The Inverse Problem Method on the Half-Line and a Nested System of Riccati Equations

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Abstract

A mixed problem for the nonlinear Bogoyavlenskii system on the half-line is studied by the inverse problem method. The solution of the mixed problem is reduced to the solution of the inverse spectral problem of recovering a forth-order differential operator on the half-line from the Weyl matrix. We derive evolution equations for the elements of the Weyl matrix and give an algorithm for the solution of the mixed problem. Evolution equations of the elements of the Weyl matrix are nonlinear. It is shown that they can be reduced to a nested system of three successively solvable matrix Riccati equations.

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1 Introduction

We study the following nonlinear mixed problem

$$u_t = -u_{xxx} + 6uu_x + 6v_x, \quad v_t = 2v_{xxx} - 6uv_x, \quad x > 0,\ t > 0, \tag{1.1}$$

$$u_{|t=0} = u_0(x), \quad v_{|t=0} = v_0(x), \tag{1.2}$$

$$\frac{\partial^{k-1}u}{\partial x^{k-1}}_{|x=0} = u_k(t), \quad \frac{\partial^{k-1}v}{\partial x^{k-1}}_{|x=0} = v_k(t), \quad k = \overline{1,3}. \tag{1.3}$$

Here $u_k, v_k,\ k = \overline{0,3}$ are continuous complex-valued functions. It was shown by Bogoyavlenskii in [5, 6] and can be verified by elementary calculations that system

(1.1) is equivalent to the Lax equation $\dot{L} = [A, L]$, where

$$Ly := y^{(4)} - 2(uy')' + wy, \quad w := v + u^2 - u'', \tag{1.4}$$

$$Ay := -4y''' + 6uy' + 3u'y,$$

i.e. system (1.1) is completely integrable. Here and below "prime" denotes differentiation with respect to x, and "dot" denotes differentiation with respect to t.

For solving the mixed problem (1.1)-(1.3) we will apply the inverse problem method in which the nonlinear problem (1.1)-(1.3) will be reduced to an inverse spectral problem for the fourth-order linear differential operator (1.4) on the half-line. The inverse problem method was intensively used in many works (see [1, 7, 8, 10, 14, 15, 18, 19] and the references therein). Most of the works are devoted to Cauchy problems on the line and are dealing with the inverse scattering problem on the line. However, mixed problems on the half-line are much more difficult problems, and there is no general theory in this direction. Even the existence of global solutions is an open question. Some aspects of the solution of mixed problems on the half-line for the KdV equation and for the nonlinear Schrödinger equation are reflected in [2, 4, 11, 12, 16, 17] and other works, where spectral problems for Sturm-Liouville and Dirac (or ZS-AKNS) operators are used.

On the other hand, a number of completely integrable nonlinear evolution equations is connected with inverse spectral problems for linear differential equations of order $n > 2$. The inverse problem theory for such operators is more complicated than for Sturm-Liouville and Dirac operators. This produces additional qualitative difficulties. For example, the Boussinesq equation [1] corresponds to spectral problems for the third-order differential operator $\ell y := y''' + uy' + vy$, and system (1.1) corresponds to spectral problems for the forth-order differential operator (1.4). The Cauchy problem for the Boussinesq equation on the line was studied in [9], and the mixed problem for the Boussinesq equation was investigated in [20].

In the present paper we study the mixed problem (1.1)-(1.3) by the inverse problem method and reduce the solution of (1.1)-(1.3) to the solution of the inverse spectral problem of recovering the fourth-order linear differential operator (1.4) on the half-line from the Weyl matrix. We derive evolution equations for the elements of the Weyl matrix and give an algorithm for the solution of problem (1.1)-(1.3). We essentially use the results on the inverse problem of recovering higher-order differential operators from the Weyl matrix from [21, 22]. We note that the existence of the global solution of (1.1)-(1.3) is equivalent to the solvability of the corresponding inverse problem (see Remark 1).

The evolution equations for the elements of the Weyl matrix are nonlinear (see (2.4)). We show that they can be reduced to a chain of three successively solvable matrix Riccati equations of dimensions 3×1, 2×1 and 1×1. Since the solution of the Korteweg-de Vries equation resp. of the Boussinesq equation by the inverse spectral method leads to a scalar Riccati equation (see [13, Sec. 4.2]) resp. to two Riccati equations of dimensions 2×1 and 1×1 in cascade (see [20]), this indicates that one can expect that, in general, nested systems of Riccati equations of dimensions $k \times 1$, $(k-1) \times 1$, $\dots$, 1×1 will appear in the solution of completely integrable systems of partial differential equations that are connected with spectral problems of order k .

2 Evolution of the Weyl matrix

Let $\mathcal{D} = \{(x,t) : \ x \geq 0, t \geq 0\}$. Denote by J the set of functions $f(x,t)$ such that the functions $\frac{\partial^{j+k}}{\partial x^j \partial t^k} f(x,t)$, $0 \leq j + 2k \leq 2$ are continuous in $\mathcal{D}$ and are integrable on the half-line $x \in (0,\infty)$ for each fixed $t > 0$. We shall say that $\{u, v\} \in \Omega$ if $u, v \in J$ and $u^2 \in L(0,\infty)$ with respect to x for each fixed $t \geq 0$. We will seek the solution of problem (1.1)-(1.3) in the class Ω.

Let $\{u, v\}$ be a solution of problem (1.1)-(1.3). For a fixed $t \geq 0$ we consider the differential equation with respect to x :

$$Ly := y^{(4)} - 2(uy')' + wy = \lambda y, \quad x > 0, \tag{2.1}$$

where $w := v + u^2 - u''$. Let $\lambda = \rho^4$. It is known that the ρ - plane can be partitioned into sectors S of angle $\frac{\pi}{4}$ $\left(\arg\rho \in \left(\frac{\nu\pi}{4}, \frac{(\nu+1)\pi}{4}\right),\ \nu = \overline{0,7}\right)$ in which the roots R_1, R_2, R_3, R_4 of the equation $R^4 - 1 = 0$ can be numbered in such a way that

$$Re(\rho R_1) < Re(\rho R_2) < \ldots < Re(\rho R_n), \quad \rho \in S. \tag{2.2}$$

Let functions $\Phi_k(x,t,\lambda)$, $k = \overline{1,4}$ be solutions of equation (2.1) satisfying the conditions $\Phi_k^{(j-1)}(0,t,\lambda) = \delta_{jk}$, $j = \overline{1,k}$ and $\Phi_k(x,t,\lambda) = O(\exp(\rho R_k x))$, $x \to \infty$, $\rho \in S$. Here and in the sequel, δ_{jk} is the Kronecker symbol. Denote $M_{kj}(t,\lambda) = \Phi_k^{(j-1)}(0,t,\lambda)$, $k < j$. The functions Φ_k and M_{kj} are called *the Weyl solutions* and *the Weyl functions*, respectively. The matrix

$$M(t,\lambda) = [M_{kj}(t,\lambda)]_{k,j=\overline{1,4}},$$

where $M_{kj}(t,\lambda) = \delta_{kj}$, $j = \overline{1,k}$, is called *the Weyl matrix* of L (see [21], [22] for details).

Denote by $M^0_{kj}(\lambda)$ the Weyl functions for the operator $L^0 y := y^{(4)} - 2(u_0 y')' + w_0 y$, where $w_0 = v_0 + u_0^2 - u_0''$. In other words, $M^0_{kj}(\lambda) = M_{kj}(0,\lambda)$ is the Weyl functions for $t = 0$. We introduce the matrix $F(t,\lambda) = [F_{kj}(t,\lambda)]_{k,j=\overline{1,4}}$ by the formulas

$$F = \begin{bmatrix} 3u_2 & 6u_1 & 0 & -4 \\ -4\lambda + 4v_1 + 4u_1^2 - u_3 & u_2 & -2u_1 & 0 \\ 4v_2 + 8u_1u_2 - u_4 & -4\lambda + 4v_1 + 4u_1^2 & -u_2 & -2u_1 \\ F_{41} & 8v_2 + 12u_1u_2 - u_4 & -4\lambda + 4v_1 - u_3 & -3u_2 \end{bmatrix}, \tag{2.3}$$

where

$$F_{41} = -2u_1(\lambda - v_1 - u_1^2 - 3u_3) + 4v_3 + 8(u_2^2) - u_5,$$

$$u_4 = -\dot{u}_1 + 6u_1u_2 + 6v_2, \quad u_5 = -\dot{u}_2 + 6u_1u_3 + 6u_2^2 + 6v_3.$$

Theorem 1. *The Weyl functions* $M_{kj}(t,\lambda)$ *satisfy the following evolution equations*

$$\left.\begin{array}{c}\dot{M}_{1j}=(F_{j1}+F_{j2}M_{12}+F_{j3}M_{13}+F_{j4}M_{14})\\ -M_{1j}(F_{11}+F_{12}M_{12}+F_{13}M_{13}+F_{14}M_{14}),\ j=2,3,4,\\ \dot{M}_{2j}=(F_{j2}+F_{j3}M_{23}+F_{j4}M_{24})-M_{2j}(F_{22}+F_{23}M_{23}+F_{24}M_{24})\\ +(-M_{1j}+M_{12}M_{2j})(F_{12}+F_{13}M_{23}+F_{14}M_{24}),\ j=3,4,\\ \dot{M}_{34}=(F_{43}+F_{44}M_{34})-M_{34}(F_{33}+F_{34}M_{34})\\ +(-M_{24}+M_{23}M_{34})(F_{23}+F_{24}M_{34})\\ +(-M_{14}+M_{12}M_{24}+M_{13}M_{34}-M_{12}M_{23}M_{34})(F_{13}+F_{14}M_{34}).\end{array}\right\}\qquad(2.4)$$

Proof. Since $\{u,v\}$ is the solution of problem (1.1)-(1.3), and $L\Phi_k=\lambda\Phi_k$, we get that for fixed t and λ, the functions $\Psi_k:=\dot{\Phi}_k-A\Phi_k$ are solutions of equation (2.1). Moreover, $\Psi_k(x,t,\lambda)=O(\exp(\rho R_k x))$ for $x\to\infty$, $\rho\in S$ in each sector S with property (2.2). Since the functions Φ_k, $k=\overline{1,4}$ form a fundamental system of solutions of equation (2.1) and $\Phi_k(x,t,\lambda)=O(\exp(\rho R_k x))$, $x\to\infty$, $\rho\in S$, one has the representation

$$\Psi_k(x,t,\lambda)=\sum_{j=1}^{k}\alpha_{kj}(t,\lambda)\Phi_j(x,t,\lambda),$$

where the coefficients $\alpha_{kj}(t,\lambda)$ do not depend on x. Hence

$$\left.\begin{array}{c}\dot{\Phi}_1=A\Phi_1+\alpha_{11}\Phi_1,\\ \dot{\Phi}_2=A\Phi_2+\alpha_{21}\Phi_1+\alpha_{22}\Phi_2,\\ \dot{\Phi}_3=A\Phi_3+\alpha_{31}\Phi_1+\alpha_{32}\Phi_2+\alpha_{33}\Phi_3.\end{array}\right\}\qquad(2.5)$$

Taking $x=0$ in (2.5) and using the conditions on the Weyl solutions Φ_k, we calculate α_{kj} :

$$\left.\begin{array}{c}\alpha_{11}=-(A\Phi_1)_{|x=0},\\ \alpha_{21}=-(A\Phi_2)_{|x=0}\,,\quad \alpha_{22}=-(A_1\Phi_2)_{|x=0}-\alpha_{21}(\Phi_1')_{|x=0},\\ \alpha_{31}=-(A\Phi_3)_{|x=0}\,,\quad \alpha_{32}=-(A_1\Phi_3)_{|x=0}-\alpha_{31}(\Phi_1')_{|x=0}\,,\\ \alpha_{33}=-(A_2\Phi_3)_{|x=0}-\alpha_{32}(\Phi_2'')_{|x=0}-\alpha_{31}(\Phi_1'')_{|x=0},\end{array}\right\}\qquad(2.6)$$

where $A_k y:=\frac{d^k}{dx^k}\Big(Ay\Big)$, $k=1,2,3$. Differentiating (2.5) with respect to x, taking $x=0$ and using (2.6), we arrive at (2.4). $\square$

3 A nested system of Riccati equations

The evolution equations (2.4) with respect to M_{kj} are nonlinear. However they can be reduced to a chain of three successively solvable Riccati equations.

We briefly remind Radon's Lemma for the matrix Riccati equation (see, for example, [3, 13]). Let us consider the following Cauchy problem for the Riccati equation:

$$\left.\begin{aligned} \dot Z = Q_{21}(t) + Q_{22}(t)Z - ZQ_{11}(t) - ZQ_{12}(t)Z, \quad t \in [t_0, t_1], \\ Z(t_0) = Z_0, \end{aligned}\right\} \tag{3.1}$$

where $Z(t), Z_0, Q_{11}(t), Q_{12}(t), Q_{21}(t)$ and $Q_{22}(t)$ are complex matrix of dimensions $m \times n, m \times n, n \times n, n \times m, m \times n,$ and $m \times m$ respectively, and $Q_{jk}(t) \in C[t_0, t_1]$.

Let the matrices $(X(t), Y(t),\ t \in [t_0, t_1])$, be the unique solution of the linear Cauchy problem

$$\left.\begin{aligned} \dot X = Q_{11}(t)X + Q_{12}(t)Y, \qquad X(t_0) = I, \\ \dot Y = Q_{21}(t)X + Q_{22}(t)Y, \qquad Y(t_0) = Z_0, \end{aligned}\right\} \tag{3.2}$$

where I is the identity $n \times n$ matrix.

Radon's Lemma. *The Cauchy problem (3.1) has a unique solution* $Z(t) \in C_1[t_0, t_1]$, *if and only if* $\det X(t) \neq 0$ *for all* $t \in [t_0, t_1]$. *Moreover,*

$$Z(t) = Y(t)(X(t))^{-1}. \tag{3.3}$$

We rewrite now the nonlinear system (2.4) with the boundary conditions $M_{kj}(t, \lambda) = M^0_{kj}(\lambda)$ as a chain of Riccati equations.

Step 1. We consider the following Cauchy problem for the Riccati equation:

$$\dot Z_1 = Q^{(1)}_{21} + Q^{(1)}_{22} Z_1 - Z_1 Q^{(1)}_{11} - Z_1 Q^{(1)}_{12} Z_1, \quad Z_1(0, \lambda) = Z^0_1(\lambda), \tag{3.4}$$

where

$$Z_1 = \begin{bmatrix} M_{12} \\ M_{13} \\ M_{14} \end{bmatrix}, \quad Z^0_1 = \begin{bmatrix} M^0_{12} \\ M^0_{13} \\ M^0_{14} \end{bmatrix}, \quad Q^{(1)}_{21} = \begin{bmatrix} F_{21} \\ F_{31} \\ F_{41} \end{bmatrix}, \quad Q^{(1)}_{11} = F_{11},$$

$$Q^{(1)}_{22} = \begin{bmatrix} F_{22} & F_{23} & F_{24} \\ F_{32} & F_{33} & F_{34} \\ F_{42} & F_{43} & F_{44} \end{bmatrix}, \quad Q^{(1)}_{12} = [F_{12}, F_{13}, F_{14}].$$

Then, according to (3.3),

$$Z_1(t, \lambda) = Y_1(t, \lambda)\Big(X_1(t, \lambda)\Big)^{-1}, \tag{3.5}$$

where the pair $\{X_1(t, \lambda), Y_1(t, \lambda)\}$ is the solution of the Cauchy problem for the linear system

$$\left.\begin{aligned} \dot X_1 = Q^{(1)}_{11} X_1 + Q^{(1)}_{12} Y_1, \quad X_1(0, \lambda) = 1, \\ \dot Y_1 = Q^{(1)}_{21} X_1 + Q^{(1)}_{22} Y_1, \quad Y_1(0, \lambda) = Z^0_1(\lambda). \end{aligned}\right\}$$

Step 2. We consider the following Cauchy problem for the Riccati equation:

$$\dot Z_2 = Q^{(2)}_{21} + Q^{(2)}_{22} Z_2 - Z_2 Q^{(2)}_{11} - Z_2 Q^{(2)}_{12} Z_2, \quad Z_2(0, \lambda) = Z^0_2(\lambda), \tag{3.6}$$

where

$$Z_2 = \begin{bmatrix} M_{23} \\ M_{24} \end{bmatrix}, \ Z_2^0 = \begin{bmatrix} M_{23}^0 \\ M_{24}^0 \end{bmatrix}, \ Q_{21}^{(2)} = \begin{bmatrix} F_{32} - F_{12}M_{13} \\ F_{42} - F_{12}M_{14} \end{bmatrix}, \ Q_{11}^{(2)} = F_{22} - F_{12}M_{12},$$

$$Q_{22}^{(2)} = \begin{bmatrix} F_{33} - F_{13}M_{13} & F_{34} - F_{14}M_{13} \\ F_{43} - F_{13}M_{14} & F_{44} - F_{14}M_{14} \end{bmatrix}, \quad Q_{12}^{(2)} = [F_{23} - F_{13}M_{12}, F_{34} - F_{14}M_{12}].$$

Then

$$Z_2(t,\lambda) = Y_2(t,\lambda)\Big(X_2(t,\lambda)\Big)^{-1}, \tag{3.7}$$

where the pair $\{X_2(t,\lambda), Y_2(t,\lambda)\}$ is the solution of the Cauchy problem for the linear system

$$\left.\begin{aligned} \dot X_2 &= Q_{11}^{(2)}X_2 + Q_{12}^{(2)}Y_2, \quad X_2(0,\lambda) = 1, \\ \dot Y_2 &= Q_{21}^{(2)}X_2 + Q_{22}^{(2)}Y_2, \quad Y_2(0,\lambda) = Z_2^0(\lambda). \end{aligned}\right\}$$

Step 3. We consider the following Cauchy problem for the Riccati equation:

$$\dot Z_3 = Q_{21}^{(3)} + Q_{22}^{(3)}Z_3 - Z_3Q_{11}^{(3)} - Z_3Q_{12}^{(3)}Z_3, \quad Z_3(0,\lambda) = Z_3^0(\lambda), \tag{3.8}$$

where

$$Z_3 = M_{34}, \quad Z_3^0 = M_{34}^0,$$

$$Q_{21}^{(3)} = F_{43} - F_{23}M_{24} - (M_{14} - M_{12}M_{24})F_{13}, \quad Q_{11}^{(3)} = 0,$$

$$Q_{22}^{(3)} = F_{44} - F_{33} - F_{24}M_{24} + F_{23}M_{23}$$

$$+(M_{13} - M_{12}M_{23})F_{13} - (M_{14} - M_{12}M_{24})F_{14},$$

$$Q_{12}^{(3)} = F_{34} - F_{24}M_{23} - (M_{13} - M_{12}M_{23})F_{14}.$$

Then

$$Z_3(t,\lambda) = Y_3(t,\lambda)\Big(X_3(t,\lambda)\Big)^{-1}, \tag{3.9}$$

where the pair $\{X_3(t,\lambda), Y_3(t,\lambda)\}$ is the solution of the Cauchy problem for the linear system

$$\left.\begin{aligned} \dot X_3 &= Q_{11}^{(3)}X_3 + Q_{12}^{(3)}Y_3, \quad X_3(0,\lambda) = 1, \\ \dot Y_3 &= Q_{21}^{(3)}X_3 + Q_{22}^{(3)}Y_3, \quad Y_3(0,\lambda) = Z_3^0(\lambda). \end{aligned}\right\}$$

4 Solution of the mixed problem

Using the evolution equations (2.4) and the solution of the inverse problem of recovering L from the Weyl matrix, we get the following algorithm for the solution of the problem (1.1)-(1.3).

Algorithm 1. Let the functions $u_k, v_k,\ k = \overline{0,3}$ be given.
1) Calculate the matrix $F(t,\lambda)$ by (2.3).
2) Construct the Weyl matrix $M^0(\lambda)$ for $t = 0$.
3) Solve the Cauchy problems (3.4), (3.6) and (3.8) by (3.5), (3.7) and (3.9), and find the Weyl matrix $M(t,\lambda)$ for all $t \geq 0$.
4) Construct the functions $u(x,t)$ and $v(x,t)$ by solving the inverse problem from the Weyl matrix (see [21]-[22]).

Remark 1. The solution of the mixed problem (1.1)-(1.3) does not exist for arbitrary functions u_k, v_k appearing in the boundary conditions. For the solvability of (1.1)-(1.3) there must be connections between u_k and v_k. These connections can be described in terms of the Weyl matrix. It can be shown that the existence of the global solution of the mixed problem (1.1)-(1.3) is equivalent to the solvability of the inverse problem (see [23]). In other words, the solution of the mixed problem (1.1)-(1.3) exists if and only if (i) the solutions of the Cauchy problems (3.4), (3.6) and (3.8) exist for $t > 0$; (ii) for each fixed t the matrix $M(t,\lambda)$, constructed by Algorithm 1, is the Weyl matrix for a certain differential operator L of the form (1.4) (i.e. $M(t,\lambda)$ satisfies necessary and sufficient conditions for the solvability of the inverse problem from the Weyl matrix given in [21, 22]).

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