Mathematical Beliefs and Their Impact on Mathematics Teaching and Learning of Mathematics[1]

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I hate mathematics, since there is always only one solution (John, Grade 3)

1. Introduction to the theme

Let's try something. Imagine that you must complete a psychological test, namely:

Continue the following sequence of numbers: 0, 1, 2, 1, 2, 3, ...

The question may at first appear somewhat trivial, and we are certain to believe that psychology would expect nothing less that we enter the number 2 to complete the sequence. Any other solution would probably lessen our I.Q. in the eyes of the psychologist.

As mathematicians, we should be slightly angered by such tests, because we know that we can continue the number sequence arbitrarily. From a mathematical point of view, the question above involves the problem of extrapolation. What we are looking for is

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a real-valued function f: R → R which is known at the positions 0,...,5, namely f(0) = 0, f(1) = 1, f(2) = 2 etc.

When we only concern ourselves with polynomial solutions, we know that we can prescribe any value for the function at position 6. This amounts to a determination of a polynomial p of grade 6. We have to solve an inhomogeneous system of equations $Ax = b$ with seven unknowns which are coefficients of the polynomial p. The kernel of the system of linear equations consists of the polynomials vanishing at the points 0, 1, 2, 1, 2, 3.

What is there to learn from such a trivial example?

(1) Where psychologists expect one answer, the mathematician has a more sophisticated understanding of the same question. For him there is more than one straightforward answer.

(2) On the other hand, nevertheless each mathematician would be aware of the expectations of the psychologists in the test and he normally would offer the 'canonical' solution in such a context without any comments.

These circumstances have a lot to tell us about mathematics and mathematics education. If mathematicians and psychologists view a problem differently, the question arises to which extent mathematicians, didacticians and mathematics teachers consent with one another on mathematical problems, mathematical objects and mathematics as a whole.

Obviously, one could take a trip back through the history of mathematics and would find many situations in which opinions on mathematical problems and, thus, their views on mathematics have been controversely discussed by mathematicians. For example, the

answer of Gauss to the famous question whether parallel lines can intersect, namely the invention of euclidean geometries seems to him not to be an adequate or acceptable solution for many contemporaries. As he noted once, he was afraid of the shouting of his colleagues.

One might think that in our context of school mathematics, the differences are not so big and will not affect the discussion of the curriculum. However, it was DIONNE (1984) who showed that differences with respect to perceptions of teaching mathematics can be convincingly and impressively illuminated. She asked teachers to distribute 30 points among the constituting aspects T = Toolbox, S = System, and P = Process whereby these aspects are defined in the following sense:

T Mathematics seen as a set of skills *(traditional perception)*

Doing mathematics is
- doing calculations
- using rules
- using procedures
- using formulas

S Mathematics seen as logic and rigour *(formalist perception)*

Doing mathematics is
- writing rigorous proofs
- using a precise and rigorous language
- using unifying concepts

P Mathematics seen a constructive process *(constructivist perception)*

Doing mathematics is

- developing thinking processes
- building rules and formulas from experiences on reality
- finding relations between different notions

Although the procedure seems to be a very rough method, it is easy to convince oneself that there is a broad spectrum of perceptions of mathematics. The following dates, which the author raised during a continuing education course, may support this idea:

	T	S	P
Teachers at Comprehens. Schools (1995) (N = 19)	10,76	11,16	8,08
Teachers at Gymnasium (1994) (N = 14)	12,79	10,07	7,14
Students at University (N = 15)	6,40	11,80	11,80

In a forthcoming paper (PEHKONEN & TÖRNER 1997) we approach the measuring process on a quite different way and let the teachers (D, H1, H2, J, K, L) mark their self-estimations on real resp. ideal teaching within an equilateral triangle (arrows pointing to ideal teaching, figure 1). Again it is obvious that the teachers in question have a quite different estimation of the relevance of constituting factors of mathematics.

It was often claimed that such facts have consequences for learning and teaching mathematics. So did the mathematician Reuben Hersh (see the foreword of the survey article of THOMPSON 1992):

> *One's conceptions of what mathematics is affects one's conception of how it should be presented. One's manner of presenting it is an indication of what one believes to be most essential in it ... The issue, then, is not, What is the best way to teach? but, What is mathematics really all about?*

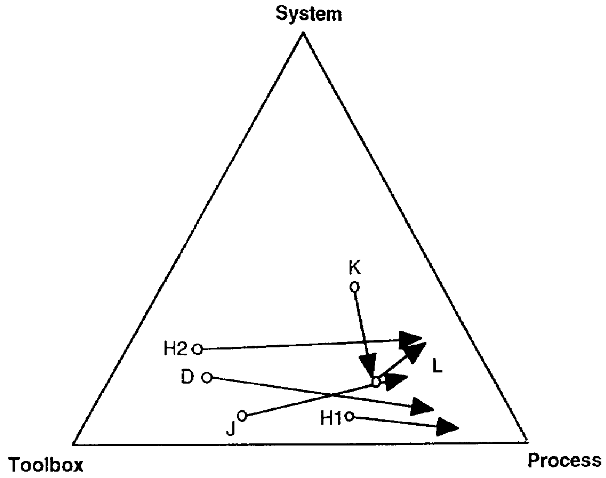

Figure 1.

Ernest (1989) expressed it in a slightly different way:

> *"I believe that the quality of any mathematics education program is a function of three variables:*
>
> - *what we teach;*
> - *how we teach it;*
> - *why we do what we do*
>
> *It is the first two of these that have been the focus of the recent change initiatives. The third, why we do what we do, is only just beginning to be recognized as an important component affecting change. Why we do what we do in classrooms is very much dependent on what we know and believe about mathematics, about the teaching and learning of mathematics, and about the nature of our particular task as mathematics educators. This very ideosyncratic bag of knowledge and beliefs can be referred to as a our personal theory of mathematics education, a term derived from George Kelly's Personal Construct Theory..."*

2. Definitions of Beliefs

The explanations have made clear that one can see mathematics from different points of view and that they contain a subjective framework whose meaning is not to be underestimated. In pedagogy the term *subjective theories* is used, reflecting that the theory contexts in which the single individual presents a non-neglectable variable. Subjective theories are also meaningful to the theory of learning and teaching mathematics. Here were speak of beliefs which seem to play a major role; they are being researched in depth especially in the American literature (see the articles of THOMPSON 1992, PEHKONEN & TÖRNER 1996 as well as the database on the Internet).

To this BEN-PERETZ & AL. (1986, 1) say:

> *It has become just as evident in recent years that teacher education focussing on the behavioural aspect while neglecting the teachers' knowledge and views will fail, as has been showed by the development of microteaching towards an ever more cognitive education of teachers to be.*

Interest in beliefs and belief systems has come mainly from cognitive psychology, whereas most of the interest in attitudes and affect has come from social psychology. The differences among the varying definitions of beliefs find their roots namely in how far beliefs are not only considered to be cognitive constructions, but rather to which extent they contain emotional dimensions. However, we will not start here a detailed discussion of beliefs and attitudes and their connections. For the following it is sufficient to assume that on one hand beliefs are subjective cognitive constructs

which are of some acceptance in the community. On the other hand they may be loaded with some emotions and which may be ultimately associated with some behavioural dispositions.

As it will turn out by the following considerations it is not easy to distinguish in general between beliefs and belief systems. It has often observed that beliefs don't appear isolated, but are often grouped together like spaghettis. Thus it may be helpful to separate beliefs as far as possible; however, one should not expect that there are 'atomic' beliefs. Nevertheless, it is worthwhile to describe the relevant beliefs under discussion as detailed as possible.

On the cognitive level we can assume that the subjective knowledge of mathematics and teaching mathematics comprises ideas in several different categories:

(1) Beliefs about mathematics
(2) Beliefs about learning mathematics
(3) Beliefs about teaching mathematics
(4) Beliefs about ourselves as practitioners of mathematics (self-concept as a mathematics practitioner: a self-evaluation of one's abilities and causal attribution to individual success and failure)

At the same time, the category, "(1) beliefs about mathematics," comprises a wide spectrum of beliefs which, at least, includes the following components: (1a) beliefs about the nature of mathematics as such, (1b) the subject of mathematics (as taught in school or at the university), (1c) beliefs about the nature of mathematical tasks and problems, (1d) beliefs about the origin of mathematical knowledge and (1e) beliefs about the relationship between mathematics and empiricism (in particular about the applicability and utility of mathematics).

The cognitive component of beliefs can comprise a wide spectrum of single, integral parts, *emotions* and *evaluations*, which are connected with beliefs as well as behavioural dispositions and intentions induced by them and may be very complex. There are easily understood affections associated with each component (1) through (4) as well as (1a) to (1e) and so on.

Therefore, in contrast to mathematics regarded as a world of gaining experience and acting there is a "world" of attitudes which we will characterize as a "mathematical world view". A "mathematical world view", as defined above, is a system of attitudes towards (integral parts of) mathematics. It is a hypothetical construction which, concerning attitudes towards mathematics, is yet to be proven and, therefore, of no empirical, but rather of heuristic value.

In the following we are talking about the three different sides of beliefs which include the cognitive side (C-side), the emotional side (A-side) and the behaviouristic side (B-side).

It was not coincidental that beliefs in the United States turned out a new field of educational research in the 80s. It was the time in which greater attention was dedicated to problem solving and recognized that the purely cognitive components of the curriculum frameworks for the analysis of mathematical behaviour did a poor job of predicting problem-solving processes of students (cf. HART 1989).

3. Some Widespread Beliefs

In the last section we mentioned the importance of beliefs. However, beliefs are nearly everywhere in daily life. They can be described as a compound of cognitive elements, e.g. uncertified knowledge, folklore, public opinions, privately drawn conclusions and uncautiouslessly drawn generalizations. Moreover, there are associated with emotional components loaded with fear or delight and finally, some beliefs are also loaded with behavioural dispositions. There are at least five functions:

- beliefs help to organize and structure the complex variety around us filling holes in a cognitive net
- beliefs help to adapt information and accommodate new facts to old knowledge
- beliefs help to orientate in a otherwise chaotic environment
- beliefs fulfill a self-assertion function; they protect the feelings of self-esteem by rejecting or ignoring unpleasant truths
- beliefs fulfill some self-portrayal function; they serve as an avenue in which one can express his/her own convictions.

However, the question whether or not the present beliefs are to our advantage or disadvantage when accepting our environment is a decisive one. Metaphorically speaking, beliefs can help us to find a path which is not marked on our map, but they can also make us blind when facing facts which should not be overlooked.

Next we will show that some of the beliefs are rooted in beliefs on the nature of mathematics. Many teachers are of the opinion that the field of mathematics is an objective science and that they accept certain positions as truths not being aware that these convictions can only be estimated as beliefs. Thus, we will take a closer examination on beliefs in the field of mathematics teaching and learning which can be derived from mathematical beliefs.

One of the fundamental questions in mathematics is the problem whether mathematics is an objective science (see GOODMAN 1979). If one proceeds further into the philosophy of mathematics, he or she must make note of the fact that this apparent objectivity is clouded by question marks upon closer examination. This claim is not made by those ignorant of mathematics, but rather by highly respected mathematicians. I would like to quote BERTRAND RUSSEL (see the article of GOODMAN):

> *I wanted certainty in the kind of way in which people want religious faith. I thought that certainty is more likely to be found in mathematics than elsewhere. But I discovered that many mathematical demonstrations, which my teachers expected me to accept, were full of fallacies, and that, if certainty were indeed discoverable in mathematics, it would be in a new field of mathematics, with more solid foundations than that had hitherto been thought secure. But as the work proceeded, I was continually reminded of the fable about the elephant and the tortoise. Having constructed an elephant tottering, and proceeded to construct a tortoise to the elephant from falling. But the tortoise was no more secure than the elephant, and after some twenty years of very arduous toil, I came to the conclusion that there was nothing more that I could do in the way of making mathematical knowledge indubitable.*

It was PHILIP J. DAVIS (1972) who wrote an article entitled 'Fidelity in Mathematical Discourse: Is One and One Really Two?' starting his article with the following conclusion:

> *The twentieth century has not yet delineated definitively the working principles and the broad articles of faith of what has come to be called 'Platonic mathematics'. Among these principles might be listed:*
>
> 1. *The belief in the existence of certain ideal mathematical entities such as the real number system.*
> 2. *The belief in certain modes of deduction.*
> 3. *The belief that if a mathematical statement makes sense, then it can be proven true or false.*
> 4. *The belief that fundamentally, mathematics exists apart from the human beings that do mathematics. Pi is in the sky.*

In the following we would like to show how these beliefs regarding the nature of mathematics form the classroom structure more or less in a hidden fashion. On the one hand, widely expanded beliefs, which are only partly mentioned in the literature, are shown to be consequences while other positions, on the other hand, are clearly only half-heatedly represented in the classroom so that an ambiguous impression arises. The reality of the classroom and theory postulate are separated from one another and are unbridgeable. In some places the reader may attest that our explanations sometimes ring a little ironic. In light of this, let us begin with belief 4.

Mathematics is regarded as an objective reality which simultaneously exists in heaven. Mathematics is in this sense somewhat divine. It is more human to be disturbing and involves the inherent danger of making mistakes.

Belief 4: ...*The belief that fundamentally, mathematics exists apart from the human beings that do mathematics. Pi is in the sky.*

Such a view of mathematics makes the following positions clear:

(4.1) (C) "Only geniuses are capable of discovering or creating mathematics." (SCHOENFELD 1985)

(4.2) (C) "Mathematics is created only by very prodigious and creative people; other people just try to learn what is handed down." (GAROFALO 1989)

whereby a certain margin of freedom associated with formation is represented in 4.2 regarding the few human 'priests', namely the ones who are extremely talented in mathematics. On the other hand, the following statement also seems plausible:

(4.3) "One cannot actually invent mathematics. He or she can only uncover mathematics; he or she can only discover mathematics."

Because mathematics is only accessible to the privileged few, the normal student is unable to get along without assistance and is, in a sense, dependent upon such assistance:

(4.4) "One can not engage in doing mathematics by him/herself. One requires instruction."

(4.5) "Dependence on the teacher is important in mathematics. However, if the teacher cannot 'clarify', there is little chance that the student will understand mathematics."

Can one be angry with the many students for not being able to see the reason to allow themselves to proceed with the material further in depth?

GAROFALO: *The students who hold this belief think that they can never be more than copiers or reproducers of other people's mathematics. They cannot imagine doing or producing mathematics on their own.*

It is not surprising that such a situation can be labelled as unjust:

(4.6) "Mathematics is unjust, because some students are privileged, while most however are at a disadvantage."

Eventually it must be registered under the premise set in (4) that, just like in other areas, the heavenly 'sanctity' of mathematics could be easily impaired and disadvantaged by human interaction. Therefore:

(4.7) "Wherever humans are active in mathematics, caution is advised."

Finally, the utilization of the computer in such cases as proving ideas must not be recognized as proper. It must perhaps even appear outrageous:

(4.8) "A computer can only play the role of a helping tool. A proof with a computer is no real proof."

After all, statement (4.2) has received world-wide agreement from the results of the teacher questionnaire at TIMSS! The items make perfectly clear that there exists only a short distance between a mathematical belief and its consequences pertaining to learning and teaching mathematics. The quoted beliefs also have differing characters: (4.7) just as (4.6) are highly emotional, while (4.8) and (4.3) reflect more cognitive assessments. Finally (4.4) and (4.5) address a behaviouristic component.

We shall devote ourselves to the third statement from DAVIS:

Belief 3: *"... that if a mathematical statement makes sense, then it can be proven true or false."*

The connections between belief 3 and the following statements are apparent:

(3.1) "The goal of doing mathematics is to obtain 'right answers'."

(3.2) "Deduction is of highest importance in mathematics. Thus structures and proofs play a major role in mathematics."

Because there exists only one truth but many untruths, we must establish in mathematics that:

(3.3) "A mathematical task/exercise has only one solution."

Taking statement (3.3), the student is consequently retained from providing a simple answer to this statement and encouraged

to paraphrase the result in one sentence in which the answer can be underlined. Why not! However, such a perception of the solving process also produces by-products. This overemphasis of the correct answer inevitably depreciates the actual solution process and appears only as a necessary evil in arriving at a conclusion and can be forgotten afterwards.

Eventually the student realizes: a little off the target and everything is wrong.

Why should someone be angry with John (3rd grade) if he says:

(3.4) "I hate mathematics, because there is always one solution."

or if statements from other students consequently appear related:

(3.5) "Don't try too hard since there is only one solution."

So that mathematics can remain practicable in the school, there lies only one way out for teachers in regards to curriculum and that is to grasp the understanding of mathematics:

(3.6) "Mathematics is computation." (FRANK 1988)

Eventually the observation that one should come to a conclusion as fast as possible is connected with these statements. Here, the following statements refer to the application of formulas:

(3.7) "Formulas are important, but their derivations are not." (GAROFALO 1989)

These formulas must be fixed in one's memory; therefore,

(3.8) "Mathematical thinking consists of being able to learn, remember, and apply facts, rules, formulas, and procedures." (GAROFALO 1989)

or as Brown & AL. (1988) state it,

(3.9) "Learning mathematics is mostly memorizing (yes: 50 %, no: 29 %)." (NAEP)

and when this strategy proves itself to be true, the student gains the impression and succumbs to the belief:

(3.10) "Almost all mathematics problems can be solved by direct application of the facts, rules, formulas, and procedures shown by the teacher or given in the textbook." (GAROFALO 1989)

or as Brown & AL. again say,

(3.11) "There is always a rule to follow in solving mathematics problems" (yes: 83 %, no: 8%).

We shall refrain from discussing the individual belief statements according to the primary weights of their aspects (affective, behavioural, cognitive). We shall, instead, turn to the second statement by DAVIS (1972):

Belief 2: *"... in certain modes of deduction."*

We had already presented a statement in (3.1) which conforms to this design:

(2.1) "Deduction is of highest importance in mathematics. Thus, structures and proofs play a major role in mathematics."

It is obvious that one should make use of logic; it is certainly worthy of mention that one should not allow him/herself to succumb to vivid misconceptions or conclusions.

(2.2) "Thoughts of plausibility are more dangerous because one can succumb to making errors."

We know deduction can be bone-breaking work, which a mathematician must carry out. Is there still room for the following motto in such a basic opinion which places more importance on the finished product rather than on the process or doing mathematics?

... *we try to explain rather than to deduce* (STRANG 1976)

One positively arrives at the opinion:

(2.3) "Mathematics differentiates itself from the other sciences by flawlessness."

But the following conclusion is not far off:

(2.4) "There is, in this sense, nothing worse than making mistakes in mathematics. Mistakes are damned."

Mathematics is in such an understanding quite merciless and inexorable. Can it be attractive for the majority of students? Who can be blamed to make no mistakes? If one wants to put this program into practice in the school, he or she must inevitably moderate certain demands in mathematics. Then the following must apply:

(2.5) "Mathematics problems (in school) should be quickly solvable in just a few steps." (FRANK 1988)

Belief 1: "*... The belief in the existence of certain ideal mathematical entities such as the real number system.*"

Such an opinion also affects the classroom and is, to a certain degree, counterproductive. Are there actual, real numbers as solutions of textbook tasks? Naturally this is excluding the very few traditional numbers, namely the square roots of $\sqrt{2}$, π and the Euler number e .

(1.1) "Tasks in textbooks always come out even."

Let me describe a real situation from an university course which the author had attended as a student. A cubic polynomial is given. Of course there is a formula which hardly can be memorized. However, the formula can be look up in a table which should be sufficient for the daily life of a mathematician. I still remember my old physics professor who told us: If I ever give you such a task, look first under the numbers -2, -1, 0, 1, 2, for a solution. That should do it.

I think that it is no better in the school. If the solution provided by the student reads 16,54735, then somewhere something is wrong. Perhaps the mistake was made on the part of the student or there is a typo in the textbook. But such a belief possesses an emotional message:

(1.2) "Formal mathematics has little or nothing to do with real thinking or problem solving." (SCHOENFELD 1985)

and in consequence

(1.3) "School mathematics has nothing to do with real life."

These emotional aspects experiencing a restricted 'version' of mathematics should not be underestimated. Students were invited to visit an exhibition on mathematics at university where they could experience an approach based on material out of the daily life. Further, they were activated to play with and manipulate the objects in the exhibition. Afterwards they were asked to comment on their experience. One student (Linda, Grade 7) wrote:

"Is that really mathematics? ... because it is fun."

And Jennifer (Grade 7) wrote:

"I was pleased to know that one could do everything by himself."

These observations show how students experience or, from another point of view, suffer from mathematics in a sterile or passive classroom. The last student found it noticeable that she could open herself to the world of mathematics by herself. It is only the complaints of the students which we hold to be applicable. From a series of interviews we are able to quote a secondary school teacher. His vision of teaching mathematics regards the planability and the exclusion of unforeseeable conclusions which, in his eyes, only disrupt classroom teaching. He described his views of mathematics using a metaphor in which he envisions himself in the role of a kindergartener and understands the class to be a field trip with the children:

> *"... leads small children by the hand through a garden without leaving the path in order not to detect unexpected things."*

In turn, the question how such a teacher likewise experienced mathematics at the university during his/her education can be permitted.

The above realizations have made clear that beliefs about mathematics are closely connected with those beliefs about teaching and learning mathematics. Here, consistent justifications are always plausibly represented.

4. Where beliefs originate and can have an effect

It has become clear, however, that beliefs are nonobservable theoretical entities. Nevertheless they can be postulated to account for certain observable relations in human behaviour. Up until now there has been a micro-understanding of beliefs. However results are missing which describe and explain a global understanding of beliefs and their impact. In the preceding section we attempted to

determine that many pieces of evidence speak for the fact that individual beliefs are coupled (spaghetti effect) with beliefs of learning and teaching mathematics, etc. If one proceeds one step further, the question of a structuring of beliefs arises: Are these beliefs coupled by the emotional side or are they coupled more by the cognitive side? What role does the technical structure of mathematics play?

Finally, further key questions are to be named:

- How do beliefs originate?
- How can beliefs be changed?

There are, indeed, some results in the didactical literature of mathematics (see TÖRNER & PEHKONEN 1996), but definitive conclusions remain yet to be resolved. The conceptual framework of the TIMSS appears to be helpful as model for use as an approach for further studies. TIMSS differentiates between:

- Intended Curricula[2]
- Implemented Curricula[3]
- Attained Curricula[4]

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[2] The question of who makes curriculum decisions is a fundamental and timeless issue ... The array of participants who officially designated or who function through default to make curriculum decisions is complex enough, but the question centers around not only who makes them, but also what type of curriculum decision is under discussion.

[3] Teachers fulfill a variety of functions regarding the creation and implementation of curriculum materials, their curriculum 'texts' ... The interpretation of curriculum materials allows teachers to express their individual approaches to teaching, as well as their responses to the needs of their specific classroom situation.

[4] There are ... potential learners who will respond to something called a curriculum, a curriculum they will perceive quite differently from the way it was perceived by all those who had something to do with producing or developing it. In its movement from wherever it had its beginnings to where these learners encounter it, this curriculum changed profoundly from whatever it was at the outset. To call it a curriculum is a mistake; it was many curricula, each successive one changing more profoundly than a larva changes in becoming a moth.

and we add a further category:

- Achieved Curricula

This framework was introduced within the process of times (see the definition in the footnotes (ROBITAILLE & AL. 1993)). That which can be said in this research for curricula can also be said to convey the general sense regarding beliefs. In this respect we differentiate

- Intended Beliefs
- Implemented Beliefs
- Attained Beliefs
- Achieved Beliefs

We assume that each curriculum reflects the philosophies behind it within itself (see Ernest 1991). For this to be true, beliefs behind an Intended Curriculum must not necessarily be consistent. It can also be a matter of a well-balanced spectrum of perceptions.

If one understands teaching to be a process of input-output involving some steps, the following diagram (figure 2) should be self-explanatory. The main patterns are the subsequent reductions of the Intended Beliefs to the Implemented Beliefs, Implemented to the Attained Beliefs and so on in the daily lessons.

At these interfaces it may be assumed that specific beliefs are generated and form.

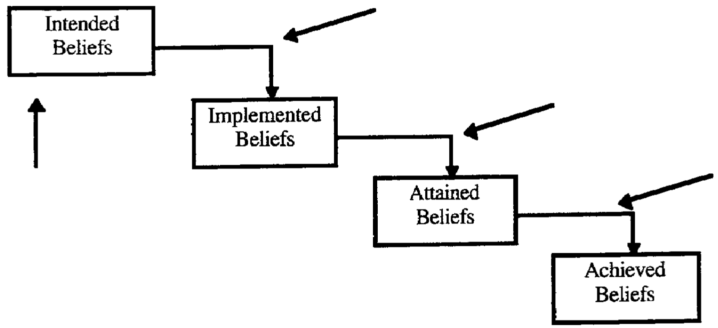

Figure 2. The interrelated dependence of beliefs.

It is unknown how these processes can be influenced. Like in other points in mathematic didactics there are basic, controversial assumptions which confront each other, of which here are two:

- Beliefs can slowly change in that the subject is offered countless information, which do not follow conformity with the presented belief (continuous model).

- But it is also entirely possible that the beliefs drastically change (see, for example, Tobin 1990). It requires only confrontation with a pragmatic situation or a pragmatic fact in order to change a belief (discrete model).

In the second model it remains questionable how coupled beliefs react to one another in such a process of change.

5. Conclusion

The observations up until now have hopefully made clear that attention must be given to beliefs of learning and teaching mathematics. Inherent in beliefs are the following points:

PEHKONEN AND TÖRNER (1996) mentioned in particular four aspects which justify a close investigation of beliefs and belief systems.

- Mathematical beliefs as a regulating system
- Mathematical beliefs as an indicator
- Mathematical beliefs as an inertia force
- Mathematical beliefs as a prognostic tool

What should be gained by this presentation:

- to become aware of beliefs
- to learn to understand how they are generated
- to learn to understand how they have impact on us
- to learn to understand how they can be changed.

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