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The sum of squared logarithms inequality

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1 Introduction

The **sum of squared logarithms inequality** (SSLI) arose as scientific issue in 2012 while proving the optimality result [7]

$$\inf_{Q\in \mathrm{SO}(n)} \|\mathrm{sym}\operatorname{Log} Q^T F\|^2 \;=\; \inf_{Q\in \mathrm{SO}(n)} \inf_{\substack{Y\in\mathbb{R}^{n\times n}\\ \exp(Y)=Q^T F}} \|\mathrm{sym}\, Y\|^2 \;=\; \|\log\sqrt{F^T F}\|^2 . \qquad (1)$$

Here $Y = \operatorname{Log} X$ denotes all solutions of the matrix exponential equation $\exp(Y) = X$, $\|\,.\,\|$ denotes the Frobenius matrix norm, and $\mathrm{sym}\, X := \frac{1}{2}(X + X^T)$. The optimal rotation in (1) is given by the orthogonal factor of $F = R\cdot U = R\cdot\sqrt{F^T F}$ in the **polar decomposition** of F [1]. Thus (1) is a fundamentally new characterization of the polar decomposition.

For $n = 3$, the SSLI-inequality can be written as follows: let $x_1, x_2, x_3, y_1, y_2, y_3 > 0$ be positive real numbers such that

$$\begin{aligned} x_1 + x_2 + x_3 &\le y_1 + y_2 + y_3 ,\\ x_1 x_2 + x_1 x_3 + x_2 x_3 &\le y_1 y_2 + y_1 y_2 + y_2 y_3 ,\\ x_1 x_2 x_3 &= y_1 y_2 y_3 . \end{aligned}$$

Then the sum of their squared logarithms satisfy the following inequality:

$$(\log x_1)^2 + (\log x_2)^2 + (\log x_3)^2 \;\le\; (\log y_1)^2 + (\log y_2)^2 + (\log y_3)^2 .$$

In 2013 Bîrsan, Neff and Lankeit in [2] found a proof for $n \in \{2, 3\}$. In 2015, Neff and Pompe [8] proved the SSLI for $n = 4$, based on a new idea that supports more functions than only log but did not extend to higher dimensions without further complications. This line of thought has been recently taken up in [9] to yield a complete classification for arbitrary n.

For arbitrary n the SSLI can be stated as follows

Theorem (Sum of squared logarithms inequality) *For all natural numbers n and all positive numbers $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n > 0$ such that*

$$\sum_{i_1<\ldots<i_k} x_{i_1} x_{i_2} \ldots x_{i_k} \;\le\; \sum_{i_1<\ldots<i_k} y_{i_1} y_{i_2} \ldots y_{i_k} \quad \text{for all } k \in \{1, \dots, n-1\}$$

and

$$x_1 x_2 \ldots x_n \;=\; y_1 y_2 \ldots y_n ,$$

it follows

$$\sum_{i=1}^{n} (\log x_i)^2 \;\le\; \sum_{i=1}^{n} (\log y_i)^2 .$$

Replacing the assumption $x_1 x_2 \ldots x_n = y_1 y_2 \ldots y_n$ by $x_1 x_2 \ldots x_n \le y_1 y_2 \ldots y_n$ easily admits counterexamples.

The general proof of the theorem was found in May 2015 [3] after P. Neff offered one ounce of fine gold [5] for a solution to the problem on the internet platform *MathOverflow*.

2 Elementary symmetric polynomials

For given (complex) numbers $z_1, \dots, z_n$ the elementary symmetric polynomials

$$e_k := \sum_{i_1<\ldots<i_k} z_{i_1} z_{i_2} \ldots z_{i_k}$$

are the coefficents of the normalized polynomial h with the roots $z_1, \dots, z_n$, i.e.

$$h(t) = (t - z_1)\cdot\ldots\cdot(t - z_n) = t^n - e_1\, t^{n-1} + e_2\, t^{n-2} + \ldots + (-1)^n e_n .$$

The function mapping the roots onto the coefficients is invertible: The fundamental theorem of Algebra guarantees the existence of a unique inversion $\varphi\colon \mathbb{R}^n_+ \to M$. This function φ is even continous and, at all vectors of coefficients corresponding to different roots, differentiable.

For all $z \in M$ let $f(z) = \sum_{i=1}^n (\log z_i)^2$; note that $f(z) \in \mathbb{R}$. We can restate the SSLI in terms of f:

If $e_1, \dots, e_n, \tilde e_1, \dots, \tilde e_n > 0$ are positive real numbers with

$$e_k \le \tilde e_k \quad \text{for all } k \in \{1, \dots, n-1\} \qquad \text{and} \qquad e_n = \tilde e_n ,$$

then $f\big(\varphi(e_1, \dots, e_n)\big) \le f\big(\varphi(\tilde e_1, \dots \tilde e_n)\big)$.

3 Sketch of proof

The main idea of the proof has already been pursued in prior attempts to prove the inequality: instead of directly working with the function $f(z) := \sum_{i=1}^n (\log z_i)^2$ on the set M of roots, we consider the composition $f \circ \varphi$ which depends on the elements $e \in T$ of a suitable set of coefficients $T \subseteq \mathbb{R}^n_+$. Of course, we have to choose T in a way such that $(f\circ\varphi)(e) \in \mathbb{R}$ for all $e \in T$.

The proof of the SSLI can now be divided into two steps:

1.) We show that $\dfrac{\partial (f\circ\varphi)}{\partial e_k} \ge 0$.

2.) We find a path $\gamma\colon [0, 1] \to \varphi(T)$ with $\gamma(0) = x$, $\gamma(1) = y$

$$\text{such that } \frac{\mathrm{d}}{\mathrm{d}s} e_k\big(\gamma(s)\big) \ge 0 \quad \text{for all } s \in (0, 1) \text{ and } k \in \{1, \dots, n-1\}$$

$$\text{as well as } \frac{\mathrm{d}}{\mathrm{d}s} e_n\big(\gamma(s)\big) = 0 \quad \text{for all } s \in (0, 1) .$$

We show 1.) for all e such that $\varphi(e)$ has no multiple roots. Instead of chosing the path required in 2.) in the set of roots $\varphi(T)$ as in prior attemps, we operate on the set of coefficients: Consider the path $e^s = (e^s_1, \dots, e^s_n) \subseteq \mathbb{R}^n_+$ for $s \in [0, 1]$ with

$$e^s_k \;:=\; (1 - s)\, e_k(x) + s\, e_k(y) .$$

Then $e^0 = e(x)$ and $e^1 = e(y)$ as well as $e_k(x) < e_k(y)$ for all $k \in \{1, \dots, n-1\}$ and $e_n(x) = e_n(y)$. The special thing is: although the roots corresponding to e^0 und e^1, given by $x_1, \dots, x_n$ und $y_1, \dots, y_n$, are positive reals, the roots of e^s are possibly complex numbers! Furthermore e^s has multiple roots only at finitely many s and 1.) applies to the rest.

4 Application to nonlinear elasticity

Let $U \in \mathrm{Sym}^+(n)$, where $\mathrm{Sym}^+(n) \subset \mathbb{R}^{n\times n}$ denotes the set of positive definite symmetric $n\times n$-matrices. Then U is orthogonally diagonalizable with real eigenvalues $\lambda_1, \dots, \lambda_n > 0$. The k-th invariant $I_k(U)$ of U is defined as the k-th elementary symmetric polynomial of the vector $\lambda(U) = (\lambda_1, \dots, \lambda_n)$, i.e. $I_k(U) := e_k(\lambda(U))$.

Since $\|\log U\|^2 = \sum_{i=1}^n \big(\log \lambda_i(U)\big)^2$, the SSLI can be equivalently expressed in terms of these invariants of positive definite symmetric matrices.

Theorem *Let $U, \widetilde U \in \mathrm{Sym}^+(n)$. If $I_k(U) \le I_k(\widetilde U)$ for all $k \in \{1, \dots, n-1\}$ and* $\det U = \det \widetilde U$, *then* $\|\log U\|^2 \le \|\log \widetilde U\|^2$, *where* $\log$ *is the principal matrix logarithm on $\mathrm{Sym}^+(n)$ and $\|\,.\,\|$ denotes the Frobenius matrix norm.*

The theorem can be applied directly to the **quadratic Hencky energy**

$$W_{\mathrm{H}}(F) \;=\; \mu\,\|\mathrm{dev}_n \log U\|^2 + \frac{\kappa}{2}\,[\mathrm{tr}(\log U)]^2 \;=\; \mu\,\|\log U\|^2 + \frac{\lambda}{2}\,[\log(\det U)]^2 ,$$

which was introduced into the theory of nonlinear elasticity in 1929 by H. Hencky [4, 8], cf. [6]. Here, $F \in \mathrm{GL}^+(n)$ is the deformation gradient, $\mathrm{GL}^+(n)$ is the set of invertible $n\times n$-matrices with positive determinant, $U = \sqrt{F^T F}$ is the right stretch tensor and $\mathrm{dev}_n \log U = \log U - \frac{1}{n}\,\mathrm{tr}(\log U)\cdot\mathbb{1}$ is the deviatoric part of the Hencky strain tensor $\log U$.

In terms of the quadratic Hencky energy, the theorem can be stated as follows:

Corollary *Let $F, \widetilde F \in \mathrm{GL}^+(n)$ with $U = \sqrt{F^T F}$ and $\widetilde U = \sqrt{\widetilde F^T \widetilde F}$. If* $\det U = \det \widetilde U$ *and $I_k(U) \le I_k(\widetilde U)$ for all $k \in \{1, \dots, n-1\}$, then $W_H(F) \le W_H(\widetilde F)$.*

According to this corollary W_H satisfies a version of Truesdell's **empirical inequalities** [10, pages 158, 171].

5 References

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