Uniqueness of Strong Solutions in Infinitesimal Perfect Gradient-Plasticity with Plastic Spin

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Abstract. A strain gradient plasticity model is motivated based on infinitesimal kinematics. The free energy is augmented by the curl of the non-symmetric plastic strain as a measure for the plastic incompatibility. Flow rules are derived and uniqueness of classical solutions is established.

Key words: gradient plasticity, plastic spin, dislocation density.

1 Introduction

Here we discuss a model of infinitesimal strain gradient plasticity including phenomenological Prager type linear kinematical hardening and nonlocal kinematical hardening due to dislocation interaction. Based on the additive decomposition of the displacement gradient into non-symmetric elastic and plastic distortions the formulation features a thermodynamically admissible model of infinitesimal plasticity involving only the Curl of the infinitesimal plastic distortion p. The model is invariant w.r.t. superposed rigid infinitesimal rotations of the reference, intermediate and spatial configuration but the model is not spin-free due to the nonlocal dislocation interaction and cannot be reduced to a dependence on the infinitesimal plastic strain tensor $\varepsilon_p = \operatorname{sym} p$. Uniqueness of strong solutions of the infinitesimal model is obtained if two non-classical boundary conditions on the non-symmetric plastic distortion p are introduced: $\operatorname{skew} \dot{p}.\tau = 0$ on the microscopically hard boundary $\Gamma_D \subset \partial\Omega$ and $[\operatorname{Curl} p].\tau = 0$ on $\partial\Omega$, where τ are the tangential vectors at the boundary $\partial\Omega$.

There is an abundant literature on gradient plasticity formulations, in most cases letting the yield-stress depend also on some higher derivative of a scalar measure of accumulated plastic distortion [3]. Experimentally, the dependence of the yield stress on plastic gradients is well-documented [2] and may become important for very small samples.

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From a numerical point of view the incorporation of plastic gradients serves the purpose of removing the mesh-sensitivity, either in the softening case, or, more difficult to observe numerically, already in classical Prandtl–Reuss plasticity (shear bands and slip lines with ill-defined band width).

This IUTAM-meeting has shown that gradient plasticity is of high current interest [4–6], but rigorous mathematical studies are still rare. Reddy [14] treats a geometrically linear irrotational (no-spin) model of Gurtin [4], different from my proposal since only symmetric plastic strains appear.

My contribution is organized as follows: first, I introduce the model and show its thermodynamic admissibility. Then I prove that strong solutions of the obtained model with general monotone, non-associative flow-rule together with suitable boundary conditions on the non-symmetric infinitesimal plastic distortion p are unique. The existence question of a weak reformulation is treated in [11]. There, also a finite-strain parent model is given and related invariance questions are investigated in [10]. The relevant notation is found in the Appendix.

2 The Geometrically Linear Gradient Plasticity Model

The model is introduced informally by considering the well known multiplicative decomposition of the deformation gradient $F = F_e\, F_p$ into elastic and plastic parts and expanding to highest order. Thus we expand $F = 1\!\!1 + \nabla u,\ F_p = 1\!\!1 + p + \cdots$, $F_e = 1\!\!1 + e + \cdots$ and the multiplicative decomposition turns into

$$\begin{aligned} 1\!\!1 + \nabla u &= (1\!\!1 + e + \cdots)(1\!\!1 + p + \cdots) \rightsquigarrow \ \nabla u \approx e + p + \cdots\,, \\ F_e^T F_e - 1\!\!1 &= 1\!\!1 + 2\,\mathrm{sym}\, e + e^T e - 1\!\!1 \rightsquigarrow\ 2\,\mathrm{sym}\, e = 2\,\mathrm{sym}(\nabla u - p)\,. \end{aligned} \tag{1}$$

Hence one obtains to highest order the *additive decomposition* [7] of the displacement gradient $\nabla u = e + p$ into nonsymmetric elastic and plastic distortion. Here $\mathrm{sym}\, e = \mathrm{sym}(\nabla u - p)$ the *infinitesimal elastic lattice strain*, $\mathrm{skew}\, e = \mathrm{skew}(\nabla u - p)$ the *infinitesimal elastic lattice rotation* and $\kappa_e = \nabla\, \mathrm{axl}(\mathrm{skew}\, e)$ the *infinitesimal elastic lattice curvature* and p the *infinitesimal plastic distortion*. We assume the quadratic energy to be given by

$$\begin{aligned} W(\nabla u, p, \mathrm{Curl}\, p) &= W_{\mathrm{e}}^{\mathrm{lin}}(\nabla u - p) + W_{\mathrm{ph}}(p) + W_{\mathrm{curl}}^{\mathrm{lin}}(\mathrm{Curl}\, p)\,, \\ W_{\mathrm{e}}^{\mathrm{lin}}(\nabla u - p) &= \mu \|\,\mathrm{sym}(\nabla u - p)\|^2 + \frac{\lambda}{2} \mathrm{tr}\left[\nabla u - p\right]^2, \\ W_{\mathrm{ph}}^{\mathrm{lin}}(p) &= \mu\, H_0 \|\,\mathrm{dev}\,\mathrm{sym}\, p\|^2\,, \quad W_{\mathrm{curl}}^{\mathrm{lin}}(\mathrm{Curl}\, p) = \frac{\mu\, L_c^2}{2} \|\,\mathrm{Curl}\, p\|^2\,. \end{aligned} \tag{2}$$

The used free energy coincides with that in [9, p. 1783] apart for the local kinematical hardening contribution. Note that the *infinitesimal plastic distortion* $p : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{M}^{3\times 3}$ need *not* be *symmetric*, but that only its symmetric part, the *in-*

finitesimal plastic strain[1] sym p, contributes to the local elastic energy expression. The *infinitesimal plastic rotation* skew p does not locally contribute to the elastic energy neither contributes to the local plastic self-hardening but appears in the non-local hardening. The resulting elastic energy is invariant under infinitesimal rigid rotations $\nabla u \mapsto \nabla u + \overline{A}$, $\overline{A} \in \mathfrak{so}(3)$ of the body. The invariance of the curvature contribution needs the homogeneity of the rotations.

Provided that the infinitesimal plastic distortion p is known, (2) defines a linear elasticity problem with pre-stress for the displacement u. It remains to provide an evolution law for p which is consistent with thermodynamics. To this end we use a nonlocal (integral) version of the second law of thermodynamics.

For any "nice" subdomain $\mathcal{V} \subseteq \Omega$ consider for fixed $t_0 \in \mathbb{R}$ the rate of change of energy storage due to inelastic processes

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}} \int_{\mathcal{V}} W(\nabla u(x,t_0), p(x,t), \operatorname{Curl} p(x,t))\, \mathrm{dV} = \\
&\quad \int_{\mathcal{V}} 2\mu \left\langle \operatorname{sym}(\nabla u - p), -\frac{\mathrm{d}}{\mathrm{dt}} p \right\rangle + \lambda \operatorname{tr}\left[\operatorname{sym}(\nabla u - p)\right] \operatorname{tr}\left[-\frac{\mathrm{d}}{\mathrm{dt}} p\right] \\
&\qquad + 2\mu\, H_0 \left\langle \operatorname{dev}\operatorname{sym} p, \operatorname{dev}\operatorname{sym} \frac{\mathrm{d}}{\mathrm{dt}} p \right\rangle + \mu\, L_c^2 \left\langle \operatorname{Curl} p, \operatorname{Curl} \frac{\mathrm{d}}{\mathrm{dt}} p \right\rangle \mathrm{dV} \\
&\quad = \int_{\mathcal{V}} 2\mu \left\langle \operatorname{sym}(\nabla u - p), -\frac{\mathrm{d}}{\mathrm{dt}} p \right\rangle + \lambda \operatorname{tr}\left[\operatorname{sym}(\nabla u - p)\right] \left\langle 1\!\!1, -\frac{\mathrm{d}}{\mathrm{dt}} p \right\rangle \\
&\qquad - 2\mu\, H_0 \left\langle \operatorname{dev}\operatorname{sym} p, -\frac{\mathrm{d}}{\mathrm{dt}} p \right\rangle + \mu\, L_c^2 \left\langle \operatorname{Curl} p, \operatorname{Curl} \frac{\mathrm{d}}{\mathrm{dt}} p \right\rangle \mathrm{dV} \\
&\quad = \int_{\mathcal{V}} \left\langle 2\mu \operatorname{sym}(\nabla u - p) + \lambda \operatorname{tr}\left[\nabla u - p\right] 1\!\!1 - 2\mu\, H_0 \operatorname{dev}\operatorname{sym} p, -\frac{\mathrm{d}}{\mathrm{dt}} p \right\rangle \\
&\qquad + \mu\, L_c^2 \left\langle \operatorname{Curl}[\operatorname{Curl} p], \frac{\mathrm{d}}{\mathrm{dt}} p \right\rangle + \sum_{i=1}^{3} \operatorname{Div} \underbrace{\mu\, L_c^2 \left(\frac{\mathrm{d}}{\mathrm{dt}} p^i \times (\operatorname{curl} p)^i \right)}_{\text{“extra energy flux”}\; q(p_t, \operatorname{Curl} p)} \mathrm{dV}\,. \qquad (3)
\end{aligned}
$$

Here and in the following, p^i denotes the i.th row of the matrix p.[2] Choosing constitutively as extra energy flux

$$
q^i = \mu\, L_c^2 \left(\frac{\mathrm{d}}{\mathrm{dt}} p^i \times (\operatorname{curl} p)^i \right), \; i = 1, 2, 3\,, \qquad (4)
$$

shows that the extended (nonlocal) form of the reduced dissipation inequality at constant temperature [8] may be evaluated as follows

$$
0 \geq \int_{\mathcal{V}} \frac{\mathrm{d}}{\mathrm{dt}} W(\nabla u(x,t_0), p(x,t), \operatorname{Curl} p(x,t)) - \operatorname{Div} q(p_t, \operatorname{Curl} p)\, \mathrm{dV}
$$

[1] The notation $\varepsilon_p \in \operatorname{Sym}(3)$ is reserved to the purely local theory and the irrotational theory.

[2] The extra energy flux term is needed to account for the possible nonlocal exchange of energy across $\partial\mathcal{V}$.

$$= \int_{\mathcal{V}} \left\langle 2\mu\,\mathrm{sym}(\nabla u - p) + \lambda \mathrm{tr}\left[\nabla u - p\right] 1\!\!1 - 2\mu H_0\,\mathrm{dev\,sym}\,p, -\frac{\mathrm{d}}{\mathrm{dt}} p \right\rangle$$
$$+ \mu L_c^2 \left\langle \mathrm{Curl}[\mathrm{Curl}\,p], \frac{\mathrm{d}}{\mathrm{dt}} p \right\rangle \mathrm{dV}$$
$$= \int_{\mathcal{V}} \left\langle \underbrace{2\mu\,\mathrm{sym}(\nabla u - p) + \lambda \mathrm{tr}\left[\nabla u - p\right] 1\!\!1 - 2\mu\, H_0\,\mathrm{dev\,sym}\,p - \mu\, L_c^2\,\mathrm{Curl}[\mathrm{Curl}\,p]}_{=:\Sigma}, -\frac{\mathrm{d}}{\mathrm{dt}} p \right\rangle \mathrm{dV}$$
$$= \int_{\mathcal{V}} \left\langle \sigma - 2\mu\, H_0\,\mathrm{dev\,sym}\,p - \mu\, L_c^2\,\mathrm{Curl}[\mathrm{Curl}\,p], -\frac{\mathrm{d}}{\mathrm{dt}} p \right\rangle \mathrm{dV}\,, \tag{5}$$

where Σ is the *linearized Eshelby stress tensor* in disguise which is the driving force for the plastic evolution. Taking

$$\frac{\mathrm{d}}{\mathrm{dt}} p = \mathfrak{f}(\Sigma)\,, \tag{6}$$

where the function $\mathfrak{f} : \mathbb{M}^{3\times 3} \mapsto \mathbb{M}^{3\times 3}$ with $\mathfrak{f}(0) = \{0\}$ satisfies the *monotonicity in zero* condition

$$\forall;\ \Sigma \in \mathbb{M}^{3\times 3}: \quad \left\langle \mathfrak{f}(\Sigma) - \mathfrak{f}(0), \Sigma - 0 \right\rangle = \left\langle \mathfrak{f}(\Sigma), \Sigma \right\rangle \geq 0\,, \tag{7}$$

ensures the correct sign in (5) (positive dissipation) and thus the plastic evolution law (6) is thermodynamically admissible. In the large scale limit $L_c = 0$ this is just the *class of pre-monotone type* defined by Alber [1]. The driving term Σ has the dimension of stress and $\mathrm{Div}(p_t \times \mathrm{Curl}\,p) = 0$ for purely elastic processes $p_t \equiv 0$.

In the case of associated plasticity the function $\mathfrak{f}$ may be obtained as subdifferential $\partial\chi$ of a convex function χ. To this end, let us define the elastic domain in stress-space $K := \{\Sigma \in \mathbb{M}^{3\times 3} \mid \|\,\mathrm{dev}\,\Sigma\| \leq \sigma_{\mathrm{y}}\}$ with yield stress σ_{y}, corresponding indicator function

$$\chi(\Sigma) = \begin{cases} 0 & \|\,\mathrm{dev}\,\Sigma\| \leq \sigma_{\mathrm{y}} \\ \infty & \text{else}\,, \end{cases} \tag{8}$$

and subdifferential in the sense of convex analysis

$$\partial\chi(\Sigma) = \begin{cases} 0 & \|\,\mathrm{dev}\,\Sigma\| < \sigma_{\mathrm{y}} \\ \mathbb{R}_0^+ \frac{\mathrm{dev}\,\Sigma}{\|\,\mathrm{dev}\,\Sigma\|} & \|\,\mathrm{dev}\,\Sigma\| = \sigma_{\mathrm{y}} \\ \emptyset & \|\,\mathrm{dev}\,\Sigma\| > \sigma_{\mathrm{y}}\,. \end{cases} \tag{9}$$

Choosing dev Σ *instead of* dev sym Σ *in* (8) *allows for plastic spin.*

The remaining divergence term which has to be evaluated in order for an *a priori global energy inequality*

$$\int_{\Omega} \frac{\mathrm{d}}{\mathrm{dt}} W(\nabla u(x, t_0), p(x,t), \mathrm{Curl}\,p(x,t))\,\mathrm{dV} \leq 0 \tag{10}$$

to hold over the entire body is given by the *global insulation condition*

$$\int_\Omega \sum_{i=1}^{3} \operatorname{Div}\left(\frac{\mathrm{d}}{\mathrm{dt}} p^i \times (\operatorname{curl} p)^i\right) \mathrm{dV} = \int_{\partial\Omega} \sum_{i=1}^{3} \left\langle \frac{\mathrm{d}}{\mathrm{dt}} p^i \times (\operatorname{curl} p)^i), \mathbf{n} \right\rangle \mathrm{dS} = 0\,. \tag{11}$$

The last condition is satisfied, e.g., if in each point of the boundary $\partial\Omega$ the *localized insulation condition* holds, i.e.,

$$0 = \left\langle \frac{\mathrm{d}}{\mathrm{dt}} p^i \times (\operatorname{curl} p)^i), \mathbf{n} \right\rangle, \quad x \in \partial\Omega\,, \quad i = 1, 2, 3\,, \tag{12}$$

which may be satisfied by postulating[3]

$$\begin{aligned} p(x,t).\tau &= p(x,0).\tau\,, \quad x \in \Gamma_D \quad \left(\Rightarrow \frac{\mathrm{d}}{\mathrm{dt}} p(x,t).\tau = 0\right), \\ \operatorname{Curl} p(x,t).\tau &= 0\,, \quad x \in \partial\Omega \setminus \Gamma_D\,. \end{aligned} \tag{13}$$

3 Strong Infinitesimal Gradient Plasticity with Plastic Spin

The infinitesimal strain gradient plasticity model reads now: find

$$\begin{aligned} u \in H^1([0,T]; H_0^1(\Omega, \Gamma_D, \mathbb{R}^3))\,, &\quad \operatorname{sym} p \in H^1([0,T]; L^2(\Omega, \mathfrak{sl}(3))\,, \\ \operatorname{Curl} p(t) \in L^2(\Omega, \mathbb{M}^{3\times3})\,, &\quad \operatorname{dev}\operatorname{Curl}\operatorname{Curl} p(t) \in L^2(\Omega, \mathbb{M}^{3\times3})\,, \end{aligned} \tag{14}$$

such that

$$\begin{aligned} \operatorname{Div}\sigma &= -f\,, \quad \sigma = 2\mu \operatorname{sym}(\nabla u - p) + \lambda \operatorname{tr}\left[\nabla u - p\right] 1\!\!1\,, \\ \dot p &\in \partial\chi(\Sigma^{\text{lin}})\,, \quad \Sigma^{\text{lin}} = \Sigma_{\text{e}}^{\text{lin}} + \Sigma_{\text{sh}}^{\text{lin}} + \Sigma_{\text{curl}}^{\text{lin}}\,, \\ \Sigma_{\text{e}}^{\text{lin}} &= 2\mu \operatorname{sym}(\nabla u - p) + \lambda \operatorname{tr}\left[\nabla u - p\right] 1\!\!1 = \sigma\,, \\ \Sigma_{\text{sh}}^{\text{lin}} &= -2\mu\, H_0 \operatorname{dev}\operatorname{sym} p\,, \quad \Sigma_{\text{curl}}^{\text{lin}} = -\mu\, L_c^2 \operatorname{Curl}(\operatorname{Curl} p)\,, \\ u(x,t) &= u_{\text{d}}(x)\,, \quad p(x,t).\tau = p(x,0).\tau\,, \quad x \in \Gamma_D\,, \\ 0 &= [\operatorname{Curl} p(x,t)].\tau\,, \quad x \in \partial\Omega \setminus \Gamma_D\,, \quad p(x,0) = p^0(x)\,. \end{aligned} \tag{15}$$

In general, $\Sigma_{\text{curl}}^{\text{lin}}$ is *not symmetric even if p is symmetric*. Thus, the plastic inhomogeneity is responsible for the plastic spin contribution in this rotationally

[3] It is not immediately obvious how a boundary condition on p at Γ_D can be posed. In Gurtin [6, 2.17] it is shown that the *microscopically hard condition* $\dot p.\tau_{|\Gamma_D} = 0$ has a precise physical meaning: there is no flow of the Burgers vector across the boundary Γ_D.

invariant formulation. Since $\partial\chi$ is monotone, the formulation is thermodynamically admissible. This remains true if we replace $\partial\chi$ with a general flow function $\mathfrak{f}:\mathbb{M}^{3\times 3}\mapsto\mathbb{M}^{3\times 3}$ which is only pre-monotone. The mathematically suitable space for symmetric p is the classical space $H_{\text{curl}}(\Omega):=\{v\in L^2(\Omega)\,,\ \operatorname{Curl} v\in L^2(\Omega)\}$. The boundary conditions on the plastic distortion p serve only the purpose to fix ideas.

In the large scale limit $L_c\to 0$ we recover a classical elasto-plasticity model with local kinematic hardening of Prager-type. Observe that the term $\Sigma_{\text{curl}}^{\text{lin}} = -\mu\, L_c^2 \operatorname{Curl}(\operatorname{Curl} p)$ acts as *nonlocal kinematical backstress* and constitutes a crystallographically motivated alternative to merely phenomenologically motivated backstress tensors. The term $-2\mu\, H_0 \operatorname{dev}\operatorname{sym} p$ is a *symmetric local kinematical backstress*. The model is therefore able to represent linear kinematic hardening[4] and Bauschinger-like phenomena. Moreover, the driving stress Σ is non-symmetric due to the presence of the second order gradients, while the local contribution σ, due to elastic lattice strains, remains symmetric.

Additionally, the infinitesimal local contributions are fully rotationally invariant (isotropic and objective) with respect to the transformation $(\nabla u, p)\mapsto(\nabla u + A(x), p + A(x))$ and the nonlocal dislocation potential is still invariant with respect to the infinitesimal rigid transformation $(\nabla u, p)\mapsto(\nabla u+\overline{A}, p+\overline{A})$ where $\overline{A}, A(x)\in\mathfrak{so}(3)$.

4 Uniqueness of Strong Solutions

Assume that strong solutions to the model (15) exist. I will show that these solutions are already unique. The aim of this paragraph is, moreover, to study the influence of the different boundary conditions for the plastic distortion p on the possible uniqueness. In that way it is intended to identify the weakest boundary condition which suffices for uniqueness. Possible boundary conditions (which are sufficient for the global insulation condition) are

$$\text{pure micro-free:}\quad \operatorname{Curl} p.\tau = 0\,,\quad x\in\partial\Omega\,,$$

$$\text{micro-hard/free:}\quad \begin{cases}\operatorname{Curl} p.\tau &= 0\,,\quad x\in\partial\Omega\setminus\Gamma_D \qquad \text{micro-free}\\ \dot{p}.\tau &= 0\,,\quad x\in\Gamma_D \qquad \text{micro-hard}\end{cases}$$

$$\text{spin micro-hard/free:}\quad \begin{cases}\operatorname{Curl} p.\tau &= 0\,,\quad x\in\partial\Omega \qquad \text{micro-free}\\ [\operatorname{skew}\dot{p}].\tau &= 0\,,\quad x\in\Gamma_D \qquad \text{spin micro-hard}\end{cases}$$

[4] Purely phenomenological Prager linear kinematic hardening can also be written as the system

$$\dot{\varepsilon}_p\in\partial\chi(\sigma-b)\,,\quad \dot{b}=2\mu\, H_0\dot{\varepsilon}_p\,, \tag{16}$$

with b the symmetric backstresss tensor and $H_0>0$ the constant hardening modulus. Assuming $b(x,0)=2\mu\, H_0\varepsilon_p(x,0)$ and integration yields the format given in (15).

$$\text{pure micro-hard:} \quad \dot{p}.\tau = 0\,, \quad x \in \partial\Omega\,,$$

$$\text{global insulation condition:} \quad \int_{\partial\Omega} \sum_{i=1}^{3} \left\langle \dot{p}^i \times (\operatorname{curl} p)^i\,, \mathbf{n} \right\rangle \mathrm{dS} = 0\,. \tag{17}$$

We note that the *global insulation condition is not additively stable*, i.e., the difference of two solutions $p_1 - p_2$ which satisfy each individually the insulation condition need not satisfy the insulation condition. Thus the global insulation condition is not a good candidate for establishing uniqueness.[5]

Here we follow closely the uniqueness proof given in [1, p.32], using the a priori energy estimate and the monotonicity for the difference of two solutions. We allow in this part the generality of a monotone flow function $\mathfrak{f}$ instead of $\partial\chi$. Assume that two strong solutions (u_1, p_2) and (u_2, p_2) of (15) exist (satisfying the same boundary and initial conditions), notably

$$\begin{aligned} \sigma_1.\mathbf{n} = \sigma_2.\mathbf{n} = 0\,, &\quad x \in \partial\Omega \setminus \Gamma_D\,, \\ u_1 = u_2 = u_{\mathrm{d}}\,, &\quad x \in \Gamma_D\,. \end{aligned} \tag{18}$$

Insert the difference of the solutions into the total energy W, integrate over Ω and consider the time derivative

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{dt}} \int_\Omega W(\nabla(u_1 - u_2), p_1 - p_2, \operatorname{Curl}(p_1 - p_2))\,\mathrm{dV} \\ &= \int_\Omega \left\langle DW_{\mathrm{e}}^{\mathrm{lin}}(\nabla(u_1 - u_2), p_1 - p_2), \nabla\dot{u}_1 - \nabla\dot{u}_2 \right\rangle \\ &\quad - \left\langle DW_{\mathrm{e}}^{\mathrm{lin}}(\nabla(u_1 - u_2), p_1 - p_2), \dot{p}_1 - \dot{p}_2 \right\rangle \\ &\quad + \left\langle DW_{\mathrm{ph}}^{\mathrm{lin}}(p_1 - p_2), \dot{p}_1 - \dot{p}_2 \right\rangle \\ &\quad + \left\langle DW_{\mathrm{curl}}^{\mathrm{lin}}(\operatorname{Curl}(p_1 - p_2)), \operatorname{Curl} \frac{\mathrm{d}}{\mathrm{dt}}(p_1 - p_2) \right\rangle \mathrm{dV} \\ &= \int_\Omega \langle \sigma(\nabla(u_1 - u_2), p_1 - p_2), \nabla\dot{u}_1 - \nabla\dot{u}_2 \rangle \\ &\quad - \langle \sigma(\nabla(u_1 - u_2), p_1 - p_2), \dot{p}_1 - \dot{p}_2 \rangle \\ &\quad + \langle 2\mu\, H_0 \operatorname{dev}\operatorname{sym}(p_1 - p_2), \dot{p}_1 - \dot{p}_2 \rangle \\ &\quad + \left\langle \mu\, L_c^2 \operatorname{Curl}(p_1 - p_2), \operatorname{Curl} \frac{\mathrm{d}}{\mathrm{dt}}(p_1 - p_2) \right\rangle \mathrm{dV} \\ &= -\int_\Omega \left\langle \underbrace{\operatorname{Div} \sigma(\nabla(u_1 - u_2), p_1 - p_2)}_{=0}, \dot{u}_1 - \dot{u}_2 \right\rangle \mathrm{dV} \end{aligned}$$

[5] In the spirit of Gurtin [5] the insulation condition is motivated by imposing boundary conditions that result in a "null expenditure of microscopic power".

$$
\begin{aligned}
&+\int_{\partial\Omega}\underbrace{\langle\sigma(\nabla(u_1-u_2),p_1-p_2).\mathbf{n},(u_1-u_2)_t\rangle}_{=0\text{ with (18)}}\,\mathrm{dS}\\
&-\langle\sigma(\nabla(u_1-u_2),p_1-p_2),\dot p_1-\dot p_2\rangle\\
&\quad+\int_\Omega\langle 2\mu\,H_0\,\mathrm{dev}\,\mathrm{sym}(p_1-p_2),\dot p_1-\dot p_2\rangle\\
&\quad+\left\langle\mu\,L_c^2\;\mathrm{Curl}(p_1-p_2),\mathrm{Curl}\,\frac{\mathrm{d}}{\mathrm{dt}}(p_1-p_2)\right\rangle\mathrm{dV}\\
&=-0+0+\int_\Omega\langle 2\mu\,H_0\,\mathrm{dev}\,\mathrm{sym}(p_1-p_2),\dot p_1-\dot p_2\rangle\\
&\quad-\langle\sigma(\nabla(u_1-u_2),p_1-p_2),\dot p_1-\dot p_2\rangle\\
&\quad+\left\langle\mu\,L_c^2\;\mathrm{Curl}\,\mathrm{Curl}(p_1-p_2),\frac{\mathrm{d}}{\mathrm{dt}}(p_1-p_2)\right\rangle\mathrm{dV}\\
&=\int_\Omega\left\langle\Sigma_2^{\mathrm{lin}}-\Sigma_1^{\mathrm{lin}},\dot p_1-\dot p_2\right\rangle\mathrm{dV}=-\int_\Omega\left\langle\Sigma_2^{\mathrm{lin}}-\Sigma_1^{\mathrm{lin}},\dot p_2-\dot p_1\right\rangle\mathrm{dV}\\
&=-\int_\Omega\left\langle\Sigma_2^{\mathrm{lin}}-\Sigma_1^{\mathrm{lin}},\mathfrak{f}(\Sigma_2^{\mathrm{lin}})-\mathfrak{f}(\Sigma_1^{\mathrm{lin}})\right\rangle\mathrm{dV}\le 0\,,
\end{aligned}\tag{19}
$$

due to the monotonicity of $\mathfrak{f}$. Hence, after integrating the last inequality in time we obtain also for the difference of two solutions

$$
\begin{aligned}
&\int_\Omega W(\nabla(u_1-u_2)(t)),(p_1-p_2)(t),\mathrm{Curl}(p_1-p_2)(t))\,\mathrm{dV}\\
&\quad\le\int_\Omega W(\nabla(u_1-u_2)(0)),(p_1-p_2)(0),\mathrm{Curl}(p_1-p_2)(0))\,\mathrm{dV}=0\,.
\end{aligned}\tag{20}
$$

Thus we have

$$
\begin{aligned}
&\int_\Omega\|\,\mathrm{sym}(\nabla(u_1-u_2)(t)-(p_1-p_2)(t)\|^2\,\mathrm{dV}=0\,,\\
&\int_\Omega\|\,\mathrm{dev}\,\mathrm{sym}(p_1-p_2)(t)\|^2\,\mathrm{dV}=0\,,\\
&\int_\Omega\|\,\mathrm{Curl}(p_1-p_2)(t)\|^2\,\mathrm{dV}=0\,.
\end{aligned}\tag{21}
$$

Since $p_1,p_2\in\mathfrak{sl}(3)$ it follows that $\mathrm{sym}(p_1-p_2)=0$ almost everywhere, i.e., $p_1-p_2\in\mathfrak{so}(3)$. Moreover, from the micro-hard boundary condition $\dot p_1.\tau=\dot p_2.\tau=0$ we obtain $p_1(x,t).\tau=p_2(x,t).\tau=p(x,0).\tau$ which implies that $(p_1-p_2).\tau=0$ on Γ_D for two linear independent tangential directions τ. Since a skew-symmetric matrix $A\in\mathfrak{so}(3)$ has either rank two or rank zero (in which case it is zero) we conclude that $p_1-p_2=0$ on the Dirichlet-boundary Γ_D due to the skew-symmetry of the difference. However Curl controls all first partial derivatives on skew-symmetric matrices [12], i.e. it holds locally

$$\forall\, A(x) \in \mathfrak{so}(3): \quad \|\operatorname{Curl} A(x)\|^2 \geq \frac{1}{2}\,\|\nabla A(x)\|^2\,, \tag{22}$$

therefore $p_1 - p_2 = 0$ by Poincaré's inequality. Thus from Korn's first inequality we obtain uniqueness also for he displacement u. □

As a result: apart for the pure micro-free condition and the global insulation condition all mentioned boundary conditions in (17) ensure uniqueness of classical solutions. In the case of the pure micro-free condition, the skew-symmetric part of the difference of two solutions remains indetermined up to a constant skew-symmetric matrix. The spin micro-hard condition has the advantage of not imposing a Dirichlet boundary condition on the symmetric plastic strain tensor $\operatorname{sym} p$ which is also not present in the classical theory. Thus it is a candidate for the desired weakest boundary condition.

5 Irrotational Strong Infinitesimal Gradient Plasticity

For completeness let us also give the infinitesimal strain gradient plasticity system without plastic spin which is included as a special case in (15). To this end we specify the elastic domain in stress-space $K := \{\Sigma \in \mathbb{M}^{3\times 3} \mid \|\operatorname{dev}\operatorname{sym}\Sigma\| \leq \sigma_{\mathrm{y}}\}$ with yield stress σ_{y}, corresponding indicator function

$$\chi^{\mathrm{sym}}(\Sigma) = \begin{cases} 0 & \|\operatorname{dev}\operatorname{sym}\Sigma\| \leq \sigma_{\mathrm{y}} \\ \infty & \text{else}\,, \end{cases} \tag{23}$$

and subdifferential in the sense of convex analysis

$$\partial\chi^{\mathrm{sym}}(\Sigma) = \begin{cases} 0 & \|\operatorname{dev}\operatorname{sym}\Sigma\| < \sigma_{\mathrm{y}} \\ \mathbb{R}_0^+ \frac{\operatorname{dev}\operatorname{sym}\Sigma}{\|\operatorname{dev}\operatorname{sym}\Sigma\|} & \|\operatorname{dev}\operatorname{sym}\Sigma\| = \sigma_{\mathrm{y}} \\ \emptyset & \|\operatorname{dev}\operatorname{sym}\Sigma\| > \sigma_{\mathrm{y}}\,. \end{cases} \tag{24}$$

Thus, $\partial\chi^{\mathrm{sym}}(\Sigma)$ is symmetric, in which case p will remain symmetric, whenever the initial condition for p is symmetric. Hence, we may rename $\varepsilon_p := \operatorname{sym} p$ in the following. As boundary condition on p we use the candidate for the weakest boundary condition which ensures uniqess, i.e. the spin micro hard condition. It turns out that the local condition on the skew-symmetric part is automatically satisfied. The model reads: find

$$u \in H^1([0,T]; H_0^1(\Omega, \Gamma_D, \mathbb{R}^3))\,, \quad \varepsilon_p \in H^1([0,T]; L^2(\Omega, \mathfrak{sl}(3))\,,$$
$$\operatorname{Curl}\varepsilon_p(t) \in L^2(\Omega, \mathbb{M}^{3\times 3})\,, \quad \operatorname{dev}\operatorname{sym}\operatorname{Curl}\operatorname{Curl}\varepsilon_p(t) \in L^2(\Omega, \mathbb{M}^{3\times 3})\,, \tag{25}$$

such that

$$
\begin{aligned}
\operatorname{Div}\sigma &= -f\,, \quad \sigma = 2\mu\,(\varepsilon-\varepsilon_p)+\lambda\,\mathrm{tr}\,[\varepsilon]\,1\!\!1\,,\\
\dot{\varepsilon}_p &\in \partial\chi^{\,\mathrm{sym}}(\Sigma^{\mathrm{lin}})\,, \quad \Sigma^{\mathrm{lin}} = \Sigma^{\mathrm{lin}}_{\mathrm{e}}+\Sigma^{\mathrm{lin}}_{\mathrm{sh}}+\Sigma^{\mathrm{lin}}_{\mathrm{curl}}\,,\\
\Sigma^{\mathrm{lin}}_{\mathrm{e}} &= 2\mu\,(\varepsilon-\varepsilon_p)+\lambda\,\mathrm{tr}\,[\varepsilon]1\!\!1 = \sigma\,,\\
\Sigma^{\mathrm{lin}}_{\mathrm{sh}} &= -2\mu\,H_0\varepsilon_p\,, \quad \Sigma^{\mathrm{lin}}_{\mathrm{curl}} = -\mu\,L_c^2\,\mathrm{Curl}(\mathrm{Curl}\,\varepsilon_p)\,,\\
u(x,t) &= u_{\mathrm{d}}(x)\,, \quad 0 = [\mathrm{Curl}\,\varepsilon_p(x,t)].\tau\,, \quad x\in\partial\Omega\,,\\
\varepsilon_p(x,0) &= \varepsilon_p^0(x)\in \mathrm{Sym}(3)\cap\mathfrak{sl}(3)\,. \qquad (26)
\end{aligned}
$$

Again, classical solutions, if they exist, are unique. For this result to hold the higher order boundary conditions on ε_p are not needed!

6 Discussion

The classical elasto-perfectly plastic Prandtl–Reuss model with kinematic hardening has been extended to include a weak nonlocal interaction of the plastic distortion by introducing the dislocation density in the Helmholtz free energy. The evolution equation for plasticity follows by an application of the secod law of thermodynamics in the formulation proposed by Maugin [8] together with sufficient conditions guaranteeing the insulation condition.

With Gurtin and Anand [5] on gradient plasticity I can say: "Our goal is a theory that allows for constitutive dependencies on (the dislocation density tensor) G, but that otherwise does not depart drastically from the classical theory." This has been achieved, since

- The *large scale limit* $L_c \to 0$ with *zero local hardening* $H_0 = 0$ does coincide with the classical *Prandtl–Reuss* model with deviatoric von Mises flow rule.
- The *large scale limit* $L_c \to 0$ does determine the plastic distortion to be *irrotational*, i.e., only $\varepsilon_p := \mathrm{sym}\, p$ appears (zero plastic spin).
- A weak reformulation of the model for $L_c > 0$ is *well-posed*. Existence and uniqueness are obtained in suitable Hilbert-spaces [11]. Uniqueness of classical solutions is also guaranteed.
- The model for $L_c > 0$ does contain *maximally second order derivatives* in the evolution law.
- The model for $L_c > 0$ is *linearized materially and spatially covariant* and *thermodynamically consistent* (in the extended sense).
- The model for $L_c > 0$ is *isotropic* with respect to both, the *referential and intermediate configuration.*
- The model for $L_c > 0$ does contain *first order boundary conditions at the hard Dirichlet boundary* $\Gamma_D \subset \partial\Omega$ for the plastic distortion p only in terms of the plastic spin skew p there.

- The symmetric plastic strains $\varepsilon_p := \operatorname{sym} p$ remain free of first order (essential Dirichlet) boundary conditions as in classical elasto-plasticity.
- The model for $L_c > 0$ does contain *second order boundary conditions* on p like $\operatorname{Curl} p.\tau = 0$ at the total external boundary $\partial\Omega$, motivated from thermodynamics and insulation conditions.

The proposed gradient plasticity model approximates formally the classical model in the large scale limit $L_c = 0$ since then the plastic distortion p remains symmetric and no boundary conditions are set. Plastic spin is purely a feature of the nonlocality of the model. Summarizing, for the elasto-plastic infinitesimal strain gradient model with spin the following has been obtained: uniqueness of strong solutions with micro-free/hard boundary conditions.

Currently, the dislocation based plasticity model is being implemented, however, only for the irrotational case (26) without plastic spin [13]. There, boundary conditions on the symmetric plastic strain ε_p need not be imposed.

Appendix: Notation

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and let Γ be a smooth subset of $\partial\Omega$ with non-vanishing 2-dimensional Hausdorff measure. We denote by $\mathbb{M}^{3\times 3}$ the set of real 3×3 second order tensors, written with capital letters. The standard Euclidean scalar product on $\mathbb{M}^{3\times 3}$ is given by $\langle X, Y\rangle_{\mathbb{M}^{3\times 3}} = \operatorname{tr}\left[XY^T\right]$, and thus the Frobenius tensor norm is $\|X\|^2 = \langle X, X\rangle_{\mathbb{M}^{3\times 3}}$ (we use these symbols indifferently for tensors and vectors). The identity tensor on $\mathbb{M}^{3\times 3}$ will be denoted by $\mathbb{1}$, so that $\operatorname{tr}[X] = \langle X, \mathbb{1}\rangle$. We let Sym and PSym denote the symmetric and positive definite symmetric tensors respectively. We adopt the usual abbreviations of Lie-algebra theory, i.e. $\mathfrak{so}(3) := \{X \in \mathbb{M}^{3\times 3} \; |X^T = -X\}$ are skew symmetric second order tensors and $\mathfrak{sl}(3) := \{X \in \mathbb{M}^{3\times 3} \; |\operatorname{tr}[X] = 0\}$ are traceless tensors. We set $\operatorname{sym}(X) = \frac{1}{2}(X^T + X)$ and $\operatorname{skew}(X) = \frac{1}{2}(X - X^T)$ such that $X = \operatorname{sym}(X) + \operatorname{skew}(X)$. For $X \in \mathbb{M}^{3\times 3}$ we set for the deviatoric part $\operatorname{dev} X = X - \frac{1}{3}\operatorname{tr}[X]\,\mathbb{1} \in \mathfrak{sl}(3)$. For a second order tensor X we let $X.e_i$ be the application of the tensor X to the column vector e_i and X^i denotes the i.th row of X. The curl of a three by three matrix is defined to be the vector curl applied on the i.th row, written in the i.th row, i.e.,

$$\operatorname{curl}\begin{pmatrix} p^{11} & p^{12} & p^{13} \\ p^{21} & p^{22} & p^{23} \\ p^{31} & p^{32} & p^{33} \end{pmatrix} = \begin{pmatrix} \operatorname{curl}[p^{11}, p^{12}, p^{13}] \\ \operatorname{curl}[p^{21}, p^{22}, p^{23}] \\ \operatorname{curl}[p^{31}, p^{32}, p^{33}] \end{pmatrix}.$$

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