relevant stress measure can be conveniently related to the local configurational driving forces on the inherent inhomogeneities introduced by the local change of reference through the plastic variable $F_p$. Moreover, this flow rule is in a natural way invariant under the change of plastic variable $F_p^{-1}\mapsto F_p$, as indeed it should be inconsequential which form of independent variable we take.

In order to formulate a viscoplastic regularization of the above evolution equation (9) the traditional approach proceeds as follows: instead of $\chi$ take the following function $\chi_\eta$:

$$\chi_\eta(\Sigma_E) = \begin{cases} 0 & \Sigma_E\in\mathcal{E} \\ \frac{1}{2\eta}\|\Sigma_E - P_{\mathcal{E}}.(\Sigma_E)\|^2 & \Sigma_E\notin\mathcal{E}. \end{cases} \tag{18}$$

Here $P_{\mathcal{E}}$ denotes the orthogonal projection onto the convex set $\mathcal{E}$ which is uniquely defined. Obviously, in the limit $\eta\to 0$ we recover the rate-independent evolution equation, at least formally. As it stands, $\partial\chi_\eta$ is just the well known Yosida approximation (linear viscosity) of the subdifferential $\partial\chi$ and it holds true Han&Reddy[19, p.184] that

$$\forall\, H\in\mathbb{M}^{3\times 3}:\;\langle\partial\chi_\eta(\Sigma_E), H\rangle = \frac{1}{\eta}\langle\Sigma_E - P_{\mathcal{E}}(\Sigma_E), H\rangle \tag{19}$$

and, moreover, $\partial\chi_\eta$ is a monotone function due to the convexity of $\chi_\eta$. Unfortunately, the ordinary differential evolution equation $-F_p \frac{d}{dt}\left[F_p^{-1}\right] = \partial\chi_\eta(\Sigma_E)$ does not possess the advantageous monotonicity properties with respect to the plastic variable $F_p$, in marked contrast to the properties of the infinitesimal flow rule according to (7) and $\operatorname{tr}[\varepsilon_p] = 0$. Reformulating the last equation we have

$$\frac{d}{dt}\left[F_p^{-1}\right] = -F_p^{-1}\,\partial\chi_\eta(\Sigma_E) = -\frac{1}{\eta} F_p^{-1}\,\left(\Sigma_E - P_{\mathcal{E}}.(\Sigma_E)\right)\ . \tag{20}$$

In particular, we see that $\partial\chi_\eta$ being Lipschitz continuous in $\Sigma_E$ does not entail that the right-hand side is Lipschitz continuous altogether with respect to $F_p^{-1}$ since $F_p^{-1}$ enters again multiplicatively. However, this is one of the main features which made the Yosida approximation so valuable in infinitesimal plasticity. For multiplicative plasticity it is typically the case that either the right-hand side is not Lipschitz continuous (rate independent case), or the right-hand side is neither monotone nor possesses potential structure.

# 6 Polyconvexity Conditions in Finite Plasticity

In order to investigate the boundary value problem which arises in the formulation of (3) if one freezes the plastic variable $F_p$ it is convenient to place this in the context of the direct methods of variations.

In the purely elastic case it is usually a convexity condition, the polyconvexity condition in the sense of Ball [3], that is used together with some coerciveness condition to ensure that a minimization problem has at least one solution. Let us recall this notion. We say that

**Definition 6.1 (Polyconvexity)**
*The free energy density $W(x,F)$ is polyconvex whenever there exists a (possibly non-unique) function $P(x,X,Y,Z) : \mathbb{R}^3 \times \mathbb{M}^{3\times 3} \times \mathbb{M}^{3\times 3} \times \mathbb{R}^+ \mapsto \mathbb{R}$ such that $P(x, \, , \, , \,)$ is convex for each $x \in \mathbb{R}^3$ and*

$$W(x,F) = P(x, F, \operatorname{Adj} F, \det[F])\ .$$

**Example 6.2**
*The Neo-Hooke energy density*

$$W(F) = \frac{\mu}{p\,(\sqrt{3})^{p-2}}\,\|F\|^p + \frac{\lambda}{4}\,\det[F]^2 - \frac{2\mu+\lambda}{2}\,\ln\det[F]$$

*is polyconvex for $p \geq 1$.*

**Corollary 6.3 (Polyconvexity and Ellipticity)**
*It is well known that*
*every smooth strictly polyconvex free energy density $W(x,F)$ is automatically ensuring overall Legendre-Hadamard ellipticity of the corresponding boundary value problem in the sense that*

$$\forall\, F \in \mathrm{GL}^+(3,\mathbb{R}) :\ \forall\, \xi,\eta \in \mathbb{R}^3 :\ D_F^2 W(x,F).(\xi\otimes\eta, \xi\otimes\eta) \geq c^+ \|\xi\|^2\,\|\eta\|^2.$$

**Example 6.4**
*Free energies defined on the Hencky strain tensor* $\ln C$ *such as*

$$W(F) = \frac{\mu}{4} \| \operatorname{dev} \ln C \|^2 + \frac{3\lambda + 2\mu}{3} \operatorname{tr} [\ln C]^2$$

*are in general not elliptic and therefore not polyconvex, see* NEFF*[38]. However, these energies are very popular among engineers due to certain advantages in a numerical implementation. Quadratic expressions in* $E$ *such as the St.Venant-Kirchhoff density*

$$W(F) = \mu \|E\|^2 + \frac{\lambda}{2} \operatorname{tr} [E]^2$$

*are neither elliptic nor polyconvex, they loose ellipticity at finite elastic compression. Energy densities which are convex functions of generalized strain measures* LUBARDA*[28, p. 33 ] are equally non elliptic.*

**Lemma 6.5 (Polyconvexity and Multiplicative Decomposition)**
*Let* $W(F)$ *be polyconvex and assume that* $F_p \in L^\infty(\Omega, \mathrm{GL}^+(3, \mathbb{R}))$ *is given. Then the function*

$$\tilde{W}(x, F) := W(F\, F_p^{-1}(x)) \det[F_p(x)]$$

*is itself polyconvex.*

**Proof.** This is accomplished by a direct check of the polyconvexity condition. Since $W$ is polyconvex, we know that there is some function $P$ such that $W(F) = P(F, \operatorname{Adj} F, \det[F])$ with $P$ convex. This yields

$$\begin{aligned}
\tilde{W}(x, F) &= W(F\, F_p^{-1}(x)) \det[F_p(x)] \\
&= P(F\, F_p^{-1}(x), \operatorname{Adj} F\, F_p^{-1}(x), \det[F\, F_p^{-1}(x)]) \det[F_p(x)] \\
&= P(F\, F_p^{-1}(x), \operatorname{Adj} F_p^{-1}(x) \operatorname{Adj} F, \det[F] \det[F_p^{-1}(x)]) \det[F_p(x)].
\end{aligned}$$

Now define

$$\tilde{P}(x, X, Y, Z) := P(X\, F_p^{-1}(x), \operatorname{Adj} F_p^{-1}(x)\, Y, Z \det[F_p^{-1}(x)]) \det[F_p(x)] \ .$$

It is easy to see that $\tilde{P}(x, \cdot, \cdot, \cdot)$ is a convex function since $P$ is. The essence is that $F_p$ introduces merely local inhomogeneity into the formulation. See also NEFF[38].

**Definition 6.6 (Coercivity)**
*We say that* $W$ *leads to a p-coercive problem whenever*

$$\int_\Omega W(\nabla\varphi)\, dx \le K_1 \Rightarrow \|\varphi\|_{1,p,\Omega} \le K_2 \ .$$

*Thus p-coercivity implies that a finite elastic energy level necessitates a finite value of the* $W^{1,p}(\Omega)$*-norm of the deformation* $\varphi$*.*

**Example 6.7**
*It is easily seen that the Neo-Hooke energy density is $p$-coercive for $p \geq 1$ if $\lambda > 0$. In this case the term $\frac{\lambda}{4}\det[F]^2 - \frac{2\mu+\lambda}{2}\ln\det[F]$ is pointwise bounded from above. An application of Poincaré's inequality completes the argument if Dirichlet boundary conditions are prescribed. However, energies defined on the Hencky strain tensor $\ln C$ such as*

$$W(F) = \frac{\mu}{4}\|\operatorname{dev}\ln C\|^2 + \frac{3\lambda + 2\mu}{3}\operatorname{tr}[\ln C]^2$$

*are, typically, not coercive for any $p$, perhaps indicating a more serious deficiency. See* Neff *[38, p.185].*

# 7 On the Choice of the Elastic Free Energy

Freezing $F_p$ is typically involved in computations of elasto-plasticity where this is called the elastic trial step. It seems to be a reasonable requirement in finite plasticity that the elastic trial step at frozen plastic variable $F_p$ should lead to a well posed elastic minimization problem as long as $F_p$ is invertible and sufficiently smooth. Whether this is indeed the case depends entirely on the chosen elastic free energy.

The only general method in finite elasticity to ascertain well posedness is based on the direct methods of the calculus of variations. The successful application relies on polyconvexity and coercivity. It is expedient to impose these conditions a priori to guarantee that the elastic trial step can be treated adequately.

In view of the invariance property expressed in Lemma 6.5 it suffices to specify some polyconvex free energy $W(F)$ and then to substitute the elastic part $F_e$ instead of $F$ to get a polyconvex minimization problem at frozen $F_p$. Thus, the multiplicative decomposition of the deformation gradient and polyconvexity are mutually compatible. It can easily be seen that $p$-coercivity is preserved as well under the multiplicative decomposition. Following this approach, elastic energies based on the Hencky tensor $\ln C$ or based on $E$ cannot be used. Formulations based on an additive decomposition $E = E_e + E_p$ which have been advocated by Green&Naghdi [16] and Naghdi [36], are also excluded since ellipticity of the elastic trial step may again be lost.

# 8 Existence Results in the General Finite Case - the Flow Based Approach.

We therefore freeze the plastic variable $F_p$ and analyse the elastic trial step. The corresponding boundary value problem has a variational structure in the sense that the equilibrium part of (3) is formally equivalent to the minimization problem

$$\begin{aligned} &\forall\, t \in [0,T]: \quad I(\varphi(t), F_p^{-1}(t)) \mapsto \min, \quad \varphi(t) \in g(t) + W_\circ^{1,p}, \\ &I(\varphi, F_p^{-1}) = \int_\Omega W(\nabla\varphi\, F_p^{-1}) \det[F_p(x)]\, dx, \end{aligned} \tag{21}$$

$$W(F_e) = \frac{\mu}{p\,(\sqrt{3})^{p-2}}\,\|F\|^p + \frac{\lambda}{4}\,\det[F_e]^2 - \frac{2\mu+\lambda}{2}\,\ln\det[F_e]\,.$$

We have the following preliminary result for fixed time:

**Theorem 8.1 (Existence for the static elastic trial step)**
*Assume that $F_p \in L^\infty(\Omega, \mathrm{GL}^+(3,\mathbb{R}))$ and $g \in W^{1,p}(\Omega,\mathbb{R}^3)$ with $p \geq 2$ is given. Then the elastic minimization problem* (21) *admits at least one minimizer.*

**Proof.** We sketch the proof and apply the direct methods of variations. The elastic free energy is of polyconvex Neo-Hooke type. By the invariance of polyconvexity under the multiplicative decomposition the elastic minimization problem is still polyconvex. The energy is also coercive over $W^{1,p}(\Omega,\mathbb{R}^3)$, cf. Example(6.7). Thus infimizing sequences $\varphi_k \in W^{1,p}(\Omega,\mathbb{R}^3)$ exist and admit weakly converging subsequences. The functional $I$ is lower semicontinuous due to its polyconvexity. Hence the weak limit $\varphi \in W^{1,p}(\Omega,\mathbb{R}^3)$ minimizes $I$. For the details, see Neff [38].

However, no statement is made as to how this solution varies if $F_p$ is varied or how it changes if the boundary data $g$ are varied. Due to the nonlinear nature of the problem at hand general theories of this kind cannot be expected to hold.

What would be most convenient is to assume that the solution of the minimization problem depends continuously on $F_p$ and the boundary data, i.e., elastic stability with respect to the data at least locally. This can be achieved by assuming that the minimizer lies in a uniform potential well for all plastic variables in a certain given set. By a uniform potential well we mean

**Definition 8.2 (Uniform potential well)**
*Assume that $\varphi$ is a global mi-nimizer of* (21)*. Whenever there exists a nondecreasing function $\gamma^+(s) > 0$, (e.g. $\gamma^+(s) = c^+\,|s|^2$), such that*

$$\begin{aligned}\forall\, 0 < s \leq s_0 : \inf_{\|h\|_{W^{1,p}_\circ}=1} \int_\Omega W(\nabla(\varphi + s\,h)\,F_p^{-1})\,\det[F_p]\,dx \\ \geq \int_\Omega W(\nabla\varphi\,F_p^{-1})\,\det[F_p]\,dx + \gamma^+(s)\,,\end{aligned}$$

*we say that $\varphi$ lies in a uniform potential well.*

A sufficient condition for $\varphi$ to lie in a uniform potential well is that the global minimizer of (21) is locally unique. This definitely does not imply that $W$ needs to be convex since there might be other local or global minimizers or stationary points, see Neff [38, p.173] and Ball&Marsden [4]. Under these circumstances it can be shown, that it is possible to define a local solution operator $\varphi = T(F_p, g)$, such that

$$\begin{aligned}\inf_{\varphi \in g + W^{1,p}_\circ} \int_\Omega W(\nabla\varphi F_p^{-1})\,\det[F_p(x)]dx \\ = \int_\Omega W(\nabla T(F_p,g)F_p^{-1})\,\det[F_p(x)]dx\,,\end{aligned} \tag{22}$$

which is Hoelder continuous if $p \geq 6$, but may in general not be Lipschitz continuous. Be that as it may, by introducing $T$ we can dispose of the boundary value problem and concentrate on the flow rule (**the flow based approach**). In order to approach the flow problem we introduce a further modification. Instead of considering $\Sigma_E$ we replace $\Sigma_E$ with a space averaged $\widehat{\Sigma}_E^\varepsilon(x)$ where $\varepsilon$ indicates the average over some small $\varepsilon$-ball $B_\varepsilon(x) := \{y \in \Omega \mid \|y - x\| \leq \varepsilon\}$ centred at $x \in \Omega$. In this fashion we introduce a nonlocal dependence into the model. This ensures at the same time that the averaged Eshelby stresses are smoothly distributed which would not be necessarily true for the non averaged quantities due to a general lack of regularity for the nonlinear elliptic problem. This type of averaging preserves frame indifference and could even be argued for on physical grounds. It is also necessary to remove the possible singularity inherent in the elastic free energy through $-\ln \det[F_e]$ if $\det[F]$ approaches zero. This can be done in a consistent manner by replacing $-\ln$ with a smooth convex function $h : \mathbb{R} \mapsto \mathbb{R}$ such that $h \geq 0,\ h'(1) = -1$.

With these assumptions and modifications it is possible to prove a local existence result for a fully viscoplastic formulation of the model. The modified nonlocal model reads then

$$
\begin{aligned}
&\int_\Omega W(F_e)\,\det[F_p]\,dx \mapsto \min. \quad \text{w.r.t. } \varphi \text{ at given } F_p\,,\\
&W(F_e) = \frac{\mu}{p\,(\sqrt{3})^{p-2}}\,\|F\|^p + \frac{\lambda}{4}\,\det[F_e]^2 - \frac{2\mu+\lambda}{2}\,h(\det[F_e])\,, \qquad (23)\\
&\Sigma_E = F_e^T D_{F_e} W(F_e)\det[F_p] - W(F_e)\,\det[F_p]\,1\!\!1\,,\\
&\frac{d}{dt}\left[F_p^{-1}\right](t) \in -F_p^{-1}(t)\,\partial\chi(\widehat{\Sigma}_E)\,,\\
&\varphi_{|\partial\Omega}(t,x) = g(t,x) \quad x \in \partial\Omega, \quad F_p^{-1}(0) = F_{p_0}^{-1}\,, \quad F_{p_0} \in \mathrm{GL}^+(3,\mathbb{R})\,,
\end{aligned}
$$

where the nonlinear flow potential $\chi : \mathbb{M}^{3\times 3} \mapsto \mathbb{R}$ is assumed to have a local Lipschitz subdifferential $\partial\chi$, e.g. of the form (25). It is possible to prove the following result:

**Theorem 8.3 (Local existence for nonlocal model)**
*Let $W$ be as above with $p = 6$ and let $g \in C^1(\mathbb{R}^+; W^{1,\infty}(\Omega, \mathbb{R}^3))$; moreover, assume that solutions of the elastic minimization problem at fixed plastic variable $F_p$ lie in a uniform potential well. Then, there exists a time $T > 0$ such that (23) admits a (possibly non unique) local solution $F_p^{-1} \in C^1([0,T]; C(\overline{\Omega}, \mathrm{GL}^+(3,\mathbb{R}))$ and $\varphi \in C([0,T], W^{1,p}(\Omega, \mathbb{R}^3))$.*

**Proof.** Consider the following iterative scheme:

$$
\begin{aligned}
&\int_\Omega W(\nabla\varphi^{n+1}(x,t)\,F_p^{-n}(x,t))\,\det[F_p^n]\,dx \mapsto \min.\text{w.r.t. } \varphi^{n+1} \text{ at given } F_p^n\,,\\
&\frac{d}{dt}\left[F_p^{-1,n+1}\right](t) \in -F_p^{-1,n+1}(t)\,\partial\chi(\widehat{\det[F_p^n]\Sigma^n})\,, \qquad (24)\\
&\Sigma^n = F_p^{-T,n}\,\nabla\varphi^{T,n+1}\,D_{F_e} W(\nabla\varphi^{n+1} F_p^{-n}) - W(\nabla\varphi^{n+1}\,F_p^{-n})\,1\!\!1\,,\\
&\widehat{\Sigma}(x) := \frac{1}{|B_\varepsilon(x)|}\int_{y\in B_\varepsilon(x)} \Sigma(y)\,dy\,,
\end{aligned}
$$

$$\varphi^{n+1}_{|\partial\Omega}(t,x) = g(t,x) \quad x \in \partial\Omega, \quad F_p^{-1,n+1}(0) = F_{p_0}^{-1}, \quad F_{p_0} \in \mathrm{GL}^+(3,\mathbb{R})\,.$$

The direct methods of variations (Theorem 8.1) show the existence of a minimizer $\nabla\varphi^{n+1}$. By the elastic stability assumptions we have that $\nabla\varphi^{n+1}(x,t)$ is well defined. The evolution equation at given $\nabla\varphi^{n+1}(x,t)$ has a unique local solution $F_p^{n+1}(x,t)$ due to Banach's fixed point principle (linear ordinary differential equation). This defines an operator

$$\begin{aligned}
&P:\ C([0,T];C(\overline{\Omega},\mathrm{GL}^+(3,\mathbb{R}))) \mapsto C([0,T];C(\overline{\Omega},\mathrm{GL}^+(3,\mathbb{R})))\,,\\
&\frac{d}{dt}\left[P.F_p^n\right]^{-1}(t) \in -\left[P.F_p^n\right]^{-1}(t)\,\partial\chi(\widehat{\Sigma}_E^n)\,, \qquad \text{(ODE)}\\
&P:\ F_p^n \mapsto F_p^{n+1}\,.
\end{aligned}$$

It is then possible to show that this operator is indeed compact. First, since the solutions of the boundary value problem $\nabla\varphi^{n+1}$ can be independently bounded in $L^2(\Omega)$ and the averaging procedure for fixed $\varepsilon > 0$ delivers smooth solutions $F_p^{n+1}$, the operator $P$ is continuous. Gronwall's inequality applied to the ODE together with an application of the Arcela-Ascoli theorem shows that $P$ maps bounded sets into equicontinuous sets and individual arguments are transformed to Hoelder continuous functions. Schauder's fixpoint principle for continuous, compact operators yields then the existence of at least one fixed point of $P$. This proves the claim, for details compare Neff [38]. Observe in passing that $F_p^{n+1}$ is not to be confused with a time incremental update but is a new solution over the whole time interval.

**Remark 8.4 (Flow Based Versus Time Incremental Formulation)**
*In what we like to call the* **flow based** *approach priority is given to the plastic flow rule, and the elastic balance equation is rather treated as a (static) side condition. This is the approach followed in this contribution. In the extreme case of rigid plastic behaviour the elastic problem is completely discarded, which is a well known strategy in the literature. Similar ideas are used on the numerical side, where the resulting discretized initial boundary value problem is interpreted as a differential algebraic equation (DAE) and the algebraic constraint corresponds to the fulfilment of the elastic balance equation.*

*In the opposite case, the* **time incremental formulation**, *priority is given to the elastic balance equation. The flow rule is implicitly discretized and the updated plastic variable is inserted into the balance equation. The resulting field problem in general looses ellipticity in the finite problem but may retain a variational structure in the case of an associated flow rule. Here, the flow rule acts merely as a side condition,* Carstensen&Hackl *[8]. In the infinitesimal case the time incremental formulation leads to the deformation theory of Hencky plasticity retaining ellipticity.*

**Remark 8.5**
*The above result (Theorem 8.3) is only a very weak statement in view of the following: we had to assume local elastic stability (uniform potential well) of the minimization problem related to the elastic trial step. This can in general not be proved. In addition we needed to modify the formulation to a nonlocal one through the space averaged elastic Eshelby stresses. And due to the use of*

*Schauder's principle we get only a local solution which might not even be locally unique. In summary, we see that the solution of the elastic trial step was easily found but that the properties of this solution are not sufficient to couple it as such with the flow problem. No attempt has been made to consider the limit behaviour $\varepsilon \to 0$.*

In order to overcome these serious technical difficulties which were entirely due to the nature of the finite elastic free energy we shall now introduce a new model which reflects more closely one aspect of the physics of the problem at least for metals. The chosen Neo-Hooke elastic free energy is in principle appropriate for arbitrarily large elastic strains. However, the prevailing deformation mode for metals is known to comply to small elastic strains in most practical cases.

# 9 A Model for Small Elastic Strains

In the three-dimensional case it is easily seen that small elastic strains, i.e., $\|F_e^T F_e - 1\!\!1\|$ pointwise small, imply that $F_e$ is approximately a rotation $R_e \in \mathrm{SO}(3)$. If we assume that $R_e$ is known, all quantities can be linearized with respect to $R_e = \mathrm{polar}(F_e)$. This is a nonlinear constraint. It is possible to relax this static constraint into a dynamic evolution equation such that a rotation $R_e$ is determined which coincides approximately with $\mathrm{polar}(F_e)$ whenever $F_e$ is approximately a rotation. The static constraint $R_e = \mathrm{polar}(F_e)$ turns out to be a global attractor of the evolution equation. These modifications significantly simplify the mathematical structure without loosing the main ingredients of finite multiplicative visco-plasticity notably frame indifference and invariance with respect to superposed rotations of the so called intermediate configuration are preserved. In addition, the model allows for finite elastic rotations, finite plastic deformations and overall finite deformations. Let us introduce the considered new 3D-model which modifies the exposition in Neff [41, 37] to include in a consistent manner non-isochoric plasticity, i.e., $\det[F_p] \neq 1$.

In the quasi-static setting without body forces we are led to study the following system of coupled partial differential and evolution equations for the deformation $\varphi : [0,T] \times \overline{\Omega} \mapsto \mathbb{R}^3$, the plastic variable $F_p : [0,T] \times \overline{\Omega} \mapsto \mathrm{GL}^+(3,\mathbb{R})$ and the independent elastic rotation $R_e : [0,T] \times \overline{\Omega} \mapsto \mathrm{SO}(3)$:

$$
\begin{aligned}
&\int_\Omega W(F_e, R_e)\, \det[F_p]\, dx \mapsto \min . \quad \text{w.r.t. } \varphi \text{ at given } R_e, F_p\,, \\
&0 = \mathrm{Div}\; D_F\left[W(F_e,R_e)\,\det[F_p]\right] = \mathrm{Div}\left[S_1(F_e,R_e)\,\det[F_p]\right]\,, \\
&W(F_e,R_e) = \frac{\mu}{4}\,\|F_e^T R_e + R_e^T F_e - 2 1\!\!1\|^2 + \frac{\lambda}{8}\,\mathrm{tr}\left[F_e^T R_e + R_e^T F_e - 2 1\!\!1\right]^2\,, \\
&F_e = \nabla\varphi\, F_p^{-1}\,, \\
&\Sigma_E = F_e^T D_{F_e} W(F_e,R_e) \det[F_p] - W(F_e,R_e)\,\det[F_p] 1\!\!1\,, \\
&S_1(F_e,R_e) = R_e\left[\mu(F_e^T R_e + R_e^T F_e - 2 1\!\!1) + \lambda\,\mathrm{tr}\left[F_e^T R_e - 1\!\!1\right] 1\!\!1\right] F_p^{-T}\,, \\
&\varphi_{|_\Gamma}(t,x) = g(t,x) \quad x \in \Gamma\,, \qquad \qquad \text{(P3)} \\
&\frac{d}{dt}\left[F_p^{-1}\right](t) \in -F_p^{-1}(t)\, \partial\chi(\Sigma_E)\,,
\end{aligned}
$$

$$\frac{d}{dt}R_e(t) = \nu^+ \text{ skew}\left(F_e R_e^T\right) R_e(t)\,, \quad \nu^+ = \nu^+(F_e, R_e) \in \mathbb{R}^+\,,$$
$$F_p^{-1}(0) = F_{p_0}^{-1}\,, \quad F_{p_0} \in \mathrm{GL}^+(3,\mathbb{R}), \quad R_e(0) = R_e^0, \qquad R_e^0 \in \mathrm{SO}(3)\,.$$

The term $\nu^+ := \nu^+(F_e, R_e)$ represents a scalar valued penalty function introducing elastic viscosity. $F_{p_0}^{-1}$ and $R_e^0$ are the initial conditions for the plastic variable and elastic rotation part, respectively. If we identify $F_p = 1\!\!1 + p$ and set $R_e \equiv 1\!\!1$, the model settles also down to (7).

In the (vanishing elastic viscosity) limit $\nu^+ \to \infty$, the model (P3) approaches formally the problem

$$\begin{aligned}
&\int_\Omega W_\infty(U_e)\,\det[F_p]\,dx \mapsto \text{stationary w.r.t. } \varphi \text{ at given } F_p\,,\\
&0 = \mathrm{Div}\; D_F\left[W_\infty(F_e)\,\det[F_p]\right]\,,\\
&W_\infty(U_e) = \mu\,\|U_e - 1\!\!1\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\left[U_e - 1\!\!1\right]^2\,,\\
&\frac{d}{dt}\left[F_p^{-1}\right](t) \in -F_p^{-1}(t)\,\partial\chi(\Sigma_{E,\infty})\,,\\
&\Sigma_{E,\infty} = U_e^T\,D_{U_e}W_\infty(U_e)\det[F_p] - W_\infty(U_e)\,\det[F_p]1\!\!1\,,
\end{aligned} \tag{Biot}$$

with $U_e = (F_e^T F_e)^{\frac{1}{2}}$ the elastic stretch and $U_e - 1\!\!1$ the elastic Biot strain tensor. The system (Biot) is an exact model for small elastic strains and finite plastic deformations. The transition from (P3) to (Biot) is not entirely trivial since it is not just the replacement of $R_e$ by $R_e = \mathrm{polar}(F_e)$. Moreover, note the tacit change from minimization in (P3) to stationarity in (Biot). For a detailed account of the derivation and the properties of the new model (P3) we refer to Neff [37, 41]. Now in Neff [39, 42] it has been shown that the model (P3) in a viscous form is locally wellposed. Moreover, first numerical computations Neff&Wieners [44] confirm the general applicability of the model (P3) for structural applications compared with standard models.
To grasp the main idea in the above approximation we look at the corresponding evolution equation for $R_e$. The following theorem can be proved.

**Theorem 9.1 (Exact dynamic polar decomposition)**
*Let*
$F_e \in \mathrm{GL}^+(3,\mathbb{R})$ *and assume that* $R_0 \in \mathrm{SO}(3)$ *is given with*
$\|R_0 - \mathrm{polar}(F_e)\|^2 < 8$*. Then the evolution equation*

$$\frac{d}{dt}R_e(t) = \nu^+ \text{ skew}\left(F_e R_e^T\right) R_e(t), \quad R_e(0) = R_0\,,$$

*has a unique global in time solution* $R_e(t) \in \mathrm{SO}(3)$ *which converges to*

$$R_\infty = \mathrm{polar}(F_e)\;.$$

**Proof.** See Neff [37, 41].

The guiding idea is then to relax the algebraic constraint $R_e = \mathrm{polar}(F_e)$ inherent in the formulation with $U_e$ in (Biot) into an associated evolution equation which locally approximates this constraint.

In practice the flow rule for the (independent) elastic rotations $R_e$ introduces merely reversible nonlinear viscoelastic behaviour restricted to multiaxial deformations well below an assumed elastic yield limit. In contrast to micropolar theories an additional field equation for $R_e$ is avoided.

# 10 Existence and Uniqueness for Small Elastic Strains

Now we specify the flow potential $\chi$ and the function $\nu^+$. For von Mises type $J_2$-viscoplasticity with elastic domain $\mathcal{E}$ (see (11)) and yield stress $\sigma_{\mathrm{y}}$, we take as visco-plastic potential $\chi : \mathbb{M}^{3\times 3} \mapsto \mathbb{R}$ of generalized Norton-Hoff overstress type the following function:

$$\chi(\Sigma_E) = \begin{cases} 0 & \Sigma_E \in \mathcal{E} \\ \frac{1}{(r+1)(m+1)\,\eta}\left(1 + (\|\operatorname{dev}(\operatorname{sym}\Sigma_E)\| - \sigma_{\mathrm{y}})^{r+1}\right)^{m+1} & \Sigma_E \notin \mathcal{E} \end{cases},$$

where $\eta > 0$ is a viscosity parameter and $r, m \geq 0$. An easy calculation shows that this leads to the single valued locally Lipschitz continuous subdifferential

$$\partial\chi(\Sigma_E) = \frac{1}{\eta}\left(1 + [\|\operatorname{dev}(\operatorname{sym}\Sigma_E)\| - \sigma_{\mathrm{y}}]_+^{r+1}\right)^m \times \\ [\|\operatorname{dev}(\operatorname{sym}\Sigma_E)\| - \sigma_{\mathrm{y}}]_+^r \,\frac{\operatorname{dev}(\operatorname{sym}\Sigma_E)}{\|\operatorname{dev}(\operatorname{sym}\Sigma_E)\|}\,. \tag{25}$$

The parameter $r$ allows to adjust the smoothness of the flow rule when passing the boundary of the elastic domain $\mathcal{E}$. The special choice $m = 0, r = 1$ corresponds precisely to the Yosida approximation (19). With $r > 5$ and $m \geq 1$ it is clear that $\partial\chi \in C^5(\mathbb{M}^{3\times 3}, \mathbb{M}^{3\times 3})$. A typical range for $m$ in engineering applications is $m \in \{0, \ldots, 80\}$. For $m \to \infty$ we recover formally ideal rate independent plasticity. For simplicity, we choose the positive parameter $\nu^+$ in the elastic flow part according to the same level of viscosity as is used in the plastic flow part and set formally

$$\nu^+ = \frac{1}{\eta}\left(1 + \left[\|\operatorname{skew}(F_e R_e^T)\| - 0\right]_+^{r+1}\right)^m \times \\ [\|\operatorname{skew}(B)\| - 0]_+^r \,\frac{1}{\|\operatorname{skew}(B)\|}\,, \tag{26}$$

similar to (25). This choice makes the flow rule altogether a $C^5$ function. In this setting we can prove the following result:

**Theorem 10.1 (Local existence and uniqueness)**
*Suppose for the*
*displacement boundary data $g \in C^1(\mathbb{R}^+, H^{5,2}(\Omega, \mathbb{R}^3))$. Then there exists a time $T > 0$ such that the initial boundary value problem* (P3) *with* (25) *and* (26) *admits a unique local solution*

$$\varphi \in C([0,T], H^{5,2}(\Omega, \mathbb{R}^3))\,,$$
$$(F_p, R_e) \in C^1([0,T], H^{4,2}(\Omega, SL(3,\mathbb{R})), H^{4,2}(\Omega, SO(3)))\,.$$

**Proof.** See NEFF [39]. We basically repeat the ideas of Theorem 8.3. At frozen variables $(F_p, R_e)$ the (elastic) equilibrium system in (P3) is a linear, second order, strictly Legendre-Hadamard elliptic boundary value problem with non-constant coefficients. The nonlinearity has been shifted into the appended evolution equation. This system has variational structure in the sense that the equilibrium part of (P3) is formally equivalent to the minimization problem

$$
\begin{aligned}
&\forall\, t \in [0,T]: \quad I(\varphi(t), F_p^{-1}(t), R_e(t)) \mapsto \min, \quad \varphi(t) \in g(t) + H_\circ^{1,2}\,, \\
&I(\varphi, F_p^{-1}, R_e) = \int_\Omega W(\nabla\varphi\, F_p^{-1}, R_e)\, \det[F_p]\, dx\,, \qquad (27)\\
&W(F_e, R_e) = \frac{\mu}{4}\, \|F_e^T R_e + R_e^T F_e - 2 1\!\!1\|^2 + \frac{\lambda}{8}\, \mathrm{tr}\left[F_e^T R_e + R_e^T F_e - 2 1\!\!1\right]^2 .
\end{aligned}
$$

The main task in proving that (P3) is well posed consists of showing uniform estimates for solutions of linear, elliptic systems whose coefficients are time dependent and do not induce a pointwise positive bilinear form. This problem does not arise in infinitesimal elasto-viscoplasticity (7) with $\mathrm{tr}\,[\varepsilon_p] = 0$, since there, the elasticity tensor $\mathcal{D}$ is assumed to be a constant positive definite 4th. order tensor.

We are first concerned with the static situation where $(F_p, R_e)$ are assumed to be known. We prove existence, uniqueness and regularity of solutions to the related (elastic) boundary value problem. In addition we elucidate in which manner these solutions depend on $(F_p, R_e)$. These investigations rely heavily on a theorem recently proved by the author extending Korn´s first inequality to nonconstant coefficients.

**Theorem 10.2 (Extended Korn's first inequality)**
*Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $\Gamma \subset \partial\Omega$ be a smooth part of the boundary with nonvanishing 2-dimensional Lebesgue measure. Define $H_\circ^{1,2}(\Omega, \Gamma) := \{\phi \in H^{1,2}(\Omega) \mid \phi_{|\Gamma} = 0\}$ and let $F_p, F_p^{-1} \in C^1(\overline{\Omega}, GL^+(3,\mathbb{R}))$ be given with $\det[F_p(x)] \geq \mu^+ > 0$. Moreover suppose that for the* **dislocation density** $\mathrm{Curl}\, F_p \in C^1(\overline{\Omega}, \mathbb{M}^{3\times3})$. *Then*

$$
\begin{aligned}
&\exists\, c^+ > 0 \quad \forall\, \phi \in H_\circ^{1,2}(\Omega, \Gamma): \\
&\qquad \|\nabla\phi\, F_p^{-1}(x) + F_p^{-T}(x)\, \nabla\phi^T\|^2_{L^2(\Omega)} \geq c^+\, \|\phi\|^2_{H^{1,2}(\Omega)}\,.
\end{aligned}
$$

*Clearly this result generalizes the classical Korn's first inequality*

$$
\exists\, c^+ > 0 \quad \forall\, \phi \in H_\circ^{1,2}(\Omega, \Gamma): \quad \|\nabla\phi + \nabla\phi^T\|^2_{L^2(\Omega)} \geq c^+\, \|\phi\|^2_{H^{1,2}(\Omega)}\,,
$$

*which is just our result with $F_p = 1\!\!1$.*

**Proof.** See NEFF [43]. Recently it was possible to significantly relax the continuity assumptions necessary for this theorem to hold. Precisely in the case $\Gamma = \partial\Omega$ it can be shown that

$$
\begin{aligned}
&\exists\, c^+ > 0 \quad \forall\, \phi \in H_\circ^{1,2}(\Omega) \\
&\qquad \|\nabla\phi\, F_p^{-1}(x) + F_p^{-T}(x)\, \nabla\phi^T\|^2_{L^2(\Omega)} \geq c^+\, \|\phi\|^2_{H^{1,2}(\Omega)}\,,
\end{aligned}
$$

if only $F_p \in L^\infty(\Omega, \mathrm{GL}^+(3,\mathbb{R}))$ with $\det[F_p(x)] \geq \mu^+ > 0$ and $\mathrm{Curl}\, F_p \in L^2(\Omega, \mathbb{M}^{3\times3})$. In addition one has to assume that with $F_p = R_p\, U_p$, the polar factorization of $F_p$, the orthogonal part $R_p$ has locally jumps of maximal height $C^+ \frac{\lambda_{min,\Omega}(U_p)}{\lambda_{max,\Omega}(U_p)}$, where $C^+$ is a given non small constant. This allows $F_p$ to be quite discontinuous. In the general case of mixed boundary data one needs, moreover, that $F_p$ is smooth in an arbitrarily small boundary layer NEFF [40].

Now, the minimization problem (27) can be easily solved by applying the direct methods of variations. We show that $I$ is strictly convex over the affine space $\{g + H_\circ^{1,2}(\Omega)\}$. This is done by computing the second derivative. We have

$$
\begin{aligned}
& D^2_\varphi I(\varphi, F_p^{-1}, R_e).(\phi,\phi) = \\
& \int_\Omega \left( \frac{\mu}{2} \, \|F_p^{-T} \nabla\phi^T R_e + R_e^T \nabla\phi F_p^{-1}\|^2 + \lambda \operatorname{tr}\left[R_e^T \nabla\phi F_p^{-1}\right]^2 \right) \det[F_p(x)] \, dx \\
& \geq \int_\Omega \frac{\mu}{2} \, \|(R_e F_p)^{-T} \nabla\phi^T + \nabla\phi (R_e F_p)^{-1}\|^2 \det[F_p(x)] \, dx \\
& \geq \mu \, c^+(F_p, R_e, \Omega) \, \|\phi\|^2_{1,2,\Omega} \, ,
\end{aligned} \tag{28}
$$

by applying Theorem 10.2 with $F_p := R_e F_p$. Since the Lamé constant $\mu > 0$ we see that $D^2_\varphi I(\varphi, F_p^{-1}, R_e).(\phi,\phi)$ is uniformly positive. Hence $I(\varphi, F_p^{-1}, R_e)$ is strictly convex.

We can write the evolution part of (P3) in the following block diagonal form with $A = (F_p^{-T}, R_e)$:

$$
\frac{d}{dt} \begin{pmatrix} F_p^{-T}(t) \\ R_e(t) \end{pmatrix} = \begin{pmatrix} -\partial\chi(\Sigma_E(t))^T & 0 \\ 0 & \nu^+ \operatorname{skew}(B(t)) \end{pmatrix} \begin{pmatrix} F_p^{-T}(t) \\ R_e(t) \end{pmatrix} . \tag{29}
$$

Thus, the system (P3) is equivalent to

$$
\frac{d}{dt} A(t) = \hat{h} \left( \nabla_x T(A(t), g(t)), A(t) \right) \, A(t) \, , \tag{30}
$$

with $\hat{h} : \mathbb{M}^{3\times3} \times \mathbb{M}^{3\times3} \mapsto \mathbb{M}^{6\times6}$

$$
\hat{h} \left( \nabla_x T(A(t), g(t)), A(t) \right) = \begin{pmatrix} -\partial\chi(\Sigma_E(t))^T & 0 \\ 0 & \nu^+ \operatorname{skew}(B(t)) \end{pmatrix} , \tag{31}
$$

where $\Sigma_E$ and $B$ are expressions depending on $A = (F_p^{-T}, R_e)$ and on

$$
F = \nabla_x \varphi = \nabla_x T(A, g) \, , \tag{32}
$$

where $\varphi = T(A, g)$ is the unique solution of the (elastic) elliptic boundary value problem at given $A$ whose existence has been established by making use of the extended Korn inequality. By use of refined elliptic regularity results EBENFELD [15] it can be shown that $T$ is indeed locally Lipschitz (note that in the general finite case of Theorem 8.3 we had instead to **assume** that $T$ is well defined and locally Hoelder continuous). It remains to show that the right hand side as a function of $A$ is locally Lipschitz in some properly defined Banach space allowing to apply the well known local existence and uniqueness theorem. This part then is standard since the subdifferential is sufficiently smooth.

In a recent numerical study NEFF&WIENERS [44] the approximation inherent in (P3) has been completely justified for structural applications.

## 11 Discussion

In the first part of the contribution we saw that finite elasto-plasticity based on the multiplicative decomposition allows the successful application of the direct methods of variations to the static elastic trial step. This feature is generally lost for additive ansatzes.

The only additional requirement necessary is polyconvexity of the elastic free energy, which can be easily met. However, elastic free energies defined on Hencky strains are not polyconvex, leading to a loss of ellipticity.

Polyconvexity in itself, however, is not sufficient to treat the coupled problem. Certain mathematical problems can be circumvented in the case of small elastic strains but finite deformations. The investigated model for small elastic strains combines mathematical simplifications with additional physical mechanisms. The introduced evolution equation for 'elastic' rotations $R_e$ leads to deformation induced texture evolution and $R_e$ can conceptually be interpreted as the elastic part of the total rotation of grains in a polycrystal, see NEFF [41].

All presented mathematical results are obtained for essentially fully rate dependent viscous models. Our investigations suggest that introducing viscous behaviour in the finite deformation regime is still enough to regularize the initial boundary value problem. Yet, the smoothness of the plastic variables may deteriorate in finite time leading to bifurcation or fracture. In the viscous case, however, this is entirely a problem of the smoothness of the elastic moduli set by the internal variables. It remains to investigate which type of mechanism could prevent this catastrophic loss of smoothness.

The flow based approach hinges on the assumption that the flow rule is locally Lipschitz thus excluding rate independent material behaviour. It may therefore turn out that the qualitative picture for rate independent behaviour changes dramatically, e.g. microstructure could immediately develop even for smooth data. Numerical calculations based on the time incremental formulation seem to indicate this possibility, cf. CARSTENSEN&HACKL [8] and ORTIZ&REPETTO [46].

First mathematical ideas suggest that a physically motivated backstress evolution, based on augmenting the thermodynamic potential with quantities measuring the local incompatibility, such as $\|F_p^T \operatorname{Curl} F_p\|$, could prevent the above mentioned failure process and at the same time introducing a length scale into the model.

The ensuing coupled system, however, is drastically changed: instead of the ordinary differential evolution equation, one has to solve a degenerate parabolic system for the evolution of the plastic variable. The standard numerical treatment of elasto-plasticity does not any longer apply.

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