Aufgabe 1

a)

$\text{Ran}\,[S_oI-A \quad -B] < n \longrightarrow$ S_o is an eigenvalue and an input decoupling zero.

$\text{Rank}\begin{pmatrix} S_oI-A \\ C \end{pmatrix} < n \longrightarrow$ S_o is an eigenvalue and an output decoupling zero.

Input/output decoupling zeros and transmission zeros both are invariant zeros. Decoupling zeros are part of the eigenvalues. On the other hand, transmission zeros are not eigenvalues.

b) Eigenvalues are:- $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = -3$.

Due to the positive eigenvalues (λ_1, λ_2) the system is neither stable nor asymptotic stable.

System is stable if $Re\{\lambda_i\} \leq 0$.

System is asymptotic stable if $Re\{\lambda_i\} < 0$.

c)

Rosenbrock system matrix:-

$$P(s) = \begin{pmatrix} sI-A & -B \\ C & D \end{pmatrix} \Longrightarrow P(s) = \begin{bmatrix} s-1 & 0 & 0 & 0 & 0 \\ 0 & s+3 & 0 & 0 & -b \\ 0 & 0 & s-2 & 0 & -1 \\ 0 & 0 & c & 0 & 0 \end{bmatrix}$$

The invariant zeros are those frequencies which meet one of the following criteria:

ⓐ $\det P(s_0) = 0$ for $m = n$

ⓑ $\text{Rank } P(s_0) < \max \text{Rank } P(s)$ for $m \neq n$

$$\text{Max rank}(P(s_0)) = 4$$

For $s_0 = 1, -3 \Longrightarrow \text{Rank}(P(s_0)) < 4$

Invariant zeros are: $s_0 = 1, -3$

d) System is full observable, if and only if the observability matrix Q_b is of rank n.

$$\text{Rank} \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \text{Rank} \begin{bmatrix} 0 & 0 & c \\ 0 & 0 & 2c \\ 0 & 0 & 4c \end{bmatrix} \overset{!}{=} n \Longrightarrow \text{Rank } Q_b = 1$$

System is not full observable.

e) System is not stabilizable because at least one of the unstable modes (when $s_0 = 1$) is uncontrollable.

To realize a full state feedback, system states should be either measurable or observable. According to C matrix, 2 states are not measurable. Therefore, the observability should be checked. If the unmeasurable states are observable, then a fully state feedback can be realized. According to Hautus, system state (when $s_0 = 1$) is not observable.

Therefore it is not possible to realize a full state feedback

f) LQR is able to stabilize the system, because the system is assumed to be fully controllable.

Yes, the controlled system is asymptotically stable. Due to Ljapunow equation if Q & R are positive definit, then P is positive definit. Therefore the controlled system is asymptotically stable.

Problem 2

DETTMANN

a) If (A, C) is fully observable, then the eigenvalues of $A_{obs} = [A - LC]$ are arbitiary placeable.

KALMAN-criterion

$$\text{Rank} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 1 \\ -3 & 9 & 8 \\ 0 & 0 & 1 \\ 33 & -57 & -10 \\ 0 & 0 & 1 \end{bmatrix} = 3 \equiv n$$

$\Rightarrow$ fully observable

$\Rightarrow$ arbitiary placeable.

b) $$A_{obs} = A - LC$$
$$= \begin{bmatrix} 1 & 4 & 6 \\ 4 & -5 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} l_2 & 0 \\ l_1 & l_2 \\ 0 & l_1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1-l_2 & 4+l_2 & 6-2\lambda_2 \\ 4-l_1 & -5+l_1 & -l_2-2l_1 \\ 0 & 0 & 1-l_1 \end{bmatrix}$$

eigenvalues: $\left| \lambda I - A_{obs} \right| \stackrel{!}{=} 0$

$$\Rightarrow \lambda^3 + (3+l_2)\lambda^2 + (10 l_1 + l_1 l_2 - l_1^2 - 25)\lambda + \ldots$$
$$+ 21 + 5 l_1^2 + l_1 l_2 - 26 l_1 - l_2 \stackrel{!}{=} 0$$

Problem 2

desired (given) eigenvalues: $\lambda_1 = -1$, $\lambda_2 = -3$, $\lambda_3 = -4$

$$\Rightarrow (\lambda+1)(\lambda+3)(\lambda+4) \stackrel{!}{=} 0$$

$$\Rightarrow \lambda^3 + 8\lambda^2 + 19\lambda + 12 \stackrel{!}{=} 0$$

Comparison of coefficients:

$$\lambda^2: \; 8 \stackrel{!}{=} 3 + l_2 \quad \Rightarrow \quad l_2 = 5$$

$$\lambda^1: \; 19 \stackrel{!}{=} 10 l_1 + 5 l_1 - l_1^2 - 25$$

$$\Rightarrow l_1^2 - 15 l_1 + 44 = 0$$

$$\Rightarrow l_1 = 4 \;\vee\; l_1 = 11$$

$$\lambda^0: \; 12 \stackrel{!}{=} 21 + 5 l_1^2 + 5 l_1 - 26 l_1 - 5$$

$$\Rightarrow l_1^2 - \frac{21}{5} l_1 + \frac{4}{5} = 0$$

$$\Rightarrow l_1 = 4 \;\vee\; l_1 = 0{,}2$$

$$\Rightarrow l_1 = 4 \;\wedge\; l_2 = 5$$

c) Stability determined by eigenvalues

$$\Rightarrow |\lambda I - A| \stackrel{!}{=} 0$$

$$\Rightarrow \begin{vmatrix} \lambda-1 & -4 & -6 \\ -4 & \lambda+5 & 0 \\ 0 & 0 & \lambda-1 \end{vmatrix} \stackrel{!}{=} 0$$

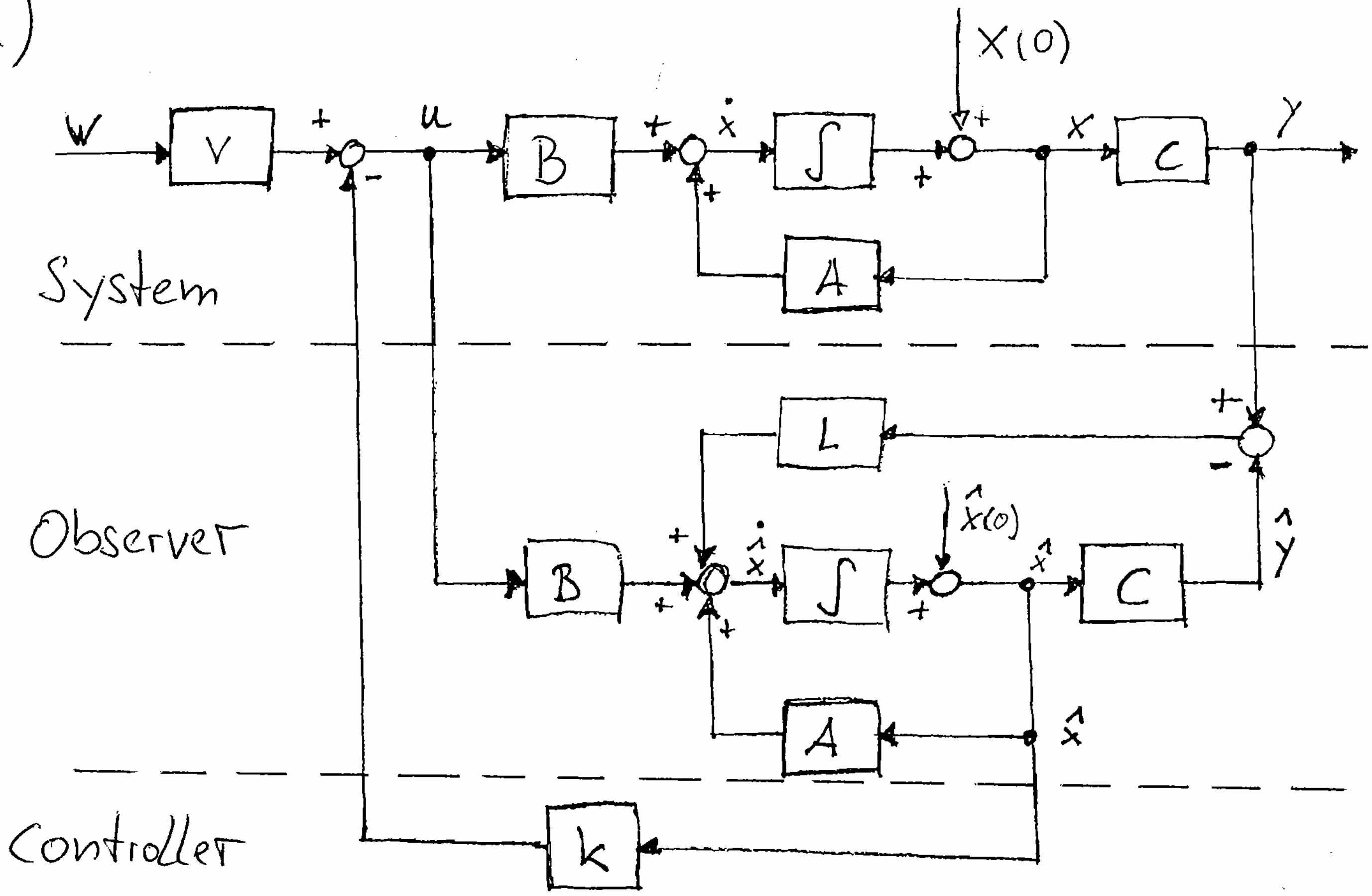
$$\Rightarrow (\lambda-1)(\lambda+5)\underline{(\lambda-1)} - \underline{(\lambda-1)}(-4)(-4) \stackrel{!}{=} 0$$

$\lambda_1 = +1$ is eigenvalue!

$\Rightarrow$ In the minimum one eigenvalue with positive real part $\Rightarrow$ <u>not</u> asymp. stable.

Problem 2

d)



$\hat{x}$:	observed state vector	$n \times 1$
y :	output vector	$m \times 1$
u :	input "	$r \times 1$
w :	reference "	$k \times 1$
V :	reference input matrix	$r \times k$
A :	system matrix	$n \times n$
B :	input "	$n \times r$
C :	measurement/output matrix	$m \times n$
L :	observer feedback matrix	$n \times m$
K :	state feedback matrix	$r \times n$

Problem 3

a) $$\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -p & -q & \lambda - r \end{vmatrix} = \lambda^3 - r\lambda^2 - q\lambda - p$$

characteristic equation: $\underbrace{1}_{a_3}\lambda^3 \underbrace{-r}_{a_2}\lambda^2 \underbrace{-q}_{a_1}\lambda \underbrace{-p}_{a_0} = 0$

Determine the stability of the system with Hurwitz criterium.

(i) all coefficients a_i must be positive.

(ii) $$H = \begin{vmatrix} a_2 & a_0 & 0 \\ a_3 & a_1 & 0 \\ 0 & a_2 & a_0 \end{vmatrix}$$ (H_1, H_2, H_3) all determinants $H_i > 0$

for (i) $a_3 = 1 > 0$, $a_2 = -r > 0$, $a_1 = -q > 0$, $a_0 = -p > 0$

$\Rightarrow r < 0,\ p < 0,\ q < 0$

for (ii) $|H_1| = a_2 = -r > 0 \Rightarrow r < 0$

$|H_2| = a_2 a_1 - a_0 a_3 = qr + p > 0 \Rightarrow qr > -p$

$|H_3| = a_0 \cdot |H_2| \Rightarrow a_0 > 0\ \ |H_2| > 0 \Rightarrow p < 0,\ qr > -p$

$\Rightarrow$

$\Rightarrow$

$\Rightarrow r < 0,\ p < 0,\ qr > -p$

$\Rightarrow$ The system is asymptotically stable, if the following conditions are satisfied:

1. $r < 0$,
2. $p < 0$,
3. $q < 0$, and
4. $qr > -p$

Problem 3

b) With $q=-1$, $p=0$ and $r=2$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\det(\lambda I - A) = \lambda^3 - r\lambda^2 - q\lambda - p = \lambda^3 - 2\lambda^2 + \lambda = 0$$

Eigenvalues: $\lambda_1 = 0$ $\quad \lambda_{2,3} = +1$ (unstable)

Stabilizability creterium (Hautus)

$$\text{Rank}\left[(\lambda_i I - A) \quad B \right] = n, \quad \text{for all unstable } \lambda_i$$

For $\lambda_{2,3} = +1$

$$\text{Rank}\begin{bmatrix} 1 & -1 & 0 & b_1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & b_2 \end{bmatrix} = 3, \text{ if } b_2 \neq 0$$

$\Rightarrow$ The system can be stabilized by state feedback when $b_2 \neq 0$.

c) Characteristic polynomial of the controlled system:

$$\det(\lambda I - (A - BK)) = \begin{vmatrix} \lambda + 2K_1 & 2K_2 - 1 & 2K_3 \\ 0 & \lambda & -1 \\ K_1 - 1 & K_2 + 1 & \lambda + K_3 \end{vmatrix}$$

$$= \lambda^3 + (2K_1 + K_3)\lambda^2 + (2K_3 + K_2 + 1)\lambda + 3K_1 + 2K_2 - 1$$

Desired characteristic polynomial:

$$(\lambda + 10)(\lambda - 10)\lambda = \lambda^3 - 100\lambda$$

Compare the coefficients:

$$\left.\begin{aligned} 2K_1 + K_3 &= 0 \\ 2K_3 + K_2 + 1 &= -100 \\ 3K_1 + 2K_2 - 1 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} &2K_1 = -K_3, \quad 2K_3 + K_2 = -101, \\ &3K_1 + 2K_2 = 1 \end{aligned}$$

Problem 3

3/4

d) with $p=1$, $q=1$, $r=-1$

$$A=\begin{bmatrix}0&1&0\\0&0&1\\1&1&-1\end{bmatrix}, \quad \det(\lambda I-A)=\lambda^3+\lambda^2-\lambda-1$$

$$=(\lambda-1)(\lambda+1)(\lambda+1)$$

Eigenvalues : $\lambda_1=1 \quad \lambda_{2,3}=-1$

(i) Left eigenvectors: $w_i^T(\lambda_i I-A)=0$

for $\lambda_1=1 \qquad w_1=\begin{bmatrix}w_{11}\\w_{12}\\w_{13}\end{bmatrix}$

$$[w_{11}\ w_{12}\ w_{13}]\begin{bmatrix}1&-1&0\\0&1&-1\\-1&-1&2\end{bmatrix}=0 \Rightarrow \begin{array}{l}w_{11}-w_{13}=0\\-w_{11}+w_{12}-w_{13}=0\\-w_{12}+2w_{13}=0\end{array}$$

let $w_{11}=1 \Rightarrow \underline{\underline{w_1=\begin{bmatrix}1\\2\\1\end{bmatrix}}}$

for $\lambda_{2,3}=-1 \qquad w_2=\begin{bmatrix}w_{21}\\w_{22}\\w_{23}\end{bmatrix}$

$$[w_{21}\ w_{22}\ w_{23}]\begin{bmatrix}-1&-1&0\\0&-1&-1\\-1&-1&0\end{bmatrix}=0 \Rightarrow \begin{array}{l}-w_{21}-w_{23}=0\\-w_{21}-w_{22}-w_{23}=0\\-w_{22}=0\end{array}$$

let $w_{21}=1 \Rightarrow \underline{\underline{w_2=\begin{bmatrix}1\\0\\-1\end{bmatrix}}}$

(ii) Right eigenvectors: $(\lambda_i I-A)v_i=0$

for $\lambda_1=1 \qquad v_1=\begin{bmatrix}v_{11}\\v_{12}\\v_{13}\end{bmatrix}$

$$\begin{bmatrix}1&-1&0\\0&1&-1\\-1&-1&2\end{bmatrix}\begin{bmatrix}v_{11}\\v_{12}\\v_{13}\end{bmatrix}=0 \Rightarrow \begin{array}{l}v_{11}-v_{12}=0\\v_{12}-v_{13}=0\\-v_{11}-v_{12}+2v_{13}=0\end{array}$$

let $v_{11}=1$

$\Rightarrow \underline{\underline{v_1=\begin{bmatrix}1\\1\\1\end{bmatrix}}}$

Problem 3

d) Right eigenvector for $\lambda_{2,3} = -1$ $\quad v_2 = \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix}$

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = 0 \Rightarrow \begin{matrix} -v_{21} - v_{22} = 0 \\ -v_{22} - v_{23} = 0 \\ -v_{21} - v_{22} = 0 \end{matrix} \qquad \text{let } v_{21} = 1 \Rightarrow v_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

e) Fully controllable:

all left eigenvectors fulfill $w_i^T B \neq 0$ for all λ_i

$$w_1^T B = [1 \;\; 2 \;\; 1] \begin{bmatrix} b_1 \\ 0 \\ b_2 \end{bmatrix} \neq 0 \Rightarrow b_1 \neq -b_2$$

$$w_2^T B = [1 \;\; 0 \;\; -1] \begin{bmatrix} b_1 \\ 0 \\ b_2 \end{bmatrix} \neq 0 \Rightarrow b_1 \neq b_2$$

$\Rightarrow$ For $b_1 \neq b_2$ and $b_1 \neq -b_2$ the system is fully controllable.

Fully observable:

all right eigenvectors fulfill $C v_i \neq 0$ for all λ_i

$$C v_1 = [0 \;\; c_1 \;\; c_2] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \neq 0 \Rightarrow c_1 \neq -c_2$$

$$C v_2 = [0 \;\; c_1 \;\; c_2] \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \neq 0 \Rightarrow c_1 \neq c_2$$

$\Rightarrow$ For $c_1 \neq c_2$ and $c_1 \neq -c_2$ the system is fully observable.

4a) $Q_S = [B \quad AB] = \begin{bmatrix} 1 & 5 \\ 1 & 2+a \end{bmatrix} \overset{!}{=} 2$

$\det Q_S = -3 + a \overset{!}{\neq} 0$

$\Rightarrow$ controllable, if $a \neq 3$

b) $Q_B = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & a \end{bmatrix}$

$Rg\ Q_B = 2$

$\Rightarrow$ observable for all a

c) $A = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}$

$G(s) = C(sI - A)^{-1}B = [0 \quad 1] \begin{bmatrix} s-1 & -4 \\ -2 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \dfrac{s+1}{s^2 - 2s - 7}$

poles: $s_{12} = 1 \pm \sqrt{8}$

zeros: $s_0 = -1$

eigenvalues: system is fully controllable and observable ($a=1$), so eigenvalues are identical to poles.

d) $\det[\lambda I - (A - Bk)] = \begin{vmatrix} \lambda - 1 + k_1 & k_2 - 4 \\ k_1 - 2 & \lambda - 1 + k_2 \end{vmatrix}$

$= \lambda^2 + \lambda[-2 + k_1 + k_2] + 3k_1 + k_2 - 7$

$k_1 + k_2 - 2 > 0$

$3k_1 + k_2 - 7 > 0$

$H_1 : k_1 + k_2 - 2 > 0$

$H_2 : (3k_1 + k_2 - 7) \cdot H_1 > 0$

e) $A^T P + PA - PBR^{-1}B'P = -Q$

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \quad P_{11} = P_{22}, \quad P_{12} = P_{21}$$

$$\Rightarrow P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_1 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} p_1 & p_2 \\ p_2 & p_1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} 2 \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_1 \end{bmatrix}$$

$$= -\begin{bmatrix} 24 & 8 \\ 8 & 24 \end{bmatrix}$$

$$\begin{bmatrix} p_1 + 2p_2 & p_2 + 2p_1 \\ 4p_1 + p_2 & 4p_2 + p_1 \end{bmatrix} + \begin{bmatrix} p_1 + p_2 & 4p_1 + p_2 \\ p_2 + 2p_1 & 4p_2 + p_1 \end{bmatrix} - 2\begin{bmatrix} (p_1+p_2)^2 & (p_1+p_2)^2 \\ (p_1+p_2)^2 & (p_1+p_2)^2 \end{bmatrix} = -Q$$

$$2(p_1 + 2p_2) - 2(p_1+p_2)^2 = -24 \quad \mathrm{I}$$
$$6p_1 + 2p_2 - 2(p_1+p_2)^2 = -8 \quad \mathrm{II}$$
$$2(4p_2 + p_1) - 2(p_1+p_2)^2 = -24 \quad \mathrm{III}$$

$\mathrm{II} - \mathrm{I}$: $4p_1 - 2p_2 = 16 \quad \mathrm{IV}$

$\mathrm{II} - \mathrm{III}$: $4p_1 - 6p_2 = 16 \quad \mathrm{V}$

$\mathrm{IV} - \mathrm{V}$

$$(-2+6)\,p_2 = 0$$
$$p_2 = 0$$

in IV or V:

$$4p_1 = 16$$
$$p_1 = 16$$

$$k^* = R^{-1} \cdot B' \cdot P = 2 \cdot \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 8 & 8 \end{bmatrix}}}$$

f) $Rk(Q_B) = Rk\left(\begin{bmatrix} c \\ c(A-BK) \end{bmatrix}\right) = Rk\left(\begin{bmatrix} 0 & 1 \\ 2-k_1 & 1-k_2 \end{bmatrix}\right) \overset{!}{=} n$

$\det Q_B = -(2-k_1) \overset{!}{\neq} 0 \quad , \; k_1 \overset{!}{\neq} 2$

g)

$(A-LC) = \left(\begin{bmatrix} 1 & 4-l_1 \\ 2 & 4-l_2 \end{bmatrix}\right)$

$\det(\lambda-(A-LC)) = \begin{vmatrix} \lambda-1 & l_1-4 \\ -2 & \lambda-4+l_2 \end{vmatrix}$

$= \lambda^2 + \lambda(l_2-5) + 2l_1 - l_2 - 4$

desired polynomial: $(\lambda+1)(\lambda+2) = \lambda^2 + 3\lambda + 2$

$l_2 - 5 = 3 \qquad \underline{\underline{l_2 = 8}}$

$2l_1 - l_2 - 4 = 2 \qquad \underline{\underline{l_1 = 7}}$