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Anti-plane shear in nonlinear elasticity

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1 Introduction

An **anti-plane shear (APS) deformation** [5, 4, 1] is a mapping φ of the form $\varphi(x_1,x_2,x_3)=(x_1,\ x_2,\ x_3+u(x_1,x_2))^T$ with an arbitrary scalar valued function u. Let $\alpha:=u_{,x_1}$, $\beta:=u_{,x_2}$ and $\gamma^2:=\|\nabla u\|^2=\alpha^2+\beta^2$. Then

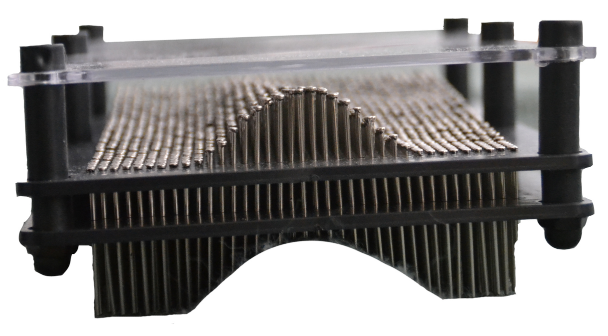
$$\nabla\varphi=F=\begin{pmatrix}1&0&0\\0&1&0\\\alpha&\beta&1\end{pmatrix},\qquad B=FF^T=\begin{pmatrix}1&0&\alpha\\0&1&\beta\\\alpha&\beta&1+\gamma^2\end{pmatrix}$$

and the three isotropic matrix invariants of the left Cauchy-Green deformation tensor B are given by

$$I_1=\operatorname{tr}B=3+\gamma^2,\quad I_2=\operatorname{tr}(\operatorname{Cof}B)=3+\gamma^2,\quad I_3=\det B=1\,.$$

Framework

For a cylinder-shaped stress-free elastic isotropic body $\Omega\subset\mathbb{R}^3$, Dirichlet boundary condition corresponding to an APS-function are applied to the lateral sides. We consider deformations that are stationary points of the energy functional $\varphi\mapsto\int_\Omega W(\nabla\varphi)\,\mathrm{d}x$.



A **global equilibrium** is a stationary point of the variational problem, i.e. a solution to the corresponding Euler-Lagrange equation $\operatorname{Div}[DW(\nabla\varphi)]=0$. In the case of isotropy, we can express $W(F)=W(I_1,I_2,I_3)$ in terms of the invariants of B, and

$$DW(F)=2\frac{\partial W}{\partial I_1}F+2\frac{\partial W}{\partial I_2}(I_1\mathbb{1}-B)F+2I_3\frac{\partial W}{\partial I_3}F^{-T}\,.$$

Then the Euler-Lagrange equations are

$$\operatorname{Div}\left(2\frac{\partial W}{\partial I_1}F+2\frac{\partial W}{\partial I_2}(I_1\mathbb{1}-B)F+2I_3\frac{\partial W}{\partial I_3}F^{-T}\right)=0\,.\qquad(1)$$

A **global APS-equilibrium** is a stationary point of the full variational problem (global equilibrium) that has the form of an APS-deformation. With

$$G(I_1,I_2):=2\frac{\partial W}{\partial I_2}\Big|_{I_3=1},\qquad H(I_1,I_2):=2\left(\frac{\partial W}{\partial I_1}+\frac{\partial W}{\partial I_2}\right)\Big|_{I_3=1},$$
$$q(I_1,I_2):=\left[2\frac{\partial W}{\partial I_3}+2\frac{\partial W}{\partial I_1}+2(2+\gamma^2)\frac{\partial W}{\partial I_2}\right]_{I_3=1},$$

the Euler-Lagrange equations (1) for an APS-deformation read

$$\begin{aligned}q_{,x_1}&=(\alpha^2G)_{,x_1}+(\alpha\beta G)_{,x_2}\,,&&\text{(I)}\\q_{,x_2}&=(\alpha\beta G)_{,x_1}+(\beta^2G)_{,x_2}\,,&&\text{(II)}\\0&=(\alpha H)_{,x_1}+(\beta H)_{,x_2}\,.&&\text{(III)}\end{aligned}$$

This system of differential equations for the scalar-valued function $u(x_1,x_2)$ is **over-determined** by two equations.

An **APS-equilibrium** is a stationary point of our variational problem with respect to the restriction of the energy functional to the class of APS-deformations:

$$\int_\Omega W(3+\|\nabla u\|^2,3+\|\nabla u\|^2,1)\,\mathrm{d}x\longrightarrow\min_u\,.$$

The single corresponding Euler-Lagrange equation is given by

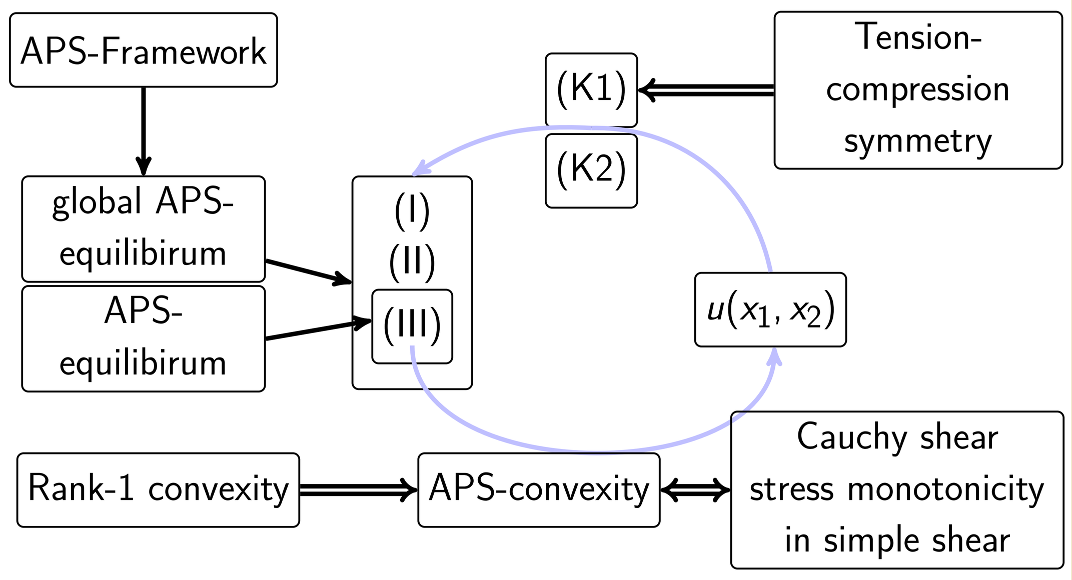
$$\operatorname{div}\left(\left[\frac{\partial W}{\partial I_1}(I_1,I_2,1)+\frac{\partial W}{\partial I_2}(I_1,I_2,1)\right]_{I_1=I_2=3+\|\nabla u\|^2}\nabla u\right)=0\quad\iff\quad\text{(III)}.$$

In particular, every global APS-equilibrium is an APS-equilibrium, but not the other way around.

Questions

- Under which conditions is every APS-equilibrium a global APS-equilibrium?
- Under which conditions does an APS-equilibrium exist?

2 Compatibility conditions



The existence of a solution to equation (III) can be ensured by the sufficient (but not necessary) condition of **APS-convexity**:

$$\nabla u\mapsto W(3+\|\nabla u\|^2,3+\|\nabla u\|^2,1)\quad\text{is convex.}\qquad(2)$$

In [5], we showed that

$$W\text{ is APS-convex}\quad\iff\quad\frac{d^2}{(dR)^2}W(3+R^2,3+R^2,1)>0\,.$$

Given a solution of equation (III) (i.e. an APS-equilibrium), Knowles' first energy function compatibility condition [3]

Knowles 1 $\quad\exists b\in\mathbb{R}:\forall\,I_1=I_2\geq3\,,I_3=1:$

$$b\frac{\partial W}{\partial I_1}(I_1,I_2,I_3)+(b-1)\frac{\partial W}{\partial I_2}(I_1,I_2,I_3)=0\qquad\text{(K1)}$$

ensures that the two other Euler-Lagrange equations (I) and (II) reduce to one single new equation

$$\frac{\partial W}{\partial I_2}(3+\gamma^2,3+\gamma^2,1)=\left[\frac{\partial q}{\partial I_1}+\frac{\partial q}{\partial I_2}\right]_{I_1=I_2=3+\gamma^2}.\qquad\text{(I+II)}$$

Expressed in its original notation, this leads to the following second compatibility condition of the energy $W(F)$:

Knowles 2 $\quad\forall\,I_1=I_2\geq3\,,I_3=1:\qquad$ (K2)

$$\frac{\partial^2W}{\partial I_1^2}+I_1\frac{\partial^2W}{\partial I_1\partial I_2}+\frac{\partial^2W}{\partial I_1\partial I_3}+(I_1-1)\frac{\partial^2W}{\partial I_2^2}+\frac{\partial^2W}{\partial I_2\partial I_3}+\frac{1}{2}\frac{\partial W}{\partial I_2}=0\,.$$

Under the constraint of incompressibility ($\det F=1$), it can be shown that the second condition (K2) is redundant [2] and that every energy function which satisfies (K1) already ensures that every APS-equilibrium is also a global APS-equilibrium.

Several important energy functions have been tested [6] for the compatibility conditions (K1) and (K2) as well as APS-convexity. For example, the volumetric-isochoric decoupled **Mooney-Rivlin energy function**

$$W(F)=\frac{\mu}{2}\left(\alpha(I_1I_3^{-\frac13}-3)+(1-\alpha)(I_2I_3^{-\frac23}-3)\right)+h(I_3)\quad(3)$$

is APS-convex, but only satisfies condition (K1). Therefore, there exists an APS-equilibrium for arbitrary APS-boundary conditions, but only in the case of incompressibility it is ensured that every APS-equilibrium is also a global equilibrium. The **Blatz-Ko energy function**

$$W(F)=\frac{\mu}{2}\left(I_1+\frac{2}{\sqrt{I_3}}-5\right),\qquad(4)$$

on the other hand, additionally satisfies condition (K2). Therefore, in the general compressible (as well as the incompressible) case, every APS-equilibrium is also a global equilibrium. In contrast, the **Veronda-Westman energy function**

$$W(F)=\mu\left(\frac{e^{\gamma(I_1-3)}-1}{\gamma}-\frac{I_2-3}{2}\right)+h(I_3)\qquad(5)$$

is APS-convex too, but satisfies neither (K1) nor (K2). Therefore, there exists an APS-equilibrium under arbitrary APS-compatible boundary conditions, but it is uncertain whether or not this is also a global equilibrium, even under the constraint of incompressibility.

3 Connections to constitutive requirements in nonlinear elasticity

An energy function W on $\mathrm{GL}^+(3)$ is called **rank-one convex** if $t\mapsto W(F+t\,\xi\otimes\eta)$ is convex on $[0,1]$ for all $F\in\mathbb{R}^{3\times3}$ and all $\xi,\eta\in\mathbb{R}^3$. Since within the class of APS-functions

$$F_1-F_2=\begin{pmatrix}1&0&0\\0&1&0\\\alpha_1&\beta_1&1\end{pmatrix}-\begin{pmatrix}1&0&0\\0&1&0\\\alpha_2&\beta_2&1\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}\otimes\begin{pmatrix}\alpha_1-\alpha_2\\\beta_1-\beta_2\\0\end{pmatrix},$$

we find that

$$\text{rank-one convexity}\implies\text{APS-convexity}\,.$$

Moreover, for **simple shear** $\varphi(x_1,x_2,x_3)=(x_1+\gamma x_2,\ x_2,\ x_3)^T$ with $\gamma\in\mathbb{R}$, APS-convexity is closely connected to the physically reasonable **monotonicity condition**

$$\frac{d}{d\gamma}\sigma_{12}(\gamma)=\frac{d^2}{(d\gamma)^2}W(3+\gamma^2,3+\gamma^2,1)>0\qquad(6)$$

on the **Cauchy shear stress** σ_{12}:

$$\frac{d}{d\gamma}\sigma_{12}(\gamma)>0\iff\text{APS-convexity.}$$

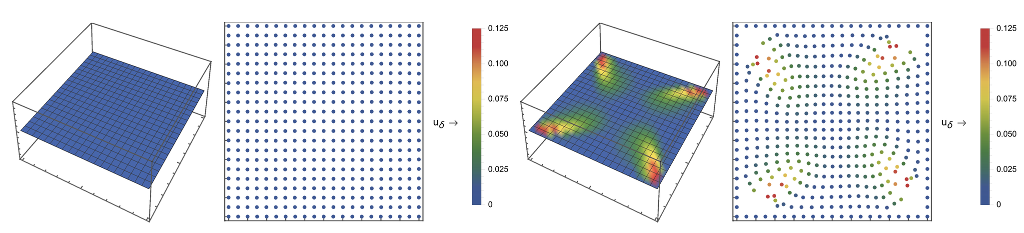
Furthermore, any **tension-compression symmetric** energy, i.e. any energy function W with

$$W(F)=W(F^{-1})\quad\iff\quad W(I_1,I_2,I_3)=W\left(\frac{I_2}{I_3},\frac{I_1}{I_3},\frac{1}{I_3}\right)\quad(7)$$

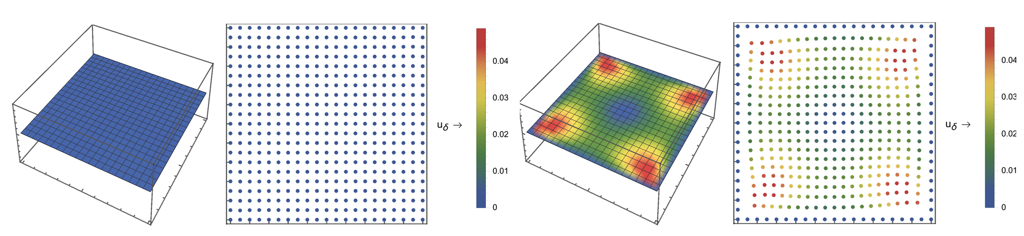
for all $F\in\mathrm{GL}^+(3)$, satisfies Knowles' first condition:

$$\text{Tension-compression symmetry}\implies\text{(K1) with }b=\tfrac12\,.$$

Finite element simulations are able to visualize the difference between an APS- and a global non-APS-equilibrium. An APS-deformation has to maintain the given grid-structure within each x_1-x_2–plane, whereas arbitrary non-APS-deformations will relocate the nodes and therefore change the original structure of the grid. In the quasi-incompressible case (bulk modulus $K\sim10^5\mu$ shear modulus), the Mooney-Rivlin energy (left) ensures an APS-equilibrium, whereas the global equilibrium for the Veronda-Westman model (right) does not have the shape of an APS-deformation:



Similarly, in the compressible case (bulk modulus $K\sim\mu$ shear modulus), the Blatz-Ko model yields an APS-equilibrium, whereas the global equilibrium with respect to the Mooney-Rivlin energy does not attain the shape of an APS-deformation:



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