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Reflection and transmission of elastic waves in non-local band-gap metamaterials

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1 Dynamical formulation of the problem

The strain energy density W and the kinetic energy J for a classical Cauchy linear elastic isotropic medium are

$$W = \mu \,\|\,\mathrm{sym}\;\nabla u\,\|^2 + \frac{\lambda}{2}\left(\mathrm{tr}\left(\mathrm{sym}\;\nabla u\,\right)\right)^2, \qquad J = \frac{\rho}{2}\,\|u_{,t}\|^2, \tag{1}$$

where λ and μ are the classical Lamé parameters, $u \in \mathbb{R}^3$ denotes the **macroscopic displacement** and ρ is the average **macroscopic mass density** of the material. The equations of motion in strong form, obtained by the classical least action principle read $\rho\, u_{,tt} = \mathrm{Div}\,\sigma = \mathrm{Div}[2\,\mu\;\mathrm{sym}\,\nabla u\, + \lambda\,\mathrm{tr}(\,\nabla u\,)\mathbb{1}]$.

The **relaxed micromorphic model** [1,4,5,7,8] couples the macroscopic displacement u and a **micro-distortion field** $P \in \mathbb{R}^{3\times 3}$ endowing the standard Mindlin-Eringen's representation with more geometric structure, reducing the curvature energy term to depend **only** on the **second order dislocation density tensor** $\alpha = -\,\mathrm{Curl}\,P$:

$$\begin{aligned} W = {} & \mu_e \,\|\,\mathrm{sym}\,(\,\nabla u\, - \,P)\|^2 + \frac{\lambda_e}{2}\,(\mathrm{tr}\,(\,\nabla u\, - \,P))^2 + \mu_c\,\|\,\mathrm{skew}\,(\,\nabla u\, - \,P)\|^2 \\ & + \mu_{\mathrm{micro}}\,\|\,\mathrm{sym}\,P\|^2 + \frac{\lambda_{\mathrm{micro}}}{2}\,(\mathrm{tr}\,P)^2 + \frac{\mu_e L_c^2}{2}\,\|\,\mathrm{Curl}\,P\|^2 . \end{aligned} \tag{2}$$

Here μ_e, μ_{micro}, λ_e and λ_{micro} are elasticity coefficients and L_c is a **characteristic length scale**. The kinetic energy is $J = \frac{\rho}{2}\,\|u_{,t}\|^2 + \frac{\eta}{2}\,\|\,P_{,t}\|^2$, with η the **micro-inertia density**. The dynamical formulation is obtained defining a joint Hamiltonian and assuming stationary action, therefore:

$$\begin{aligned} \rho\, u_{,tt} &= \mathrm{Div}\,\widetilde{\sigma} = \,\mathrm{Div}\,[2\,\mu_e\;\mathrm{sym}\,(\,\nabla u\, - \,P) + 2\,\mu_c\;\mathrm{skew}\,(\,\nabla u\, - \,P) + \lambda_e\,\mathrm{tr}\,(\,\nabla u\, - \,P)\,\mathbb{1}]\,, \\ \eta\, P_{,tt} &= 2\,\mu_e\;\mathrm{sym}\,(\,\nabla u\, - \,P) + 2\,\mu_c\;\mathrm{skew}\,(\,\nabla u\, - \,P) + \lambda_e\,\mathrm{tr}\,(\,\nabla u\, - \,P)\,\mathbb{1} \\ & \quad - [2\mu_{\mathrm{micro}}\;\mathrm{sym}\,P + \lambda_{\mathrm{micro}}\,\mathrm{tr}(\,P)\mathbb{1}] - \mu_e L_c^2\,\mathrm{Curl}\,\mathrm{Curl}\,P. \end{aligned} \tag{3}$$

This system is a **generalized tensorial Maxwell-problem** for the micro-distortion P coupled to balance of linear momentum.

2 Interface between a Cauchy medium and a micromorphic one

An **incident plane wave** traveling in the Cauchy medium impacts the interface to a relaxed micromorphic medium. Two waves are then generated: one wave reflected in the Cauchy medium and one wave transmitted in the relaxed micromorphic medium. To obtain the amplitude of the resulting waves, we impose suitable connections which are compatible with the jump duality conditions [6]. The **boundary conditions** can constrain the displacement field u in both media and the tangential micro-distortion $P \times \vec{n}$ in the relaxed micromorphic medium, where $\vec{n}$ is the unit normal to the interface. A possible choice of the boundary conditions, capable of describing phenomena of wave transmission in real mechanical metamaterials, is the **macro internal clamp with free microstructure**, that guarantees continuity of the macroscopic displacement, i.e. zero jump $[[u]] = 0$, and free motion of the microstructure at the interface.

Conservation of total energy $E = J + W$ implies

$$E_{,t} + \mathrm{div}\;H = 0, \tag{4}$$

where H is the energy flux vector that can be evaluated for a Cauchy medium and a relaxed one, respectively, as

$$H_{\mathrm{Cauchy}} = -\sigma^T \cdot u_{,t}, \qquad H_{\mathrm{relaxed}} = -\widetilde{\sigma}^T \cdot u_{,t} - \mu_e L_c^2\left(\left(\mathrm{Curl}\;P^T\right)\cdot\;P_{,t}\right) : \epsilon, \tag{5}$$

where ϵ is the Levi-Civita alternator.

Once a solution of (3) is found with the energy fluxes H_i , H_r and H_t of the incident, reflected and transmitted waves, respectively, we introduce the quantities

$$J_i = \frac{1}{\tau}\int_0^\tau H_i\,(0,t)\,dt, \qquad J_r = \frac{1}{\tau}\int_0^\tau H_r\,(0,t)\,dt, \qquad J_t = \frac{1}{\tau}\int_0^\tau H_t\,(0,t)\,dt, \tag{6}$$

where τ is the period of the traveling wave. The reflection (R) and transmission (T) coefficients are

$$R = \frac{J_r}{J_i}, \qquad T = \frac{J_t}{J_i}. \tag{7}$$

Since the system is conservative, one must have $R + T = 1$.

3 Modeling a two dimensional phononic crystal

For the calibration of the constitutive parameters of the relaxed micromorphic model we focus on the experiment proposed in [2], in which **longitudinal plane waves** are considered [3]. The structure shown in Figure 1(a), consists of a steel plate with liquid-filled holes in a square array.

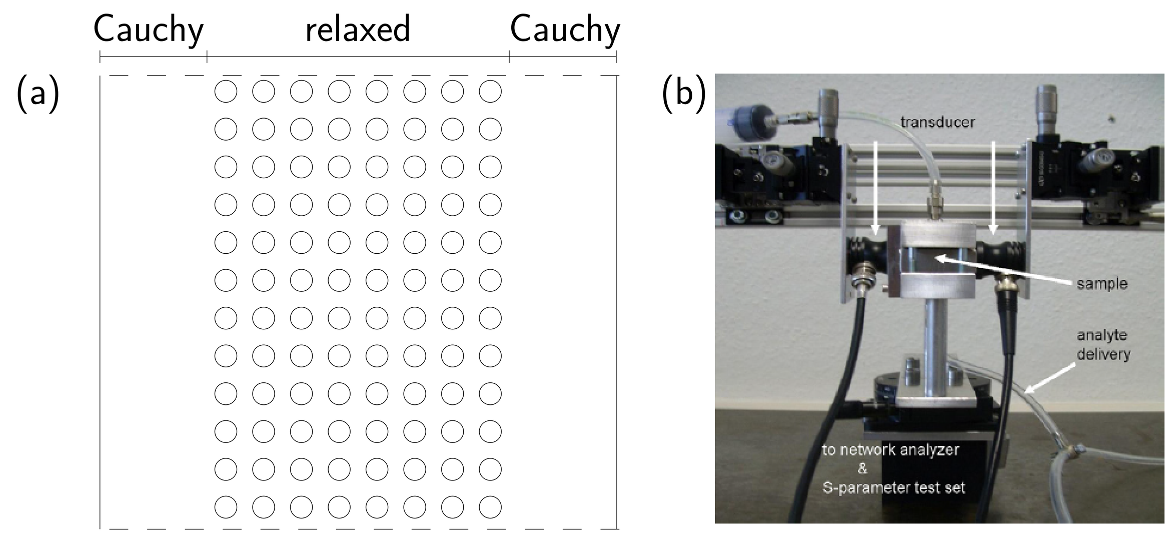

Figure 1: Schematics of the sample structure (a) and the experimental setup (b) ([2, Figure 1(b)]).

Suitably choosing the values of the parameters of the relaxed micromorphic model, we fit the profile of the transmission coefficient T proposed in [2] as function of the frequency of the traveling waves [3]. **Two band-gaps** which almost collapse to form a unique band-gap can be observed both in [2] and as a result of the simulations based upon the relaxed micromorphic model. The continuity of such extended band-gap is broken due to the presence of a **resonant peak** of transmitted energy that is seen to be related to the internal resonance of the fluid embedded in the microstructure. The degenerate limit case $L_c = 0$ lets the calculated transmission coefficient T slightly deviate from the experimentally-based one (sharp corners for $L_c = 0$ and smooth corners for $L_c > 0$). This means we are able to show the **non-locality** of the metamaterial and estimate the characteristic length L_c to be of the order of $0.5\,mm$, i.e. $\sim 1/3$ of the diameters of the holes.

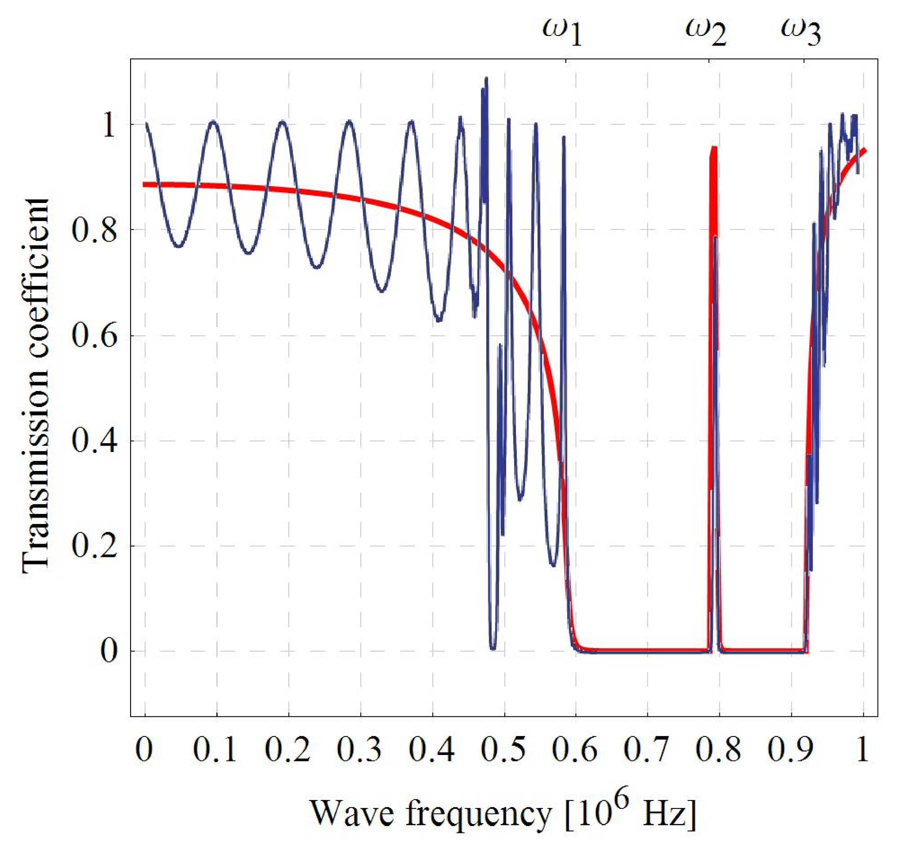

Figure 2: Comparison of the profile obtained in [2] based on a real metamaterial and the one obtained with the relaxed micromorphic model for the transmission coefficient T.

4 Conclusion

Metamaterials are artifacts composed by **microstructural elements** in periodic or quasi-periodic patterns, giving rise to materials with **unorthodox properties**. For some of these metamaterials, the presence of a microstructure allows for **local resonances** at the micro-level which globally result in **macroscopic wave-inhibition**: the energy of the incident wave remains trapped at the level of the microstructure.

The **relaxed micromorphic model** is the **only linear, isotropic, reversibly elastic, nonlocal generalized continuum model** known to date able to **predict complete frequency band gaps**. Here, we have shown that it can be calibrated against real experiments of nontrivial wave transmission problems in **phononic crystals**.

5 References

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