STRUCTURING MATHEMATICAL BELIEF STRUCTURES – SOME THEORETICAL CONSIDERATIONS ON BELIEFS, SOME RESEARCH QUESTIONS AND SOME PHENOMENOLOGICAL OBSERVATIONS

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Abstract: This article is to be understood as a theoretical-analytical contribution on the definition of mathematical beliefs and their possible structuring. The understanding of possible clusters of mathematical beliefs is our central concern. Whereas data collection concerning mathematical beliefs has received substantial attention in specialized literature, leading to the identification of a considerable number of beliefs, contributions focused on the definition and on the categorization of beliefs or the description of correlating interdependencies are comparatively rare. Alongside a short overview of various approaches, implication patterns for beliefs in relation to calculus are used as an example here. There is the impression that domain-specific beliefs are induced by certain mathematical global beliefs.

The Starting Point: The Lacking Definition of Mathematical Beliefs

In scientific contexts, terms play a functional role. Their appropriateness is especially justifiable when they facilitate the formation of pertinent research questions. This interplay between the creation of terminology on one hand and the resulting implications on the other should specifically be clarified for the field of *mathematical beliefs*. The reported observations are constituted by a phenomenological character, and while the answers should be understood only as initial explanatory attempts, they should show the categorical prolificness of the terminology coinage. For conciseness, consequences for the everyday mathematical life of students, teachers and others cannot be discussed.

In the literature, there are a great number of papers to be found concerning beliefs in mathematics as well as beliefs in the learning and teaching of mathematics (e.g., Thompson 1992), featuring in particular teachers' beliefs. In current literature, however, there is still no consensus on a unique definition of the term 'belief'. There is also no evidence that enlightening contributions can be expected in the near future in the pedagogical related sciences, especially since considering culturally differing positions cannot be ignored (e.g., Alexander & Dochy, 1995). Our mathematical didactic focus on beliefs apparently indicates a demand that is otherwise not pivotal in social sciences. This sobering conclusion however is opposed by the observation that more than a few scientific papers have included substantial results concerning mathematical beliefs without first explicitly defining the term or specifically referring to an existing definition. Possibly concepts that incorporate the various potential definitions of beliefs are being tacitly underlain from each of the respective researchers, even if only implicitly (cp. to the comparative discussion in Furinghetti & Pehkonen, 1999).

However, there are many papers focusing on processes of learning and teaching which totally overlook beliefs, although some overlapping with belief features is evident. It might be that related theoretical frameworks, i.e. theories, are widely accepted and strongly established, e.g., theory of attitudes, theory of attributions or motivations, and these theories are then applied directly without mentioning beliefs. Inasmuch interesting developments that could enrich the theoretical discussion about beliefs unfortunately do not provide direct contributions to the technical advancement of the research of beliefs.

The author must ask to what extent the search for an authoritative definition is not a question posed improperly. Perhaps this can be presented analogously with an example from mathematics. At no point is it defined in arithmetic what is to be understood by a number, and yet man has successfully worked with numbers for many centuries in spite of this. Only Dedekind's noted work (1995, originally published 1888) *What are numbers and what should they be?* led the way to an axiomatic definition of numbers. In other words, the naïve number term is anchored in the perception of the (number) fields. The definition of a (number) field is by no means monomorphical, and so there are a number of non-isomorphical realizations and models.

Finally it should be noted that several authors have themselves legitimately modified the 'definitions' of what they understood to be beliefs over time, e.g., Schoenfeld (1985, 1998).

Theoretical Framework and Significance

The integral motivation for the struggle for a definition of beliefs is the effort to separate beliefs from cognition. Schoenfeld (1985) points out that the purely cognitive components of his framework for the analysis of mathematical behavior did a poor job of predicting the problem-solving processes of students. A significant contribution to this topic was presented in the works of Abelson (1979) as well as in those of Calderhead (1996); the former included quasi characterizing stipulations that do not provide the aspired limitations between beliefs and cognition, but are all in all considered constitutive (see also Nespor, 1987). In this sense we attempt to list relevant characteristics of beliefs and to understand the demands as a whole as a definition. The author is aware of the fact that all variables (for example the context dependence) will never be able to be made explicit with a conceptual construction this interwoven. On the other hand, highly dimensional models are seldom explanatory, thus we must reduce the number of variables.

Our starting point is Schoenfeld's (1998) definition approach in which he conceives mental constructs representing the codification of people's experiences and understandings as beliefs. It is our task to define what should be considered mental constructs. In the following, we aim to present constitutive characteristics, which are possibly central for all belief definitions.

As in the theory of attitudes (Eagly & Chaiken, 1992) we first logically speak of **belief objects**. Abelson (1979) uses the term 'content set.' Basically anything that shares a direct or indirect connection to mathematics can function as a belief object. We will provide several typical examples without an attempt at completeness:

(a) mathematical facts (= objects) (e.g. binomial theorem, the definition of a square, the number Pi etc.), mathematical procedures, domains within mathematics (e.g. geometry, calculus etc.), mathematics as whole, mathematics as a discipline (school mathematics, mathematics at university, industrial mathematics, mathematics within society etc.).

(b) relations where mathematics or a subunit of mathematics (see (a)) is a substantial part (mathematics and application, mathematics and history, usefulness of mathematics).

(c) relations where mathematics as well as the individual is a substantial part (self-concept as a learner of mathematics, self-concept as a teacher of mathematics, personal anxiety and mathematics etc.).

(d) the learning of mathematics itself, the learning within a specific domain, the learning of special content or topic etc.

It is noticeable that the belief objects have various 'sizes', so that we refer to the **breadth** of a belief object.

The belief object O is associated to the actual – what we are traditionally calling - 'beliefs' whereas a large variation breadth – from a single belief to complex network of beliefs - must be presumed here. When Schoenfeld (1998) refers to 'mental constructs', one can understand by that the individual statements, suppositions, commitment and ideologies, but also attitudes, stances, comprehensive episodical knowledge, rumors, perceptions and finally even mental picture. *It is essential that they can be allowed sufficient stability.* We will refer to the multitude of these mental associations as the **range R_O of a belief related to the object O**. We remark that Rodd (1997) differentiates between epistemic and attitudinal beliefs. Regarded mathematically, we associate not only a classical set to the belief object O, but indeed a fuzzy set R_O, i.e. for the elements of this set of mental constructs we allow various degrees of membership and so R_O turns into a fuzzy set (Zimmermann, 1990). In other words, we assign a **membership degree $\mu(x)$** $\in [0,1]$ to each element x within the range of beliefs. This approach takes into account the fact that beliefs can be held *with varying degrees of certitude*. On the other hand, *activation levels of a belief* can also be modeled using the membership degree. To insure completeness, it is often remarked that beliefs in differing contexts have differing strengths. To determine the underlying influencing variables (when, why, how much etc.), however, is a central question of research.

When we likewise accept pictures or perception as mental associations, we also make possibly the integration of the known term formation 'concept image' into our terminology. Unfortunately, it has far too rarely been noticed that Tall & Vinner's (1981) concept of concept images contains constitutive elements of belief definitions. One notes, *"We shall use the term concept image to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over years through experiences of all kinds, changing as the individual meets new stimuli and matures."* or even more explicit, *"the visual representations, mental pictures, the impressions, and the experiences associated with the concept name"* (Vinner, 1991). In a rough approximation, the so-called 'concept definition' (using Vinner's terminology) plays the role of a belief object.

It is folklore and all various definitions in common that beliefs rely heavily on evaluative and affective components. For this reason we require as a further module one or more **evaluation map(s)** e_λ, defined on the range of a belief R_O and with a linguistic value scale (in fuzzy theory we speak of a 'linguistic variable' - see Zimmermann, 1990, p. 132). Possible values could be 'important' or 'minor', 'good' or 'bad' etc., however a continuous scale can also be assumed under other intentions.

There is no need to mention that every belief definition must take two basic variables into account, namely the person P who has professed the belief or to whom the belief is attributed. Finally beliefs are dependent on the time of constitution. By a **belief B** we understand a quadruple (O, R_O, μ, e_λ), whereby O is the debatable belief object, R_O represents the range of mental associations (what traditionally is called belief), μ models the activation levels or differing strengths of a belief and the evaluation map(s) is (are) represented by e_λ. Furthermore B should fulfill the following characteristics in a probabilistic sense:

(1) For each person $P' \neq P$ the range R_O of beliefs on the same belief object O is not necessarily consensual (nonconsensuality).

(2) Beliefs are likely to include a substantial amount of episodic material from either personal experience, from folklore or from propaganda which influences the evaluation map e_λ (episodic material and its evaluative impact).

(3) The range R_O of a belief is a priori not necessarily bounded (unboundedness).

(4) Beliefs are often anchored in authorities (external anchoring).

(5) Beliefs are directly or indirectly linked to the self-concept of the believer P at some level (self-linkage).

We stress: (1) Beliefs of different persons on the same beliefs objects are not necessarily consensual (nonconsensuality). Sementically, 'beliefs' as distinct from knowledge carries the connotation of disputability, and the believer is aware (or will become aware) that others may think differently. (2) It is known that e.g., especially knowledge systems are not necessarily dependent on episodical material and that the knowledge possibly carries a stamped date, which does not contridict the first point. (3) The 'openness' and 'unboundedness' apply to the amount R_O. This can be accounted for by the situation in which the process of the integration of episodic material can never be perceived as fully completed. (4) This condition can also be found in part in Abelson's (1979) work when he postulates that belief systems are in part concerned with the existence or nonexistence of certain conceptual entities. Here, these authorities might be virtual authorities in a platonic sense, might be teachers, colleagues, friends, parents etc. It should be mentioned that for transforming a belief into knowledge, the warrants of the beliefs are crucial (see Rodd, 1995). (5) This property is in some sense dual to (4): Abelson (1979) pointed out that knowledge systems usually exclude the Self, while beliefs do not.

We will forego a detailed definition of a **belief system**, but at any rate, several objects play a role in belief systems. The rational network of the objects in question then transfers itself onto the structure of beliefs resp. their ranges. Nespor also suggested that beliefs loosely bounded networks with highly variable and uncertain linkages to events, situations, and knowledge systems (Calderhead, 1996).

However, we are interessed in the question of to what extent sets of beliefs with respect to different belief objects are structured. The question of the structure of belief networks appears to us to be of greater importance. It can be assumed that via the internal network structures of beliefs enlightenment can be attained of the cognitive memory patterns and their links. At the same time, this should enable the localization of weaknesses in the acquisition of knowledge. In this sense, beliefs also have diagnostic characteristics and therefore understanding structures of belief networks is of central importance.

Possible Categorizations of Beliefs and Belief Systems

With reference to the above definiton of beliefs, we would like to present possibilities of structuring beliefs.

The personal parameter P as a variable - group-specific differentiation

Beliefs are often specified and then researched according to the various groups of subjects. Accordingly, various results are collected when surveying different groups (e.g., students, teachers, professors, etc.) about the problem field of beliefs on math. Building on this background, the central question arises as to what effect beliefs have on teaching and learning processes. Only a few isolated empirical-based results are available here, although there seems to be positive confirmation in that research.

Belief objects O as a variable - different belief dimensions

As previously mentioned concerning beliefs on mathematics (as a science, as a university subject, as a school subject, as an engineering discipline, etc.), the learning or teaching of mathematics as such also entails value judgements by the learner or the teacher and is thus in this sense directed introvertedly. A distinction according to these aspects leads to a preliminary categorization for terminology clarification. These specifications take the possible diversity of potential belief objects O into consideration. There are numerous indications that beliefs to single objects (e.g., mathematics) can hardly be discussed successfully when one ignores the relation to other objects (e.g., mathematics teaching). Thom´s quotation[1] (1973) which demonstrates in exemplary fashion that cross-links between the above-mentioned fields cannot be ignored is sufficiently well known.

The evaluation map e_λ - Green's dualistic categories

In his book *Activities of Teaching,* Green (1971) is also concerned with the question of which role beliefs play in the learning process. Alongside the obvious postulate that beliefs distinguish clusters, Green distinguishes beliefs according to two features. He refers to quasi-logical and quasi-psychological dimensions of beliefs and allocates them two polar states; in view of their quasi-logical character, beliefs can be, *primary* or, *derivative*. The quasi-psychological role can be either, *psychological primary* or alternatively be more *peripheral*. In view of the definition that we initially provided, Green differentiates with reference to possible evaluation maps e_λ. In one case, it deals with the quasi-logical scale with two possible values, namely primary or derivative. In the other case, the map e_λ measures quasi-psychological situations.

At a first glance this 2 x 2 typification appears quite convincing. However, it proves to be problematic and finally open-ended for the identification of beliefs. Only a few papers in the literature have previously offered convincing interpretations and contributions (cf. Cooney et al. 1998, Jones 1990), regarding which criteria should be correlated to each respective 'value'. An open question of research is the possible interaction patterns of the accordingly categorized beliefs.

Subject-Specific Structuring of Beliefs

At this point, we would like to return to the discussion on the differentiation of beliefs according to breadth of the belief objects O. In specialized literature, the word 'belief' is employed at times as a synonym for the terms 'philosophy' or 'ideology', in particular when a discussion focuses on general attitudes or beliefs, e.g., on mathematics as a discipline (McLeod, 1989). In view of the belief object, i.e. here of mathematics in general, we use the term *'global beliefs'*. This Top-Down-approach compliments a Bottom-Up-Analysis when referring to detailed aspects of mathematical objects. Analogous to the term subject-matter-knowledge used by Even (1993), we use the term *'subject-matter-beliefs'* which refers to the amount and organization of knowledge and beliefs per se in the mind of the subject (see also Lloyd & Wilson, 1998). However, any investigation of beliefs in the field of this subject matter will soon indicate that these two poles, namely global beliefs versus subject-matter-beliefs, are too distant to cover all aspects. We therefore propose the use of a middle 'intermediate level' which we give the term *'domain-specific beliefs'*.

Our research (Törner, 1999) shows that mathematical domains such as geometry, stochastics or calculus are always associated with specific beliefs. For example, in the case of calculus, beliefs represent views on the role of logic, application, exactness, calculation, etc. Similar dimensions are also relevant in other fields; however, there is the impression that subjective realizations differ. Domain-specific beliefs should be classed hierarchically higher than e.g. notions of the term 'derivative' or the term 'function', although on the whole they still touch on basic views on mathematics. Thus the following **research question** arises:

> *Which dependency or implication structure exists between global beliefs, domain-specific beliefs and subject-matter-beliefs? Do the sum of the beliefs from the individual fields of mathematics constitute beliefs on mathematics as a whole, or do general views tend to imprint subjective perceptions in the individual domains more?*

Sources of Information and Mode of Inquiry

In previous research, we asked six preservice upper-secondary-school teachers (in their post-graduate phase) to express their experiences with calculus lessons in the form of freely written essays. At the time of composing these essays, the students were still participants in a didactics of mathematics university course. Therefore, we had to rely on voluntary participation of the essayists and accept anonymous participation in completing the related questionnaire. The essay themes were "Calculus and me - how I experienced Calculus at school and university," "How I would have liked to have learned Calculus," and "How I would like to teach Calculus." We were subsequently presented with a total of 3 x 6 = 18 qualified, but partially anonymous statements (two to four pages each). These served as the basis of our survey in which we examined the beliefs imminent in the essays (comp. Törner, 1999). The individual results revealed the following main statements which can be allocated the status of a belief-characteristic: (1) Calculus is (reduced in school down to) calculating (not necessarily meaningfully) with functions. (2) Differential calculus is a craft - integral calculus is an art. (3) Logic is a central guideline for mathematics and in particular for calculus. (4) Exactness as a property of mathematics can be demonstrated in calculus in particular. (5) Calculus has the special task of preparing pupils for subsequent university courses. The next two statements with belief character address learning aspects. (6) Mathematical elegance and abstractness - a liking of mathematicians - mean a loss of descriptiveness and understandability. (7) The recognition of application links facilitates learning.

In the following, we assess statements (4) and (5) in the students' essays in view of a possible interrelation with general views on mathematics, whereby we must limit ourselves to the aforementioned essays as subsequent research was restricted by the partly anonymous nature of the essays.

Results

For conciseness, only a few aspects of the evaluation are listed here as an exemplary discussion. Lars made the most prominent statement on the aspects of logic in its relation to calculus. It is remarkable that his global view on mathematics is structurally dominated. We quote some excerpts: ... *'logical material' can easily be worked with ... when you have acquired the rules (e.g., the transformation of fractions into decimal numbers).* According to Lars, calculus has a similar pattern, *as ... mathematical-logical thought was developed and deepened here ...* The university seminar he visited on this strengthened his belief: ... *This began in calculus with the foundations of logic (which I found to be very helpful).* The consequence for him is a rigorous orientation to the aspects of logic: ... *if it were possible to do something on logic in school as early sixth grade* (with the eleven to twelve-year-olds). Without going into details, the beliefs on logic from Lars can be psychologically evaluated as central as well as primary.

Another student (Sascha) also underlines the central role of logic in calculus lessons. His view of mathematics was indirectly influenced by his assessment of lessons at secondary school in Germany in the mathematics courses in the Oberstufe, and it is his opinion that *the schools should pay greater attention to the demands of the mathematics students to make studying the topics later at university possible, even attractive (in the calculus course) with the aid of e.g., formal logic....*

Concerning assessment of exactness as an important feature of mathematics, which is experienced particularly, well in calculus, two further students, Nicolas and Lars, state their positions. Whereas Nicolas views exactness as an unavoidable difficulty, which can be didactically mastered, Lars views the aspect of exactness more fundamentally. Mathematics demands in his words ... *utmost precision and a lot of effort..., therefore one should start operating with exact terms as soon as possible. Calculus is suitable for this pursuit. ... For example the* ε-δ-*definition for constancy can be considered one of the greatest achievements in the cultural history of mathematics....*

Interpretation of results

The students' quotations show that domain-specific beliefs cannot be dislocated principally from global views on mathematics. A number of obvious conclusions can be made to this effect. Our mathematics lessons (and partially our university courses) do not necessarily induce a pluralistic worldview of mathematics. There are a number of reasons for this: mathematics is often taught in modules and for this reason often perceived as such. Also, from a learning psychology viewpoint, the perception of unity is more dominant than perception of broad variation. Thus, global beliefs are oriented towards a more structural-axiomatic organization of mathematics, which in turn leads to aspects of logic being allocated a central role. In this sense, a perceptive student can experience a reinforcement of his or her assessment due to the content and the methodology of the university calculus course. Under the "axiom" that school mathematics classes are a preparation for the university, school lessons are also viewed one-sidedly.

There is an impression that Perry's stages theory presented by Ernest (1991) in another context offers a possible explanation to understanding the strict dependency in Lars' beliefs: they can be understood as a dualism. From the author's viewpoint, there is evidence here of a multifaceted, pluralistic working with and understanding of mathematics. Central mathematization patterns have to balance scales with the multifaceted nature of mathematical phenomena and have to enrich each other in their interdependent nature. This ideal state could then be described in the wording of Perry as 'relativism'.

Note

1. In fact, whether one wishes it or not, all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics.

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