An iterative procedure for the computation of the maximal solution of a class of generalized Riccati differential equations

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Abstract

In this note we propose an algorithm for the computation of the stabilizing (and maximal) solution of a class of stochastic Riccati-type differential equations with time-varying coefficients. The procedure is based on the successive solution of Lyapunov differential equations.

1 Problem formulation

Let $\mathbb{R}^{m\times n}$ be the linear space of real $m \times n$ matrices and $\mathcal{S}^n \subset \mathbb{R}^{n\times n}$ be the linear subspace of symmetric $n \times n$-matrices. In this paper we consider the differential equation

$$\begin{aligned}
&\frac{d}{dt}X(t) + A^T(t)X(t) + X(t)A(t) + M(t) + \Pi_1(t)[X(t)] \\
&\qquad - \big\{X(t)B(t) + \Pi_{12}(t)[X(t)] + L(t)\big\}\big\{R(t) + \Pi_2(t)[X(t)]\big\}^{-1} \\
&\qquad\quad \times\big\{X(t)B(t) + \Pi_{12}(t)[X(t)] + L(t)\big\}^T = 0 \qquad (1.1)
\end{aligned}$$

where A, B, M, L, R are continuous functions defined on an right unbounded interval $\mathcal{I} \subseteq \mathbb{R}$ with values in $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times m}$, $\mathcal{S}^n$, $\mathbb{R}^{n\times m}$, $\mathcal{S}^m$, respectively. We assume furthermore that $\Pi\colon \mathcal{I} \to \mathcal{B}(\mathcal{S}^n, \mathcal{S}^{n+m})$ with

$$\Pi(t)[X] := \begin{pmatrix} \Pi_1(t)[X] & \Pi_{12}(t)[X] \\ \big(\Pi_{12}(t)[X]\big)^T & \Pi_2(t)[X] \end{pmatrix} \qquad (1.2)$$

is a continuous operator valued function where $\mathcal{B}(\mathcal{S}^n, \mathcal{S}^{n+m})$ is the space of linear operators defined on $\mathcal{S}^n$ taking values in $\mathcal{S}^{n+m}$.

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Throughout this paper we assume that $X \geq 0$ implies $\Pi(t)(X) \geq 0$ for all $t \in \mathcal{I}$. The unknown function in (1.1) takes values in $\mathcal{S}^n$. Equation (1.1) contains as particular cases several types of matrix Riccati differential equations involved in many control problems both in deterministic and stochastic framework. For example, if $\Pi(t)[X]$ is the null operator, then (1.1) reduces to the well-known matrix Riccati differential equation intensively investigated starting with Kalman's work in connection with linear-quadratic optimization problem in the deterministic context.

If (1.2) takes the form

$$\Pi(t)[X] = \begin{pmatrix} C(t) & D(t) \end{pmatrix}^T X \begin{pmatrix} C(t) & D(t) \end{pmatrix} \tag{1.3}$$

equation (1.1) occurs in connection with the linear-quadratic optimization problem described by the differential equation

$$dx(t) = (A(t)x(t) + B(t)u(t))\,dt + (C(t)x(t) + D(t)u(t))\,dw(t) \tag{1.4}$$

and the cost functional

$$J(u) := E\left\{x(t_f)^T Q_f x(t_f) + \int_{t_0}^{t_f} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^T \begin{pmatrix} M(t) & L(t) \\ L^T(t) & R(t) \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} dt\right\} \tag{1.5}$$

(see [2, 4, 5] and the references therein).

The problem of existence of global solutions of (1.1) was investigated among others in [2, 3, 7, 9, 10, 11] and references therein. In some of the cited papers equation (1.1) was considered as a differential equation with interest in itself and the proofs do not use the relationship between this equation and some optimization problems.

Among the global solutions of equation (1.1) a crucial role in solving the linear quadratic optimization problem is played by the maximal solution and stabilizing solution. Necessary and sufficient conditions which guarantee the existence of the maximal solution and stabilizing solution for the equation (1.1) were provided in [7] and [11], where it was also shown how this solution can be determined by the Newton-Kantorovich algorithm, whose realization requires the solution of a sequence of linearly perturbed Lyapunov differential equations.

In this paper we provide an alternative iterative procedure which allows us to compute the maximal and stabilizing solution of (1.1). At each iteration of this proposed procedure we need to solve a standard Lyapunov differential equation.

The result presented in this paper may be viewed (in some sense) as an extension of the result of [12] to the time-varying case.

The technique of the proof is an adaptation to the case of equation (1.1) of the proof of the main results in [6, 8] (see also [1, Section 6.7]).

The outline of the paper is: In section 2 we recall some definitions and results from [7] in order to make more clear the proof of the main result which is given in Section 3.

Some comments concerning the realization of the proposed algorithm can be found in Section 4.

2 Some preliminaries

Let

$$\mathcal{D}(\mathcal{R}) := \left\{(t,X) \in \mathcal{I} \times \mathcal{S}^n \mid \det\left\{R(t) + \Pi_2(t)[X]\right\} \neq 0\right\}$$

and define the operator $\mathcal{R}\colon \mathcal{D}(\mathcal{R}) \to \mathcal{S}^n$ by

$$\begin{aligned}\mathcal{R}(t,X) = {} & A^T(t)X + XA(t) + \Pi_1(t)[X] + M(t) \\ & - \left\{XB(t) + \Pi_{12}(t)[X] + L(t)\right\}\left\{R(t) + \Pi_2(t)[X]\right\}^{-1} \\ & \times \left\{XB(t) + \Pi_{12}(t)[X] + L(t)\right\}^T. \end{aligned} \tag{2.1}$$

So equation (1.1) may be written in a compact form:

$$\frac{d}{dt}X(t) + \mathcal{R}(t, X(t)) = 0.$$

As we can see the operator $\mathcal{R}$ and consequently also equation (1.1) are associated to the quadruple $\Sigma = (A, B, \Pi, \mathcal{Q})$, where (A, B, Π) are as before and

$$\mathcal{Q}(t) := \begin{pmatrix} M(t) & L(t) \\ L(t)^T & R(t) \end{pmatrix}.$$

We introduce the so-called generalized dissipation matrix $\lambda^\Sigma\colon C^1(\mathcal{I}, \mathcal{S}^n) \to \mathcal{S}^{n+m}$ associated to the quadruple Σ by

$$\lambda^\Sigma(t,X) = \begin{pmatrix} \lambda_1(t) & XB(t) + \Pi_{12}(t)[X] + L(t) \\ \left\{XB(t) + \Pi_{12}(t)[X] + L(t)\right\}^T & R(t) + \Pi_2(t)[X] \end{pmatrix}$$

where

$$\lambda_1(t) = \frac{d}{dt}X(t) + A^T(t)X(t) + X(t)A(t) + \Pi_1(t)[X(t)] + M(t)$$

and $C^1(\mathcal{I}, \mathcal{S}^n)$ being the space of C^1-functions defined on the interval $\mathcal{I}$ taking values in $\mathcal{S}^n$.

The following two subsets of

$$C_b^1(\mathcal{I}, \mathcal{S}^n) = \left\{X \in C^1(\mathcal{I}, \mathcal{S}^n) \;\middle|\; X, \frac{d}{dt}X \text{ are bounded}\right\}$$

will play an important role in the next developments:

$$\begin{aligned} \Gamma^\Sigma &= \left\{X \in C_b^1(\mathcal{I}, \mathcal{S}^n) \mid \lambda^\Sigma[X(t)] \geq 0,\ R(t) + \Pi_2(t)[X(t)] \gg 0,\ t \in \mathcal{I}\right\}, \\ \tilde{\Gamma}^\Sigma &= \left\{X \in C_b^1(\mathcal{I}, \mathcal{S}^n) \mid \lambda^\Sigma[X(t)] \gg 0,\ t \in \mathcal{I}\right\}. \end{aligned}$$

We see that Γ^Σ contains all solutions $X\colon \mathcal{I} \to \mathcal{S}^n$ of equation (1.1) with the additional property $R(t) + \Pi_2(t)[X(t)] \gg 0$.

As usual, if $H\colon \mathcal{I} \to \mathcal{S}^n$, we shall say that $H(t)$ is uniformly positive and we shall write $H(t) \gg 0$ if there exists $h > 0$ such that $H(t) \geq hI_n$ for all $t \in \mathcal{I}$.

2.1 Definition. A solution $\tilde{X}\colon \mathcal{I} \to \mathcal{S}^n$ of equation (1.1) is said to be the **maximal solution with respect to** Γ^Σ (or **maximal solution** for shortness) if $\tilde{X}(t) \geq \hat{X}(t)$ for arbitrary $\hat{X} \in \Gamma^\Sigma$.

2.2 Definition. A solution $X_s\colon \mathcal{I} \to \mathcal{S}^n$ is called a **stabilizing solution** of equation (1.1) if the zero state equilibrium of the linear differential equation

$$\frac{d}{dt}S(t) = \mathcal{L}_{A+BF_s,\Pi^*_{F_s}}(t)[S(t)]$$

is exponentially stable, where

$$\begin{aligned}\mathcal{L}_{A+BF_s,\Pi^*_{F_s}}(t)[S] = {} & [A(t)+B(t)F_s(t)]S + S[A(t)+B(t)F_s(t)]^T \\ & + \Pi^*_{F_s}(t)[S],\end{aligned} \tag{2.2}$$

$\Pi^*_{F_s}(t)$ is the adjoint operator of $\Pi_{F_s}(t)$ defined by

$$\Pi_{F_s}(t,X) = \begin{pmatrix} I_n \\ F_s(t)\end{pmatrix}^T \Pi(t)(X) \begin{pmatrix} I_n \\ F_s(t)\end{pmatrix}, \tag{2.3}$$

I_n is the identity matrix and

$$F_s(t) := -\big\{R(t) + \Pi_2(t)[X_s(t)]\big\}^{-1}\big\{X_s(t)B(t) + \Pi_{12}(t)[X_s(t)] + L(t)\big\}^T.$$

2.3 Definition. We say that the triple (A, B, Π) is **stabilizable** if there exists a bounded and continuous function $F\colon \mathcal{I} \to \mathbb{R}^{m\times n}$ such that the operator $\mathcal{L}_{A+BF,\Pi^*_F}$ generates an exponentially stable evolution. The function F will be termed **a stabilizing feedback gain**.

$\mathcal{L}_{A+BF,\Pi^*_F}$ is defined as in (2.2) with F instead of F_s. For a better understanding of the concepts of stabilizability and stabilizing solution introduced by the above definitions we remark that if (1.2) takes the particular form of (1.3) then the operator (2.2) becomes

$$\begin{aligned}\mathcal{L}_{A+BF,\Pi^*_F}(t)[S] &= [A(t)+B(t)F(t)]S + S[A(t)+B(t)F(t)]^T \\ &\quad + [C(t)+D(t)F(t)]S[C(t)+D(t)F(t)]^T\end{aligned}$$

which is the Lyapunov-type operator involved in the characterization of the exponential stability in mean square of the solutions of the closed-loop system obtained from (1.4) under state feedback $u(t) = F(t)x(t)$ (see for example [13]).

The next two results were proved in [7].

2.4 Theorem. *Assume that the quadruple Σ is such that A, B, Π, $\mathcal{Q}$ are bounded and (A, B, Π) is stabilizable. Then the following are equivalent:*

(i) Γ^Σ is not empty.

(ii) Equation (1.1) has a bounded solution $X\colon \mathcal{I} \to \mathcal{S}^n$ *which is the maximal solution with respect to* Γ^{Σ}.

Moreover, if A, B, Π, $\mathcal{Q}$ *are periodic functions with period* $\theta > 0$, *then the maximal solution* $\tilde{X}$ *is also a periodic function with period* θ. *If the coefficients of equation (1.1) does not depend upon* t *then the maximal solution* $\tilde{X}$ *is constant and solves the algebraic equation*

$$\begin{aligned} &A^T X + XA + \Pi_1(X) + M - [XB + \Pi_{12}(X) + L] \\ &\quad \times [R + \Pi_2(X)]^{-1}[XB + \Pi_{12}(X) + L]^T = 0. \end{aligned} \tag{2.4}$$

2.5 Theorem. *Assume that* A, B, Π, $\mathcal{Q}$ *are bounded functions. Then the following are equivalent:*

(i) (A, B, Π) *is stabilizable and the set* $\tilde{\Gamma}^{\Sigma}$ *is not empty.*

(ii) Equation (1.1) has a bounded and stabilizing solution $X_s\colon \mathcal{I} \to \mathcal{S}^n$ *which satisfies*

$$R(t) + \Pi_2(t)[X_s(t)] \gg 0 \quad \textit{for} \quad t \in \mathcal{I}.$$

Moreover if A, B, Π, $\mathcal{Q}$ *are periodic functions with period* θ, *then* X_s *is a* θ-*periodic function. If* $A(t) \equiv A$, $B(t) \equiv B$, $\Pi(t) \equiv \Pi$, $\mathcal{Q}(t) \equiv \mathcal{Q}$ *then* X_s *is constant and solves (2.4).*

In the next section we shall provide a procedure which allows us to construct a sequence of iterations converging to the maximal solution $\tilde{X}$ of equation (1.1).

The result of Theorem 3.3 may be viewed as an alternative proof of the implication *(i)* ⇒ *(ii)* of Theorem 2.4 given in [7].

Since in the case, when (i) in Theorem 2.5 is fulfilled, the stabilizing solution coincides with the maximal solution, it follows that the iterative procedure described in Section 3 leads to the stabilizing solution X_s of (1.1).

3 The main result

Throughout the remainder of the paper we assume that A, B, Π, $\mathcal{Q}$ are continuous and bounded functions.

3.1 Lemma. *Assume that* (A, B, Π) *is stabilizable and* $\Gamma^{\Sigma} \neq \emptyset$. *Let* $F_0\colon \mathcal{I} \to \mathbb{R}^{m\times n}$ *be a stabilizing feedback gain. Let* $X_1\colon \mathcal{I} \to \mathcal{S}^n$ *be the bounded solution of the linear differential equation*

$$\begin{aligned} &\frac{d}{dt}X_1(t) + [A(t) + B(t)F_0(t)]^T X_1(t) + X_1(t)[A(t) + B(t)F_0(t)] \\ &\qquad + \Pi_{F_0(t)}(t)[X_1(t)] + M_{F_0}(t) + \varepsilon I_n = 0 \end{aligned} \tag{3.1}$$

where $\varepsilon > 0$ *is given,* $\Pi_{F_0}(t)$ *is defined as in (2.3) and*

$$M_{F_0} = \begin{pmatrix} I_n \\ F_0(t) \end{pmatrix}^T \begin{pmatrix} M(t) & L(t) \\ L^T(t) & R(t) \end{pmatrix} \begin{pmatrix} I_n \\ F_0(t) \end{pmatrix}.$$

Set

$$F_1(t) := -\big\{R(t) + \Pi_2(t)[X_1(t)]\big\}^{-1}\big\{X_1(t)B(t) + \Pi_{12}(t)[X_1(t)] + L(t)\big\}^T.$$

Then:

(i) $X_1(t) - \hat{X}(t) \gg 0$ *for* $t \in \mathcal{I}$ *and arbitrary* $\hat{X} \in \Gamma^{\Sigma}$.

(ii) F_1 *is a stabilizing feedback gain for the triple* (A, B, Π).

Proof. Equation (3.1) may be written in compact form as

$$\frac{d}{dt}X_1(t) + \mathcal{L}^*_{A+BF_0,\Pi^*_{F_0}}(t)[X_1(t)] + M_{F_0}(t) + \varepsilon I_n = 0. \tag{3.2}$$

Since F_0 is a stabilizing feedback it follows that (3.1) admits bounded solutions defined on the whole interval $\mathcal{I}$. On the other hand if $\hat{X} \in \Gamma^{\Sigma}$ then by a Schur complement argument one deduces that $\hat{X}$ solves

$$\frac{d}{dt}\hat{X}(t) + \mathcal{R}(t, \hat{X}(t)) - \hat{M}(t) = 0 \tag{3.3}$$

where $\hat{M}(t) \geq 0$ for all $t \in \mathcal{I}$. Based on Lemma 3.4 in [7] we rewrite (3.3) as

$$\begin{aligned} &\frac{d}{dt}\hat{X}(t) + \mathcal{L}^*_{A+BF_0,\Pi^*_{F_0}}(t)[\hat{X}(t)] + M_{F_0}(t) \\ &\quad - \big[F_0(t) - \hat{F}(t)\big]^T\big\{R(t) + \Pi_2(t)[\hat{X}(t)]\big\}\big[F_0(t) - \hat{F}(t)\big] - \hat{M}(t) = 0, \end{aligned} \tag{3.4}$$

where $\hat{F}(t)$ is defined as $F_1(t)$ with X_1 replaced by $\hat{X}$. Subtracting (3.4) from (3.2) one obtains that

$$\frac{d}{dt}[X_1(t) - \hat{X}(t)] + \mathcal{L}^*_{A+BF_0,\Pi^*_{F_0}}(t)[X_1(t) - \hat{X}(t)] \ll 0.$$

Hence by Theorem 2.13 (b) in [7] we have $X_1(t) - \hat{X}(t) \gg 0$ for $t \in \mathcal{I}$.

From (i) it follows that $F_1(t)$ is well defined. Applying again [7, Lemma 3.4] we deduce that (3.3) may be written as

$$\begin{aligned} &\frac{d}{dt}\hat{X}(t) + \mathcal{L}^*_{A+BF_1,\Pi^*_{F_1}}(t)[\hat{X}(t)] + M_{F_1}(t) \\ &\quad - \big[F_1(t) - \hat{F}(t)\big]^T\big\{R(t) + \Pi_2(t)[\hat{X}(t)]\big\}\big[F_1(t) - \hat{F}(t)\big] - \hat{M}(t) = 0. \end{aligned} \tag{3.5}$$

On the other hand by direct calculation one obtains that (3.2) is equivalent to

$$\begin{aligned} &\frac{d}{dt}X_1(t) + \mathcal{L}^*_{A+BF_1,\Pi^*_{F_1}}(t)[X_1(t)] + M_{F_1}(t) + \varepsilon I_n \\ &\quad + \big[F_1(t) - F_0(t)\big]^T\big\{R(t) + \Pi_2(t)[X_1(t)]\big\}\big[F_1(t) - F_0(t)\big] = 0. \end{aligned} \tag{3.6}$$

From (3.5) and (3.6) we deduce that $X_1(t) - \hat{X}(t)$ is a bounded and uniformly positive solution of the linear differential inequality

$$\frac{d}{dt}Y(t) + \mathcal{L}^*_{A+BF_1,\Pi^*_{F_1}}(t)[Y(t)] \ll 0.$$

Involving the implication $(vii) \to (i)$ of Theorem 2.11 in [7] we deduce that the operator $\mathcal{L}_{A+BF_1,\Pi^*_{F_1}}$ generates an exponentially stable evolution on $\mathcal{S}^n$, which is equivalent to the fact that F_1 is a stabilizing feedback gain. □

3.2 Remark. Combining Lemma 3.1 (ii) with Corollary 2.12 in [7] we deduce that the zero state equilibrium of the linear differential equation

$$\frac{d}{dt}x(t) = [A(t) + B(t)F_1(t)]x(t)$$

is exponentially stable.

Starting with $X_1(t)$, $F_1(t)$ we construct iteratively sequences $\{X_k(t)\}_{k\geq 1}$ and $\{F_k(t)\}_{k\geq 1}$ as follows:

Let X_k be the unique bounded solution of the Lyapunov differential equation

$$\begin{aligned}\frac{d}{dt}X_k(t) &+ [A(t) + B(t)F_{k-1}(t)]^T X_k(t) + X_k(t)[A(t) + B(t)F_{k-1}(t)] \\ &+ M_k(t) = 0 \end{aligned} \tag{3.7}$$

where for $k \geq 2$

$$F_k(t) := -\big\{R(t) + \Pi_2(t)[X_{k-1}(t)]\big\}^{-1}\big\{X_k(t)B(t) + \Pi_{12}(t)[X_{k-1}(t)] + L(t)\big\}^T$$

and

$$M_k(t) = M_{F_{k-1}}(t) + \Pi_{F_{k-1}}(t)[X_{k-1}(t)] + \frac{\varepsilon}{k}I_n.$$

Now we prove:

3.3 Theorem. *Assume that (A, B, Π) is stabilizable and Γ^Σ is not empty. Then for arbitrary stabilizing feedback gain $F_0(t)$ and $\varepsilon > 0$ fixed, the sequence $\{X_k(t)\}_{k\geq 1}$ defined by (3.7), with $X_1(t)$ the bounded solution of (3.1), is convergent.*

If we denote

$$\tilde{X}(t) := \lim_{k\to\infty} X_k(t)$$

then $\tilde{X}(t)$ is the maximal solution of equation (1.1). Moreover if (A, B, Π) is stabilizable and $\widetilde{\Gamma}^\Sigma$ is not empty, then $\tilde{X}$ is just the stabilizing solution of (1.1).

Proof. We prove inductively the following items:

a_k) $X_k(t) - \hat{X}(t) \geq \mu_k I_n$ for every $\hat{X} \in \Gamma^{\Sigma}$ where $\mu_k > 0$ is independent of $\hat{X}$.

b_k) the zero state equilibrium of the linear differential equation

$$\frac{d}{dt}x(t) = [A(t) + B(t)F_k(t)]x(t)$$

is exponentially stable.

c_k) $X_k(t) \gg X_{k+1}(t)$ for $t \in \mathcal{I}$.

We remark that a_1) and b_1) follow from Lemma 3.1 and Remark 3.2. To prove c_1) we subtract equation (3.7), written for $k = 2$, from (3.6) and obtain

$$\begin{aligned}&\frac{d}{dt}\big[X_1(t) - X_2(t)\big] + \big[A(t) + B(t)F_1(t)\big]^T\big[X_1(t) - X_2(t)\big]\\ &\qquad + \big[X_1(t) - X_2(t)\big]\big[A(t) + B(t)F_1(t)\big] + \Delta_1(t) = 0,\end{aligned}$$

where

$$\Delta_1(t) := \frac{\varepsilon}{2}I_n + \big[F_1(t) - F_0(t)\big]^T\big\{R(t) + \Pi_2(t)[X_1(t)]\big\}\big[F_1(t) - F_0(t)\big] \gg 0.$$

Since b_1) holds, it follows that $X_1(t) - X_2(t) \gg 0$ and thus c_1) is fulfilled.

Let us assume that $a_i), b_i), c_i)$ hold for $1 \leq i \leq k-1$. To check that a_k) is fulfilled we rewrite equation (3.3) in the form

$$\begin{aligned}&\frac{d}{dt}\hat{X}(t) + \big[A(t) + B(t)F_{k-1}(t)\big]^T\hat{X}(t) + \hat{X}(t)\big[A(t) + B(t)F_{k-1}(t)\big]\\ &\quad - \big[F_{k-1}(t) - \hat{F}(t)\big]^T\big\{R(t) + \Pi_2(t)[\hat{X}(t)]\big\}\big[F_{k-1}(t) - \hat{F}(t)\big]\\ &\quad + \Pi_{F_{k-1}}[\hat{X}(t)] + M_{F_{k-1}}(t) - \hat{M}(t) = 0.\end{aligned} \tag{3.8}$$

Subtracting (3.8) from (3.7) we conclude that $X_k - \hat{X}$ is the bounded solution of the standard Lyapunov equation

$$\frac{d}{dt}Z(t) + [A(t) + B(t)F_{k-1}(t)]^T Z(t) + Z(t)[A(t) + B(t)F_{k-1}(t)] + \tilde{\Delta}_k(t) = 0$$

where

$$\begin{aligned}\tilde{\Delta}_k(t) \;&:=\; \frac{\varepsilon}{k}I_n + \Pi_{F_{k-1}}(t)[X_{k-1}(t) - \hat{X}(t)] + \hat{M}(t)\\ &\quad + \big[F_{k-1}(t) - \hat{F}(t)\big]^T\big\{R(t) + \Pi_2(t)[\hat{X}(t)]\big\}\big[F_{k-1}(t) - \hat{F}(t)\big].\end{aligned}$$

Since a_{k-1}) is verified it follows that $\tilde{\Delta}_k(t) \geq \frac{\varepsilon}{k}I_n$. This allows us to write

$$X_k(t) - \hat{X}(t) \geq \frac{\varepsilon}{k}\int_t^{\infty} \Phi_{k-1}^T(s,t)\Phi_{k-1}(s,t)\,ds$$

where $\Phi_{k-1}(s,t)$ is the fundamental matrix solution of the linear differential equation

$$\frac{d}{dt}x(t) = [A(t) + B(t)F_{k-1}(t)]x(t). \tag{3.9}$$

Since $b_{k-1})$ is fulfilled we obtain that

$$X_k(t) - \hat{X}(t) \geq \frac{\varepsilon\gamma}{k} I_n \tag{3.10}$$

for some $\gamma > 0$ independent of t and thus $a_k)$ is true.

From $a_{k-1})$ it follows that F_k is well defined. To check that $b_k)$ holds we write (3.7) and (3.3) as:

$$\begin{aligned}
&\frac{d}{dt}X_k(t) + \big[A(t) + B(t)F_k(t)\big]^T X_k(t) + X_k(t)\big[A(t) + B(t)F_k(t)\big] \\
&\quad + \big[F_k(t) - F_{k-1}(t)\big]^T \big\{R(t) + \Pi_2(t)[X_{k-1}(t)]\big\} \big[F_k(t) - F_{k-1}(t)\big] \\
&\quad + \Pi_{F_k}(t)[X_{k-1}(t)] + M_{F_k}(t) + \frac{\varepsilon}{k} I_n = 0
\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
&\frac{d}{dt}\hat{X}(t) + \big[A(t) + B(t)F_k(t)\big]^T \hat{X}(t) + \hat{X}(t)\big[A(t) + B(t)F_k(t)\big] \\
&\quad - \big[F_k(t) - \hat{F}(t)\big]^T \big\{R(t) + \Pi_2(t)[\hat{X}(t)]\big\} \big[F_k(t) - \hat{F}(t)\big] \\
&\quad + \Pi_{F_k}(t)[\hat{X}(t)] + M_{F_k}(t) - \hat{M}(t) = 0.
\end{aligned}$$

From these last two equations we deduce that $X_k - \hat{X}$ is the bounded uniformly positive solution of the Lyapunov equation

$$\frac{d}{dt}Y(t) + [A(t) + B(t)F_k(t)]^T Y(t) + Y(t)[A(t) + B(t)F_k(t)] + G_k(t) = 0 \tag{3.12}$$

where

$$\begin{aligned}
G_k(t) = &\frac{\varepsilon}{k} I_n + \hat{M}(t) + \Pi_{F_k}(t)[X_{k-1}(t) - \hat{X}(t)] \\
&+ \big[F_k(t) - F_{k-1}(t)\big]^T \big\{R(t) + \Pi_2(t)[X_{k-1}(t)]\big\} \big[F_k(t) - F_{k-1}(t)\big] \\
&+ \big[F_k(t) - \hat{F}(t)\big]^T \big\{R(t) + \Pi_2(t)[\hat{X}(t)]\big\} \big[F_k(t) - \hat{F}(t)\big] \geq \frac{\varepsilon}{k} I_n.
\end{aligned}$$

From (3.12) and (3.10) we infer that $b_k)$ holds.

It remains to prove that $c_k)$ holds. To this end we subtract equation (3.7), written for $k+1$ instead of k, from equation (3.11) and get

$$\begin{aligned}
&\frac{d}{dt}\big[X_k(t) - X_{k+1}(t)\big] + \big[A(t) + B(t)F_k(t)\big]^T \big[X_k(t) - X_{k+1}(t)\big] \\
&\quad + \big[X_k(t) - X_{k+1}(t)\big]\big[A(t) + B(t)F_k(t)\big] + \Delta_k(t) = 0,
\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}
\Delta_k(t) := &\big[F_k(t) - F_{k-1}(t)\big]^T \big\{R(t) + \Pi_2(t)[X_{k-1}(t)]\big\} \big[F_k(t) - F_{k-1}(t)\big] \\
&+ \frac{\varepsilon}{k(k+1)} I_n + \Pi_{F_k}(t)[X_{k-1}(t) - X_k(t)].
\end{aligned}$$

Since $A + BF_k$ generates an exponentially stable evolution and since c_{k-1} holds one obtains via [7, Theorem 2.13, (b)], that equation (3.13) has a unique bounded solution which additionally is uniformly positive. Therefore $X_k(t) - X_{k+1}(t) \gg 0$ and thus c_k) holds.

Further, from a_k) and c_k) we deduce that the sequence $\{X_k(t)\}_{k\geq 1}$ is convergent. Let

$$\tilde{X}(t) = \lim_{k\to\infty} X_k(t) \quad \text{for} \quad t \in \mathcal{I}. \tag{3.14}$$

In a standard way, one obtains that $\tilde{X}(t)$ is a solution of the equation (1.1). Combining (3.14) with a_k) we conclude that $\tilde{X}(t)$ is just the maximal solution of (1.1) and thus the proof ends. □

4 Comments on the procedure

As it follows from (3.7) to compute the k-th approximant X_k, $k \geq 2$ we need to find the unique bounded solution of a standard Lyapunov differential equation. However to find the first approximation $X_1(t)$ we need to solve the linear matrix differential equation (3.1), which is a perturbed Lyapunov equation. Obviously we can use instead of this solution any bounded solution of the corresponding differential inequality

$$\begin{aligned} &\frac{dX_1(t)}{dt} + [A(t) + B(t)F_0(t)]^T X_1(t) + X_1(t)[A(t) + B(t)F_0(t)] \\ &\qquad + \Pi_{F_0(t)}(t)[X_1(t)] + M_{F_0}(t) + \varepsilon I_n \leq 0 \end{aligned} \tag{4.1}$$

The next result shows how we may construct iteratively a bounded solution of (3.1).

4.1 Proposition. *Assume that (A, B, Π) is stabilizable and let F_0 be a stabilizing feedback gain. Let $\varepsilon > 0$ be such that*

$$M_{F_0}(t) + 2\varepsilon I_n \geq 0. \tag{4.2}$$

Let $\{Y_k\}_{k\geq 1}$ be the sequence defined by $Y_1 \equiv 0$ and

$$\frac{d}{dt}Y_k(t) + [A(t) + B(t)F_0(t)]^T Y_k(t) + Y_k(t)[A(t) + B(t)F_0(t)] + P_k(t) = 0 \tag{4.3}$$

for $k \geq 2$ where $P_k(t) = \Pi_{F_0}(t)[Y_{k-1}(t)] + M_{F_0}(t) + 2\varepsilon I_n$. Then $\{Y_k(t)\}_{k\geq 1}$ is an increasing sequence for every $t \in \mathcal{I}$ with $\lim_{k\to\infty} Y_k(t) = X_0(t)$ where $X_0(t)$ is the unique bounded solution of

$$\begin{aligned} &\frac{d}{dt}X_0(t) + [A(t) + B(t)F_0(t)]^T X_0(t) + X_0(t)[A(t) + B(t)F_0(t)] \\ &\qquad + \Pi_{F_0}(t)[X_0(t)] + M_{F_0}(t) + 2\varepsilon I_n = 0. \end{aligned} \tag{4.4}$$

Moreover if $k_0 \geq 1$ is such that

$$\lambda_{max}[\Pi_{F_0}(t)[Y_{k_0}(t) - Y_{k_0-1}(t)]] \leq \varepsilon \tag{4.5}$$

then $Y_{k_0}(t)$ is a bounded solution of (4.1).

Proof. It is easy to see that $Y_{k+1} - Y_k$ satisfies

$$\frac{d}{dt}[Y_{k+1}(t) - Y_k(t)] + [A(t) + B(t)F_0(t)]^T[Y_{k+1}(t) - Y_k(t)] \\ + [Y_{k+1}(t) - Y_k(t)][A(t) + B(t)F_0(t)] + \Pi_{F_0}(t)[Y_k(t) - Y_{k-1}(t)] = 0$$

for $k \geq 2$ and we deduce inductively that $Y_{k+1}(t) - Y_k(t) \geq 0$. To this end we take into account that $A(t) + B(t)F_0(t)$ generates an exponentially stable evolution and $Y_2(t) \geq 0$ because of (4.2) and (4.3). On the other hand $t \to X_0(t) - Y_k(t)$ solves

$$\frac{d}{dt}[X_0(t) - Y_k(t)] + [A(t) + B(t)F_0(t)]^T[X_0(t) - Y_k(t)] \\ + [X_0(t) - Y_k(t)][A(t) + B(t)F_0(t)] + \Pi_{F_0}(t)[X_0(t) - Y_{k-1}(t)] = 0$$

and inductively we obtain that $X_0(t) - Y_k(t) \geq 0$ for all $k \geq 1$. Therefore the sequence $\{Y_k(t)\}_{k\geq 1}$ is convergent. If $\tilde{Y}(t) = \lim_{k\to\infty} Y_k(t)$ then $\tilde{Y}(t)$ is the bounded solution of (4.4). From the uniqueness of the bounded solution of (4.4) we conclude that $\tilde{Y}(t) = X_0(t)$.

Finally equation (4.3) may be rewritten:

$$\frac{d}{dt}Y_k(t) + [A(t) + B(t)F_0(t)]^T Y_k(t) + Y_k(t)[A(t) + B(t)F_0(t)] \\ + \Pi_{F_0}(t)[Y_k(t)] + M_{F_0}(t) + \varepsilon I_n + (\Pi_{F_0}(t)[Y_{k-1}(t) - Y_k(t)] + \varepsilon I_n) = 0.$$

It is obvious that if k_0 and ε are such that (4.5) is fulfilled then $Y_{k_0}(t)$ solves (4.1) and thus the proof ends. □

4.2 Remark. The stabilizability of the triple (A, B, Π) is required in order to guarantee the boundedness of the sequence $\{Y_k\}$ constructed by (4.3). One may see that the sequence (4.3) may be constructed even if F_0 is such that $A(t) + B(t)F_0(t)$ generates an exponentially stable evolution (which is less then the stabilizability of (A, B, Π)).

In this paper we have studied the time-varying case in order to cover the periodic case. In this case all iterations are computed only on the compact interval $[0, \theta]$. Let us assume that A, B, Π, M, L, R are periodic functions with period $\theta > 0$. If (A, B, Π) is stabilizable then based on Corollary 5.11 (i) in [7] it follows that we may choose a stabilizing feedback gain which is periodic function with period θ. Therefore to compute the first term of the sequence of approximation of Y_k we may use a stabilizing feedback gain which is periodic with period θ. Choose ε such that (4.2) is satisfied for $t \in [0, \theta]$. From the uniqueness of the bounded solution of the Lyapunov equation we deduced that Y_k provided by (4.3) is a periodic function with period θ for which we have the representation

$$Y_k(t) = \Phi_0^T(\theta, t)Y_k(\theta)\Phi_0(\theta, t) + \int_t^\theta \Phi_0^T(s, t)P_k(s)\Phi_0(s, t)\, ds \quad \text{for} \quad t \in [0, \theta[\tag{4.6}$$

where $\Phi_0(s, t)$ is the fundamental matrix solution of the equation

$$\frac{d}{dt}x(t) = [A(t) + B(t)F_0(t)]x(t). \tag{4.7}$$

The initial value $Y_k(0)$ is obtained as a solution of the following Stein equation

$$Y_k(0) = \Phi_0^T(\theta, 0) Y_k(0) \Phi_0(\theta, 0) + \Gamma_k$$

where

$$\Gamma_k = \int_0^\theta \Phi_0^T(s, 0) P_k(s) \Phi_0(s, 0)\, ds.$$

Condition (4.5) to determine k_0 and to stop the iterative procedure (4.3) for obtaining X_1 must be checked along the interval $[0, \theta]$.

The initial values at $t = 0$ of the approximations $X_k(t)$, $k \geq 2$ are obtained as solutions of the following Stein equations

$$X_k(0) = \Phi_{k-1}^T(\theta, 0) X_k \Phi_{k-1}(\theta, 0) + W_{k-1} \tag{4.8}$$

where

$$W_{k-1} = \int_0^\theta \Phi_{k-1}^T(s, 0) M_k(s) \Phi_{k-1}(s, 0)\, ds,$$

$\Phi_{k-1}(s, t)$ being the fundamental matrix solution of the linear differential equation

$$\frac{d}{dt} x(t) = [A(t) + B(t) F_{k-1}(t)] x(t), \quad k \geq 2. \tag{4.9}$$

Finally we remark that if $A(t) \equiv A$, $B(t) \equiv B$, $M(t) \equiv M$, $L(t) \equiv L$, $R(t) \equiv R$, $\Pi(t) \equiv \Pi$ and if (A, B, Π) is stabilizable then from Corollary 5.11 (ii) or [7] one deduces that there exists a stabilizing feedback gain which is constant.

Under these conditions both the bounded solutions of the equations (3.7) and (4.6) are constant. Therefore in the time-invariant case we have to solve some standard algebraic Lyapunov equations to obtain the iterations Y_k and X_k.

4.3 Remark. From the representation formulae (4.6), (4.8) it follows that to obtain the iterations based on standard Lyapunov equations we need to solve linear differential equations on a linear space of dimension n (see (4.7), (4.9)). In the case of iterative procedures based on perturbed Lyapunov equations which are used in Newton-Kantorovich algorithm, we need to solve linear differential equations on a linear space of dimension $n(n+1)/2$.

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