MENTAL REPRESENTATIONS – THE INTERRELATIONSHIP OF SUBJECT MATTER KNOWLEDGE AND PEDAGOGICAL CONTENT KNOWLEDGE – THE CASE OF EXPONENTIAL FUNCTIONS

Günter Törner
Universität Duisburg
toerner@math.uni-duisburg.de

The exponential function ... it looks easier than what it is.

Introduction and the Research Field

Growth of mathematical understanding is the result of a comprehensive mental restructuring process. Thus observing and understanding growth first necessitates grasping the accompanying transformation process, which then implies taking stock of the respective initial and final states. Based on such analysis, one is capable of estimating and probably designing the results of possible learning supporting measures (or activities).

A central claim of the philosopher Wittgenstein is that thought is limited by language. Mathematics can be understood as elaborated language, whereby graphic elements are viewed as integral parts of this language. Development in this expert language is insofar relevant for mathematical thought and, vice versa, mode of use of this language presents us with insight into such thought. As Davis (1992) stressed, the 'study of mental representations' is a paramount didactical research task: "... *One knows very little about how someone thinks if one has no knowledge at all of his or her mental representations...*" In light of the above-mentioned background this thought is of central importance. It is beyond dispute that not too many results are available in the research literature, especially for topics of higher mathematics. Of course, mental representations can only be opened indirectly on the macro level. Inasmuch, even the adequate methodology of data gathering is a research topic of its own. Whereas subject matter knowledge of the persons in question can be observed through tests, such a data gathering can only provide limited results on the underlying representations. But how should we observe them?

Mental representations of mathematical objects are part of the comprehensive (student) conceptions on these objects. We distinguish between a *microstructure* and a *macrostructure* of mental representations. The papers in Confrey (1991) represent a comprehensive research contribution towards possible (micro-) mental representations of exponential functions; we however are primarily interested in researching the *macro-view* of the subject-specific network structures and their links to the outside. Besides criticism occasionally expressed here, we employ on the whole Pines' (1985, p. 101) idea of conceptual structure, namely that ... *the emphasis here is not on the ele-*

ments, although they are important to a structure, but on the way those elements are bound together.

Research Hypotheses and Research Questions

Our research is based on several initial hypotheses:

(A) The interrelationship of knowledge and beliefs. Mental representations are not primarily tied to 'hard' knowledge, but have substantial reference to 'soft' information (see e.g., Confrey, 1990). In this context, the open question of the interweaving of knowledge and beliefs becomes more important. . Our dualistic *knowledge – belief* view is probably reflected in the opposition of *formal* versus *informal knowledge.* Once again, it becomes evident that every separation of knowledge components from belief components must be considered artificial. Ernest (1989) considers the belief variable *'why what we do what we do'* especially significant for the quality of any mathematics education program. When we accept Schoenfeld's (1998) definition of 'beliefs' as '*mental constructs* that represent the codification of people's experiences and understandings', the correlation to mental representations is obvious.

If one asks for models of individual mental representations, then the guiding metaphor that serves our purpose well seems to us to be *a mathematical graph.* Both *vertices* and *edges* can represent subject-matter knowledge as well as corresponding beliefs. However, we do not believe that we are able to describe the 'mental landscape' by this alone. We cannot ignore that mathematical contents are perceived by prospective teachers on the background of (still virtual) classroom teaching situations.

(B) Subject-matter knowledge and pedagogical content knowledge. According to Shulman (1986), one differentiates between subject-matter content knowledge, pedagogical content knowledge and curricular knowledge. This approach has been scrutinized and improved by additional researchers (see in particular Cooney, 1994 and Bromme, 1994). The concept of pedagogical content knowledge has also been questioned by some researchers who have suggested that it is not a discrete category of knowledge at all, but inextricable from content knowledge itself. We consider it important to understand which role pedagogical content knowledge plays and, referring to the above-mentioned metaphor, it can be a vertice as well as an edge.

Pedagogical content knowledge is in some sense 'knowledge about', and there are many arguments (see (A)) for also considering again the respective beliefs, thus including *pedagogical content beliefs* from the beginning. One notes that mathematical subject-matter knowledge about exponential functions has a multiple-directed structure: What is a definition? What is the consequence? What is a sentence? What is a corollary? What is an application? Pedagogical content knowledge however can draw together various aspects, because they are equivalent. The proposition of a theorem can also serve as a definition, through which the previous definition receives the character of a theorem itself.

Following the categorization in Törner (2000), subject-specific information can be distinguished according to its *content extension* (C); the levels to be given here are insofar *global*, *domain specific* and *content-related.*

Thus the theoretical framework of our *research question* has been mapped out in outlines, namely registering possible mental representations of mathematical objects and, when possible, describing phenomena pointing to areas beyond the singular mathematical object.

Methodology and the Interview Participants

As the mathematics objects under consideration, the topic of 'exponential functions' was selected. This topic is first-semester subject matter, namely in the first-year-calculus lecture, and in other introductory courses, e.g., linear algebra and numerical analysis, exponential functions are dealt with again from different perspectives. In addition, exponential functions are taught in German school mathematics curriculum. The theme is 'exceedingly rich' (Confrey, 1995) ('ideal critical research site') and features a high degree of networking possibilities to other mathematical themes.

The information collection was the result of phenomenological and qualitative research. For this, open interviews were chosen. Interviews were recorded on video and later, result protocols were sent to the interviewees for comment.

Illuminating for the behavior of teachers for Schoenfeld (1998) are *'teaching-in-context'* situations in which in particular original influence factors for teaching mathematics arise. We were therefore interested in a comparable 'replying-in-context' situation, whereby we deliberately kept the interviewees uninformed about the intended interview. It was therefore not possible for the students to prepare themselves for the interview. To prevent the impression of an examination situation however, we pointed out the advantages of experiencing such a situation as a test without having to fear adverse consequences.

The persons who had volunteered to be interviewees were six prospective teacher students of the upper secondary grade level who were already in their third-year-course and had passed an intermediate examination after their fourth semester. It therefore can be assumed confidently that they will reach their teaching qualifications within three additional semesters.

Data Gathering

The six interviews were held in 2000. The participating students were presented the scope of the actual research beforehand in order to relieve especially the 45 minute conversation of the character of an exam. The contents of questions intended to initiate conversation and induce narrative reports were of the following topics:

- Where were you confronted with exponential functions for the first time? (type of introduction)

- In what way were exponential functions defined? At school? At university? What do you think of e^x?
- What do you understand by e? Which decimals of e are you familiar with?
- Name some defining properties of exponential functions (characteristics)!
- Compare the treatment of this theme in school and at university.
- Report on the inner-mathematical and outer-mathematical importance of exponential functions.
- Which historical aspects of exponential functions are you aware of?
- Are there information gaps about exponential functions that would like to fill in?

Along these technical guidelines intended to illuminate the network around the e-functions, prompting questions were asked for that were intended to provide spontaneous responses. We believe that students' conceptions can be confirmed only through methods that encourage students to be expressive and predictive:

- To what extent was each respective learning phase demonstrative from an affective perspective resp. to what extent are emotionally-charged memories noticeable in hindsight?
- What is each respective assessment based on? (through teaching personal, independent confrontation)
- Which evaluation motive in combination with the meaning of exponential functions are significant for them, and which central? Usefulness of exponential functions? Beauty of the relevant mathematical features? Mathematical centrality? High complexity?
- Did they experience the process of knowledge acquisition organically?

Following the reference in Calderhead (1996) that teachers' knowledge may be better represented in terms of metaphors and images, leeway for metaphoric assessment was provided. Thus we asked the following question:

- If the different functional classes were comparable with animals in a zoo, which species would be assigned to exponential functions?

Results

For the sake for brevity, only a few results can be referred to and summarized. For the same reason we abstain from relating the results to the 6 individual interview partners. The classification of the results into the terminology of concept images from Tall and Vinner (1981) can also not be responded to here. According to the above mentioned steps (A) and (B), we exemplary refer to the individual observed results; to prevent repetitions of the observations we categorize the comments according to content criteria whereby we limit ourselves to just a few positions.

The Euler number e. In particular the categories (A) and (B) are very strongly interwoven. Alone the decimal number e = 2.71 82 82... is known by all the students up to the numeral before the decimal point, two further students name two further numerals after the point. It is commented that there must be further numerals behind the point, one of the students gets tangled up in the infinity of decimal quotient development with the transcendental properties of e, which points to deficiencies in the understanding of real numbers. Vaguely represented is also information about equivalent mathematical definitions of the number e. The importance of this number is not doubted as such, nevertheless this circumstance is only indirectly reached: "... *Euler must have discovered this, he is famous...* ", without stating any precise details as to his historic achievement or as to his personal biography: "... *Don't ask me for dates*."

What is further emphasized below is the interwoven nature of factually correct information by the students and quite reservedly expressed imprecise knowledge featuring on the whole belief character. Concerning pedagogical content knowledge, the statement of only one student strikes the core of this issue: "... *the Euler number is not really defined, it is constructed, it is a kind of natural constant in mathematics.*"

Definition of the exponential function to the basis e from a mathematical view. The graphic representation of the e-function does not present a problem to any of the students. However, three students have to correct their statements after the interviewer made objecting comments referring to the graph of the general exponential function a^x, with respect to the variance of the basis a, and to some characteristic function points. Only one of the students pointed out that the function e^x is qualitatively something different than 2^x. The response of the interviewer as to the function value at a prominent argument, e.g., the circle constant Pi, is answered by all the students as question easily solved by a (not available) pocket computer (if only it *were* available!). It does not appear to the interviewees to be particularly exciting that in-depth and historically relevant mathematical definition processes (the fundamental research aspect) underlie this (e.g., in the case of e^{Pi} for the basis e as well as for the exponent one is dealing with infinite decimal series).

Representation of the exponential function. The observations make clear that the students uncritically associate the e-function with a monotone increasing graph *(iconic representation)*. The answers become uncommiting when it comes to stating characteristic parameters for the specific growth behavior ('exponential growth')! This representation is insofar of only limited evaluation reliability (for solving mathematical tasks for students).

Not one student articulates that exponential growth on the one side stands in contrast to exponential convergence towards zero on the other side.

All students share the reassuring view that there exists a tool, namely a pocket calculator, which represents the function *(enactive representation)*. This rather comfortable representation entails the danger of an almost negligent simplification of viewing the exponential function e^x in a naive sense as an exponential function with simply a slightly more complicated basis. It is less this naive representation as such which has to be categorized as reserved but moreover the uncritical, mathematically insensitive use of such a reductionistic view.

The observations feature another form of representation of the exponential function: the *symbolic representation* via $f' = f$. A students states: "*... This occurs only once...*" All test persons (two candidates however only after prompting) recalled the property of derivation $f' = f$. None independently noted that this differential equation could possibly characterize the e-function axiomatically, thus providing a sensible symbolic representation.

The mathematical observer is however sobered when viewing very subjective reasons associated with this property. It is the *mnemotechnical simple structure of the differential equation*, which remains in the mind: "*... I will be able to tell this to my grandchildren in fifty years...*" Thus, it is not the mathematical centrality, the beauty of the presentation or similar assessment criteria, which is predominant. "*...You simply cannot make a mistake when differentiating...*", was one comment.

The mentioned three representations of the test persons are only weakly interlocked. Only two students explain the relation between the central differential equation property $f' = f$ and the limit value definition of the Euler number e.

The equivalent approaches usually dealt with in the academic study of mathematics on the series definition or the limit value definition, possibly through an axiomatic characteristic, do not play a primary role in the reports. It is only of secondary importance to the students that mathematics makes possible various different terminological approaches for the definition of its objects.

Pedagogical content knowledge. From the author's perspective, an integral role is played by the pedagogical content knowledge as well as the pedagogical content beliefs. These pedagogical content beliefs have a networking function, they represent orientation information extending beyond the mathematical context. In particular *the relevance aspect of exponential functions* is to be mentioned here. It appears that in the presentation of mathematical contexts on the exponential functions, this aspect is only partially taken into account. Only one test person, who had however enrolled for physics as a second major proved an incomparable comprehensive and highly networked pedagogical content knowledge concerning the relevance of exponential functions.

On the relevance of the exponential function. One hardly needs to present any evidence here on the importance of the exponential function for other fields within mathematics in the narrow sense (inner-mathematical aspect) and also for other sciences drawing heavily on mathematics (outer-mathematical aspect). Inner-mathematical centrality is hardly mentioned in the statements. Due to the fact that the exponential function in school is "*... a completely different one...*" than the one in academic studies ("*... that was the e-function from school...*", we conclude that the integration has not yet been sufficiently consolidated yet. "... *As Euler, who achieved a lot in mathematics, developed the exponential function, it should be allocated a central role in mathematics"*.

All the participants initially emphasize the outer-mathematical relevance aspect, but further statements here remain often non-committing when it comes to exactly describing facts. This observation is based on the unconventional question of naming a zoo animal as a representative of the function class of all functions in the zoo.

By means of relating a zoo animal to an exponential function, information was intended to be elicited on the role of this function in comparison to other functions. The answers of the students are: "... *no small animals since the growth of the function is large... but, one could also imagine rabbits, which increase their population quickly! Isn't there the famous mathematical problem [Fibonacci problem]...* " or "... *maybe the human being, who exists everywhere...*" Another student proposed: "... *the lion!"* Then he hesitated: "... *But the lion does however not occur everywhere, only in Asia and Africa...*" Finally it was remarked: "... *the giraffe with its long steeply rising neck reminds me of graphs... or maybe also a bird which can rise above everything else...*"

It is common practice to speak of the so-called exponential growth, which links onto a differential equation condition. Ideas that alterations can be measured by derivations are not evident and point to a weakly developed basic understanding of this mathematical term. "...*The exponential function stands for exponential growth... what that exactly means I cannot say*..." The information presented here is almost wholly of belief character, e.g., when one refers to the statements of other persons or bases one own statement on the basis of another person: "...*Mr. S. [a university lecturer] reported once in an preparatory university mathematics course on growth of fish populations during the First World War... My fellow-student, who studies engineering, often employs the exponential function in the complex rabbit task of Fibonacci.*" Some narrations also have almost episodic character: "...*Our teacher always says: Look there comes an e-function again"*.

Emotional loadings of mathematical topics. Through the description of the interview topic at the beginning of the interview, a spontaneous situation comparable to that of a test was created. All participants confirmed that the information about the topic of the interview immediately caused emotional reactions within them which encompassed the entire spectrum, from 'dismay' to relief; the mean of such a qualitative distribution can more likely be accounted in the negative range.

The candidates all only had vague memories of the introduction of the subject 'exponential functions' in school curriculum. The visual image of the run of curves remained, whereas the term e^x primarily played the role of indicating a special function in their memories. Besides this iconic representation, the enactive representation of e^x as the name of a calculator button is often referred to, especially when asking about e^p. Two test persons mentioned the correlation to natural logarithms as the respective inverse function in reference with the symbolic representation. Here it cannot be overlooked that 'mathematical bridges' also emit lasting 'negative affective charges' (from the context of logarithms) onto the e-functions.

Conclusion

Overall the results the author was sobered by the interviewees' weak and modest subject matter knowledge of exponential functions. The same applies to detailed information: precise definitions, characterizing properties, importance for application, importance of the Euler number e, correct sketching of the graph a^x dependent on the basis a, the asymptotical behavior at infinity, intersection with the y-axis, etc.

When the fact that beliefs are held with varying degrees of conviction is deemed constitutive, the information that was hesitantly presented by the test persons has overtly belief character. Additional questioning tended to increase uncertainty and it is often left unanswered how possible doubts could be cleared up in a mathematically uncritical fashion. Very often, it is related to authority figures ("*...we learned that in our course...*"; "*... a student told me...*"). This indicates that the knowledge is from unreliable non-academic sources or justified by reference to learning phases in which one vaguely experienced the contents.

The experiment exposed again that the difference between knowledge and beliefs is rather theoretical, in analysis, dividing lines can hardly be drawn, even the interviewees themselves did not articulate differences between provable, theoretically present mathematical facts and approximate, imprecise memories. Gaps registered as deficits in the eye of the observer were accepted as natural by the test persons and not considered disturbing, even the fuzzy character of some information did not cause uncertainty. Inasmuch a discussion about subject matter knowledge always includes inventory taking of subject matter beliefs.

Metaphorically speaking, the experiment however provides evidence that the knots in a cognitive knowledge network carry emotional charge in which the previous experience of knowledge acquisition are stored. Cognitively neighboring elements appear to radiate onto one another with respect to their affective charges. One student interestingly reported of the e-function being burdened by the even more complicated logarithm function: *"... this pair are inseparably linked..."*

When teaching mathematics (in a beginners lecture), a generally understood 'pedagogical context knowledge' apparently is normally only allocated to a minimal role. Mathematically constituted networks which are substantial for experts (e.g., test developers) are only taught weakly. Networks receive supplementary stability only through interdisplinary work or through a specific changes of perspective. Students having multi-modal (mathematically equivalent) representations of the mathematical object 'exponential function' is the exception to the rule. The action character of knowledge (enactive representation) is often dominant. Possibly the deficits in subject matter knowledge are not only covered by an insufficient pedagogical content knowledge, but even favored because respective gaps in the metaknowledge do not result in some pressure in further questioning.

References

Bromme, R. (1994). Beyond subject matter: A psychological topology of teachers' professional knowledge. In R. Biehler, R. W. Scholz, R. Strasser, & B. Winkelmann (Eds.), *Didactics of mathematics as a scientific discipline* (pp. 73 - 88). Dordrecht: Kluwer.

Calderhead, J. (1996). Teachers: Beliefs and knowledge. In D. C. Berliner & R. Calfee, (Eds.), *Handbook of educational psychology* (pp. 709-725). New York: Simon & Schuster, Macmillan.

Confrey, J. (1990). A review of the research on student conceptions in mathematics, science and programming. In C. B. Cazden (Ed.), *Review of research in education* (Vol. 16, pp. 3-55). Washington DC: American Educational Research Association.

Confrey, J. (1991). The concept of exponential functions: A student's perspective. In L. Steffe (Ed.), *Epistemological foundations of mathematical experience* (pp. 124-159). New York: Springer.

Confrey, J., & Smith, E. (1994). Exponential functions, rates of changes, and the multiplicative unit. *Educational Studies in Mathematics, 26*, 135-164.

Cooney, T. (1994). Research and teacher education: In search of common ground. *Journal for Research in Mathematics Education, 25*, 608-636.

Davis, R. B. (1992). Reflections on where mathematics education now stands and on where it may be going. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 724-734). New York: Macmillan.

Ernest, P. (1989). The impact of beliefs on the teaching of mathematics. In C. Keitel, P. Damerow, A. Bishop, & P. Gerdes (Eds.), *Mathematics, education and society* (pp. 99-101). Paris: UNESCO Science and Technology Education Document Series No 35.

Ernest, P. (1999). Forms of knowledge in mathematics and mathematics education: Philosophical and rhetorical perspectives. *Educational Studies in Mathematics, 38*, 67-83.

Even, R., & Tirosh, D. (1995). Knowing that and knowing why: Subject-matter knowledge and knowledge about students as sources of mathematics pedagogical knowledge. *Educational Studies in Mathematics, 29* (1), 1-20.

Fennema, E., & Franke, M. L. (1992). Teachers' knowledge and its impact. In D. A. Grouws (Ed.), *Handbook of research on mathematics learning and teaching* (pp. 147-164). New York: Macmillan.

Pines, A. L. (Ed.). (1985). Towards a taxonomy of conceptual relations. In L. West & A. L. Pines (Eds.), *Cognitive structure and conceptual change* (pp. 101-116). New York: Academic Press.

Schoenfeld, A. (1998). Toward a theory of teaching-in-context. *Issues in Education,* 4 (1), 1-94.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15* (2), 4-14.

Tall, D., & Vinner, S. (1981). Concept images and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*, 151-169.

Thompson, A. G. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D.A. Grouws (Ed.), *Handbook of research on mathematics learning and teaching* (pp. 127-146). New York: Macmillan.

Törner, G. (2000). Structuring mathematical belief structures – Some theoretical considerations on beliefs, some research questions and some phenomenological observations. In M. L. Fernández, (Ed.), *Proceedings of the Twenty-Second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 499-508). Columbus, OH: ERIC Clearinghouse for Science, Mathematics and Environmental Education.