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Homogeneous Cauchy stress induced by non-homogeneous deformations

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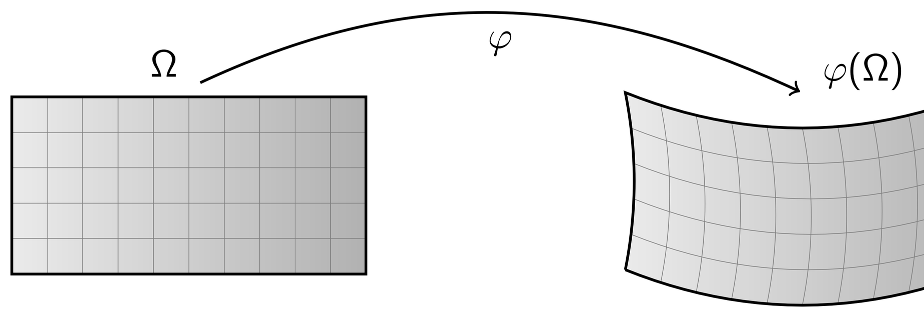
1 Introduction

For a deformation $x = \varphi(X)$, changes of length, area, and volume are governed by $F = \nabla\varphi$, $\operatorname{Cof} F = \det(F)F^{-T}$, and $J = \det(F) > 0$, respectively. From the deformation we get a displacement field $u(X) = \varphi(X) - X = x - X$.



2 Linear elastic deformations

The isotropic linear elastic energy takes the form

$$W_{lin}(\nabla u) = \mu\|\operatorname{dev}\operatorname{sym}\nabla u\|^2 + \frac{\kappa}{2}[\operatorname{tr}(\operatorname{sym}\nabla u)]^2\,, \tag{1}$$

in which $\mu > 0$ is the shear and $\kappa > 0$ the bulk modulus. The corresponding stress-strain law is

$$\sigma = 2\mu\operatorname{dev}\varepsilon + \kappa\operatorname{tr}(\varepsilon)\mathbb{1}\,, \tag{2}$$

with the infinitesimal strain tensor $\varepsilon = \operatorname{sym}\nabla u$. It is invertible if and only if $\mu > 0$ and $\kappa > 0$. Then $\sigma^{-1} : \operatorname{Sym}(3) \to \operatorname{Sym}(3)$ exists. In addition when the Cauchy stress $\sigma = \overline{T}$ is constant, the **homogeneous displacement**

$$u(X) = \left[\sigma^{-1}(\overline{T}) + \overline{A}\right] X + \overline{b}\,, \tag{3}$$

is **uniquely determined**, up to infinitesimal rigid body rotations $\overline{A} \in \mathfrak{so}(3)$ and translations $\overline{b} \in \mathbb{R}^3$.
Furthermore, there is basically only one stress tensor in linear elasticity, because in linear approximition all stress tensors are identical.

3 Nonlinear elastic deformations

In contrast to linear elasticity, in nonlinear elasticity many different stress tensors exist. For example the first Piola-Kirchhoff stress $S_1 = D_F W(F)$ and the true Cauchy stress $\sigma = S_1(F)\cdot(\operatorname{Cof} F)^{-1}$.

Questions

- Does homogeneous Cauchy stress σ imply homogeneous strain in nonlinear elasticity?
- If not, how can a homogeneous Cauchy stress be generated by non-homogeneous finite strain deformations?

We already know that

$$\sigma \text{ homogeneous} \implies \operatorname{div}_{\varphi(\Omega)}\sigma = 0\,,\ \text{“self-equilibrated field”}$$
$$\text{homogeneous strain} \implies \text{all stress tensors are homogeneous.}$$

Moreover, for a homogeneous isotropic hyperelastic body under finite strain deformation, the Cauchy stress tensor takes the form

$$\sigma(B) = \beta_0\mathbb{1} + \beta_1 B + \beta_{-1}B^{-1}, \tag{4}$$

where $B = FF^T$ is the left Cauchy-Green tensor and

$$\beta_0 = \frac{2}{\sqrt{\mathrm{I}_3}}\left(\mathrm{I}_2\frac{\partial W}{\partial \mathrm{I}_2} + \mathrm{I}_3\frac{\partial W}{\partial \mathrm{I}_3}\right),\ \beta_1 = \frac{2}{\sqrt{\mathrm{I}_3}}\frac{\partial W}{\partial \mathrm{I}_1},\ \beta_{-1} = -2\sqrt{\mathrm{I}_3}\frac{\partial W}{\partial \mathrm{I}_2}$$

are scalar functions of the principal invariants

$$\mathrm{I}_1(B) = \operatorname{tr} B = \|F\|^2,\quad \mathrm{I}_2(B) = \frac{1}{2}\left[(\operatorname{tr} B)^2 - \operatorname{tr} B^2\right] = \|\operatorname{Cof} F\|^2,$$
$$\mathrm{I}_3(B) = \det B = (\det F)^2,$$

with $W(\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3)$ as the strain energy density function describing the physical properties of the isotropic hyperelastic material. W should be stress free in the reference configuratin Ω, i.e. $\beta_0 + \beta_1 + \beta_{-1}|_{F=\mathbb{1}} = 0$.
If $\sigma : \operatorname{Sym}^+(3) \to \operatorname{Sym}(3)$ is invertible, then for constant Cauchy stress $\sigma = \overline{T}$ we have a unique left Cauchy-Green tensor $\overline{B} \in \operatorname{Sym}^+(3)$ which satisfies

$$\nabla\varphi(\nabla\varphi)^T = \overline{B} = \sigma^{-1}(\overline{T})\,. \tag{5}$$

The latter implies (formally equivalent to the infinitesimal situation) that

$$\varphi(X) = (\overline{V}\,\overline{R})\,X + \overline{b} = \left[\sqrt{\sigma^{-1}(\overline{T})}\,\overline{R}\right]\,X + \overline{b}\,, \tag{6}$$

where $\overline{R} \in \mathrm{SO}(3)$ is an arbitrary constant rotation, $\overline{b} \in \mathbb{R}^3$ is an arbitrary constant translation, and $\overline{V}$ is the left principal stretch tensor satisfying $\overline{V}^2 = \overline{B}$, that is uniquely determined by the given Cauchy stress $\sigma = \overline{T}$.

3.1 Geometric compatibility

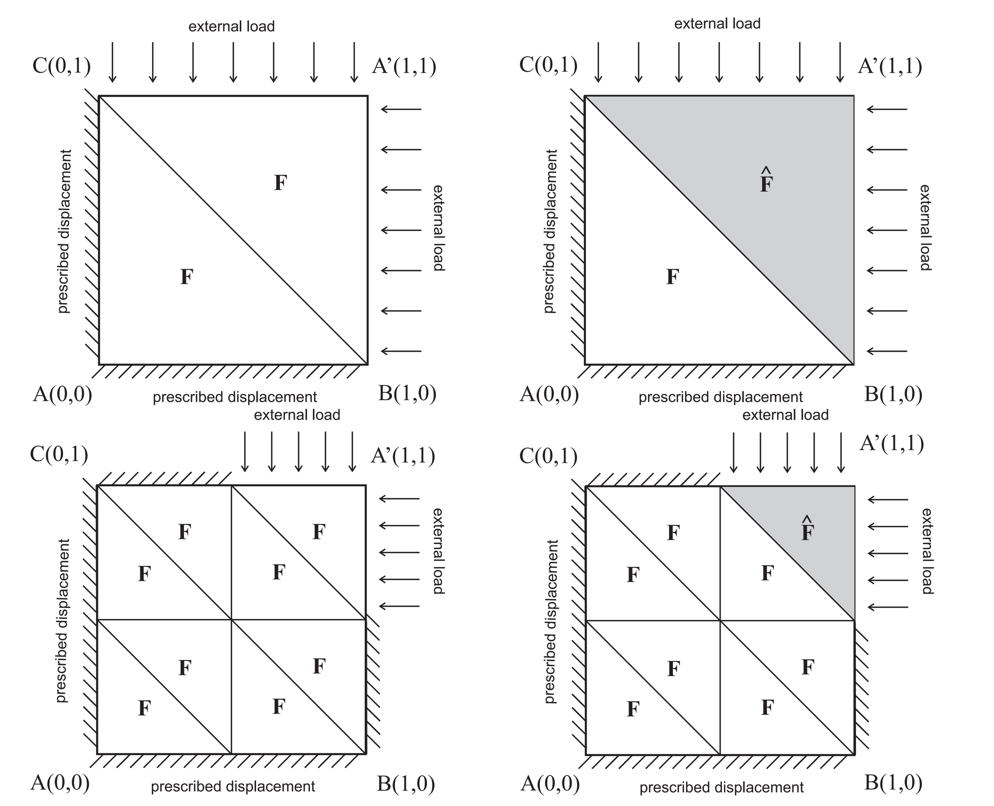
Assume that the deformation φ is continuous and the deformation gradient takes on two different values F and $\widehat{F}$ such that $B = FF^T \neq \widehat{F}\widehat{F}^T = \widehat{B}$. We already know that continuity of φ requires the **Hadamard jump condition** $F - \widehat{F} = a \otimes n$, where n is the normal vector to the interface between the two phases with deformation gradients F and $\widehat{F}$, and $\otimes$ denotes the dyadic product. The rank-one connection of F and $\widehat{F}$ is equivalent to this proposition and shows that the deformation gradient $\nabla\varphi$ can only jump along a unique straight interface.

3.1.1 Example: Compatible 2D deformations

An elastic square is partitioned into uniform right-angled triangles, such that the deformation gradient is homogeneous on every triangle. Then, if φ is continuous everywhere, and the deformation gradient is F on one set of triangles and $\widehat{F}$ on the remaining set, such that $\operatorname{rank}\left(F - \widehat{F}\right) = 1$, the two sets are separated by a single straight line.

Therefore there are no layers of the domain where these sets can alternate.

[1]



3.2 New strain energy function

We define the strain energy function

$$W = \frac{\mu}{2}\left(\mathrm{I}_3^{-1/3}\mathrm{I}_1 - 3\right) + \frac{\tilde\mu}{4}(\mathrm{I}_1 - 3)^2 + \frac{\kappa}{2}\left(\mathrm{I}_3^{1/2} - 1\right)^2, \tag{7}$$

where $\mu > 0$ is the infinitesimal shear modulus, $\kappa > 0$ is the infinitesimal bulk modulus, and $\tilde\mu > 0$ is a positive constant independent of the deformation. For this material,

$$\beta_0 = -\frac{\mu}{3}\mathrm{I}_1\mathrm{I}_3^{-5/6} + \kappa\left(\mathrm{I}_3^{1/2} - 1\right)\,,\ \beta_1 = \mu\mathrm{I}_3^{-5/6} + \tilde\mu\mathrm{I}_3^{-1/2}(\mathrm{I}_1 - 3)\,,$$
$$\beta_{-1} = 0\,,$$

depending only on the principal invariants I_1 and I_3.
And W is stress free in the reference configuration. [1] We take

$$F = \begin{bmatrix} k & sa & 0 \\ 0 & a & 0 \\ 0 & 0 & 1/a \end{bmatrix},\qquad \widehat{F} = \begin{bmatrix} k & -sa & 0 \\ 0 & a & 0 \\ 0 & 0 & 1/a \end{bmatrix},$$

where $k > 0$, $a > 0$, and $s > 0$ are positive constants. The corresponding left Cauchy-Green tensors are

$$B = \begin{bmatrix} k^2 + s^2a^2 & sa^2 & 0 \\ sa^2 & a^2 & 0 \\ 0 & 0 & 1/a^2 \end{bmatrix},\qquad \widehat{B} = \begin{bmatrix} k^2 + s^2a^2 & -sa^2 & 0 \\ -sa^2 & a^2 & 0 \\ 0 & 0 & 1/a^2 \end{bmatrix},$$

and have the same principal invariants. The associated Cauchy stress tensors are

$$\sigma(B) = \beta_0\,\mathbb{1} + \beta_1\,B\,,\qquad \sigma(\widehat{B}) = \beta_0\,\mathbb{1} + \beta_1\,\widehat{B}\,,$$

with

$$\beta_0 = -\frac{\mu}{3}k^{-5/3}\left(k^2 + s^2a^2 + a^2 + \frac{1}{a^2}\right) + \kappa\,(k - 1)\,,$$
$$\beta_1 = \mu k^{-5/3} + \tilde\mu k^{-1}\left(k^2 + s^2a^2 + a^2 + \frac{1}{a^2} - 3\right).$$

Then, if

$$\frac{\mu}{3\tilde\mu} < \left(\frac{3 - a^2 - 1/a^2}{4}\right)^{4/3} \text{ and } 0 < s < \frac{1}{a}\sqrt{3 - 4\left(\frac{\mu}{3\tilde\mu}\right)^{3/4} - a^2 - \frac{1}{a^2}},$$

there exists $k = k_0 \in (0,1)$, such that

$$\beta_1 = 0 \qquad\text{and}\qquad \boxed{\sigma(B) = \beta_0\mathbb{1} = \sigma(\widehat{B})}.$$

Thus, we obtain homogeneous Cauchy stress although we suppose a non homogeneous deformation.

4 A truly large deformation result

We know that W is strictly **rank-one convex** on $\mathrm{GL}^+(3) = \{A \in \mathbb{R}^{3\times3} \,|\, \det A > 0\}$ if it is strictly convex on all closed line segments in $\mathrm{GL}^+(3)$ with end points differing by a matrix of rank one, i.e.,

$$W\left(F + (1-\theta)\,\xi\otimes\eta\right) < \theta W\left(F\right) + (1-\theta)W\left(F + \xi\otimes\eta\right)\,, \tag{8}$$

for all $F \in \mathrm{GL}^+(3)$, $\theta \in [0,1]$ and all $\xi, \eta \in \mathbb{R}^3$ with $F + t\xi\otimes\eta \in \mathrm{GL}^+(3)$ for all $t \in [0,1]$. The strain-ernergy function W in (7) is not rank-one convex due to the presence of the $\tilde\mu$-term.

Theorem: Strict rank-one convexity implies that Cauchy stress is injective along rank-one lines, i.e.
$\sigma(F + \xi\otimes\eta) = \sigma(F) \implies \xi\otimes\eta = 0$. [5]

Remark: Similar to linear elasticity, strict rank-one convexity **does not imply** that homogeneous Cauchy stress necessitates homogeneous strain.

5 Outlook

Invertibility of the first Piola-Kirchhoff stress S_1 violates material objectivity (frame-indifference). Therefore, it cannot be imposed. However, contrary to S_1, invertibility of the Cauchy stress tensor σ is not at variance with any known physical principle, and therefore, it may be imposed as a constitutive requirement.
An example is the **exponentiated Hencky-type energy** with the left stretch tensor $V = \sqrt{FF^T}$

$$W_{eH}(\log V) = \frac{\mu}{\kappa}\,e^{\|\operatorname{dev}_3\log V\|^2} + \frac{\kappa}{2\widehat{\kappa}}\,e^{\widehat{\kappa}\,[\operatorname{tr}(\log V)]^2} \tag{9}$$
$$\sigma_{eH}(\log V) = 2\mu\,e^{\kappa\,\|\operatorname{dev}_3\log V\|^2 - \operatorname{tr}(\log V)}\cdot\operatorname{dev}_3\log V$$
$$+\,\kappa\,e^{\widehat{\kappa}\,[\operatorname{tr}(\log V)]^2 - \operatorname{tr}(\log V)}\,\operatorname{tr}(\log V)\cdot\mathbb{1}\,. \tag{10}$$

σ_{eH} is invertible, while W_{eH} is not rank-one convex. [2, 3, 4]

Moreover in finite strain elasticity strict rank-one convexity and homogeneous Cauchy stress excludes rank-one laminates, but a construction similar to the linearized case shows that there exists non-unique inhomogeneous strains for homogeneous Cauchy stress. So strict rank-one convexity is not enough to ensure that homogeneous Cauchy stress implies homogeneous strain.

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