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# Real wave propagation in the isotropic relaxed micromorphic model

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## 1 The relaxed micromorphic model

The **relaxed micromorphic model** [3–7] couples the macroscopic displacement $u$ and a **micro-distortion field** $P \in \mathbb{R}^{3\times 3}$ endowing the standard Mindlin-Eringen representation with more geometric structure, reducing the curvature energy term to depend **only** on the **second order dislocation density tensor** $\alpha = -\operatorname{Curl} P$:

$$
\begin{aligned}
\mathrm{W} = \mu_e \,\|\operatorname{sym}(\nabla u - P)\|^2 + \frac{\lambda_e}{2}\operatorname{tr}(\nabla u - P)^2 + \mu_c\,\|\operatorname{skew}(\nabla u - P)\|^2 \\
+ \mu_{\text{micro}}\,\|\operatorname{sym} P\|^2 + \frac{\lambda_{\text{micro}}}{2}(\operatorname{tr}(P))^2 + \frac{\mu_e L_c^2}{2}\,\|\operatorname{Curl} P\|^2 .
\end{aligned} \tag{1}
$$

Here $\mu_e$, $\mu_{\text{micro}}$, $\lambda_e$ and $\lambda_{\text{micro}}$ are material parameters and $L_c$ is a characteristic length scale. **Positive definiteness** of the elastic energy is equivalent to the following simple relations for the introduced parameters [7]:

$$
\mu_e > 0, \quad \mu_c > 0, \quad 2\,\mu_e + 3\,\lambda_e > 0, \quad \mu_{\text{micro}} > 0, \quad 2\,\mu_{\text{micro}} + 3\,\lambda_{\text{micro}} > 0, \quad L_c > 0. \tag{2}
$$

The kinetic energy is $\mathrm{J} = \frac{\rho}{2}\,\|u_{,t}\|^2 + \frac{\eta}{2}\,\|P_{,t}\|^2$, where $\rho$ is the average **macroscopic mass density** of the material and $\eta$ is the **micro-inertia density**.

Comparing classical linear elasticity with our new relaxed model for $L_c \to 0$ we offer an **a priori relation** between $\mu_e$, $\lambda_e$, $\mu_{\text{micro}}$ and $\lambda_{\text{micro}}$ on the one side and $\lambda_{\text{macro}}$ and $\mu_{\text{macro}}$ on the other side that we call **macroscopic consistency condition** (see [1] for the fully anisotropic case) where $\mu_{\text{macro}}$ and $\lambda_{\text{macro}}$ are defined through:

$$
\mu_{\text{macro}} := \frac{\mu_{\text{micro}}\,\mu_e}{\mu_{\text{micro}} + \mu_e}, \qquad 2\mu_{\text{macro}} + 3\lambda_{\text{macro}} := \frac{(2\mu_{\text{micro}} + 3\lambda_{\text{micro}})\,(2\mu_e + 3\lambda_e)}{(2\mu_{\text{micro}} + 3\lambda_{\text{micro}}) + (2\mu_e + 3\lambda_e)} . \tag{3}
$$

For $\mu_{\text{micro}} \to \infty$ we recover the **Cosserat micropolar model**, which means that $P \in \mathfrak{so}(3)$, and for $L_c \to 0$ we obtain classical linear elasticity with $\mu_{\text{macro}}$, $\lambda_{\text{macro}}$ from (3).

## 2 Necessary and sufficient conditions for real wave propagation

The dynamical formulation is obtained defining a joint Hamiltonian and assuming stationary action, therefore:

$$
\begin{aligned}
\rho\, u_{,tt} &= \operatorname{Div}\left[2\,\mu_e \operatorname{sym}(\nabla u - P) + 2\,\mu_c \operatorname{skew}(\nabla u - P) + \lambda_e \operatorname{tr}(\nabla u - P)\,\mathbb{1}\right], \\
\eta P_{,tt} &= 2\,\mu_e \operatorname{sym}(\nabla u - P) + 2\,\mu_c \operatorname{skew}(\nabla u - P) + \lambda_e \operatorname{tr}(\nabla u - P)\,\mathbb{1} \\
&\quad - \left[2\mu_{\text{micro}} \operatorname{sym} P + \lambda_{\text{micro}} \operatorname{tr}(P)\,\mathbb{1}\right] - \mu_e\, L_c^2 \operatorname{Curl}\operatorname{Curl} P.
\end{aligned} \tag{4}
$$

This system is a **generalized tensorial Maxwell-problem** for the micro-distortion $P$ coupled to balance of linear momentum.

In our study of wave propagation in micromorphic media we limit ourselves to the case of **plane propagating waves**. We suppose that the space dependence of all introduced kinematic fields are limited to the component $x_1$ of $x$ which is the direction of propagation of the wave. Therefore we look for solutions of (4) with **real wavenumber $k$** in the form:

$$
u(x,t) = \widehat{u}\, e^{i(k\,x_1 - \omega\, t)}, \ \widehat{u} \in \mathbb{C}^3, \qquad P(x,t) = \widehat{P}\, e^{i(k\,x_1 - \omega\, t)}, \ \widehat{P} \in \mathbb{C}^{3\times 3}. \tag{5}
$$

With this ansatz, problem (4) can be analogously expressed as an eigenvalue-problem, see [8]:

$$
\begin{aligned}
\det\left(\mathbf{B}_1(k) - \omega^2\,\mathbb{1}\right) &= 0, & \det\left(\mathbf{B}_2(k) - \omega^2\,\mathbb{1}\right) &= 0, \\
\det\left(\mathbf{B}_3(k) - \omega^2\,\mathbb{1}\right) &= 0, & \det\left(\mathbf{B}_4(k) - \omega^2\,\mathbb{1}\right) &= 0,
\end{aligned} \tag{6}
$$

where $\mathbf{B}_1(k)$, $\mathbf{B}_2(k)$, $\mathbf{B}_3(k)$ and $\mathbf{B}_4(k)$, which are the blocks of the **acoustic tensor**, are real symmetric matrices depending on the material coefficients. Therefore, the resulting eigenvalues $\omega^2$ are real. Obtaining **real wave velocity $\omega/k$** is tantamount to having $\boldsymbol{\omega^2 \geq 0}$ for all solutions of (6).

**Sylvester's criterion** states that a Hermitian matrix M is positive-definite if and only if the leading principal minors are positive. Hence, considering the bulk moduli $\kappa_e = \frac{2\,\mu_e + 3\,\lambda_e}{3}$ and $\kappa_{\text{micro}} = \frac{2\,\mu_{\text{micro}} + 3\,\lambda_{\text{micro}}}{3}$, it is possible to prove the following proposition.

**Proposition.** *The dynamic relaxed micromorphic model* (4) *admits* ***real wave velocity $\omega/k$*** *if and only if*

$$
\begin{aligned}
\mu_c \geq 0, \qquad & \mu_e > 0, & 2\,\mu_e + \lambda_e &> 0, \\
& \mu_{\text{micro}} > 0, & 2\,\mu_{\text{micro}} + \lambda_{\text{micro}} &> 0, \\
& (\mu_{\text{macro}} > 0), & 2\,\mu_{\text{macro}} + \lambda_{\text{macro}} &> 0, \\
& \kappa_e + \kappa_{\text{micro}} > 0, & 4\,\mu_{\text{macro}} + 3\,\kappa_e &> 0. \qquad \blacksquare
\end{aligned} \tag{7}
$$

In (7) the requirement $\mu_{\text{macro}} > 0$ is redundant, since it is already assumed that $\mu_e, \mu_{\text{micro}} > 0$ and we note that the **Cosserat couple modulus $\mu_c$** only needs to be non-negative for **real wave velocity $\omega/k$**. It is clear that positive definiteness of the elastic energy (2) implies (7). The set of inequalities (7) is already implied by:

$$
\mu_e > 0, \qquad \mu_{\text{micro}} > 0, \qquad \mu_c \geq 0, \qquad \kappa_e + \kappa_{\text{micro}} > 0, \qquad 2\,\mu_{\text{macro}} + \lambda_{\text{macro}} > 0. \tag{8}
$$

## 3 A comparison: classical isotropic linear elasticity

The strain energy density W and the kinetic energy J for a classical Cauchy linear elastic isotropic medium are

$$
\mathrm{W} = \mu_{\text{macro}}\,\|\operatorname{sym}\nabla u\,\|^2 + \frac{\lambda_{\text{macro}}}{2}(\operatorname{tr}(\nabla u))^2, \qquad \mathrm{J} = \frac{\rho}{2}\,\|u_{,t}\|^2, \tag{9}
$$

where $\lambda_{\text{macro}}$ and $\mu_{\text{macro}}$ are the classical Lamé parameters, $u \in \mathbb{R}^3$ denotes the **macroscopic displacement** and $\rho$ is the average **macroscopic mass density** of the material. **Positive definiteness** of the energy is equivalent to:

$$
\mu_{\text{macro}} > 0, \qquad 2\,\mu_{\text{macro}} + 3\,\lambda_{\text{macro}} > 0. \tag{10}
$$

The equations of motion in strong form, obtained by the classical least action principle read $\rho\, u_{,tt} = \operatorname{Div}[2\,\mu_{\text{macro}} \operatorname{sym}\nabla u + \lambda_{\text{macro}} \operatorname{tr}(\nabla u)\,\mathbb{1}]$. Requiring **real wave velocity $\omega/k$** is **equivalent** to the **strong ellipticity condition** and holds if and only if:

$$
\mu_{\text{macro}} > 0, \qquad 2\,\mu_{\text{macro}} + \lambda_{\text{macro}} > 0, \tag{11}
$$

which is implied by positive definiteness (10).

## 4 A further comparison: the Cosserat model

In the isotropic hyperelastic case the elastic energy density and the kinetic energy of the Cosserat model read:

$$
\begin{aligned}
\mathrm{W} &= \mu_{\text{macro}}\,\|\operatorname{sym}(\nabla u)\|^2 + \mu_c\,\|\operatorname{skew}(\nabla u - A)\|^2 + \frac{\lambda_{\text{macro}}}{2}(\operatorname{tr}(\nabla u))^2 \\
&\quad + \frac{\mu_{\text{macro}}\, L_c^2}{2}\,\|\operatorname{Curl} A\|^2, \qquad J = \frac{\rho}{2}\,\|u_{,t}\|^2 + \frac{\eta}{2}\,\|A_{,t}\|^2,
\end{aligned} \tag{12}
$$

where $A \in \mathfrak{so}(3)$. **Positive definiteness** of the elastic energy is equivalent to the following simple relations for the introduced parameters

$$
\mu_{\text{macro}} > 0, \qquad 2\,\mu_{\text{macro}} + 3\,\lambda_{\text{macro}} > 0, \qquad \mu_c > 0, \qquad L_c > 0. \tag{13}
$$

The dynamical equilibrium equations are:

$$
\begin{aligned}
\rho\, u_{,tt} &= \operatorname{Div}\left[2\,\mu_{\text{macro}} \operatorname{sym}(\nabla u - A) + 2\,\mu_c \operatorname{skew}(\nabla u - A) + \lambda_{\text{macro}} \operatorname{tr}(\nabla u - A)\,\mathbb{1}\right], \\
\eta A_{,tt} &= -\,\mu_{\text{macro}}\, L_c^2 \operatorname{Curl}\operatorname{Curl} A + 2\,\mu_c \operatorname{skew}(\nabla u - A).
\end{aligned}
$$

The **necessary** and **sufficient** condition for **real wave velocity $\omega/k$** is [8]:

$$
\mu_{\text{macro}} > 0, \qquad 2\,\mu_{\text{macro}} + \lambda_{\text{macro}} > 0, \qquad \mu_c \geq 0, \tag{14}
$$

which is implied by the positive-definiteness of the energy (13). In [2] it is shown that **strong ellipticity** for the Cosserat-micropolar model holds if and only if:

$$
2\,\mu_{\text{macro}} + \lambda_{\text{macro}} > 0, \qquad \mu_{\text{macro}} + \mu_c > 0. \tag{15}
$$

Here, we note that the **Cosserat couple modulus $\mu_c$** may even be negative. We conclude that for micropolar material models (and therefore also for micromorphic materials), **strong ellipticity** (15) does not imply **real wave velocity $\omega/k$** (14), while the reverse is true.

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